École Polytechnique

Problems and exercises in Operations Research

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[1]Some exercises have been proposed by other authors, as detailed in the text. All the solutions, however, are by the author, who takes full responsibility for their accuracy (or lack thereof).

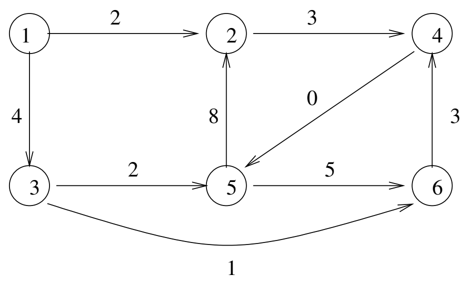
Contents

Chapter 1

Optimization on graphs

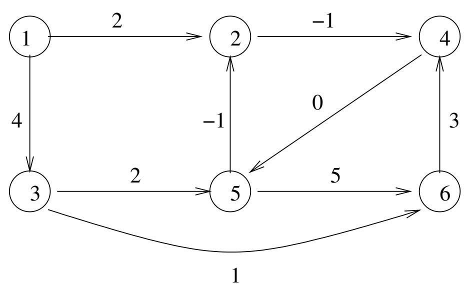
1.1 Dijkstra's algorithm

Use Dijkstra's algorithm to find the shortest path tree in the graph below using vertex 1 as source.



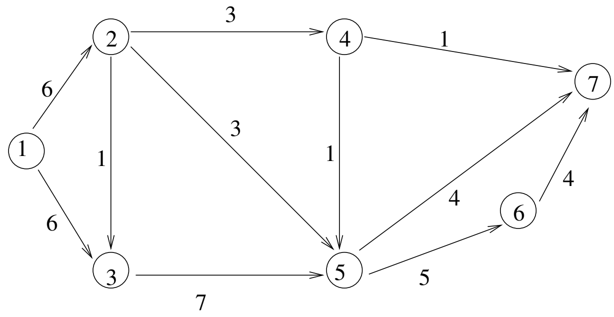
1.2 Bellman-Ford's algorithm

Check whether the graph below has negative cycles using Bellman-Ford's algorithm and 1 as a source vertex.



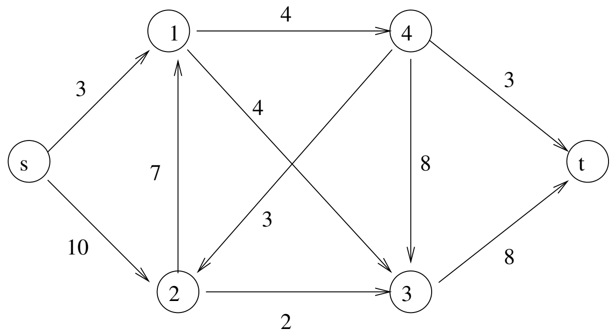
1.3 Maximum flow

Determine a maximum flow from node 1 to node 7 in the network $G = (V, A)$ below (the values on the arcs (i, j) are the arc capacities k_{ij}). Also find a cut having minimum capacity.



1.4 Minimum cut

Find the mimum cut in the graph below (arc capacities are marked on the arcs). What algorithm did you use?



1.5 Renewal plan

A small firm buys a new production machinery costing 12000 euros. In order to decrease maintenance costs, it is possible to sell the machinery second-hand and buy a new one. The maintenance costs and possible gains derived from selling the machinery second-hand are given below (for the next 5 years):

age (years)	costs (keuro)	gain (keuro)
0	2	-
1	4	7
2	5	6
3	9	2
4	12	1

Determine a renewal plan for the machinery which minimizes the total operation cost over a 5-year period. [*E. Amaldi, Politecnico di Milano*]

1.6 Connected subgraphs

Consider the complete undirected graph $K_n = (V, E)$ where $V = \{0, \ldots, n-1\}$ and $E = \{\{u, v\} \mid u, v \in V\}$. Let $U = \{i \mod n \mid i \geq 0\}$ and $F = \{\{i \mod n, (i+2) \mod n\} \mid i \geq 0\}$. Show that (a) $H = (U, F)$ is a subgraph of G and that (b) H is connected if and only if n is odd.

1.7 Strong connection

Consider the complete undirected graph $K_n = (V, E)$ and orient the edges arbitrarily into an arc set A so that for each vertex $v \in V$, $|\delta^+(v)| \geq 1$ and $|\delta^-(v)| \geq 1$. Show that the resulting directed graph $G = (V, A)$ is strongly connected.

Chapter 2

Linear programming

2.1 Graphical solution

Consider the problem

$$\begin{aligned} \min_x \quad & cx \\ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

where $x = (x_1, x_2)^T$, $c = (16, 25)$, $b = (4, 5, 9)^T$, and

$$A = \begin{pmatrix} 1 & 7 \\ 1 & 5 \\ 2 & 3 \end{pmatrix}.$$

1. Solve the problem graphically.
2. Write the problem in standard form. Identify B and N for the optimal vertex of the feasible polyhedron.

[*E. Amaldi, Politecnico di Milano*]

2.2 Geometry of LP

Consider the following LP problem.

$$\begin{aligned} \max z^* \quad & = 3x_1 + 2x_2 & & (*) \\ 2x_1 + x_2 & \leq 4 & & (2.1) \\ -2x_1 + x_2 & \leq 2 & & (2.2) \\ x_1 - x_2 & \leq 1 & & (2.3) \\ x_1, x_2 & \geq 0. \end{aligned}$$

1. Solve the problem graphically, specifying the variable values and z^* at the optimum.
2. Determine the bases associated to all the vertices of the feasible polyhedron.

