Cut elimination in classical sequent calculus modulo a super-consistent theory

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Abstract. It is known that if a theory T defined by a rewrite system is super-consistent [2], its proofs strongly normalize in natural deduction modulo [2] and that T has the cut elimination property in the intuitionistic sequent calculus [5]. In this paper we show that the super-consistency of a rewrite system also implies the cut elimination property of such theories in the classical sequent calculus.

1 Introduction

Deduction Modulo is a formalism that aims at distinguishing reasoning from computation in proofs by reasoning *modulo* some congruence. The congruence is generated by the means of rewrite rules, on terms and on propositions, making the congruence, under the assumption of confluence and termination decidable by pure computation.

Assuming rewrite rules on propositions is a key feature, allowing to express many formalism in a first-order setting *without* any additional axiom, such as higher-order logic [3,5], arithmetic [1] and much more. However, studying fundamental properties of the calculus such as cut elimination becomes a hard challenge. That is why new techniques have been developed in order to ensure cut elimination for the widest possible range of rewrite systems.

A first criterion has been developed in [5]. The authors show that if a congruence has a reducibility candidate-valued model, then any proof normalizes in natural deduction modulo this congruence. They then extend this result to cut elimination in intuitionistic sequent calculus modulo, but fail to extend the criterion to classical sequent calculus modulo. Instead of that, they propose another criterion ensuring cut elimination in the classical case.

A second approach is the notion of super-consistency developed in [2] that is a more general semantic criterion: the existence, for a given congruence, of a model for *any* truth value algebra, of which the reducibility candidate model of [5] is a special case. Therefore this criterion, called *super-consistency*, implies that of [5], hence strong normalization on first order natural deduction modulo and cut elimination in the intuitionistic sequent calculus modulo hold. But this work does not applies to classical sequent calculus modulo.

If the strong normalization theorem is well-fitted to natural deduction, the super-consistency criterion does not depend on the framework. That is why super-consistency should be enough to prove cut elimination in any framework: natural deduction as well as intuitionistic or classical sequent calculus modulo. This is exactly what we show here: if a congruence R has the super-consistency property, the cut rule is eliminable the classical sequent calculus modulo R.

After giving the definitions of the semantic tools one would need to keep the paper as much self contained as possible, we give a short introduction to the deduction modulo and the sequent calculus. We only assume familiarity with basic syntactic definitions of first-order logic. Then we present a light double negation translation on propositions and rewrite systems [5], inspired by the usual Gödel double negation translation to walk between the intuitionistic and the classical world.

The core of the paper resides in the lift of this translation to the semantic level, on the truth values algebras we need to model our theories. Then we prove that if a congruence is super-consistent, its translation is as well super-consistent: the congruence has a model in the translated algebra, so the translated theory should as well have a model in the original algebra.

Those two key results allows us to deduce that super-consistency is sufficient to prove the cut elimination property in classical sequent calculus, propagating the property of cut elimination from natural deduction modulo to intuitionistic sequent calculus and eventually to classical sequent calculus, following the work done in [5].

2 Definitions

2.1 Truth values algebra

Definition 1 (Truth values algebra (TVA, [2])). *Let* $\mathcal{B}$ *be a set, whose elements are called* truth values, $\tilde{\mathcal{B}}^+$ *be a subset of* $\mathcal{B}$, *whose elements are called* positive *truth values,* $\mathcal{A}$ *and* $\mathcal{E}$ *be subsets of* $\wp(\mathcal{B})$, $\tilde{\top}$ *and* $\tilde{\bot}$ *be elements of* $\mathcal{B}$, $\tilde{\Rightarrow}$, $\tilde{\wedge}$, *and* $\tilde{\vee}$ *be functions from* $\mathcal{B} \times \mathcal{B}$ *to* $\mathcal{B}$, $\tilde{\forall}$ *be a function from* $\mathcal{A}$ *to* $\mathcal{B}$ *and* $\tilde{\exists}$ *be a function from* $\mathcal{E}$ *to* $\mathcal{B}$. *The structure* $\tilde{\mathcal{B}} = \langle \mathcal{B}, \tilde{\mathcal{B}}^+, \mathcal{A}, \mathcal{E}, \tilde{\top}, \tilde{\bot}, \tilde{\Rightarrow}, \tilde{\wedge}, \tilde{\vee}, \tilde{\forall}, \tilde{\exists} \rangle$ *is said to be* a truth value algebra *if the set* $\tilde{\mathcal{B}}^+$ *is* closed by the intuitionistic deduction rules i.e. *if for all* a, b, c *in* $\mathcal{B}$, A *in* $\mathcal{A}$ *and* E *in* $\mathcal{E}$,

1. *if* $a \tilde{\Rightarrow} b \in \tilde{\mathcal{B}}^+$ *and* $a \in \tilde{\mathcal{B}}^+$ *then* $b \in \tilde{\mathcal{B}}^+$,
2. $a \tilde{\Rightarrow} b \tilde{\Rightarrow} a \in \tilde{\mathcal{B}}^+$,
3. $(a \tilde{\Rightarrow} b \tilde{\Rightarrow} c) \tilde{\Rightarrow} (a \tilde{\Rightarrow} b) \tilde{\Rightarrow} a \tilde{\Rightarrow} c \in \tilde{\mathcal{B}}^+$,
4. $\tilde{\top} \in \tilde{\mathcal{B}}^+$,
5. $\tilde{\bot} \tilde{\Rightarrow} a \in \tilde{\mathcal{B}}^+$,
6. $a \tilde{\Rightarrow} b \tilde{\Rightarrow} (a \tilde{\wedge} b) \in \tilde{\mathcal{B}}^+$,
7. $(a \tilde{\wedge} b) \tilde{\Rightarrow} a \in \tilde{\mathcal{B}}^+$,
8. $(a \tilde{\wedge} b) \tilde{\Rightarrow} b \in \tilde{\mathcal{B}}^+$,
9. $a \tilde{\Rightarrow} (a \tilde{\vee} b) \in \tilde{\mathcal{B}}^+$,
10. $b \tilde{\Rightarrow} (a \tilde{\vee} b) \in \tilde{\mathcal{B}}^+$,
11. $(a \tilde{\vee} b) \tilde{\Rightarrow} (a \tilde{\Rightarrow} c) \tilde{\Rightarrow} (b \tilde{\Rightarrow} c) \tilde{\Rightarrow} c \in \tilde{\mathcal{B}}^+$,

12. *the set* $a \tilde{\Rightarrow} A = \{a \tilde{\Rightarrow} e \mid e \in A\}$ *is in* $\mathcal{A}$*, and the set* $E \tilde{\Rightarrow} a = \{e \tilde{\Rightarrow} a \mid e \in E\}$ *is in* $\mathcal{E}$*,*
13. *if all elements of* A *are in* $\tilde{\mathcal{B}}^+$ *then* $\tilde{\forall}\ A \in \tilde{\mathcal{B}}^+$*,*
14. $\tilde{\forall}\ (a \tilde{\Rightarrow} A) \tilde{\Rightarrow} a \tilde{\Rightarrow} (\tilde{\forall}\ A) \in \tilde{\mathcal{B}}^+$*,*
15. *if* $a \in A$*, then* $(\tilde{\forall}\ A) \tilde{\Rightarrow} a \in \tilde{\mathcal{B}}^+$*,*
16. *if* $a \in E$*, then* $a \tilde{\Rightarrow} (\tilde{\exists}\ E) \in \tilde{\mathcal{B}}^+$*,*
17. $(\tilde{\exists}\ E) \tilde{\Rightarrow} \tilde{\forall}\ (E \tilde{\Rightarrow} a) \tilde{\Rightarrow} a \in \tilde{\mathcal{B}}^+$*.*

Notice that a truth values algebra can also be seen as a weakened Heyting Algebra, where the order becomes a pseudo-order, unsurprisingly defined as $a \leq b$ if and only if $a \tilde{\Rightarrow} b \in \tilde{\mathcal{B}}^+$. For more information, see [2].

