Operations Research — Mock examination paper
ISC612

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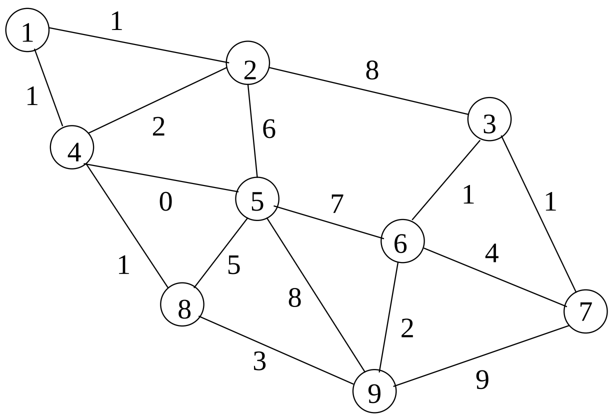
Here follows a list of OR exercises of the difficulty and type that is likely to arise in the exam.

1 Optimization on graphs

1. Apply Prim's algorithm to the graph $G = (V, E)$ below to find the spanning tree of minimum cost. Describe each step of the algorithm graphically. Now suppose the cost c_{45} on the edge $\{4, 5\}$ is set to the value 12. How can you modify the spanning tree so that it is optimal with respect to this edge cost change, without having to re-apply Prim's algorithm from the start? *[Liberti]*



2. Compute the cost of the Fundamental Cycle Basis associated to both spanning trees found in Exercise 1. Can you find a Fundamental Cycle Basis with lower cost? *[Liberti]*

3. Dijkstra's algorithm finds all shortest paths from a given root vertex to all other vertices on an *directed* graph. How do you proceed to apply it to an undirected graph? Apply Dijkstra's algorithm from root vertex 1 to the graph $G = (V, E)$ of Exercise 1. *[Liberti]*

4. Describe an algorithmic procedure to find a shortest path between two given vertices $s, t \in V$ on a weighted undirected graph $G = (V, E)$, and apply it to the graph of Exercise 1, first with $s = 1, t = 5$, then with $s = 1, t = 7$. *[Liberti]*

5. In the graph $G = (V, E)$ of Exercise 1, let the cost c_{25} on the edge $\{2, 5\}$ take the value -6. What is the shortest path from vertex 1 to vertex 6? *[Liberti]*

2 Dynamic programming

No mock question was readied in this section. Either contact Philippe Baptiste directly, or just carry out some exercises on this subject in any operations research book.

3 Linear programming

1. Give the mathematical programming formulation of an optimization problem with exactly two distinct local minima whose objective function values are 0 and (respectively) 1. *[Liberti]*

2. Give the mathematical programming formulation of an optimization problem with variables vector $x \in \mathbb{R}^3$ whose feasible region has volume $\sqrt{2}$. *[Liberti]*

3. Write a linear programming formulation of a problem with infinitely many local minima. Are these minima also global? *[Liberti]*

4. Write a linear programming formulation of a problem in canonical form with variables vector $x \in \mathbb{R}^2$ where all the vertices of the feasible region are degenerate. *[Liberti]*

5. Given a polyhedron $K = \{x \in \mathbb{R}^n \mid Ax = b \wedge x \geq 0\}$ in standard form, reformulate it to the corresponding polyhedron K' in canonical form. *[Liberti]*

6. Find an example of a linear programming problem where the simplex algorithm passes from a basic feasible solution x to a feasible solution x' where both x, x' correspond to the same vertex of the feasible polyhedron. *[Liberti]*

7. Solve the linear programming problem

$$\max 7x_1 + 8x_2 \tag{1}$$
$$x_1 + 2x_2 + x_3 = 2 \tag{2}$$
$$2x_1 + x_2 \leq 2 \tag{3}$$
$$3x_1 + x_2 \leq 3 \tag{4}$$
$$\forall i \leq 3 \; x_i \geq 0. \tag{5}$$

   Draw a picture in $\mathbb{R}^2$ of its feasible region. *[Liberti]*

8. Write the dual of problem (1)-(5). Using the Karush-Kuhn-Tucker conditions, prove that the point $(0, 0, 2)$ of the problem is not optimal. *[Liberti]*

4 Integer Programming

1. **Total unimodularity.** Are the following matrices totally unimodular or not?

$$A_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} -1 & & 1 & & -1 & & \\ & 1 & & 1 & 1 & & 1 \\ -1 & 1 & & & & & \\ & & 1 & & & 1 & 1 \\ & & & 1 & & -1 & \end{pmatrix}.$$

   [Sadykov]

2. **Convex hull.** Consider the set $X = \{x \in \mathbb{Z}^2_+ : \; x_1 - x_2 \geq -1, 2x_1 + 6x_2 \leq 15, x_1 - x_2 \leq 3, 2x_1 + 4x_2 \leq 7\}$. List and represent graphically the set of feasible points. Use this to find inequalities which describe the convex hull of X. *[Sadykov]*

3. **Gomory inequalities.** Prove that $x_2 + x_3 + 2x_4 \leq 6$ is valid for

$$X = \left\{x \in \mathbb{Z}^4_+ : \; 4y_1 + 5y_2 + 9y_3 + 12y_4 \leq 34\right\}.$$

   [Sadykov]

5 Modelling

Look at the exercises in Chapter 4 of the course's Exercise Book.