3. Specify the sequence of the bases visited by the simplex algorithm to reach the solution (choose x_1 as the first variable entering the basis).
4. Determine the value of the reduced costs relative to the basic solutions associated to the following vertices, expressed as intersections of lines in $\mathbb{R}^2$: (a) (Eq. 2.1) $\cap$ (Eq. 2.2); (b) ((Eq. 2.1) $\cap$ (Eq. 2.3), where (Eq. i) is the equation obtained by inequality (i) replacing $\leq$ with $=$.
5. Verify geometrically that the objective function gradient can be expressed as a non-negative linear combination of the active constraint gradients only in the optimal vertex (keep in mind that the constraints must all be cast in the $\leq$ form, since the optimization direction is maximization — e.g. $x_1 \geq 0$ should be written as $-x_1 \leq 0$).
6. Say for which values of the RHS coefficient b_1 in constraint (2.1) the optimal basis does not change.
7. Say for which values of the objective function coefficients the optimal vertex is $((x_1 = 0) \cap$ (Eq. 2.2)), where $x_1 = 0$ is the equation of the ordinate axis in $\mathbb{R}^2$.
8. For which values of the RHS coefficient associated to (2.2) the feasible region is (a) empty (b) contains only one solution?
9. For which values of the objective function coefficient c_1 there is more than one optimal solution?

[*M. Trubian, Università Statale di Milano*]

2.3 Simplex method

Solve the following LP problem using the simplex method:

$$\begin{array}{rrl} \min z = & x_1 - 2x_2 & \\ & 2x_1 + 3x_3 & = 1 \\ & 3x_1 + 2x_2 - x_3 & = 5 \\ & x_1, x_2, x_3 & \geq 0. \end{array}$$

Use the two-phase simplex method (the first phase identifies an initial basis) and Bland's rule (for a choice of the entering and exiting basis which ensures algorithmic convergence). [*E. Amaldi, Politecnico di Milano*]

2.4 Duality

What is the dual of the following LP problems?

1. $$\left.\begin{array}{rrcl} \min_x & 3x_1 + 5x_2 - x_3 & & \\ & x_1 - x_2 + x_3 & \leq & 3 \\ & 2x_1 - 3x_2 & \leq & 4 \\ & x & \geq & 0 \end{array}\right\} \quad (2.4)$$

2. $$\left.\begin{array}{rrcl} \min_x & x_1 - x_2 - x_3 & & \\ & -3x_1 - x_2 + x_3 & \leq & 3 \\ & 2x_1 - 3x_2 - 2x_3 & \geq & 4 \\ & x_1 - x_3 & = & 2 \\ & x_1 & \geq & 0 \\ & x_2 & \geq & 0 \end{array}\right\} \quad (2.5)$$

3.

$$\left.\begin{array}{rrcl} \max_x & x_1 - x_2 - 2x_3 + 3 & & \\ & -3x_1 - x_2 + x_3 & \le & 3 \\ & 2x_1 - 3x_2 & \ge & 4x_3 \\ & x_1 - x_3 & = & x_2 \\ & x_1 & \ge & 0 \\ & x_2 & \le & 0 \end{array}\right\} \quad (2.6)$$

2.5 Geometrical interpretation of the simplex algorithm

Solve the following problem

$$\left.\begin{array}{rrcl} \max_x & x_1 + x_2 & & \\ & -x_1 + x_2 & \le & 1 \\ & 2x_1 + x_2 & \le & 4 \\ & x_1 & \ge & 0 \\ & x_2 & \le & 0 \end{array}\right\}$$

using the simplex algorithm. Start from the initial point $\bar{x} = (1, 0)$.

2.6 Complementary slackness

Consider the problem

$$\begin{array}{rrcrl} \max & 2x_1 & + & x_2 & \\ & x_1 & + & 2x_2 & \le 14 \\ & 2x_1 & - & x_2 & \le 10 \\ & x_1 & - & x_2 & \le 3 \\ & x_1 & , & x_2 & \ge 0. \end{array}$$

1. Write the dual problem.
2. Verify that $\bar{x} = (\frac{20}{3}, \frac{11}{3})$ is a feasible solution.
3. Show that $\bar{x}$ is optimal using the complementary slackness theorem, and determine the optimal solution of the dual problem. [*Pietro Belotti, Carnegie Mellon University*]

2.7 Sensitivity analysis

Consider the problem:

$$\left.\begin{array}{rrcl} \min & x_1 - 5x_2 & & \\ & -x_1 + x_2 & \le & 5 \\ & x_1 + 4x_2 & \le & 40 \\ & 2x_1 + x_2 & \le & 20 \\ & x_1, x_2 & \ge & 0. \end{array}\right\}$$

1. Check that the feasible solution $x^* = (4, 9)$ is also optimal.
2. Which among the constraints' right hand sides should be changed to decrease the optimal objective function value, supposing this change does not change the optimal basis? Should the change be a decrease or an increase?

2.8 Dual simplex method

Solve the following LP problem using the dual simplex method.

$$\begin{array}{rll} \min & 3x_1 + 4x_2 + 5x_3 & \\ & 2x_1 + 2x_2 + x_3 & \geq 6 \\ & x_1 + 2x_2 + 3x_3 & \geq 5 \\ & x_1, x_2, x_3 & \geq 0. \end{array}$$

What are the advantages with respect to the primal simplex method?

Chapter 3

Integer programming

3.1 Piecewise linear objective

Reformulate the problem $\min\{f(x) \mid x \in \mathbb{R}_{\geq 0}\}$, where:

$$f(x) = \begin{cases} -x+1 & 0 \leq x < 1 \\ x-1 & 1 \leq x < 2 \\ \frac{1}{2}x & 2 \leq x \leq 3 \end{cases}$$

as a Mixed-Integer Linear Programming problem.

3.2 Gomory cuts

Solve the following problem using Gomory's cutting plane algorithm.

$$\left.\begin{array}{lrcl} \min & x_1 - 2x_2 & & \\ & -4x_1 + 6x_2 & \leq & 9 \\ & x_1 + x_2 & \leq & 4 \\ & x \geq 0 & , & x \in \mathbb{Z}^2 \end{array}\right\}$$

[Bertsimas & Tsitsiklis, *Introduction to Linear Optimization*, Athena Scientific, Belmont, 1997.]

3.3 Branch and Bound I

Solve the following problem using the Branch and Bound algorithm.

$$\left.\begin{array}{lrcl} \max & 2x_1 + 3x_2 & & \\ & x_1 + 2x_2 & \leq & 3 \\ & 6x_1 + 8x_2 & \leq & 15 \\ & x_1, x_2 & \in & \mathbb{Z}_+ \end{array}\right\}$$

Each LP subproblem may be solved graphically.

