Géométrie des clusters de spins dans les triangulations munies d’un modèle d’Ising

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Maps – Definition(s)

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To avoid dealing with symmetries: maps are **rooted** (an edge is marked and oriented).

\[ \mathcal{M} = \text{set of rooted planar maps} \]
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Planar map = planar graph + cyclic order of edges around each vertex.

To avoid dealing with symmetries: maps are rooted (an edge is marked and oriented).

A map $M$ defines a discrete metric space:

- points: set of vertices of $M = V(M)$.
- distance: graph distance $= d_{gr}$. 
A **triangulation** is a planar map in which all faces have degree 3.

Triangulation of size $n$ has $3n$ edges

(or equivalently $n + 2$ vertices, $2n$ faces).
Triangulations

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(or equivalently $n + 2$ vertices, $2n$ faces).

A **triangulation with a boundary** is a planar map in which all faces have degree 3, except possibly the root face.

Triangulation with a boundary of length 4.
Triangulations

A triangulation is a planar map in which all faces have degree 3.

Triangulation of size $n$ has $3n$ edges
(or equivalently $n + 2$ vertices, $2n$ faces).

What does a random triangulation of size $n$ look like (as $n$ tends to $\infty$)?
Triangulations

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What does a **random triangulation** of size $n$ look like (as $n$ tends to $\infty$)?

Simulation by I.Kortchemski

Today: local limit point of view

Scaling limit point of view
Local topology ($\sim$ Benjamini–Schramm convergence)

For $m$ a rooted planar map and $R \in \mathbb{N}^*$,

$$B_R(m) = \text{ball of radius } R \text{ around the root vertex of } m$$

**Definition:**
The local topology on $\mathcal{G}$ is induced by the distance:

$$d_{loc}(m, m') := \frac{1}{1 + \max\{R \geq 0 : B_R(m) = B_R(m')\}}$$

For all fixed $R$, there exists $n_0$ s.t.:

$$B_R(m_n) = B_R(m) \quad \text{for } n \geq n_0$$

**First examples:**

```
\begin{tikzpicture}
  \node[vertex] (0) at (0,0) {0};
  \node[vertex] (1) at (1,0) {1};
  \node[vertex] (2) at (2,0) {2};
  \node[vertex] (n) at (3,0) {n};
  \draw (0) -- (1); \draw (1) -- (2); \draw (2) -- (n);
\end{tikzpicture}
```

Root = 0
Local topology (~ Benjamini–Schramm convergence)

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B_R(m) = \text{ball of radius } R \text{ around the root vertex of } m
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First examples:

\[
\bullet \bullet \bullet \rightarrow (\mathbb{Z}_+, 0)
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\( m_n \rightarrow m \) for the local topology

\( \iff \)

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First examples:

\[ \begin{align*}
0 & \quad 1 & \quad 2 & \quad n \\
\end{align*} \quad \rightarrow (\mathbb{Z}_+, 0) \]

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\[ \begin{align*}
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Uniformly chosen root

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\begin{array}{cccc}
0 & 1 & 2 & n \\
\end{array} \quad \rightarrow \quad (\mathbb{Z}_+, 0)
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**Definition:**

The local topology on $\mathcal{G}$ is induced by the distance:

$$d_{\text{loc}}(m, m') := \frac{1}{1 + \max\{R \geq 0 : B_R(m) = B_R(m')\}}$$

$$m_n \to m \quad \text{for the local topology}$$

$$\iff$$

For all fixed $R$, there exists $n_0$ s.t.:

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**First examples:**

- Root = 0
  - $m \to (\mathbb{Z}_+, 0)$

- Uniformly chosen root
  - $m \to (\mathbb{Z}, 0)$

- Root does not matter
  - $m \to (\mathbb{Z}, 0)$
Local convergence: more complicated examples

$$\mu_n = \text{uniform measure on plane trees with } n \text{ vertices:}$$

$$\mu_4 = \frac{1}{5}$$

1/5  1/5  1/5  1/5  1/5
Local convergence: more complicated examples

\( \mu_n = \) uniform measure on plane trees with \( n \) vertices:

\[ \mu_4 = \]

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\[ n = 500 \]

\[ n = 1000 \]
Local convergence: more complicated examples

\( \mu_n \) = uniform measure on plane trees with \( n \) vertices:

\( \mu_4 = \)

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The limit is a probability distribution on infinite trees with one infinite branch [Kesten].

n = 500

n = 1000
Theorem [Angel – Schramm, ’03]

Let $\mathbb{P}_n = \text{uniform distribution on triangulations of size } n$.