We will only consider truth values algebras with special properties as it is the case in [2] and as required by super-consistency. The following has a clear analog in the Heyting Algebras (we want the semantic operators $\tilde{\forall}$ and $\tilde{\exists}$ to be total), the other are of lesser importance if the reader does not want to step into the details.

Definition 2 (Full [2]). *A truth values algebra is said to be* full *if* $\mathcal{A} = \mathcal{E} = \wp(\mathcal{B})$*,* i.e. *if* $\tilde{\forall}\ A$ *and* $\tilde{\exists}\ A$ *exist for all subsets* A *of* $\mathcal{B}$*.*

Definition 3 (Ordered truth values algebra [2]). *An* ordered truth values algebra *is a truth values algebra together with a relation* $\sqsubseteq$ *on* $\tilde{\mathcal{B}}$ *such that*

- $\sqsubseteq$ *is an order relation,*
- $\tilde{\mathcal{B}}^+$ *is upward closed,*
- $\tilde{\top}$ *is a maximal element,*
- $\tilde{\wedge}$*,* $\tilde{\vee}$*,* $\tilde{\forall}$ *and* $\tilde{\exists}$ *are monotonous,* $\tilde{\Rightarrow}$ *is left anti-monotonous and right monotonous.*

Definition 4 (Complete ordered truth values algebra [2]). *An ordered truth values algebra* $\tilde{\mathcal{B}}$ *is said to be* complete *if every subset of* $\mathcal{B}$ *has a greatest lower bound for* $\sqsubseteq$*. Notice that this implies that every subset also has a least upper bound. We write* $glb(a, b)$ *and* $lub(a, b)$ *the greatest lower bound and the least upper bound of* a *and* b *for the order* $\sqsubseteq$*.*

2.2 Models

Definition 5 ($\tilde{\mathcal{B}}$-valued structure [2]). *Let* $\mathcal{L} = \langle f_i, P_j \rangle$ *be a language in predicate logic and* $\tilde{\mathcal{B}}$ *be a truth values algebra, a* $\tilde{\mathcal{B}}$-valued structure *for the language* $\mathcal{L}$*,* $\mathcal{M} = \langle \mathcal{M}, \tilde{\mathcal{B}}, \hat{f}_i, \hat{P}_j \rangle$ *is a structure such that* $\hat{f}_i$ *is a function from* $\mathcal{M}^n$ *to* $\mathcal{M}$ *where* n *is the arity of the symbol* f_i *and* $\hat{P}_j$ *is a function from* $\mathcal{M}^n$ *to* $\tilde{\mathcal{B}}$ *where* n *is the arity of the symbol* P_i*.*

This definition extends trivially to many-sorted languages.

Definition 6 (Denotation [2]). *Let* $\tilde{\mathcal{B}}$ *be a truth values algebra,* $\mathcal{M}$ *be a* $\tilde{\mathcal{B}}$*-valued structure and* ϕ *be an assignment. The denotation* $\llbracket A \rrbracket_\phi$ *of a formula* A *in* $\mathcal{M}$ *is defined as follows*

- $\llbracket x \rrbracket_\phi = \phi(x)$*,*

- $[\![f(t_1, ..., t_n)]\!]_\phi = \hat{f}([\![t_1]\!]_\phi, ..., [\![t_n]\!]_\phi)$,
- $[\![P(t_1, ..., t_n)]\!]_\phi = \hat{P}([\![t_1]\!]_\phi, ..., [\![t_n]\!]_\phi)$,
- $[\![\top]\!]_\phi = \tilde{\top}$,
- $[\![\bot]\!]_\phi = \tilde{\bot}$,
- $[\![A \Rightarrow B]\!]_\phi = [\![A]\!]_\phi \tilde{\Rightarrow} [\![B]\!]_\phi$,
- $[\![A \wedge B]\!]_\phi = [\![A]\!]_\phi \tilde{\wedge} [\![B]\!]_\phi$,
- $[\![A \vee B]\!]_\phi = [\![A]\!]_\phi \tilde{\vee} [\![B]\!]_\phi$,
- $[\![\forall x\ A]\!]_\phi = \tilde{\forall}\ \{[\![A]\!]_{\phi+\langle x,e\rangle} \mid e \in \mathcal{M}\}$,
- $[\![\exists x\ A]\!]_\phi = \tilde{\exists}\ \{[\![A]\!]_{\phi+\langle x,e\rangle} \mid e \in \mathcal{M}\}$.

Notice that the denotation of a formula containing quantifiers may be undefined, but it is always defined if the truth value algebra is full.

Definition 7 (Model [2] [2]). *A formula A is said to be* valid *in a $\tilde{\mathcal{B}}$-valued structure $\mathcal{M}$, and the $\tilde{\mathcal{B}}$-valued structure $\mathcal{M}$ is said to be* a model of A, $\mathcal{M} \models A$, *if for all assignments ϕ, $[\![A]\!]_\phi$ is defined and is a positive truth value.*

The $\tilde{\mathcal{B}}$-valued structure $\mathcal{M}$ is said to be a model of *a theory $\mathcal{T}$ if it is a model of all the axioms of $\mathcal{T}$.*

2.3 Rewrite system

Definition 8 (Proposition rewrite rule). *We call* proposition rewrite rule *any rule $P \rightarrow Q$ rewriting atomic propositions P into an arbitrary proposition Q such that $\mathcal{FV}(Q) \subseteq \mathcal{FV}(P)$.*

Definition 9 (Proposition rewrite system). *We define a* proposition rewrite system *as an orthogonal, hence confluent set of proposition rewrite rules.*

2.4 Deduction Modulo

Deduction Modulo is a formalism that aims at distinguishing reasoning from computation in proofs by reasoning *modulo* some congruence[3], in replacing many deduction steps with blind calculation.

This may be explicited by a rephrasing of the inference rules, capturing the computational aspect of the proof in the congruence. We give here some samples of the rules, see for instance [5] for the full systems.