3.4 Branch and Bound II

Solve the following problem using the Branch and Bound algorithm.

$$\begin{aligned}
\max z^* &= 3x_1 + 4x_2 \\
2x_1 + x_2 &\leq 6 \\
2x_1 + 3x_2 &\leq 9 \\
x_1, x_2 &\geq 0, \text{ intere}
\end{aligned}$$

Each LP subproblem may be solved graphically.

3.5 Knapsack Branch and Bound

An investment bank has a total budget of 14 million euros, and can make 4 types of investments (numbered 1,2,3,4). The following tables specifies the amount to be invested and the net revenue for each investment. Each investment must be made in full if made at all.

Investment	1	2	3	4
Amount	5	7	4	3
Net revenue	16	22	12	8

Formulate an integer linear program to maximize the total net revenue. Suggest a way, beside the simplex algorithm, to solve the continuous relaxation of the problem, and use it within a Branch-and-Bound algorithm to solve the problem. [*E. Amaldi, Politecnico di Milano*]

Chapter 4

Easy modelling problems

4.1 Compact storage of similar sequences

One practical problem encountered during the DNA mapping process is that of compactly storing extremely long DNA sequences of the same length which do not differ greatly. We consider here a simplified version of the problem with sequences of 2 symbols only (0 and 1). The *Hamming distance* between two sequences $a, b \in \mathbb{F}_2^n$ is defined as $\sum_{i=1}^{n} |a_i - b_i|$, i.e. the number of bits which should be flipped to transform a into b. For example, on the following set of 6 sequences below, the distance matrix is as follows:

1. `011100011101`
2. `101101011001`
3. `110100111001`
4. `101001111101`
5. `100100111101`
6. `010101011100`

	1	2	3	4	5	6
1	0	4	4	5	4	3
2	-	0	4	3	4	5
3	-	-	0	5	2	5
4	-	-	-	0	3	6
5	-	-	-	-	0	5
6	-	-	-	-	-	0

As long as the Hamming distances are not too large, a compact storage scheme can be envisaged where we only store one complete sequence and all the differences which allow the reconstruction of the other sequences. Explain how this problem can be formulated to find a spanning tree of minimum cost in a graph. Solve the problem for the instance given above. [*E. Amaldi, Politecnico di Milano*]

4.2 Communication of secret messages

Given a communication network the probability that a secret message is intercepted along a link connecting node i to j is p_{ij}. Explain how you can model the problem of broadcasting the secret message to every node minimizing the interception probability as a minimum spanning tree problem on a graph. [*E. Amaldi, Politecnico di Milano*]

4.3 Mixed production

A firm is planning the production of 3 products A_1,A_2,A_3. In a month production can be active for 22 days. In the following tables are given: maximum demands (units=100kg), price ($/100Kg), production costs (per 100Kg of product), and production quotas (maximum amount of 100kg units of product that would be produced in a day if all production lines were dedicated to the product).

Product	A_1	A_2	A_3
Maximum demand	5300	4500	5400
Selling price	$124	$109	$115
Production cost	$73.30	$52.90	$65.40
Production quota	500	450	550

1. Formulate an AMPL model to determine the production plan to maximize the total income.
2. Change the mathematical program and the AMPL model to cater for a fixed activation cost on the production line, as follows:

Product	A_1	A_2	A_3
Activation cost	$170000	$150000	$100000

3. Change the mathematical program and the AMPL model to cater for both the fixed activation cost and for a minimum production batch:

Product	A_1	A_2	A_3
Minimum batch	20	20	16

[*E. Amaldi, Politecnico di Milano*]

4.4 Production planning

A firm is planning the production of 3 products A_1, A_2, A_3 over a time horizon of 4 months (january to april). Demand for the products over the months is as follows:

Demand	January	February	March	April
A_1	5300	1200	7400	5300
A_2	4500	5400	6500	7200
A_3	4400	6700	12500	13200

Prices, production costs, production quotas, activation costs and minimum batches (see Ex. 4.3 for definitions of these quantities) are:

Product	A_1	A_2	A_3
Unit prices	$124	$109	$115
Activation costs	$150000	$150000	$100000
Production costs	$73.30	$52.90	$65.40
Production quotas	500	450	550
Minimum batches	20	20	16

There are 23 productive days in january, 20 in february, 23 in march and 22 in april. The activation status of a production line can be changed every month. Minimum batches are monthly.

Moreover, storage space can be rented at monthly rates of \$3.50 for A_1, \$4.00 for A_2 and \$3.00 for A_3. Each product takes the same amount of storage space. The total available volume is 800 units.

Write a mathematical program to maximize the income, and solve it with AMPL. [*E. Amaldi, Politecnico di Milano*]

4.5 Transportation

An Italian transportation firm should carry some empty containers from its 6 stores (in Verona, Perugia, Rome, Pescara, Taranto and Lamezia) to the main national ports (Genoa, Venice, Ancona, Naples, Bari). The container stocks at the stores are the following:

	Empty containers
Verona	10
Perugia	12
Rome	20
Pescara	24
Taranto	18
Lamezia	40

The demands at the ports are as follows:

	Container demand
Genoa	20
Venice	15
Ancona	25
Naples	33
Bari	21

Transportation is carried out by a fleet of lorries. The transportation cost for each container is proportional to the distance travelled by the lorry, and amounts to 30 euro / km. Every lorry can carry at most 2 containers. Distances are as follows:

	Genoa	Venice	Ancona	Naples	Bari
Verona	290 km	115 km	355 km	715 km	810 km
Perugia	380 km	340 km	165 km	380 km	610 km
Rome	505 km	530 km	285 km	220 km	450 km
Pescara	655 km	450 km	155 km	240 km	315 km
Taranto	1010 km	840 km	550 km	305 km	95 km
Lamezia	1072 km	1097 km	747 km	372 km	333 km

Write a mathematical program to find the minimal cost transportation policy and solve it with AMPL. [*E. Amaldi, Politecnico di Milano*]