$\mathbb{P}_n \xrightarrow{(d)} \text{UIPT, for the local topology}$

$\text{UIPT} = \text{Uniform Infinite Planar Triangulation}$

$= \text{measure supported on infinite planar triangulations.}$
Local limit of large uniformly random triangulations

**Theorem** [Angel – Schramm, ’03]

Let $\mathbb{P}_n$ = uniform distribution on triangulations of size $n$.

$$\mathbb{P}_n \xrightarrow{(d)} \text{UIPT}, \quad \text{for the local topology}$$

UIPT = Uniform Infinite Planar Triangulation
= measure supported on infinite planar triangulations.

Some properties of the UIPT:

- The UIPT has almost surely one end [Angel – Schramm, 03]
- Volume (nb. of vertices) and perimeters of balls known to some extent.
  $$\mathbb{E} [|B_R(T_\infty)|] \sim \frac{2}{7} R^4$$  [Angel 04, Curien – Le Gall 12]
- The simple random walk is recurrent [Gurel-Gurevich + Nachmias, 13]

**Universality**: we expect the same behavior for other “reasonable” models of maps.

In particular, we expect the volume growth to be 4.

(proved for quadrangulations [Krikun 05], simple triangulations [Angel 04])
Intermezzo: why should we care about local limits?

Suppose that a sequence of random graphs $G_n$ admits a local weak limit $G_\infty$, then,

\[ f(G_n) \overset{\text{proba}}{\longrightarrow} f(G_\infty) \quad \text{for any } f \text{ which is continuous for } d_{loc}. \]

E.g.: \[ f = |B_R(.)| \]
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E.g.: $f = |B_R(.)|$

Main idea: The limiting object is often “nicer”.

Hence, it is easier to compute $f(G_\infty)$, from which we can deduce the behavior of $f(G_n)$. 
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For graphs, it has been formalized as the objective method [Aldous-Steele 94].
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For graphs, it has been formalized as the **objective method** [Aldous-Steele 94].

**Two example for maps:**

- one-endedness in the UIPT:
  Allows to give an explicit description of what can happen when the map gets disconnected.

- spatial Markov property

Simulation by T.Budd
II - Local limits of Ising-weighted triangulations
Escaping universality: adding matter

First, **Ising model** on a finite deterministic planar triangulation $T$:

**Spin configuration** on $T$:

$$\sigma : V(T) \rightarrow \{-1, +1\} = \{\circ, +\}.$$ 

**Ising model** on $T$: take a random spin configuration with probability:

$$P(\sigma) \propto e^{\beta J \sum_{v \sim v'} 1 \{\sigma(v) = \sigma(v')\}}$$

$\beta > 0$: inverse temperature.
$J = \pm 1$: coupling constant.
$h = 0$: no magnetic field.
Escaping universality: adding matter

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**Combinatorial formulation**: \( P(\sigma) \propto \nu^{m(\sigma)} \)

with \( m(\sigma) = \) number of monochromatic edges \( (\nu = e^{\beta J}) \).
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**Combinatorial formulation**: $P(\sigma) \propto \nu^{m(\sigma)}$  
with $m(\sigma) = \text{number of monochromatic edges}$ ($\nu = e^{\beta J}$).

Next step: Sample a triangulation of size $n$ **together** with a spin configuration, with probability $\propto \nu^{m(T,\sigma)}$.

$$\mathbb{P}_n^{\nu} \left( \{(T, \sigma)\} \right) = \frac{\nu^{m(T,\sigma)} \delta_{|e(T)|=3n}}{\mathcal{Z}_n}.$$  

$\mathcal{Z}_n = \text{normalizing constant}$. 

$$m(\sigma) = 5$$
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Combinatorial formulation: $P(\sigma) \propto \nu^{m(\sigma)}$

with $m(\sigma) = \text{number of monochromatic edges} \ (\nu = e^{\beta J})$.

