Algorithmic equality in Heyting Arithmetic modulo

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Projet LogiCal

Introduction

Proving 1+1=2

- Axiomatic Heyting Arithmetic
  $\rightarrow$ Leibniz's axiom scheme + deduction

- Heyting Arithmetic modulo - previous version
  $\rightarrow$ Leibniz's rewrite rule + few steps of deduction remaining

- The new Heyting Arithmetic modulo
  $\rightarrow 1 + 1 = 2 \equiv \top$, no deduction

Deduction modulo

Deduction modulo = reasoning + computation

- Natural deduction rules
- A congruence $\equiv$

Some rules in deduction modulo :

$$\frac{}{\Gamma \vdash_{\equiv} B} \text{Ax if } A \in \Gamma \text{ and } A \equiv B$$

$$\frac{\Gamma \vdash_{\equiv} C \qquad \Gamma \vdash_{\equiv} A}{\Gamma \vdash_{\equiv} B} \Rightarrow\text{e if } C \equiv A \Rightarrow B$$

$$\frac{\Gamma \vdash_{\equiv} A \qquad \Gamma \vdash_{\equiv} B}{\Gamma \vdash_{\equiv} C} \wedge\text{i if } C \equiv A \wedge B$$

Theories in deduction modulo

An axiomatic theory is a set of axioms.

A modulo theory is a set of axioms and a congruence defined as the reflexive, transitive and symmetric closure of a set of rewrite rules.

A purely computational theory is a modulo theory where the set of axioms is empty.

Goals

- To have a purely computational presentation of Heyting Arithmetic
- To take advantage of the decidability of equality in Arithmetic

Axiomatic Heyting Arithmetic

Induction

$$(P\{x := 0\} \wedge \forall y\ (P\{x := y\} \Rightarrow P\{x := S(y)\})) \Rightarrow \forall n\ P\{x := n\}$$

Equality

$\forall x\ x = x$

$\forall x\ \forall y\ x = y \Rightarrow P(x) \Rightarrow P(y)$

$\forall x\ 0 = S(x) \Rightarrow \bot$

$\forall x\ \forall y\ (S(x) = S(y) \Rightarrow x = y)$

Addition and multiplication

$\forall y\ (0 + y = y)$

$\forall x\ \forall y\ (S(x) + y = S(x + y))$

$\forall y\ (0 \times y = 0)$

$\forall x\ \forall y\ (S(x) \times y = x \times y + y)$

1+1=2 in axiomatic Heyting Arithmetic

Axioms needed in the proof :

$\forall y \ (0 + y = y)$

$\forall x \ \forall y \ (S(x) + y = S(x + y))$

$\forall x \ \forall y \ x = y \Rightarrow P(x) \Rightarrow P(y)$

$$\dfrac{\dfrac{}{\vdash_{HA} 0 + y = y}\ \text{Ax}}{\vdash_{HA} 0 + S(0) = S(0)}\ \forall\text{e}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Instance of Leibniz's axiom scheme}}{\vdash_{HA} x = y \Rightarrow S(0) + S(0) = S(x) \Rightarrow S(0) + S(0) = S(y)}\ \text{Ax}}{\vdash_{HA} 0 + S(0) = S(0) \Rightarrow S(0) + S(0) = S(0 + S(0)) \Rightarrow S(0) + S(0) = S(S(0))}\ \forall\text{ex2}}{\vdash_{HA} S(0) + S(0) = S(0 + S(0)) \Rightarrow S(0) + S(0) = S(S(0))}\ \Rightarrow\text{e} \quad \dfrac{\dfrac{}{\vdash_{HA} \forall x \ S(x) + y = S(x + y)}\ \text{Ax}}{\vdash_{HA} S(0) + S(0) = S(0 + S(0))}\ \forall\text{ex2}}{\vdash_{HA} S(0) + S(0) = S(S(0))}\ \Rightarrow\text{e}$$

A first attempt of Heyting Arithmetic modulo : HA_{DW}

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A first attempt of Heyting Arithmetic modulo : HA_{DW}

Induction

$$x \in f_{z,y_1,\ldots,y_n,P}(y_1,\ldots,y_n) \longrightarrow P\{z := x\}$$
$$N(n) \longrightarrow \forall f\ (0 \in f \Rightarrow \forall y\ (N(y) \Rightarrow y \in f \Rightarrow S(y) \in f) \Rightarrow n \in f)$$

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$$0 + y \longrightarrow y$$
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A first attempt of Heyting Arithmetic modulo : HA_{DW}

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$$N(n) \longrightarrow \forall f\ (0 \in f \Rightarrow \forall y\ (N(y) \Rightarrow y \in f \Rightarrow S(y) \in f) \Rightarrow n \in f)$$

Equality

$$Pred(0) \longrightarrow 0 \qquad Null(0) \longrightarrow \top$$
$$Pred(S(x)) \longrightarrow x \qquad Null(S(x)) \longrightarrow \bot$$

$$y = z \longrightarrow \forall p\ (y \in p \Rightarrow z \in p)$$

Addition and multiplication

$$0 + y \longrightarrow y \qquad S(x) + y \longrightarrow S(x + y)$$
$$0 \times y \longrightarrow 0 \qquad S(x) \times y \longrightarrow x \times y + y$$