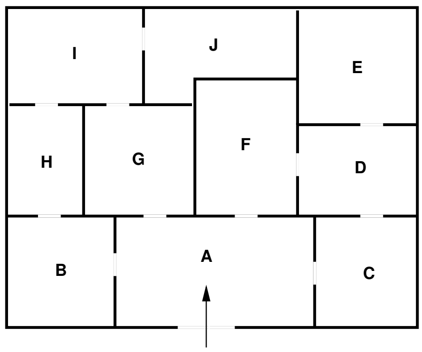
4.6 Project planning with precedences

A project consists of the following 7 activities, whose length in days is given in brackets: A (4), B (3), C (5), D (2), E (10), F (10), G (1). The following precedences are also given: $A \to G, D$; $E, G \to F$;

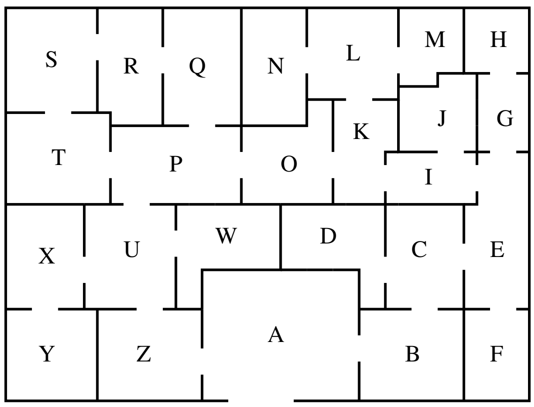
$D, F \to C$; $F \to B$. Each day of work costs 1000 euros; furthermore a special machinery must be rented from the beginning of activity A to the end of activity B at a daily cost of 5000 euros. Formulate this as an LP problem and suggest an algorithm for solving it. [*F. Malucelli, Politecnico di Milano*]

4.7 Museum guards

A museum director must decide how many guards should be employed to control a new wing. Budget cuts have forced him to station guards at each door, guarding two rooms at once. Formulate a mathematical program to minimize the number of guards. Solve the problem on the map below using AMPL.



Also solve the problem on the following map.



[*P. Belotti, Carnegie Mellon University*]

4.8 Inheritance

A rich aristocrat passes away, leaving the following legacy:

- A Caillebotte picture: 25000$
- A bust of Diocletian: 5000$
- A Yuan dinasty chinese vase: 20000$

- A 911 Porsche: 40000$
- Three diamonds: 12000$ each
- A Louis XV sofa: 3000$
- Two very precious Jack Russell race dogs: 3000$ each (the will asserts that they may not be separated)
- A sculpture dated 200 A.D.: 10000$
- A sailing boat: 15000$
- A Harley Davidson motorbike: 10000$
- A piece of furniture that once belonged to Cavour: 13.000$,

which must be shared between the two sons. What is the partition that minimizes the difference between the values of the two parts? Formulate a mathematical program and solve it with AMPL. [*P. Belotti, Carnegie Mellon*]

4.9 Carelland

The independent state of Carelland mainly exports four goods: steel, engines, electronic components and plastics. The Chancellor of the Exchequer (a.k.a. the minister of economy) of Carelland wants to maximize exports and minimize imports. The unit prices on the world markets for steel, engines, electronics and plastics, expressed in the local currency (the Klunz) are, respectively: 500, 1500, 300, 1200. Producing 1 steel unit requires 0.02 engine units, 0.01 plastics units, 250 Klunz in other imported goods and 6 man-months of work. Producing 1 engine unit requires 0.8 steel units, 0.15 electronics units, 0.11 plastics units, 300 Klunz in imported goods and 1 man-year. One electronics unit requires: 0.01 steel units, 0.01 engine units, 0.05 plastics units, 50 Klunz in imported goods and 6 man-months. One plastics unit requires: 0.03 engine units, 0.2 steel units, 0.05 electronics units, 300 Klunz in imported goods and 2 man-years. Engine production is limited to 650000 units, plastics production to 60000 units. The total available workforce is 830000 each year. Steel, engines, electronics and plastics cannot be imported. Write a mathematical program that maximizes the gross internal product and solve the problem with AMPL. [*G. Carello, Politecnico di Milano*]

4.10 CPU Scheduling

10 tasks must be run on 3 CPUs at 1.33, 2 and 2.66 GHz (each processor can run only one task at a time). The number of elementary operations of the tasks (expressed in billions of instructions (BI)) is as follows:

process	1	2	3	4	5	6	7
BI	1.1	2.1	3	1	0.7	5	3

Schedule tasks to processors so that the completion time of the last task is minimized. Solve the problem with AMPL.

4.11 Dyeing plant

A fabric dyeing plant has 3 dyeing baths. Each batch of fabric must be dyed in each bath in the order: first, second, third bath. The plant must colour five batches of fabric of different sizes. Dyeing batch i in bath j takes a time s_{ij} expressed in hours in the matrix below:

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 1.5 & 1 \\ 3 & 1.2 & 1.3 \\ 2 & 2 & 2 \\ 2.1 & 2 & 3 \end{pmatrix}.$$

Schedule the dyeing operations in the baths so that the ending time of the last batch is minimized.

4.12 Parking

On Dantzig Street cars can be parked on both sides of the street. Mr. Edmonds, who lives at number 1, is organizing a party for around 30 people, who will arrive in 15 cars. The length of the i-th car is λ_i, expressed in meters as follows:

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
λ_i	4	4.5	5	4.1	2.4	5.2	3.7	3.5	3.2	4.5	2.3	3.3	3.8	4.6	3

In order to avoid bothering the neighbours, Mr. Edmonds would like to arrange the parking on both sides of the street so that the length of the street occupied by his friends' cars should be minimum. Give a mathematical programming formulation and solve the problem with AMPL.