Next step: Sample a triangulation of size $n$ together with a spin configuration, with probability $\propto \nu^{m(T, \sigma)}$.

Remark: This is a probability distribution on triangulations with spins. But, forgetting the spins gives a probability a distribution on triangulations without spins different from the uniform distribution.
Escaping universality: new asymptotic behavior

Counting exponent for undecorated maps:

number of (undecorated) maps of size $n \sim \kappa \rho^{-n} n^{-5/2}$

(e.g.: triangulations, quadrangulations, general maps, simple maps,...)

where $\kappa$ and $\rho$ depend on the combinatorics of the model.
Escaping universality: new asymptotic behavior

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where \( \kappa \) and \( \rho \) depend on the combinatorics of the model.

Generating series of Ising-weighted triangulations:

\[
Z(\nu, t) = \sum_{T \text{ triangulation}} \sum_{\sigma:V(T) \to \{-1,+1\}} \nu^{m(T,\sigma)} t^{e(T)}.
\]

**Theorem** [Bernardi – Bousquet-Mélou 11]

For every \( \nu > 0 \), \( Z(\nu, t) \) is algebraic and satisfies

\[
[t^3n]Z(\nu, t) \sim \begin{cases} 
\kappa \rho_{\nu_c}^{-n} n^{-7/3} & \text{if } \nu = \nu_c = 1 + \frac{1}{\sqrt{7}}, \\
\kappa \rho_{\nu}^{-n} n^{-5/2} & \text{if } \nu \neq \nu_c.
\end{cases}
\]

See also [Boulatov – Kazakov 1987], [Bousquet-Melou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].

This suggests a **different behavior** of the underlying maps for \( \nu = \nu_c \).
Local convergence of triangulations with spins

**Theorem** [A. – Ménard – Schaeffer, 21]

Let \( \mathbb{P}_n^\nu \) = \( \nu \)-Ising weighted probability distribution for triangulations of size \( n \):

\[
\mathbb{P}_n^\nu \overset{(d)}{\longrightarrow} \nu\text{-IIPT}, \quad \text{for the local topology with spins}
\]

\( \nu\text{-IIPT} = \nu\text{-Ising Infinite Planar Triangulation} \)
Local convergence of triangulations with spins

**Theorem** [A. – Ménard – Schaeffer, 21]

Let $\mathbb{P}^\nu_n = \nu$–Ising weighted probability distribution for triangulations of size $n$:

$$\mathbb{P}^\nu_n \xrightarrow{(d)} \nu$-$\text{IIPT}$, \quad \text{for the local topology with spins}

$\nu$-$\text{IIPT} = \nu$-Ising Infinite Planar Triangulation

Related result by [Chen, Turunen, 20] for a slightly different model.
Local convergence of triangulations with spins

**Theorem** [A. – Ménard – Schaeffer, 21]
Let $\mathbb{P}^\nu_n = \nu$–Ising weighted probability distribution for triangulations of size $n$:

$$\mathbb{P}^\nu_n \xrightarrow{(d)} \nu$$_{IPT}, \quad \text{for the local topology with spins}

$\nu$-IPT = $\nu$-Ising Infinite Planar Triangulation

Related result by [Chen, Turunen, 20] for a slightly different model.

Some properties of the $\nu$-IPT:
- One-ended a.s.
- Simple random walk is recurrent.
- Geometry of the clusters ?
- Volume (nb. of vertices) and perimeters of balls ??
Local convergence of triangulations with spins

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Let \( P^{\nu}_n \) = \( \nu \)-Ising weighted probability distribution for triangulations of size \( n \):

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Related result by [Chen, Turunen, 20] for a slightly different model.

Some properties of the \( \nu\text{-IIPT} \):
- One-ended a.s.
- Simple random walk is recurrent.
- Geometry of the clusters ?
- Volume (nb. of vertices) and perimeters of balls ???

**Non-universality**: we expect a **different** behavior for \( \nu = \nu_c \)

In particular, we expect the volume growth to be different from 4.