Natural deduction modulo

The implication elimination rule of natural deduction:

$$\frac{\Gamma \vdash_{\equiv} A \Rightarrow B \qquad \Gamma \vdash_{\equiv} A}{\Gamma \vdash_{\equiv} B}$$

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[3] In this paper, we restrict to congruences expressed by the reflexive, symmetric and transitive closure of proposition rewrite systems defined bellow.

becomes in natural deduction modulo:

$$\frac{\Gamma \vdash_{\equiv} C \qquad \Gamma \vdash_{\equiv} A}{\Gamma \vdash_{\equiv} B}\; C \equiv A \Rightarrow B$$

The other rules are built on the same pattern upon natural deduction.

Sequent Calculus modulo

We here give directly the rules of classical sequent calculus modulo. The subscript under the turnstyle is present to remain us that we now work modulo a congruence.

$$\frac{}{A \vdash_{\equiv} B}\text{ Axiom if } A \equiv B$$

$$\frac{\Gamma, A \vdash_{\equiv} \Delta \qquad \Gamma \vdash_{\equiv} B, \Delta}{\Gamma \vdash_{\equiv} \Delta}\text{ Cut if } A \equiv B$$

$$\frac{\Gamma, A_1, A_2 \vdash_{\equiv} \Delta}{\Gamma, A \vdash_{\equiv} \Delta}\text{ Contr-left if } A \equiv A_1 \equiv A_2$$

$$\frac{\Gamma \vdash_{\equiv} A_1, A_2, \Delta}{\Gamma \vdash_{\equiv} A, \Delta}\text{ Contr-right if } A \equiv A_1 \equiv A_2$$

$$\frac{\Gamma \vdash_{\equiv} \Delta}{\Gamma, A \vdash_{\equiv} \Delta}\text{ Weak-left}$$

$$\frac{\Gamma \vdash_{\equiv} \Delta}{\Gamma \vdash_{\equiv} A, \Delta}\text{ Weak-right}$$

$$\frac{\Gamma \vdash_{\equiv} A, \Delta \qquad \Gamma, B \vdash_{\equiv} \Delta}{\Gamma, C \vdash_{\equiv} \Delta}\ \Rightarrow\text{-left if } C \equiv (A \Rightarrow B)$$

$$\frac{\Gamma, A \vdash_{\equiv} B, \Delta}{\Gamma \vdash_{\equiv} C, \Delta}\ \Rightarrow\text{-right if } C \equiv (A \Rightarrow B)$$

$$\frac{\Gamma, A, B \vdash_{\equiv} \Delta}{\Gamma, C \vdash_{\equiv} C, \Delta}\ \wedge\text{-left if } C \equiv (A \wedge B)$$

$$\frac{\Gamma \vdash_{\equiv} A, \Delta \qquad \Gamma \vdash_{\equiv} B, \Delta}{\Gamma \vdash_{\equiv} C, \Delta}\ \wedge\text{-right if } C \equiv (A \wedge B)$$

$$\frac{\Gamma, A \vdash_{\equiv} \Delta \qquad \Gamma, B \vdash_{\equiv} \Delta}{\Gamma, C \vdash_{\equiv} \Delta}\ \vee\text{-left if } C \equiv (A \vee B)$$

$$\frac{\Gamma \vdash_{\equiv} B\Delta}{\Gamma \vdash_{\equiv} C, \Delta}\ \vee\text{-right-1 if } C \equiv (A \vee B)$$

$$\frac{\Gamma \vdash_{\equiv} A\Delta}{\Gamma \vdash_{\equiv} C, \Delta}\ \vee\text{-right-2 if } C \equiv (A \vee B)$$

$$\frac{}{\Gamma, A \vdash_{\equiv} \Delta}\ \bot\text{-left if } A \equiv \bot$$

$$\frac{\Gamma, C \vdash_{\equiv} \Delta}{\Gamma, B \vdash_{\equiv} \Delta}\ \langle x, A, t\rangle\ \forall\text{-left if } B \equiv (\forall x\ A) \text{ and } C \equiv Ax := t$$

$$\frac{\Gamma \vdash_{\equiv} A, \Delta}{\Gamma \vdash_{\equiv} B, \Delta}\ \langle x, A\rangle\ \forall\text{-right if } B \equiv (\forall x\ A) \text{ and } x \notin \mathcal{FV}(\Gamma\Delta)$$

$$\frac{\Gamma, A \vdash_{\equiv} \Delta}{\Gamma, B \vdash_{\equiv} \Delta}\ \langle x, A\rangle\ \exists\text{-left if } B \equiv (\exists x\ A) \text{ and } x \notin \mathcal{FV}(\Gamma\Delta)$$

$$\frac{\Gamma \vdash_{\equiv} C, \Delta}{\Gamma \vdash_{\equiv} B, \Delta}\ \langle x, A, t\rangle\ \exists\text{-right if } B \equiv (\exists x\ A) \text{ and } C \equiv Ax := t$$

2.5 Super-consistency

Definition 10 (Model for deduction modulo). *Let $\mathcal{T}, \equiv$ be a theory in deduction modulo. The $\tilde{\mathcal{B}}$-valued structure $\mathcal{M}$ is said to be* a model of *the theory $\mathcal{T}, \equiv$ if all axioms of $\mathcal{T}$ are valid in $\mathcal{M}$ and for all terms or formulas A and B such that $A \equiv B$ and assignment ϕ, $[\![A]\!]_\phi$ and $[\![B]\!]_\phi$ are defined and $[\![A]\!]_\phi = [\![B]\!]_\phi$.*

Remark 1. We can also restrict A and B in the previous definition to be $P \to A \in R$, R being the rewrite system.

Definition 11 (Super-consistent [2]). *A theory $\mathcal{T}, \equiv$ in deduction modulo is* super-consistent *if it has a $\tilde{\mathcal{B}}$-valued model for any full, ordered and complete truth values algebra $\tilde{\mathcal{B}}$. A rewrite system R is super-consistent if the congruence it generates is.*

3 Properties

We recall the proposition already established that we will use later.

Proposition 1 ([2]). *If a theory is super-consistent then its proofs strongly normalize in natural deduction modulo.*

Proposition 2 ([5]). *It proofs strongly normalizes in natural deduction modulo then we can eliminate the cut rule in intuitionistic sequent calculus modulo.*

Proposition 3. *Let $\tilde{\mathcal{B}} = \langle \mathcal{B}, \tilde{\mathcal{B}}^+, \mathcal{A}, \mathcal{E}, \tilde{\top}, \tilde{\bot}, \tilde{\Rightarrow}, \tilde{\wedge}, \tilde{\vee}, \tilde{\forall}, \tilde{\exists} \rangle$ be a TVA. Let $\mathcal{M}$ be a $\mathcal{B}$-valued structure and ϕ be an assignment.*
If P is intuitionistically provable then $[\![P]\!]_\Phi$ is in $\tilde{\mathcal{B}}^+$.

Proof. By Definition 1, $\tilde{\mathcal{B}}^+$ is closed by the intuitionistic deduction rules.