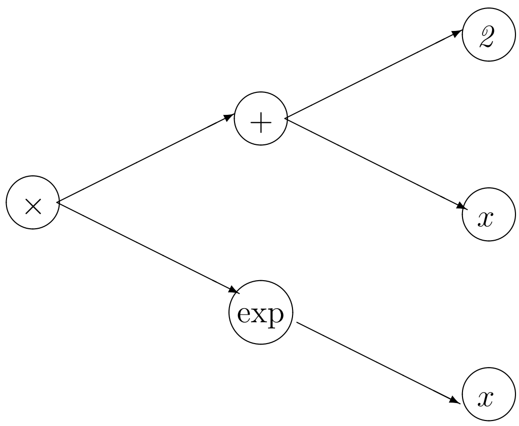
How does the program change if on exactly one of the street sides the cars should not occupy more than 15m?

Chapter 5

Difficult modelling problems

5.1 Checksum

An *expression parser* is a program that reads mathematical expressions (input by the user as strings) and evaluates their values on a set of variable values. This is done by representing the mathematical expression as a directed binary tree. The leaf nodes represent variables or constants; the other nodes represent binary (or unary) operators such as arithmetic (+, -, *, /, power) or transcendental (sin, cos, tan, log, exp) operators. The unary operators are represented by a node with only one arc in its outgoing star, whereas the binary operators have two arcs. The figure below is the binary expression tree for $(x + 2)e^x$.



The expression parser consists of several subroutines.

- `main()`: the program entry point;
- `parse()`: reads the string containing the mathematical expression and transforms it into a binary expression tree;
- `gettoken()`: returns and deletes the next semantic token (variable, constant, operator, brackets) from the mathematical expression string buffer;
- `ungettoken()`: pushes the current semantic token back in the mathematical expression string buffer;
- `readexpr()`: reads the operators with precedence 4 (lowest: +,-);

- `readterm()`: reads the operators with precedence 3 (*, /);
- `readpower()`: reads the operators with precedence 2 (power);
- `readprimitive()`: reads the operators of precedence 1 (functions, expressions in brackets);
- `sum(term` a`, term` b`)`: make a tree $+ \begin{smallmatrix} \nearrow & a \\ \searrow & b \end{smallmatrix}$;
- `difference(term` a`, term` b`)`: make a tree $- \begin{smallmatrix} \nearrow & a \\ \searrow & b \end{smallmatrix}$;
- `product(term` a`, term` b`)`: make a tree $* \begin{smallmatrix} \nearrow & a \\ \searrow & b \end{smallmatrix}$;
- `fraction(term` a`, term` b`)`: make a tree $/ \begin{smallmatrix} \nearrow & a \\ \searrow & b \end{smallmatrix}$;
- `power(term` a`, term` b`)`: make a tree $\wedge \begin{smallmatrix} \nearrow & a \\ \searrow & b \end{smallmatrix}$;
- `minus(term` a`)`: make a tree $- \to a$;
- `logarithm(term` a`)`: make a tree make a tree $\log \to a$;
- `exponential(term` a`)`: make a tree make a tree $\exp \to a$;
- `sine(term` a`)`: make a tree make a tree $\sin \to a$;
- `cosine(term` a`)`: make a tree make a tree $\cos \to a$;
- `tangent(term` a`)`: make a tree make a tree $\tan \to a$;
- `variable(var` x`)`: make a leaf node x;
- `number(double` d`)`: make a leaf node d;
- `readdata()`: reads a table of variable values from a file;
- `evaluate()`: computes the value of the binary tree when substituting each variable with the corresponding value;
- `printresult()`: print the results.

For each function we give the list of called functions and the quantity of data to be passed during the call.

- main: readdata (64KB), parse (2KB), evaluate (66KB), printresult(64KB)
- evaluate: evaluate (3KB)
- parse: gettoken (0.1KB), readexpr (1KB)
- readprimitive: gettoken (0.1KB), variable (0.5KB), number (0.2KB), logarithm (1KB), exponential (1KB), sine (1KB), cosine (1KB), tangent (1KB), minus (1KB), readexpr (2KB)
- readpower: power (2KB), readprimitive (1KB)
- readterm: readpower (2KB), product (2KB), fraction (2KB)
- readexpr: readterm (2KB), sum (2KB), difference (2KB)

- gettoken: ungettoken (0.1KB)

Each function call requires a bidirectional data exchange between the calling and the called function. In order to guarantee data integrity during the function call, we require that a checksum operation be performed on the data exchanged between the pair (calling function, called function). Such pairs are called *checksum pairs*. Since the checksum operation is costly in terms of CPU time, we limit these operations so that no function may be involved in more than one checksum pair. Naturally though, we would like to maximize the total quantity of data undergoing a checksum.

1. Formulate a mathematical program to solve the problem, and solve the given instance with AMPL.

2. Modify the model to ensure that `readprimitive()` and `readexpr()` are a checksum pair. How does the solution change?

5.2 Eight queens

Formulate an integer linear program to solve the problem of positioning eight queens on the chessboard so that no queen is under threat by any other queen. Solve this program with AMPL. [*P. Belotti, Carnegie Mellon University*]

5.3 Production management

A firm which produces only one type of product has 40 workers. Each one of them produces 20 units per month. The demand varies during the semester according to the following table:

Month	1	2	3	4	5	6
Demand (units)	700	600	500	800	900	800

In order to increase/decrease production depending on the demand, the firm can offer some (paid) extra working time (each worker can produce at most 6 additional units per month at unit cost of 5 euros), use a storage space (10 euros/month per unit of product), employ or dismiss personnel (the number of employed workers can vary by at most ± 5 per month at an additional price of 500 euros per employment and 700 euros per dismissal).

At the outset, the storage space is empty, and we require that it should be empty at the end of the semester. Formulate a mathematical program that maximizes the revenues, and solve it with AMPL. How does the objective function change when all variables are relaxed to be continuous? [*E. Amaldi, Politecnico di Milano*]

5.4 The travelling salesman problem

A travelling salesman must visit 7 customers in 7 different locations whose (symmetric) distance matrix is:

	1	2	3	4	5	6	7
1	-	86	49	57	31	69	50
2		-	68	79	93	24	5
3			-	16	7	72	67
4				-	90	69	1
5					-	86	59
6						-	81

Formulate a mathematical program to determine a visit sequence starting at ending at location 1, which minimizes the travelled distance, and solve it with AMPL. Knowing that the distances obey a triangular inequality and are symmetric, propose a suitable heuristic method.