Watabiki’s conjecture: \( \frac{7 + \sqrt{97}}{4} \sim 4.21\ldots \)
III - Clusters in the $\nu$-IIPT
Ferromagnetic Ising model on $\mathbb{Z}^2$: clusters

Simulations by R.Cerf:

$\nu < \nu_c$

$\nu = \nu_c$

$\nu > \nu_c$

One infinite cluster
Ferromagnetic Ising model on $\mathbb{Z}^2$: clusters

Simulations by R. Cerf:

$\nu < \nu_c$

$\nu = \nu_c$

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CLE$_3$

One infinite cluster

[Chelkak+Smirnov 2012]
Ferromagnetic Ising model on $\mathbb{Z}^2$: clusters

Simulations by R. Cerf:

$\nu < \nu_c$

$\nu = \nu_c$

$\nu > \nu_c$

Same universality class as critical percolation.
CLE$_6$ ??

CLE$_3$

One infinite cluster

[Smirnov 2001]
for critical percolation

[Chelkak+Smirnov 2012]
Theorem [A. – Ménard, 22+]

Under $P_{\nu}^{\infty}$, the cluster of the root vertex is:

- finite almost surely for $\nu \leq \nu_c$
- infinite with ("explicit"!) positive probability for $\nu > \nu_c$.
Clusters in the $\nu$-IIPT: phase transition

**Theorem** [A. – Ménard, 22+]

Under $\mathbb{P}_\nu$, the cluster of the root vertex is:
- finite almost surely for $\nu \leq \nu_c$
- infinite with ("explicit"!) positive probability for $\nu > \nu_c$.

\[
\mathbb{P}_\nu(\lvert \mathcal{C} \rvert = \infty) \sim \kappa (\nu - \nu_c)^{1/4}
\]

Percolation critical exponent:

On $\mathbb{Z}^2$, exponent $= 1/8$ [Onsager 1944], [Yang 1952].
Clusters in the $\nu$-IIPT: cluster size exponents

**Theorem** [A. – Ménard, 22+]

Denote by $\mathcal{C}$ the spin cluster of the root vertex.

<table>
<thead>
<tr>
<th></th>
<th>for $\nu &lt; \nu_c$</th>
<th>for $\nu = \nu_c$</th>
<th>for $\nu &gt; \nu_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{P}_\infty^\nu (</td>
<td>\mathcal{C}</td>
<td>\geq n)$</td>
<td>$\propto n^{-1/7}$</td>
</tr>
<tr>
<td>$\mathbb{P}_\infty^\nu (</td>
<td>\partial \mathcal{C}</td>
<td>= p)$</td>
<td>$\propto p^{-2}$</td>
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</tbody>
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Denote by $\mathcal{C}$ the spin cluster of the root vertex. 

$\mathcal{C}$ and $\partial \mathcal{C}$ are shown in the diagrams.
The special case $\nu = 1$: UIPT with critical percolation

Recall that for a triangulation $T$ with spin configuration $\sigma$, $\mathbb{P}_n^{\nu}\left(\{(T, \sigma)\}\right) = \frac{\nu^m(T, \sigma) \delta_{|e(T)|=3n}}{Z_n}$.

For $\nu = 1$, all configurations (= trig. + spins) have the same probability
\[ \Leftrightarrow \text{uniform triangulation of size } n \text{ where spins are independent and } +/\text{ with probability } 1/2. \]
\[ \Leftrightarrow \text{uniform triangulation of size } n \text{ with a percolation of parameter } 1/2 \text{ on its vertices.} \]
The special case \( \nu = 1 \): UIPT with critical percolation

Recall that for a triangulation \( T \) with spin configuration \( \sigma \),
\[
P_\nu^n \left( \{(T, \sigma)\} \right) = \frac{\nu^m(T, \sigma) \delta_{|e(T)|=3n}}{Z_n}.
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For \( \nu = 1 \), all configurations (= trig. + spins) have the same probability:

\( \Leftrightarrow \) uniform triangulation of size \( n \) where spins are independent and +/- with probability 1/2.