4 Light double negation translation: definitions

4.1 Syntactic translation

Definition 12 (Light double negation translation of a proposition [5]). *Let A be a proposition. We define $A^\neg$ as:*
$A^\neg = A$ if A is atomic,
$\top^\neg = \top$,
$\bot^\neg = \bot$,
$(A \Rightarrow B)^\neg = \neg\neg A^\neg \Rightarrow \neg\neg B^\neg$,
$(A \wedge B)^\neg = \neg\neg A^\neg \wedge \neg\neg B^\neg$,
$(A \vee B)^\neg = \neg\neg A^\neg \vee \neg\neg B^\neg$,
$(\forall x\ A)^\neg = \forall x\ (\neg\neg A^\neg)$,
$(\exists x\ A)^\neg = \exists x\ (\neg\neg A^\neg)$,

Remark 2. The more usual Gödel double negation translation [5] of A is $\neg\neg A^\neg$.

Definition 13 (Light double negation translation of a rewrite system [5]). *Let $R = \{A_i \to P_i\}$ be a proposition rewrite system. We define its light $\neg\neg$ translation, written $\overset{\neg}{R}$, as $\{A_i \to P_i^\neg\}$*

4.2 Semantic translation of a TVA

We shift the double negation translation at the semantic level, so as to be able to switch between the direct model and the double negation translated model as well as between the direct rewrite system and its light double negation translation.

Definition 14. *Let* $\tilde{\mathcal{B}} = \langle \mathcal{B}, \tilde{\mathcal{B}}^+, \wp(\mathcal{B}), \wp(\mathcal{B}), \tilde{\top}, \tilde{\bot}, \tilde{\Rightarrow}, \tilde{\wedge}, \tilde{\vee}, \tilde{\forall}, \tilde{\exists} \rangle$ *be a full TVA. we build* $\overset{\neg}{\mathcal{B}} = \langle \mathcal{B}, \overset{\neg}{\mathcal{B}}^+, \wp(\mathcal{B}), \wp(\mathcal{B}), \overset{\neg}{\top}, \overset{\neg}{\bot}, \overset{\neg}{\Rightarrow}, \overset{\neg}{\wedge}, \overset{\neg}{\vee}, \overset{\neg}{\forall}, \overset{\neg}{\exists} \rangle$*, that we call the light double negation translation of* $\tilde{\mathcal{B}}$*, as follows*[4] *:*

- *The domain* $\mathcal{B}$ *is the same*
- $a \in \overset{\neg}{\mathcal{B}}^+$ *iff* $\tilde{\neg}\tilde{\neg}a \in \tilde{\mathcal{B}}^+$
- $\overset{\neg}{\top} \triangleq \tilde{\top}$
- $\overset{\neg}{\bot} \triangleq \tilde{\bot}$
- $a \overset{\neg}{\Rightarrow} b \triangleq (\tilde{\neg}\tilde{\neg}a \;\tilde{\Rightarrow}\; \tilde{\neg}\tilde{\neg}b)$
- $a \overset{\neg}{\wedge} b \triangleq (\tilde{\neg}\tilde{\neg}a \;\tilde{\wedge}\; \tilde{\neg}\tilde{\neg}b)$
- $a \overset{\neg}{\vee} b \triangleq (\tilde{\neg}\tilde{\neg}a \;\tilde{\vee}\; \tilde{\neg}\tilde{\neg}b)$
- $\overset{\neg}{\forall}A \triangleq (\tilde{\forall}\; \tilde{\neg}\tilde{\neg}A)$
- $\overset{\neg}{\exists}A \triangleq (\tilde{\exists}\; \tilde{\neg}\tilde{\neg}A)$

5 Light double negation translation: properties

Proposition 4. *If* $\tilde{\mathcal{B}}$ *is a full TVA then its light double negation translation* $\overset{\neg}{\mathcal{B}}$ *is a full TVA.*

Proof. We shall first verify that $\overset{\neg}{\mathcal{B}}$ is a valid structure, which is pretty straightforward given that $\tilde{\mathcal{B}}$ is. Remember that the domain of $\overset{\neg}{\mathcal{B}}$ is $\mathcal{B}$, the same as $\tilde{\mathcal{B}}$.

We must check that $\overset{\neg}{\Rightarrow}, \overset{\neg}{\wedge}, \overset{\neg}{\vee}$ are functions from $\mathcal{B} \times \mathcal{B}$ to $\mathcal{B}$. Letting $a \in \mathcal{B}$ and $b \in \mathcal{B}$, we naturally have $\tilde{\neg}\tilde{\neg}a \in \mathcal{B}$ and $\tilde{\neg}\tilde{\neg}b \in \mathcal{B}$, hence $\tilde{\neg}\tilde{\neg}a \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg}b = a \overset{\neg}{\Rightarrow} b \in \mathcal{B}$, as well for the other operators.

We must check that $\overset{\neg}{\forall}, \overset{\neg}{\exists}$ are function from $\wp(\mathcal{B})$ to $(\mathcal{B})$. Let $E \in \wp(\mathcal{B})$ then $\tilde{\neg}\tilde{\neg}E = \{\tilde{\neg}\tilde{\neg}e \mid e \in E\} \in \wp(\mathcal{B})$ and $\tilde{\exists}\; \tilde{\neg}\tilde{\neg}E = \overset{\neg}{\exists}E \in \mathcal{B}$. $\overset{\neg}{\forall}$ works the same.

Then we check that $\overset{\neg}{\mathcal{B}}$ satisfies the criteria of Definition 1. Let a, b, c in $\mathcal{B}$, A in $\mathcal{A} = \wp(\mathcal{B})$ and E in $\mathcal{E} = \wp(\mathcal{B})$:

1. if $a \in \overset{\neg}{\mathcal{B}}^+$ and $a \;\overset{\neg}{\Rightarrow}\; b \in \overset{\neg}{\mathcal{B}}^+$ then $b \in \tilde{\mathcal{B}}^+$
   By definition of $\overset{\neg}{\mathcal{B}}^+$, $\tilde{\neg}\tilde{\neg}a \in \tilde{\mathcal{B}}^+$ and $\tilde{\neg}\tilde{\neg}(a \;\overset{\neg}{\Rightarrow}\; b) \in \tilde{\mathcal{B}}^+$ i.e. $\tilde{\neg}\tilde{\neg}(\tilde{\neg}\tilde{\neg}a \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg}b) \in \tilde{\mathcal{B}}^+$. We have to conclude $\tilde{\neg}\tilde{\neg}b \in \tilde{\mathcal{B}}^+$. For any atomic proposition A, B, the following proposition is an intuitionistic tautology:

$$\neg\neg A \Rightarrow \neg\neg(\neg\neg A \Rightarrow \neg\neg B) \Rightarrow \neg\neg B$$

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[4] the semantic negation is defined as $\tilde{\neg}A \triangleq A \tilde{\Rightarrow} \tilde{\bot}$

Interpreting A by a and B by b in $\tilde{\mathcal{B}}$, we get by Proposition 3 that:

$$\tilde{\neg}\tilde{\neg}a \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg}(\tilde{\neg}\tilde{\neg}a \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg}b) \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg}b \in \tilde{\mathcal{B}}^+$$

It remains then to use twice the corresponding property 1 of Definition 1 for $\tilde{\mathcal{B}}$ to get directly the desired conclusion.