5.5 Optimal rocket control 1

A rocket of mass m is launched at sea level and has to reach an altitude H within time T. Let $y(t)$ be the altitude of the rocket at time t and $u(t)$ the force acting on the rocket at time t in the vertical direction. Assume $u(t)$ may not exceed a given value b, that the rocket has constant mass m throughout, and that the gravity acceleration g is constant in the interval $[0, H]$. Discretizing time $t \in [0, T]$ in n intervals, propose a linear program to determine, for each $k \leq n$, the force $u(t_k)$ acting on the rocket so that the total consumed energy is minimum. Solve the problem with AMPL with the following data: $m = 2140$kg, $H = 23$km, $T = 1$min, $b = 60000$N, $n = 20$.

5.6 Double monopoly

In 2021, all the world assets are held by the AA (Antidemocratic Authorities) bank. In 2022, a heroical judge manages to apply the ancient (but still existing) anti-trust regulations to the AA bank, right before being brutally decapitated by its vicious corporate killers. A double monopoly situation is thus established with the birth of the BB (Bastard Business) bank. In a whirlpool of flagrantly illegal acts that are approved thanks to the general public being hypnotized by the highly popular television programme "The International Big Brother 26", the AA bank manages to insert a codicil in the anti-trust law that ensures that BB may not draw any customer without the prior approval of AA. However, in an era where the conflict of interest is not only accepted but even applauded, AA's lawyers are the same as BB's, so the same codicil is inserted in the law in favour of BB. When the law is finally enforced, as usually happens when two big competitors share the market, AA and BB get together and develop some plans to make as much money as they can without damaging each other too much. It decided that each bank must, independently of the codicil, make at least one investment. When operations begin, the following situation occurs: there are 6 big customers and a myriad of small private individuals with their piddly savings accounts which are progressively deprived of everything they have and which can, to the end of this exercise (but let's face it — to every other end, too), be entirely disregarded. The effect of the codicil and of the bank agreement brings about a situation where the quantity of money earned by AA in a given investment is exactly the same as the quantity of money lost by BB for not having made the same investment (and vice versa). The following 6×6 matrix $A = (a_{ij})$ represents the revenues and losses of the two banks:

$$A = \begin{pmatrix} 1 & -2 & 1 & 3 & -5 & 2 \\ 4 & 1 & 1 & 3 & 2 & -1 \\ 1 & 10 & -6 & 3 & 1 & 4 \\ 2 & 2 & 1 & 3 & 3 & -8 \\ -12 & 1 & 3 & -4 & 2 & 1 \\ 3 & 3 & -1 & 4 & 2 & 2 \end{pmatrix}.$$

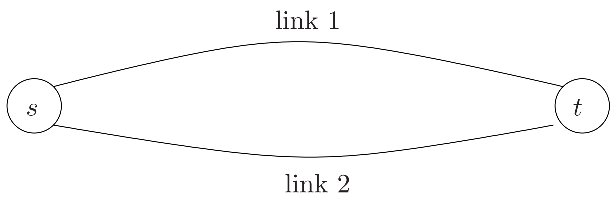
a_{ij} represents the revenue of AA and loss of BB if AA invests *all* of its budget in client i and BB *all* of its budget in client j. The banks may naturally decide to get more than one client, but without exceeding their budgets (which amount to exactly the same amount: 1 fantastillion dollars). Even to the bank managers it appears evident that AA's optimal strategy is to maximize the expected revenues and BB's to minimize the expected loss. Write two linear programs: one to model the expected revenues of AA and the other to model the losses of BB. Comment on the relations between the models.

Chapter 6

Telecommunication networks

6.1 Packet routing

There are n data flows that must be routed from a source node s to a destination node t following one of two possible links, with capacity $u_1 = 1$ and $u_2 = 2$ respectively.



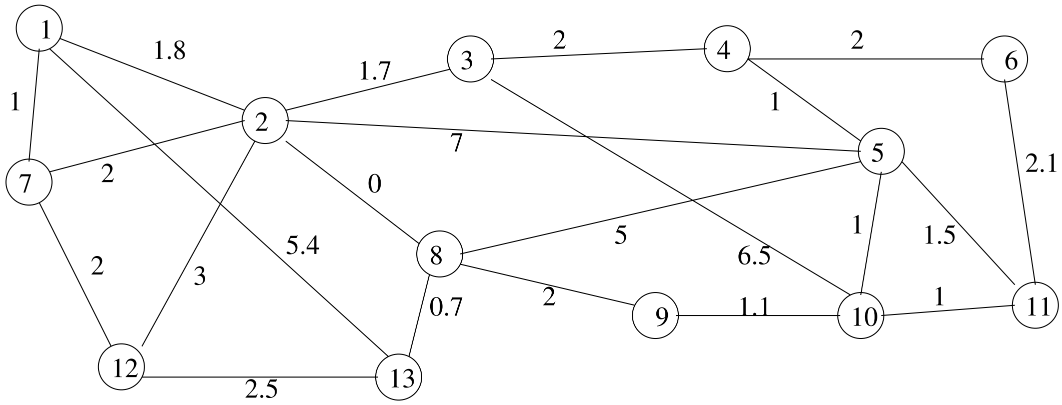
The company handling link 2 is 30% more expensive than the company handling link 1. The table below specifies the demands to be routed and the cost on link 1.

Demand	Required capacity (Mbps)	Cost on link 1
1	0.3	200
2	0.2	200
3	0.4	250
4	0.1	150
5	0.2	200
6	0.2	200
7	0.5	700
8	0.1	150
9	0.1	150
10	0.6	900

Formulate a mathematical program to minimize the routing cost of all the demands. How would you change the model to generalize to a situation with m possible parallel links between s and t?