\( \Leftrightarrow \) uniform triangulation of size \( n \) with a percolation of parameter 1/2 on its vertices.

Percolation on the UIPT much studied:

\( p_c = 1/2 + \text{no infinite cluster at } p_c \) [Angel 04]

\( \mathbb{P}_\infty^1 = \text{UIPT with critical percolation} \)

Percolation of the UIPT via its clusters in [Bernardi, Curien, Miermont, 17].
The special case $\nu = 1$: UIPT with critical percolation

Recall that for a triangulation $T$ with spin configuration $\sigma$, $\mathbb{P}^\nu_n\left(\{(T, \sigma)\}\right) = \frac{\nu^m(T, \sigma) \delta_{|e(T)|=3n}}{Z_n}$.

For $\nu = 1$, all configurations (= trig. + spins) have the same probability
⇔ uniform triangulation of size $n$ where spins are independent and +/- with probability 1/2.
⇔ uniform triangulation of size $n$ with a percolation of parameter 1/2 on its vertices.

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$p_c = 1/2 + \text{no infinite cluster at } p_c$ [Angel 04]
$\mathbb{P}_\infty^1 = \text{UIPT with critical percolation}$

Percolation of the UIPT via its clusters in [Bernardi, Curien, Miermont, 17].

Our results reinforce the idea that:
Ising model in high-temperature (i.e. $\nu < \nu_c$) $\sim$ Critical percolation
Idea of the proof I: Gasket decomposition
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Idea of the proof I: Gasket decomposition

Weight of a triangulation with spins $t := \nu^m(t) t|E(t)|$
Idea of the proof I: Gasket decomposition


Weight of a triangulation with spins \( t := \nu^m(t) |E(t)| \)

Total weight of triangulations with spin cluster \( \mathcal{C} \):

\[
\equiv \prod_{f \in \text{Faces}(\mathcal{C})} (\nu t)^{\deg(f)/2} \sum_l \text{Neck}(\deg(f), l) Q_l(\nu, t)
\]
Idea of the proof I: Gasket decomposition


Weight of a triangulation with spins $t := \nu^m(t) \, t|E(t)|$

Total weight of triangulations with spin cluster $\mathcal{C}$:

$$\equiv \prod_{f \in \text{Faces}(\mathcal{C})} (\nu \, t)^{\deg(f)/2} \sum_{l} \text{Neck}(\deg(f), l) \, Q_l(\nu, t)$$
Idea of the proof I: Gasket decomposition


Weight of a triangulation with spins \( t := \nu^{m(t)} \cdot t|E(t)| \)

Total weight of triangulations with spin cluster \( C \):

\[
\prod_{f \in \text{Faces}(C)} (\nu t)^{\deg(f)/2} \sum_{l} \text{Neck}(\deg(f), l) Q_l(\nu, t)
\]

Triangulations with boundary length \( l \) and \textbf{monochromatic} boundary conditions
Idea of the proof I: Gasket decomposition

Weight of a triangulation with spins $t := \nu^m(t) t |E(t)|$

Total weight of triangulations with spin cluster $C$:

$$\equiv \prod_{f \in \text{Faces}(C)} (\nu t)^{\deg(f)/2} \sum_l \text{Neck}(\deg(f), l) Q_l(\nu, t)$$

$$\left(\frac{\deg(f) + l - 1}{l}\right) t^{\deg(f)+l}$$
Idea of the proof I: Gasket decomposition

Weight of a triangulation with spins $t := \nu^m(t) t|E(t)|$

Total weight of triangulations with spin cluster $\mathcal{C}$:

$$\equiv \prod_{f \in \text{Faces}(\mathcal{C})} (\nu t)^{\deg(f)/2} \sum_l \text{Neck}(\deg(f), l) Q_l(\nu, t)$$

$$\equiv \prod_{f \in \text{Faces}(\mathcal{C})} q^{\deg(f)}(\nu, t)$$

\[A\]

\[B\]
**Idea of the proof I: Gasket decomposition**


Weight of a triangulation with spins \( t := \nu^m(t) \, t|E(t)| \)

