A bijection between fractional trees and $d$-angulations

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LIX – CNRS

Young workshop in arithmetics and combinatorics – June, 22th 2011
Definition of planar maps

- **Planar map** = planar connected graph embedded properly in the sphere up to a direct homomorphism of the sphere.
- **Rooted planar map** = an oriented edge is marked.
- **with a planar embedding** = the “outer face” is chosen.
Triangulations, quadrangulations, ... 

Faces = connected components of the plane without the edges of the map. Triangulation, quadrangulation, pentagulation, $d$-angulation, ... = map whose faces are all of degree 3, 4, 5, $d$, ... 

Girth = length of the shortest cycle.

From now on, only $d$-angulations of girth $d$. 
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One of the main question when studying some families of maps:

How many maps belong to this family?

- Tutte ’60s: recursive decomposition
- Matrix integrals: t’Hooft ’74, Brézin, Itzykson, Parisi and Zuber ’78,
- Representation of the symmetric group: Goulden and Jackson ’87,
- Bijective approach with labeled trees: Cori-Vauquelin ’81, Schaeffer ’98, Bouttier, Di Francesco and Guitter ’04, Bernardi, Chapuy, Fusy, Miermont, …
- Bijective approach with blossoming trees: Schaeffer ’98, Schaeffer and Bousquet-Mélou ’00, Poulalhon and Schaeffer ’05, Fusy, Poulalhon and Schaeffer ’06.
Rooted simple triangulations

The number of rooted simple triangulations with $2n$ faces, $3n$ edges and $n + 2$ vertices is equal to:

$$\frac{2(4n - 3)!}{n!(3n - 1)!} = \frac{1}{n} \cdot \frac{2}{(4n - 2)} \binom{4n - 2}{n - 1}.$$

Blossoming tree = rooted plane tree where each node (= inner vertex) carries exactly two leaves.

Theorem (Poulalhon and Schaeffer ‘05)

There exists a one-to-one correspondence between the set of balanced plane trees with $n$ nodes and two leaves adjacent to each node, and the set of rooted simple triangulations of size $n$. 
Closure of a blossoming tree

Root of the tree is not involved in the local closure $\Rightarrow$ the tree is balanced.

$n$ trees correspond to the same rooted triangulation.
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How to describe the inverse construction? with orientations.
Orientations

**Orientation** of a planar map = an orientation is given to each edge

We want to consider orientations where the outdegree of each vertex is prescribed
→ general theory of α-orientation (Felsner).

For triangulations:

$$3\text{-orientation} =
\begin{cases}
\text{out}(v) = 3 & \text{for each } v \text{ not in the root face} \\
\text{out}(v) = 0 & \text{otherwise}
\end{cases}$$

**Theorem (Schnyder '89, Felsner '04)**

Each rooted triangulation of girth 3 admits a unique minimal 3-orientation, ie. a 3-orientation without counterclockwise cycle.
Moreover there exists a directed path from any vertices to the root face : the orientation is accessible.
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And for $d$-angulations?

$k$-fractional orientation = orientation of the expended map where each edge is replaced by $k$ copies.

$$j/k$$-orientation =

$$\begin{align*}
\text{out}(v) &= j & \text{for each } v \text{ not in the root face} \\
\text{out}(v) &= k & \text{otherwise}
\end{align*}$$

Theorem (Bernardi and Fusy ’11)

Any rooted $d$-angulation of girth $d$ admits a unique minimal $\frac{d}{d-2}$-orientation such that the root face is a clockwise cycle. Moreover this orientation is accessible.
**d-fractional trees**

*d-fractional tree* = rooted plane tree where each edge carries a flow (possibly in two directions) such that:

- sum of the flows in the edge = \( d - 2 \),
- for each node \( u \), \( \text{out}(u) = d \),
- for each leaf \( l \), \( \text{out}(l) = 0 \),
- there exists a directed path from each node to the root.

\[\Rightarrow\] Trees not stable by rerooting, do not lead to nice combinatorial equalities.

\[\Rightarrow\] Cyclic closure operation

*d-fractional forest* = simple rooted cycle of length \( d \), on which are grafted *d-fractional trees.*
**$d$-fractional trees**

A **$d$-fractional tree** is a rooted plane tree where each edge carries a flow (possibly in two directions) such that:

- The sum of the flows in the edge is $d - 2$.
- For each node $u$, $\text{out}(u) = d$.
- For each leaf $l$, $\text{out}(l) = 0$.
- There exists a directed path from each node to the root.

Trees not stable by rerooting do not lead to nice combinatorial equalities.

**Cyclic closure operation**

A **$d$-fractional forest** is a simple rooted cycle of length $d$, on which are grafted $d$-fractional trees.
Closure of a $d$-fractional forest

**Theorem**

There exists a one-to-one constructive correspondence between $d$-fractional forests with $n$ nodes and rooted $d$-angulations of girth $d$ with $n$ vertices.
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Proof of the theorem

- Induction on the number of faces of $M$.
- There exists a saturated clockwise edge $e$ on the outer face:
  1. if $M \setminus e$ is still accessible: delete $e$.
  2. otherwise, there exists such a partition:
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Generalization

“Theoretical proof” in quadratic time: relying on it, we can give a direct method to identify the closure edges.

⇒ Opening algorithm in linear time.

- Method generalizes directly to $p$-gonal $d$-angulations (ie. map with faces of degree $d$ but root face of degree $p$).
- Enumerative consequences: recursive decomposition of the $d$-fractional trees
  ⇒ Equations for the generating series of $d$-angulations.

General framework to obtain a bijection between maps endowed with a minimal accessible orientation and blossoming trees.

⇒ Yield enumerative results when the blossoming trees can be enumerated.
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That’s all . . . Thank you!