6.2 Network Design

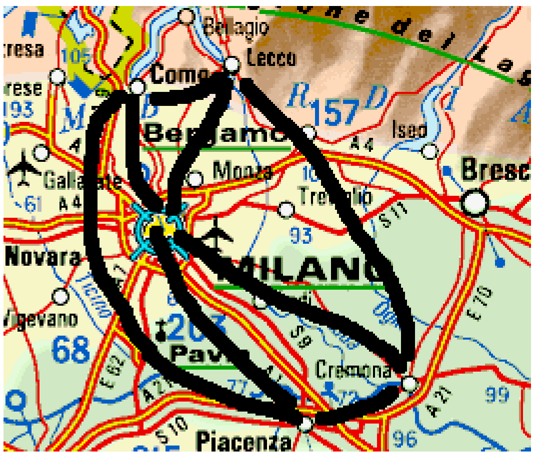
Orange is the unique owner and handler of the telecom network in the figure below.



The costs on the links are proportional to the distances $d(i, j)$ between the nodes, expressed in units of 10km. Because of anti-trust regulations, Orange must delegate to SFR and Bouygtel two subnetworks each having at least two nodes (with Orange handling the third part). Orange therefore needs to design a backbone network to connect the three subnetworks. Transforming an existing link into a backbone link costs $c = 25$ euros/km. Formulate a mathematical program to minimize the cost of implementing a backbone connecting the three subnetworks, and solve it with AMPL. How does the solution change if Orange decides to partition its network in 4 subnetworks instead of 3?

6.3 Network Routing

The main telephone network backbone connecting the different campuses of the Politecnico di Milano (at Milano, Como, Lecco, Piacenza, Cremona) has grown over the years to its present state without a clear organized plan. Politecnico asked its main network provider to optimize the routing of all the traffic demands to see whether the installed capacity is excessive. The network topology is as depicted below.



For each link there is a pair (u, c) where u is the link capacity (Mb/s) and c the link length (km).

1. Como, Lecco: (200, 30)
2. Como, Milano: (260, 50)
3. Como, Piacenza: (200, 110)
4. Lecco, Milano: (260, 55)

5. Lecco, Cremona: (200, 150)
6. Milano, Piacenza: (260, 72)
7. Milano, Cremona: (260, 90)
8. Piacenza, Cremona: (200, 100)

The traffic demands to be routed (in Mb/s) are as follows.

1. Como, Lecco: 20
2. Como, Piacenza: 30
3. Milano, Como: 50
4. Milano, Lecco: 40
5. Milano, Piacenza: 60
6. Milano, Cremona: 25
7. Cremona, Lecco: 35
8. Cremona, Piacenza: 30

The current routing is as given below.

1. Como → Lecco
2. Como → Milano → Piacenza
3. Milano → Lecco → Como
4. Milano → Como → Lecco
5. Milano → Piacenza
6. Milano → Piacenza → Cremona
7. Cremona → Milano → Piacenza → Como → Lecco
8. Cremona → Milano → Como → Lecco → Milano → Piacenza

Formulate a mathematical program to find an optimal network routing. [with *P. Belotti, Carnegie Mellon University*]

Chapter 7

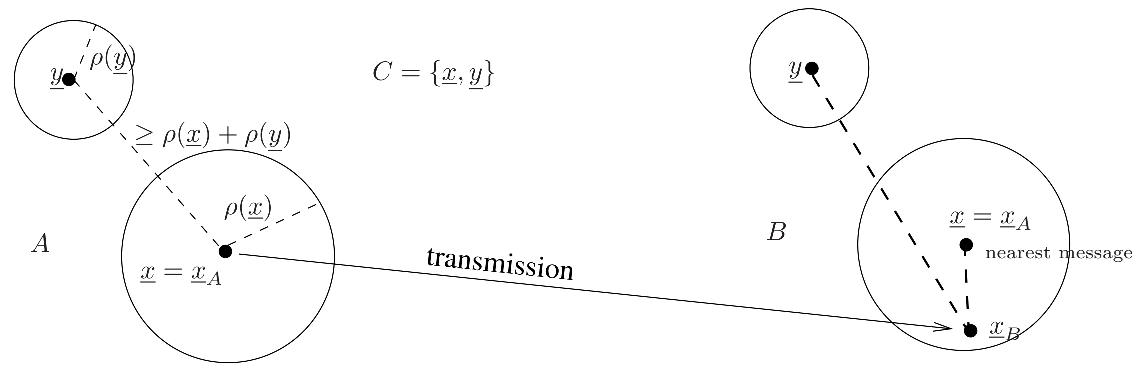
Nonlinear programming

7.1 Error correcting codes

A message sent by A to B is represented by a vector $\underline{z} = (\underline{z}_1, \ldots, \underline{z}_m) \in \mathbb{R}^m$. An *Error Correcting Code* (ECC) is a finite set C (with $|C| = n$) of messages with an associated function $\rho : C \to \mathbb{R}$, such that for each pair of distinct messages $\underline{x}, \underline{y} \in C$ the inequality $||\underline{x} - \underline{y}|| \geq \rho(\underline{x}) + \rho(\underline{y})$ holds. The *correction radius* of code C is given by

$$R_C = \min_{\underline{x} \in C} \rho(\underline{x}),$$

and represents the maximum error that can be corrected by the code. Assume both A and B know the code C and that their communication line is faulty. A message $\underline{x}_A \in C$ sent by A gets to B as $\underline{x}_B \notin C$ because of the faults. Supposing the error in $\underline{x}_B$ is strictly less than R_C, B is able to reconstruct the original message $\underline{x}_A$ looking for the message $\underline{x} \in C$ closest to $\underline{x}_B$ as in the figure below.

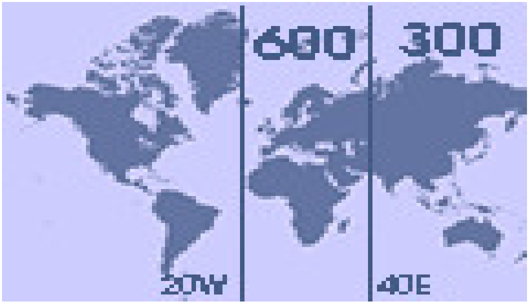


Formulate a (nonlinear) mathematical program to build an ECC C of 10 messages in $\mathbb{R}^3$ (where all message components are in $[0, 1]$) so that the correction radius is maximized. What kind of mathematical program is this? What solver are you going to use? Implement your program in AMPL and compute the error correcting code C, ρ.