2. $a \overset{\neg}{\Rightarrow} b \overset{\neg}{\Rightarrow} a \in \overset{\neg}{\mathcal{B}}^+$
   We want to prove that $\tilde{\neg}\tilde{\neg}(a \overset{\neg}{\Rightarrow} b \overset{\neg}{\Rightarrow} a) = \tilde{\neg}\tilde{\neg}(\tilde{\neg}\tilde{\neg}a \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg}(\tilde{\neg}\tilde{\neg}b \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg}a))) \in \tilde{\mathcal{B}}^+$. We use the fact that any interpretation of the intuitionistic tautology

   $$(A \Rightarrow B \Rightarrow A) \Rightarrow (\neg\neg(\neg\neg A \Rightarrow \neg\neg(\neg\neg B \Rightarrow \neg\neg A))$$

   is in $\tilde{\mathcal{B}}^+$ (by Proposition 3), in particular when A is interpreted by a and B by b. Since $a \tilde{\Rightarrow} b \tilde{\Rightarrow} a \in \tilde{\mathcal{B}}^+$ by property 2 of Definition 1, we conclude by property 1 of Definition 1 that $\tilde{\neg}\tilde{\neg}(\tilde{\neg}\tilde{\neg}a \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg}(\tilde{\neg}\tilde{\neg}b \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg}a))) \in \tilde{\mathcal{B}}^+$. Thus $a \overset{\neg}{\Rightarrow} b \overset{\neg}{\Rightarrow} a \in \overset{\neg}{\mathcal{B}}^+$.

3. $(a \overset{\neg}{\Rightarrow} b \overset{\neg}{\Rightarrow} c) \overset{\neg}{\Rightarrow} (a \overset{\neg}{\Rightarrow} b) \overset{\neg}{\Rightarrow} a \overset{\neg}{\Rightarrow} c \in \overset{\neg}{\mathcal{B}}^+$
   Shown as previously using $(a \tilde{\Rightarrow} b \overset{\neg}{\Rightarrow} c) \tilde{\Rightarrow} (a \tilde{\Rightarrow} b) \tilde{\Rightarrow} a \tilde{\Rightarrow} c \in \tilde{\mathcal{B}}^+$ and the tautology
   $((A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C) \Rightarrow$
   $\neg\neg(\neg\neg(\neg\neg A \Rightarrow \neg\neg(\neg\neg B \Rightarrow \neg\neg C)) \Rightarrow \neg\neg(\neg\neg(\neg\neg A \Rightarrow \neg\neg B) \Rightarrow \neg\neg(\neg\neg A \Rightarrow \neg\neg C)))$

4. $\overset{\neg}{\top} \in \overset{\neg}{\mathcal{B}}^+$
   Shown as previously using $\tilde{\top} \in \tilde{\mathcal{B}}^+$ and the tautology $\top \Rightarrow \neg\neg\top$.

5. $\overset{\neg}{\bot} \overset{\neg}{\Rightarrow} a \in \overset{\neg}{\mathcal{B}}^+$
   Shown as previously using $\tilde{\bot} \tilde{\Rightarrow} a \in \tilde{\mathcal{B}}^+$ and the tautology
   $(\bot \Rightarrow A) \Rightarrow (\neg\neg(\neg\neg\bot \Rightarrow \neg\neg A))$

6. $a \overset{\neg}{\Rightarrow} b \overset{\neg}{\Rightarrow} (a \overset{\neg}{\wedge} b) \in \overset{\neg}{\mathcal{B}}^+$
   Shown as previously using $a \tilde{\Rightarrow} b \tilde{\Rightarrow} (a \tilde{\wedge} b) \in \tilde{\mathcal{B}}^+$ and the tautology
   $(A \Rightarrow B \Rightarrow A \wedge B) \Rightarrow \neg\neg(\neg\neg A \Rightarrow \neg\neg(\neg\neg B \Rightarrow \neg\neg(\neg\neg A \wedge \neg\neg B)))$

7. $(a \overset{\neg}{\wedge} b) \overset{\neg}{\Rightarrow} a \in \overset{\neg}{\mathcal{B}}^+$
   Shown as previously using $(a \tilde{\wedge} b) \tilde{\Rightarrow} a \in \tilde{\mathcal{B}}^+$ and the tautology
   $(A \wedge B \Rightarrow A) \Rightarrow \neg\neg(\neg\neg(\neg\neg A \wedge \neg\neg B) \Rightarrow \neg\neg A)$

8. $(a \overset{\neg}{\wedge} b) \overset{\neg}{\Rightarrow} b \in \overset{\neg}{\mathcal{B}}^+$
   Shown as previously using $(a \tilde{\wedge} b) \tilde{\Rightarrow} b \in \tilde{\mathcal{B}}^+$ and the tautology
   $(A \wedge B \Rightarrow B) \Rightarrow \neg\neg(\neg\neg(\neg\neg A \wedge \neg\neg B) \Rightarrow \neg\neg B)$

9. $a \overset{\neg}{\Rightarrow} (a \overset{\neg}{\vee} b) \in \overset{\neg}{\mathcal{B}}^+$
   Shown as previously using the tautology
   $(A \Rightarrow A \vee B) \Rightarrow \neg\neg(\neg\neg A \Rightarrow \neg\neg(\neg\neg A \vee \neg\neg B))$ and $a \tilde{\Rightarrow} (a \tilde{\vee} b) \in \tilde{\mathcal{B}}^+$

10. $b \ \overset{\neg}{\Rightarrow} \ (a \ \overset{\neg}{\vee} \ b) \in \overset{\neg}{\mathcal{B}}^+$
    Shown as previously using $b \mathrel{\tilde{\Rightarrow}} (a \mathbin{\tilde{\vee}} b) \in \tilde{\mathcal{B}}^+$ and the tautology
    $(A \Rightarrow A \vee B) \Rightarrow \neg\neg(\neg\neg A \Rightarrow \neg\neg(\neg\neg A \vee \neg\neg B))$

11. $(a \ \overset{\neg}{\vee} \ b) \ \overset{\neg}{\Rightarrow} \ (a \ \overset{\neg}{\Rightarrow} \ c) \ \overset{\neg}{\Rightarrow} \ (b \ \overset{\neg}{\Rightarrow} \ c) \ \overset{\neg}{\Rightarrow} \ c \in \overset{\neg}{\mathcal{B}}^+$
    Shown as previously using $(a \mathbin{\tilde{\vee}} b) \mathrel{\tilde{\Rightarrow}} (a \mathrel{\tilde{\Rightarrow}} c) \mathrel{\tilde{\Rightarrow}} (b \mathrel{\tilde{\Rightarrow}} c) \mathrel{\tilde{\Rightarrow}} c \in \tilde{\mathcal{B}}^+$ and the tautology
    $(A \vee B \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C) \Rightarrow$
    $\neg\neg\,(\neg\neg\,(\neg\neg A \vee \neg\neg B) \Rightarrow \neg\neg\,(\neg\neg(\neg\neg A \Rightarrow \neg\neg C) \Rightarrow \neg\neg(\neg\neg(\neg\neg B \Rightarrow \neg\neg C) \Rightarrow \neg\neg C)))$

12. the set $a \ \overset{\neg}{\Rightarrow} \ A = \{a \ \overset{\neg}{\Rightarrow} \ e \mid e \in A\}$ is in $\mathcal{A}$ and the set $E \ \overset{\neg}{\Rightarrow} \ a = \{e \ \overset{\neg}{\Rightarrow} \ a \mid e \in E\}$ is in $\mathcal{E}$. Immediate since $\mathcal{E} = \mathcal{A} = \wp(\mathcal{B})$.