Total weight of triangulations with spin cluster \( C \):

\[
\equiv \prod_{f \in \text{Faces}(C)} (\nu \, t)^{\text{deg}(f)/2} \sum_l \text{Neck}(\text{deg}(f), l) \, Q_l(\nu, t)
\]

\[
\equiv \prod_{f \in \text{Faces}(C)} q_{\text{deg}(f)}(\nu, t), \text{ where } q_k(\nu, t) = (\nu \, t)^{k/2} \, 1_{\{k=3\}} + (\nu \, t^3)^{k/2} \cdot \sum_{l \geq 0} \binom{k+l-1}{k-1} \, t^l \, Q_l^+(\nu, t)
\]
Idea of the proof II: Boltzmann maps

Boltzmann map associated to \((q_k) = \) 
Probability distribution on the set of rooted planar maps such that:

\[ P_{\text{bol}}(m) \propto \prod_{f \in F(m)} q^{\deg(f)} \text{ for any rooted planar map } m \]

\( P_{\text{bol}} \) is admissible if 
\[ \sum_{m \in \mathcal{M}} \prod_{f \in F(m)} q^{\deg(f)} < \infty \]
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Properties of the random map depends on the properties of $(q_k)$.

[Marckert-Miermont, Miermont-Le Gall, Borot-Bouttier-Guitter, Budd, Bernardi-Curien-Miermont, Marzouk]
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The Bouttier – Di Francesco – Guitter bijection (a.k.a the BDG bijection).
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$\mathbb{P}^{bol}$ is critical if $\mathbb{E}^{bol}(|m|) < \infty$ and $\mathbb{E}^{bol}(|m|^2) = \infty$. 
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\(P^{\text{bol}}\) is critical if \(\mathbb{E}^{\text{bol}}(|m|) < \infty\) and \(\mathbb{E}^{\text{bol}}(|m|^2) = \infty\).

- \(P^{\text{bol}}\) is regular critical if \(P^{\text{bol}}(\text{degree of a typical face } > k)\) decreases exponentially.

- \(P^{\text{bol}}\) is non-regular critical if \(P^{\text{bol}}(\text{degree of a typical face } > k)\) decreases polynomially.
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\nu > \nu_c
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- \(P^{\text{bol}}\) is regular critical if \(P^{\text{bol}}(\text{degree of a typical face} > k)\) decreases exponentially.

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Idea of the proof III: singularity analysis via rational parametrization

\[ Q^+(\nu, t, y) := \sum_{l \geq 0} Q^+_l (\nu, t) y^l, \quad \text{where} \quad Q^+_l := \sum_{(T, \sigma) = l} t^{|T|} \nu^m(T, \sigma) \]

**Theorem** [A. – Ménard, 22+]

Study of the singular developments of \( Q^+ \) in \( t \) and in \( y \).
Idea of the proof III:
singularity analysis via rational parametrization

\[ Q^+(\nu, t, y) := \sum_{l \geq 0} Q^+_l (\nu, t) y^l, \]

where \( Q^+_l := \sum_{(T, \sigma)} t^{|T|} \nu^m(T, \sigma) \)

Theorem [A. – Ménard, 22+]
Study of the singular developments of \( Q^+ \) in \( t \) and in \( y \).

Sketch of the proof:

- Obtained in [AMS 21] an algebraic equation for \( Q^+ \), by Tutte’s invariants method [Bernardi, Bousquet-Mélou].
- We use the rational parametrization (for \( t \)) given in [Bernardi, Bousquet-Mélou] for \( Q_1 \).
- With Maple, we compute a rational parametrization (for \( y \)) for different values of \( \nu \).
- We interpolate the coefficients given in the different parametrizations.
- With the rational parametrizations (and Maple), can compute the asymptotics.