7.2 Airplane maintenance

Boeing needs to build 5 maintenance centers for the Euro-Asian area. The construction cost for each center is 300 million euros in the European area (between 20°W and 40°E) and 150 million euros in the

Asian area (between 40°E and 160°E), as shown below.



Each center can service up to 60 airplanes each year. The centers should service the airports with the highest number of Boeing customers, as detailed in the table below (airport name, geographical coordinates, expected number of airplanes/year needing maintenance).

Airport	**Coordinates**		**N. of planes**
London Heathrow	51°N	0°W	30
Frankfurt	51°N	8°E	35
Lisboa	38°N	9°W	12
Zürich	47°N	8°E	18
Roma Fiumicino	41°N	12°E	13
Abu Dhabi	24°N	54°E	8
Moskva Sheremetyevo	55°N	37°E	15
Vladivostok	43°N	132°E	7
Sydney	33°S	151°E	32
Tokyo	35°N	139°E	40
Johannesburg	26°S	28°E	11
New Dehli	28°N	77°E	20

The total cost is given by the construction cost plus the expected servicing cost, which depends linearly on the distance an airplane needs to travel to reach the maintenance center (weighted by 50 euro/km). We assume earth is a perfect sphere, so that the shortest distance between two points with geographical coordinates (δ_1, φ_1) and (δ_2, φ_2) is given by:

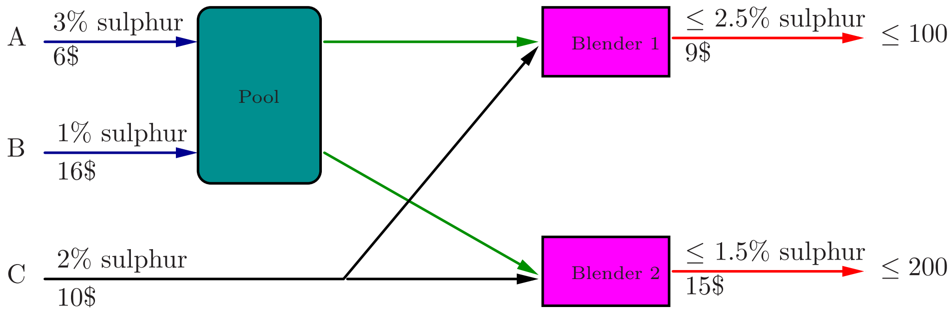
$$d(\delta_1, \varphi_1, \delta_2, \varphi_2) = 2r \text{ asin}\sqrt{\sin^2\left(\frac{\delta_1 - \delta_2}{2}\right) + \cos\delta_1 \cos\delta_2 \sin^2\left(\frac{\varphi_1 - \varphi_2}{2}\right)},$$

where r, the earth radius, is 6371km.

Formulate a (nonlinear) mathematical program to minimize the operation costs.

7.3 Pooling problem

The blending plant in a refinery is composed by a pool and two mixers as shown in the picture below.



Two types of crude A, B whose unitary cost and sulphur percentage are 6$, 16$ and 3%, 1% respectively enter the pool through two input valves. The output of the pool is then carried to the mixers, together with some crude of type C (unit cost 10%, sulphur percentage 2%) which enters the plant through a third input valve. Mixer 1 must produce petrol containing at most 2.5% sulphur, which will be sold at a unitary price of 9$. Mixer 2 must produce a more refined petrol containing at most 1.5% sulphur, sold at a unitary price of 15$. The maximum market demand for the refined petrol 1 is of 100 units, and of 200 units for refined petrol 2. Formulate a (nonlinear) mathematical program to maximize revenues. Does your program describe a convex programming problem? Propose and implement (with AMPL and CPLEX) a heuristic algorithm to find a feasible, and hopefully good, solution to the problem.

[Haverly, *Studies of the behaviour of recursion for the pooling problem*, ACM SIGMAP Bulletin **25**:19-28, 1978]

7.4 Optimal rocket control 2

A rocket of mass m is launched at sea level and has to reach an altitude H within time T. Let $y(t)$ be the altitude of the rocket at time t and $u(t)$ the force acting on the rocket at time t in the vertical direction. Assume: (a) $u(t)$ may not exceed a given value b; (b) the rocket has initial mass $m = m_0 + c$ (where c is the mass of the fuel) and loses $\alpha u(t)$ kg of mass (burnt fuel) each second; (c) the gravity acceleration g is constant in the interval $[0, H]$. Discretizing time $t \in [0, T]$ in n intervals, propose a (nonlinear) mathematical program to determine, for each $k \leq n$, the force $u(t_k)$ acting on the rocket so that the total consumed energy is minimum.

7.5 Economies of scale

A production site that manufactures a given type of machinery is ordering parts for the next week. Each piece of produced machinery requires a certain number d_i of parts of each type, out of a set of $n = 10$ part numbers. Moreover, each unit of each part number requires a certain amount of storage space s_i and is sold at a certain base price b_i, as follows:

i	1	2	3	4	5	6	7	8	9	10
d_i	10	8	4	9	12	18	10	5	1	100
s_i (m^3)	1	3	1.5	2	0.6	0.6	0.9	4	5	0.5
b_i (EUR)	5.60	2.30	1.20	3.45	5.12	1.00	1.43	8.50	6.32	4.01

The supplier can deliver orders of at most 4000 units per part. The total storage space at the production site is $S = 6250\text{m}^3$. Because of the economies of scale, the actual unit price paid p_i (for $i \in \{1, \ldots, n\}$) consists of a fixed part (the base price b_i) and a (negative) variable part p_i that depends on the ordered quantity x:

$$\forall x \geq 1 \quad p_i(x) = b_i + (\frac{b_i/2}{\sqrt{x_i}} - b_i/2).$$

The sales forecasts for the next quarter determine a market demand of at least 30 machinery units.

1. Write a mathematical program that helps you quantify part orders so as to minimize the total cost, meet the market demands and not exceed storage space.
2. What kind of mathematical program is this? What kind of solver is required in order to obtain a locally optimal solution? And a globally optimal one?
3. Use AMPL to compute an answer. What are the savings due to the economies of scale? Is the storage space completely filled?
4. What is the maximum market demand that can be met?