13. if all elements of A are in $\overset{\neg}{\mathcal{B}}^+$ then $\overset{\neg}{\forall} \ A \in \overset{\neg}{\mathcal{B}}^+$
    By hypothesis, all elements of $\tilde{\neg}\tilde{\neg}A$ (which are by definition of the form $\tilde{\neg}\tilde{\neg}a$ for some $a \in A$) are in $\tilde{\mathcal{B}}^+$. Hence, by property 13 of Definition 1, $\tilde{\forall} \ \tilde{\neg}\tilde{\neg}A \in \tilde{\mathcal{B}}^+$. We use the tautology $P \Rightarrow \neg\neg P$ as previously to show that this implies $\overset{\neg}{\forall} \ A \in \overset{\neg}{\mathcal{B}}^+$.

14. $\overset{\neg}{\forall} \ (a \ \overset{\neg}{\Rightarrow} \ A) \ \overset{\neg}{\Rightarrow} \ a \ \overset{\neg}{\Rightarrow} \ (\overset{\neg}{\forall} \ A) \in \overset{\neg}{\mathcal{B}}^+$
    Shown as previously using $\tilde{\forall} \ (a \mathrel{\tilde{\Rightarrow}} A) \mathrel{\tilde{\Rightarrow}} a \mathrel{\tilde{\Rightarrow}} (\tilde{\forall} \ A) \in \tilde{\mathcal{B}}^+$ and the provable proposition
    $(\forall x(B \Rightarrow P(x)) \Rightarrow (B \Rightarrow \forall x P(x))) \Rightarrow$
    $\neg\neg\,(\neg\neg\,(\forall x \neg\neg(\neg\neg B \Rightarrow \neg\neg P(x))) \Rightarrow \neg\neg\,(\neg\neg B \Rightarrow \neg\neg(\forall x \neg\neg P(x))))$
    where we give the following interpretation: we choose $\mathcal{M}$ such that $\{\hat{P}(d) \mid d \in \mathcal{M}\} = A$ (consider for instance $\mathcal{M} = A$ and $\hat{P}$ to be the identity).

15. if $a \in A$, then $(\overset{\neg}{\forall} \ A) \ \overset{\neg}{\Rightarrow} \ a \in \overset{\neg}{\mathcal{B}}^+$
    Shown as previously using $(\tilde{\forall} \ A) \mathrel{\tilde{\Rightarrow}} a \in \tilde{\mathcal{B}}^+$ and the provable proposition
    $((\forall x P(x)) \Rightarrow B) \Rightarrow \neg\neg\,(\neg\neg(\forall x(\neg\neg P(x)))) \Rightarrow (\neg\neg B))$

16. if $a \in E$, then $a \ \overset{\neg}{\Rightarrow} \ (\overset{\neg}{\exists} \ E) \in \overset{\neg}{\mathcal{B}}^+$
    Shown as previously using $a \mathrel{\tilde{\Rightarrow}} (\tilde{\exists} \ E) \in \tilde{\mathcal{B}}^+$ and the provable proposition
    $(B \Rightarrow (\exists x Q(x))) \Rightarrow \neg\neg\,(\neg\neg B \Rightarrow \neg\neg(\exists x(\neg\neg Q(x))))$

17. $(\overset{\neg}{\exists} \ E) \ \overset{\neg}{\Rightarrow} \ \overset{\neg}{\forall} \ (E \ \overset{\neg}{\Rightarrow} \ a) \ \overset{\neg}{\Rightarrow} \ a \in \overset{\neg}{\mathcal{B}}^+$
    Shown as previously using $(\tilde{\exists} \ E) \mathrel{\tilde{\Rightarrow}} \tilde{\forall} \ (E \mathrel{\tilde{\Rightarrow}} a) \mathrel{\tilde{\Rightarrow}} a \in \tilde{\mathcal{B}}^+$ and the provable proposition
    $(\exists x Q(x) \Rightarrow ((\forall x Q(x) \Rightarrow A) \Rightarrow A)) \Rightarrow$
    $\neg\neg\,(\neg\neg\exists x(\neg\neg Q(x)) \Rightarrow \neg\neg((\neg\neg\forall x(\neg\neg Q(x)) \Rightarrow (\neg\neg A)) \Rightarrow (\neg\neg A)))$

Proposition 5. *If* $(\tilde{\mathcal{B}}, \sqsubseteq)$ *is a full and ordered TVA then* $(\overset{\neg}{\mathcal{B}}, \sqsubseteq)$ *is a full and ordered TVA.*

Proof. We check all the conditions of Definition 3.

- $\sqsubseteq$ is an order relation on the domain of $\overset{\neg}{\mathcal{B}}$ because $\overset{\neg}{\mathcal{B}}$ shares the same domain as $\tilde{\mathcal{B}}$.

- $\overset{\neg}{\mathcal{B}}^{+}$ is upward closed
Let $x \in \overset{\neg}{\mathcal{B}}^{+}$ and $y \in \mathcal{B}$ such that $x \sqsubseteq y$. Then, $\tilde{\neg} y \sqsubseteq \tilde{\neg} x$ ($\tilde{\Rightarrow}$ is left anti-monotonous) and $\tilde{\neg}\tilde{\neg} x \sqsubseteq \tilde{\neg}\tilde{\neg} y$. But $x \in \overset{\neg}{\mathcal{B}}^{+}$ means that $\tilde{\neg}\tilde{\neg} x \in \tilde{\mathcal{B}}^{+}$. Henceforth, $\tilde{\neg}\tilde{\neg} y \in \tilde{\mathcal{B}}^{+}$, which means that $y \in \overset{\neg}{\mathcal{B}}^{+}$.

- $\overset{\neg}{\top}$ (resp. $\overset{\neg}{\bot}$) is a maximal (resp. minimal) element
This holds since $\overset{\neg}{\top} = \tilde{\top}$, $\overset{\neg}{\bot} = \tilde{\bot}$, $\tilde{\top}$ is maximal and $\tilde{\bot}$ is minimal.