1+1=2 in HA_{DW}

The needed rules :

$0 + y \longrightarrow y$

$S(x) + y \longrightarrow S(x + y)$

$y = z \longrightarrow \forall p\ (y \in p \Rightarrow z \in p)$

$$\dfrac{\dfrac{\dfrac{\overline{1 + 1 \in p \vdash_{HA_{DW}} 2 \in p}\ \text{Ax, } 1 + 1 \in p \equiv 2 \in p}{\vdash_{HA_{DW}} 1 + 1 \in p \Rightarrow 2 \in p}\ \Rightarrow\text{i}}{\vdash_{HA_{DW}} \forall p\ 1 + 1 \in p \Rightarrow 2 \in p}\ \forall\text{i}}{\vdash_{HA_{DW}} 1 + 1 = 2}\ \text{rewrite - Leibniz}$$

A New Heyting Arithmetic modulo : $HA_{\longrightarrow}$

Induction

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Equality

$\forall x\ x = x$

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A New Heyting Arithmetic modulo : $HA_{\longrightarrow}$

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A New Heyting Arithmetic modulo : $HA_{\longrightarrow}$

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$$N(n) \longrightarrow \forall f\ (0 \in f \Rightarrow \forall y\ (N(y) \Rightarrow y \in f \Rightarrow S(y) \in f) \Rightarrow n \in f)$$

Equality

$$0 = 0 \longrightarrow \top \qquad S(x) = 0 \longrightarrow \bot$$
$$0 = S(x) \longrightarrow \bot \qquad S(x) = S(y) \longrightarrow x = y$$

$$\forall x\ \forall y\ x = y \Rightarrow P(x) \Rightarrow P(y)$$

Addition and multiplication

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$1{+}1{=}2$ in $\mathrm{HA}_{\longrightarrow}$

The needed rules :

$0 = 0 \longrightarrow \top$

$S(x) = S(y) \longrightarrow x = y$

$0 + y \longrightarrow y$

$S(x) + y \longrightarrow S(x + y)$

$$\frac{}{\vdash_{\equiv} 1 + 1 = 2}\ \text{Ax},\ 1 + 1 = 2 \equiv \top$$

$$1 + 1 = 2 \ \longrightarrow\ 2 = 2 \ \longrightarrow\ 1 = 1 \ \longrightarrow\ 0 = 0 \ \longrightarrow\ \top$$

$HA_{\longrightarrow}$ is a conservative extension of HA.

- HA
- HA_R
- HA_N
- HA_K
- $HA_{\longrightarrow}$

Induction

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Equality

$\forall x\ x = x$

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From HA to $HA_{\longrightarrow}$: HA_R

Induction

$$(P\{x := 0\} \wedge \forall y\ (P\{x := y\} \Rightarrow P\{x := S(y)\})) \Rightarrow \forall n\ P\{x := n\}$$

Equality

$$0 = 0 \longrightarrow \top \qquad S(x) = 0 \longrightarrow \bot$$
$$0 = S(x) \longrightarrow \bot \qquad S(x) = S(y) \longrightarrow x = y$$

Addition and multiplication

$$0 + y \longrightarrow y \qquad S(x) + y \longrightarrow S(x + y)$$
$$0 \times y \longrightarrow 0 \qquad S(x) \times y \longrightarrow x \times y + y$$

From HA to HA$_{\longrightarrow}$: HA$_N$

Induction

$N(0)$
$\forall x\ N(x) \Rightarrow N(S(x))$
$\forall n\ N(n) \Rightarrow (P\{x := 0\} \wedge \forall y\ (N(y) \Rightarrow P\{x := y\} \Rightarrow P\{x := S(y)\})) \Rightarrow P\{x := n\}$

Equality

$$0 = 0 \longrightarrow \top \qquad S(x) = 0 \longrightarrow \bot$$
$$0 = S(x) \longrightarrow \bot \qquad S(x) = S(y) \longrightarrow x = y$$

Addition and multiplication

$$0 + y \longrightarrow y \qquad S(x) + y \longrightarrow S(x + y)$$
$$0 \times y \longrightarrow 0 \qquad S(x) \times y \longrightarrow x \times y + y$$

From HA to $HA_{\longrightarrow}$: HA_K

Induction

$$\forall x \forall y_1 ... \forall y_n \; (x \in f_{z,y_1,\dots,y_n,P}(y_1, \dots, y_n) \Leftrightarrow P\{z := x\})$$
$$\forall n \; (N(n) \Leftrightarrow \forall f \; (0 \in f \Rightarrow \forall y \; (N(y) \Rightarrow y \in f \Rightarrow S(y) \in f) \Rightarrow n \in f))$$

Equality

$$0 = 0 \longrightarrow \top \qquad S(x) = 0 \longrightarrow \bot$$
$$0 = S(x) \longrightarrow \bot \qquad S(x) = S(y) \longrightarrow x = y$$

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$HA_{\longrightarrow}$ has cut elimination property.

Proof by super-consistency.

Conclusion

$\mathsf{HA}_{\longrightarrow}$ is a conservative extension of HA that uses the decidability of equality to simplify the proofs.
Leibniz is not defining equality anymore but is a consequence.

Future work :
This open the perspective of defining equality in an algorithmic way in inductive types with decidable equality.