Same strategy used in a slightly different context by [Chen, Turunen]
Idea of the proof III:
singularity analysis via rational parametrization

\[ Q^+(\nu, t, y) := \sum_{l \geq 0} Q_l^+ (\nu, t)y^l, \quad \text{where } Q_l^+ := \sum t^{|T|} \nu^m(T, \sigma) \]

\[ t^3 = U \frac{(1 + \nu)U - 2}{32\nu^3(1 - 2U)^2} \]

\[ y = \frac{8\nu(1 - 2U)}{U((1 + \nu) \cdot U - 2)} \cdot \frac{V(V + 1)}{V^3 + \frac{9(1 + \nu) \cdot U^2 - 2(3 + 10\nu)U + 8\nu}{U((1 + \nu) \cdot U - 2)} \cdot \frac{V^2 - \frac{9(1 + \nu) \cdot U - 2(2\nu + 3)}{U((1 + \nu) \cdot U - 2)} \cdot V - 1}{}} \]

\[ \hat{Q}^+(\nu, U, V) = U \cdot \frac{(1 + \nu) \cdot U - 2)(1 - \nu)}{(V + 1)^3 \cdot P(\nu, U)} \]

\[ \times \left( \frac{V^3 + \frac{9(1 + \nu) \cdot U^2 - 2(3 + 10\nu) \cdot U + 8\nu}{U \cdot (1 + \nu) \cdot U - 2)} \cdot \frac{V^2 - \frac{9(1 + \nu) \cdot U - 2(2\nu + 3)}{U \cdot ((1 + \nu) \cdot U - 2)} \cdot V - 1}{}} \right) \]

\[ \times \left( \frac{V^2 + \frac{5(1 + \nu) \cdot U^2 - 2(3\nu + 2) \cdot U + 2\nu}{U \cdot (1 + \nu) \cdot U - 2)} \cdot 2V - \frac{P(\nu, U)}{U \cdot ((1 + \nu) \cdot U - 2)(1 - \nu))} \right) \]
Idea of the proof IV: Computations + Maple = 💚
We compute explicitely:

$$P_\infty (|\text{cluster}| < \infty) = \sum_{c \in M} P_\infty (\text{cluster} = c)$$
Idea of the proof IV: Computations + Maple = ❤️

We compute explicitly:

\[ P_\nu^\infty (|\text{cluster}| < \infty) = \sum_{c \in \mathcal{M}} P_\nu^\infty (\text{cluster} = c) = 1 \quad \text{for } \nu \leq \nu_c. \]
Idea of the proof IV: Computations + Maple = ⋆

We compute explicitely:

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\mathbb{P}_\infty^{\nu} (|\text{cluster}| < \infty) = \sum_{c \in \mathcal{M}} \mathbb{P}_\infty^{\nu} (\text{cluster} = c) = 1 \quad \text{for} \quad \nu \leq \nu_c.
\]

For \( \nu > \nu_c \), we obtain an expression with an integral.
Additional results:

We obtain similar tail estimates for the size of the clusters for related models:

- **Ising-weighted Boltzmann triangulations**
  
  We recover in particular the results obtained in [Bernardi, Curien, Miermont]. Connections with some results obtained in [Borot, Bouttier, Guitter] and [Borot, Bouttier, Duplantier].

- Expected size of the cluster for Ising-weighted triangulations of size \( n \).

\[
\mathbb{E}_n^\nu (|\text{cluster}|) \sim \begin{cases} 
  c(\nu) n^{3/4} & \text{for } \nu < \nu_c \\
  c(\nu_c) n^{5/6} & \text{for } \nu = \nu_c \\
  c(\nu) n & \text{for } \nu > \nu_c 
\end{cases}
\]

- Geometry of cluster interfaces, via looptrees [Curien, Kortchemski 15].
IV - Link with Liouville Quantum Gravity and KPZ relation
Motivations from statistical physics

Originally the Ising model was studied on regular lattices such as $\mathbb{Z}^2$ [Ising, Onsager].
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Why do we study it on random metric spaces?
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In general relativity, the underlying space is not Euclidian anymore but is a Riemannian space, whose curvature describes the gravity.