- $\overset{\neg}{\wedge}$ is monotonous
Let a, b and c be elements of $\mathcal{B}$ such that $a \sqsubseteq b$. We also have $\tilde{\neg}\tilde{\neg} a \sqsubseteq \tilde{\neg}\tilde{\neg} b$ as proved above. As $\tilde{\wedge}$ is monotonous, $\tilde{\neg}\tilde{\neg} a \tilde{\wedge} \tilde{\neg}\tilde{\neg} c \sqsubseteq \tilde{\neg}\tilde{\neg} b \tilde{\wedge} \tilde{\neg}\tilde{\neg} c$ which gives by definition $a \overset{\neg}{\wedge} c \sqsubseteq b \overset{\neg}{\wedge} c$.

- $\overset{\neg}{\vee}, \overset{\neg}{\forall}$ and $\overset{\neg}{\exists}$ are proved monotonous in the same way, remembering that $A \sqsubseteq A'$ means by extension that for any $a \in A$, there exists a $a' \in A'$ such that $a \sqsubseteq a'$.

- $\overset{\neg}{\Rightarrow}$ is left anti-monotonous and right monotonous
Let a, b, c be elements of $\mathcal{B}$, assume that $a \sqsubseteq b$. We have $\tilde{\neg}\tilde{\neg} a \sqsubseteq \tilde{\neg}\tilde{\neg} b$, and since $\tilde{\Rightarrow}$ is left anti-monotonous, $\tilde{\neg}\tilde{\neg} b \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg} c \sqsubseteq \tilde{\neg}\tilde{\neg} a \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg} c$, i.e.$b \overset{\neg}{\Rightarrow} c \sqsubseteq a \overset{\neg}{\Rightarrow} c$. As well, since $\tilde{\Rightarrow}$ is right monotonous, we have $\tilde{\neg}\tilde{\neg} c \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg} a \sqsubseteq \tilde{\neg}\tilde{\neg} c \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg} b$, i.e. $c \overset{\neg}{\Rightarrow} a \sqsubseteq c \overset{\neg}{\Rightarrow} b$.

Theorem 1. *If $\tilde{\mathcal{B}}$ is a full, ordered and complete TVA then its light double negation translation $\overset{\neg}{\mathcal{B}}$ is a full, ordered and complete TVA.*

Proof. By Proposition 5 if $(\tilde{\mathcal{B}}, \sqsubseteq)$ is a full and ordered TVA then $(\overset{\neg}{\mathcal{B}}, \sqsubseteq)$ is a full and ordered TVA. As the completion property [2] is built upon $\sqsubseteq$ and the domain $\mathcal{B}$ that are both shared by $\tilde{\mathcal{B}}$ and $\overset{\neg}{\mathcal{B}}$, if $\tilde{\mathcal{B}}$ is a full, ordered and complete TVA then $\overset{\neg}{\mathcal{B}}$ is a full, ordered and complete TVA.

Proposition 6. *Let $\tilde{\mathcal{B}}$ be a full, ordered and complete TVA and $\overset{\neg}{\mathcal{B}}$ be the light double negation translation of $\tilde{\mathcal{B}}$. Consider a $\tilde{\mathcal{B}}$-valued structure $\mathcal{M}$, it is as well a $\overset{\neg}{\mathcal{B}}$-valued structure.*

Denote $\llbracket . \rrbracket$ the denotation $\mathcal{M}$ generates in $\tilde{\mathcal{B}}$ and $\llbracket . \rrbracket^{\neg}$ the denotation it generates in $\overset{\neg}{\mathcal{B}}$. Then for any proposition A, any assignment ϕ, $\llbracket \overset{\neg}{A} \rrbracket_{\phi} = \llbracket A \rrbracket^{\neg}_{\phi}$

Proof. The fact that $\mathcal{M}$ is also a $\overset{\neg}{\mathcal{B}}$-valued structure is a mere check of Definition 5, since $\overset{\neg}{\mathcal{B}}$ and $\tilde{\mathcal{B}}$ share the same domain. The second statement is proved by induction on the structure of A, where we omit the valuation ϕ, that plays no role:

- if A is atomic then by construction $\llbracket A \rrbracket^{\neg} = \llbracket A \rrbracket = \llbracket \overset{\neg}{A} \rrbracket$

- $\top$
  $\llbracket \top^{\neg} \rrbracket = \llbracket \top \rrbracket = \tilde{\top} = \llbracket \top \rrbracket^{\neg}$

- $\bot$
  $\llbracket \bot^{\neg} \rrbracket = \llbracket \bot \rrbracket = \tilde{\bot} = \llbracket \bot \rrbracket^{\neg}$

- $A \Rightarrow B$
  By induction hypothesis we have $\llbracket A^{\neg} \rrbracket = \llbracket A \rrbracket^{\neg}$ and $\llbracket B^{\neg} \rrbracket = \llbracket B \rrbracket^{\neg}$. Hence $\llbracket (A \Rightarrow B)^{\neg} \rrbracket = \llbracket \neg\neg A^{\neg} \Rightarrow \neg\neg B^{\neg} \rrbracket = \tilde{\neg}\tilde{\neg}\llbracket A^{\neg} \rrbracket \tilde{\Rightarrow} \tilde{\neg}\tilde{\neg}\llbracket B^{\neg} \rrbracket = \llbracket A \rrbracket^{\neg} \overset{\neg}{\Rightarrow} \llbracket B \rrbracket^{\neg} = \llbracket A \Rightarrow B \rrbracket^{\neg}$

- $A \vee B$
  By induction hypothesis we have $\llbracket A^{\neg} \rrbracket = \llbracket A \rrbracket^{\neg}$ and $\llbracket B^{\neg} \rrbracket = \llbracket B \rrbracket^{\neg}$. Hence $\llbracket (A \vee B)^{\neg} \rrbracket = \llbracket \neg\neg A^{\neg} \vee \neg\neg B^{\neg} \rrbracket = \tilde{\neg}\tilde{\neg}\llbracket A^{\neg} \rrbracket \tilde{\vee} \tilde{\neg}\tilde{\neg}\llbracket B^{\neg} \rrbracket = \llbracket A \rrbracket^{\neg} \overset{\neg}{\vee} \llbracket B \rrbracket^{\neg} = \llbracket A \Rightarrow B \rrbracket^{\neg}$

- $A \wedge B$
  By induction hypothesis we have $\llbracket A^{\neg} \rrbracket = \llbracket A \rrbracket^{\neg}$ and $\llbracket B^{\neg} \rrbracket = \llbracket B \rrbracket^{\neg}$. Hence $\llbracket (A \wedge B)^{\neg} \rrbracket = \llbracket \neg\neg A^{\neg} \wedge \neg\neg B^{\neg} \rrbracket = \tilde{\neg}\tilde{\neg}\llbracket A^{\neg} \rrbracket \tilde{\wedge} \tilde{\neg}\tilde{\neg}\llbracket B^{\neg} \rrbracket = \llbracket A \rrbracket^{\neg} \overset{\neg}{\wedge} \llbracket B \rrbracket^{\neg} = \llbracket A \Rightarrow B \rrbracket^{\neg}$

- $\forall x\ A$
  By induction hypothesis we have $\llbracket A^{\neg} \rrbracket = \llbracket A \rrbracket^{\neg}$. Hence $\llbracket (\forall x\ A)^{\neg} \rrbracket = \tilde{\forall}\ \{\ \tilde{\neg}\tilde{\neg}\llbracket A^{\neg} \rrbracket_{\langle x,d \rangle} \mid d \in \mathcal{M}\ \} = \tilde{\forall}\ \{\ \tilde{\neg}\tilde{\neg}\llbracket A \rrbracket^{\neg}_{\langle x,d \rangle} \mid d \in \mathcal{M}\ \} = \tilde{\forall}\ \tilde{\neg}\tilde{\neg}\{\ \llbracket A \rrbracket^{\neg}_{\langle x,d \rangle} \mid d \in \mathcal{M}\ \} = \overset{\neg}{\forall}\ \{\ \llbracket A \rrbracket^{\neg}_{\langle x,d \rangle} \mid d \in \mathcal{B}\ \} = \llbracket \forall x\ A \rrbracket^{\neg}$

- $\exists x\ A$
  By induction hypothesis we have $\llbracket A^{\neg} \rrbracket_{\langle x,d \rangle} = \llbracket A \rrbracket^{\neg}_{\langle x,d \rangle}$ for any $d \in \mathcal{M}$. Hence $\llbracket (\exists x\ A)^{\neg} \rrbracket = \tilde{\exists}\ \{\ \tilde{\neg}\tilde{\neg}\llbracket A^{\neg} \rrbracket_{\langle x,d \rangle} \mid d \in \mathcal{M}\ \} = \tilde{\exists}\ \{\ \tilde{\neg}\tilde{\neg}\llbracket A \rrbracket^{\neg}_{\langle x,d \rangle} \mid d \in \mathcal{M}\ \} = \tilde{\exists}\ \tilde{\neg}\tilde{\neg}\{\ \llbracket A \rrbracket^{\neg}_{\langle x,d \rangle} \mid d \in \mathcal{M}\ \} = \overset{\neg}{\exists}\ \{\ \llbracket A \rrbracket^{\neg}_{\langle x,d \rangle} \mid d \in \mathcal{M}\ \} = \llbracket \exists x\ A \rrbracket^{\neg}$

5.1 stability of super-consistency

In this section we show that the super-consistency criterion over a rewrite system is stable by light double negation translation.

Proposition 7. *Let R be a rewrite system and $\overset{\neg}{R}$ be its double negation translation. If R has a $\overset{\neg}{\mathcal{B}}$-model then $\overset{\neg}{R}$ has a $\tilde{\mathcal{B}}$-model.*

Proof. The rewrite rules $P \rightarrow F$ of R and $P \rightarrow F^{\neg}$ of $\overset{\neg}{R}$ are in ejection. From Definitions 10 and Remark 1, R has a $\overset{\neg}{\mathcal{B}}$-model means $\llbracket P \rrbracket^{\neg} = \llbracket F \rrbracket^{\neg}$.
By Proposition 6 we know that $\llbracket P \rrbracket^{\neg} = \llbracket P \rrbracket$ and $\llbracket F \rrbracket^{\neg} = \llbracket F^{\neg} \rrbracket$ so we get $\llbracket P \rrbracket = \llbracket F^{\neg} \rrbracket$.

Theorem 2. *Let R be a super-consistent rewrite system and $\overset{\neg}{R}$ be its double negation translation of R. $\overset{\neg}{R}$ is super-consistent.*

Proof. Let $\tilde{\mathcal{B}}$ be a TVA:

1. we build $\overset{\neg}{\mathcal{B}}$. it is a TVA by Proposition 4.
2. R has a $\overset{\neg}{\mathcal{B}}$-model because R is super-consistent.
3. $\overset{\neg}{R}$ has a $\tilde{\mathcal{B}}$-model by Proposition 7).

We have proved that for any TVA $\tilde{\mathcal{B}}$, $\overset{\neg}{R}$ a $\tilde{\mathcal{B}}$-model thus $\overset{\neg}{R}$ is super-consistent.

6 Super-consistency and classical sequent calculus

6.1 From intuitionistic to classical deduction modulo

We quote two additional propositions from [5] one need to prove the main result of this paper. The main idea of these propositions is to translate the corresponding results on Gödel double negation translation for the usual sequent calculus, using the fact that if $A \equiv_R B$ then $A^{\neg} \equiv_{\overset{\neg}{R}} B^{\neg}$.

Proposition 8 ([5]). *Let R be a rewrite system and $\overset{\neg}{R}$ be its light double negation translation. Let $\Gamma \vdash_{\equiv} \Delta$ be a sequent and $\Gamma' \vdash_{\equiv} \Delta'$ be its usual Gödel double negation translation (which is $\neg\neg\Gamma^{\neg}$).*

- *If $\Gamma \vdash_{\equiv} \Delta$ has a proof in the classical sequent calculus modulo R then $\Gamma', \neg\Delta^{\neg} \vdash_{\equiv}$ has a proof in the intuitionistic sequent calculus modulo $\overset{\neg}{R}$*

- *If $\Gamma' \vdash_{\equiv} \Delta'$ has a cut free proof in the intuitionistic sequent calculus modulo $\overset{\neg}{R}$ then $\Gamma \vdash_{\equiv} \Delta$ has a cut free proof in the classical sequent calculus modulo R.*

6.2 Cut elimination in classical sequent calculus modulo

Theorem 3. *If a theory $\mathcal{T}$ is defined by a super-consistent rewrite system R then $\mathcal{T}$ has the cut elimination property in classical sequent calculus.*

Proof. Let $\Gamma \vdash_{\equiv} \Delta$ be a provable sequent in the classical sequent calculus modulo R. We know by Proposition 8 that $\Gamma' \vdash_{\equiv} \Delta'$ has a proof in the intuitionistic sequent calculus modulo $\overset{\neg}{R}$. As $\overset{\neg}{R}$ is super-consistent (Theorem 2), $\Gamma' \vdash_{\equiv} \Delta'$ has a *cut free* proof in the intuitionistic sequent calculus modulo $\overset{\neg}{R}$ (Proposition 2). By Proposition 8 we conclude that $\Gamma \vdash_{\equiv} \Delta$ has a *cut free* proof in the classical sequent calculus modulo R.

7 Conclusion

We have proved that the super-consistency criterion over a rewrite system defining a theory is sufficient to get the cut elimination property of this theory in classical sequent calculus modulo. Super-consistency appears to be the right criterion to deal with when one wants to know about the cut elimination property of a modulo theory, as the property holds whatever the framework is. It would be interesting to extend deduction modulo to such frameworks such as the Calculus of Structures [7] or Linear Logic [6] to investigate whether super-consistency and its properties apply to those frameworks.

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