One of the main challenge of modern physics is to make two theories consistent:

- quantum mechanics (which governs microscopic scales)
- general relativity (which governs macroscopic scales)
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One of the main challenge of modern physics is to make two theories consistent:

- quantum mechanics (which governs microscopic scales)
- general relativity (which governs macroscopic scales)

One attempt to reconcile these two theories, is the Liouville Quantum gravity which replaces the deterministic Riemannian space by a random metric space.
Liouville Quantum Gravity

For $\gamma \in (0, 2)$, $\gamma$-Liouville Quantum Gravity (or $\gamma$-LQG) = measure on a surface defined as the “exponential of the Gaussian Free Field” [Polyakov, 1981], [Duplantier, Sheffield 2011].
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Maps without matter converge to \( \sqrt{\frac{8}{3}} \)-LQG [Miermont 13, Le Gall 13, Miller-Sheffield 16+16+17, Holden-Sun 20].

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Simulation of the Brownian map by T.Budd

Simulation of a large simple triangulation embedded in the sphere by circle packing. Software CirclePack by K.Stephenson.

Simulation of \( \sqrt{\frac{8}{3}} \)-LQG by T.Budd
**Liouville Quantum Gravity**

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Maps without matter converge to $\sqrt{8/3}$-LQG [Miermont 13, Le Gall 13, Miller-Sheffield 16+16+17, Holden-Sun 20].

Other statistical models on random maps are believed to converge towards $\gamma$-LQG:
For critical Ising model on maps, $\gamma = \sqrt{3}$ (for non-critical Ising, $\gamma = \sqrt{8/3}$).
**Decorated $\gamma$-LQG**

Statistical models on random maps are believed to converge towards $\gamma$-LQG:

- Established for maps without matter, $\gamma = \sqrt{8/3}$.
- Conjectured for critical Ising model on maps, $\gamma = \sqrt{3}$
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What about the clusters? And their boundary?

Recall the behaviour in the Euclidean case:

\[ \nu < \nu_c \]

Critical Ising model

CLE$_6$ ?

CLE$_3$
**Decorated $\gamma$-LQG**

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What about the clusters? And their boundary?

Recall the behaviour in the Euclidean case:

We expect the same behaviour but on the corresponding $\gamma$-LQG.

For **critical percolation on uniform triangulations**, proved by [Holden-Sun 20], building on earlier works [Bernardi-Holden-Sun 18] and [Gwynne-Holden-Sun 21].
**Decorated $\gamma$-LQG and KPZ**

The KPZ relation [Knizhnik, Polyakov, Zamolodchikov, 1988], [Duplantier, Sheffield 2011]:

\[
x = \frac{\gamma^2}{4} \Delta^2 + \left(1 - \frac{\gamma^2}{4}\right) \Delta.
\]

links the Eucliden conformal weight $x$ of a fractal to its quantum counterpart $\Delta$.

i.e. We could “transfer” volume and perimeter exponents from **deterministic** to **random** geometry and vice versa.


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**Exponents for the perimeter $|\partial \mathcal{C}|$:**

For $\nu < \nu_c$  
KPZ, $\gamma = \sqrt{8/3}$  
Dimension of SLE$_6$

For $\nu = \nu_c$  
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Exponents for the perimeter $|\partial \mathcal{C}|$:

- For $\nu < \nu_c$ \textbf{KPZ}, $\gamma = \sqrt{8/3}$ \textbf{Dimension of SLE}_6 \cite{Beffara 08}
- For $\nu = \nu_c$ \textbf{KPZ}, $\gamma = \sqrt{3}$ \textbf{Dimension of SLE}_3

\textbf{All exponents match!}

Exponents for the volume $|\mathcal{C}|$:

- For $\nu < \nu_c$ \textbf{KPZ}, $\gamma = \sqrt{8/3}$ \textbf{Dimension of the gasket of CLE}_6 \cite{Miller, N.Sun, Watson 14}
- For $\nu = \nu_c$ \textbf{KPZ}, $\gamma = \sqrt{3}$ \textbf{Dimension of the gasket of CLE}_3
Perspectives

- Singularity with respect to the UIPT for $\nu \neq 1$.
- Geometry of the map via its clusters (especially for $\nu > \nu_c$).
- Convergence to the Brownian map for $\nu \neq \nu_c$. 
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- Volume growth exponent $> 4$ for $\nu = \nu_c$.
- Find a bijection!
  Bijections with walks in the 1/4-plane for a “mating of trees” approach?
  And extend results in [Gwynne, Holden, Sun, 20]?
Perspectives

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Thank you for your attention!