Hamiltonian Paths in Two Classes of Grid Graphs

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Abstract

In this paper, we give the necessary and sufficient conditions for the existence of Hamiltonian paths in $L-$alphabet and $C-$alphabet grid graphs. We also present a linear-time algorithm for finding Hamiltonian paths in these graphs.

Key words: Hamiltonian path, Hamiltonian cycle, Grid graph, Alphabet grid graph, Rectangular grid graph.
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1. Introduction

A Hamiltonian path in a graph $G(V, E)$ is a simple path that includes every vertex in V. The problem of deciding whether a given graph G has a Hamiltonian path is a well-known NP-complete problem [2, 3]. Rectangular grid graphs first appeared in [5], where Luccio and Mugnia tried to solve the Hamiltonian path problem. The Hamiltonian path problem was studied for grid graphs in [4], where the authors gave the necessary and sufficient conditions for the existence of Hamiltonian paths in rectangular grid graphs and proved that the problem for general grid graphs is NP-complete. Also, the authors in [8] presented sufficient conditions for a grid graph to be Hamiltonian and proved that all finite grid graphs of positive width have Hamiltonian line graphs. Chen *et al.* [1] improved the algorithm of [4] and presented a parallel algorithm for the problem in mesh architecture. In [7, 6], Salman

et al. determined the classes of alphabet graphs which contain Hamilton cycles. In this paper, we obtain the necessary and sufficient conditions for a $L-$alphabet and $C-$alphabet graphs to have Hamiltonian paths. Also, we present a linear-time algorithm for finding Hamiltonian paths in these graphs.

2. Preliminaries

In this section, we present some definitions and previously established results on the Hamiltonian path problem in grid graphs which appeared in [1, 4, 6, 7].

The *two-dimensional integer grid* G^∞ is an infinite graph with the vertex set of all the points of the Euclidean plane with integer coordinates. In this graph, there is an edge between any two vertices of unit distance. For a vertex v of this graph, let v_x and v_y denote x and y coordinates of its corresponding point. We color the vertices of the two-dimensional integer grid by black and white colors. A vertex v is colored *white* if $v_x + v_y$ is even, and is colored *black* otherwise. A *grid graph* G_g is a finite vertex-induced subgraph of the two-dimensional integer grid. In a grid graph G_g, each vertex has degree at most four. Clearly, there is no edge between any two vertices of the same color. Therefore, G_g is a bipartite graph. Note that any cycle or path in a bipartite graph alternates between black and white vertices. A *rectangular grid graph* $R(m,n)$ (or R for short) is a grid graph whose vertex set is $V(R) = \{v \mid 1 \leq v_x \leq m,\ 1 \leq v_y \leq n\}$. In the figures, we assume that $(1,1)$ is the coordinates of the vertex in the lower left corner. The size of $R(m,n)$ is defined to be $m \times n$. $R(m,n)$ is called *odd-sized* if $m \times n$ is odd, and is called *even-sized* otherwise. $R(m,n)$ is called a *k-rectangle* if $n = k$.

The following lemma states a result about the Hamiltonicity of even-sized rectangular graphs.

Lemma 2.1. [1] *$R(m,n)$ has a Hamiltonian cycle if and only if it is even-sized and $m,n > 1$.*

Two different vertices v and v' in $R(m,n)$ are called *color-compatible* if either both v and v' are white and $R(m,n)$ is odd-sized, or v and v' have different colors and $R(m,n)$ is even-sized. Without loss of generality, we assume $s_x \leq t_x$.

For $m,n \geq 3$, a $L-$alphabet graph $L(m,n)$ (or L for short) and a $C-$alphabet graph $C(m,n)$ (or C for short) are subgraphs of $R(3m-2, 5n-4)$ induced

by $V(R)\backslash\{V|v_x = m+1,\ldots,3m-2$ and $v_y = n+1,\ldots,5n-4\}$, and $V(R)\backslash\{V|v_x = m+1,\ldots,3m-2$ and $v_y = n+1,\ldots,4n-4\}$, respectively. These alphabet graphs are shown in Figure 1 for $m=4$ and $n=3$.
An alphabet graph is called *odd-sized* if its corresponding rectangular graph is odd-sized, and is called *even-sized* otherwise.
In the following by $L(m,n)$ we mean an $L-$alphabet grid graph $L(m,n)$, and by $C(m,n)$ we mean an $C-$alphabet grid graph $C(m,n)$. We use $P(A(m,n),s,t)$ to indicate the problem of finding a Hamiltonian path between vertices s and t in grid graph $A(m,n)$, and use $(A(m,n),s,t)$ to indicate the grid graph $A(m,n)$ with two specified distinct vertices s and t of it, where A is rectangular grid graph, $L-$alphabet graph or $C-$alphabet graph. $(A(m,n),s,t)$ is *Hamiltonian* if there is a Hamiltonian path between s and t in $A(m,n)$. In this paper, since *even* $\times$ *odd* $L-$alphabet ($C-$alphabet) graph and *odd* $\times$ *even* $L-$alphabet graph are isomorphic, then we only consider *even* $\times$ *odd* $L-$alphabet ($C-$alphabet) graphs.
An even-sized grid graph contains the same number of black and white vertices. Hence, the two end-vertices of any Hamiltonian path in the graph must have different colors. Similarly, in an odd-sized grid graph the number of white vertices is one more than the number of black vertices. Therefore, the two end-vertices of any Hamiltonian path in such a graph must be white. Hence, the color-compatibility of s and t is a necessary condition for $(R(m,n),s,t)$ to have Hamiltonian. Furthermore, Itai *et al.* [4] showed that if one of the following conditions hold, then $(R(m,n),s,t)$ is not Hamiltonian:

(F1) $R(m,n)$ is a 1-rectangle and either s or t is not a corner vertex (Figure 2(a)).

(F2) $R(m,n)$ is a 2-rectangle and (s,t) is a nonboundary edge, i.e. (s,t) is an edge and it is not on the outer face (Figure 2(b)).

(F3) $R(m,n)$ is isomorphic to a 3-rectangle $R'(m,n)$ such that s and t are mapped to s' and t', and:

1. m is even,
2. s' is black, t' is white,
3. $s'_y = 2$ and $s'_x < t'_x$ (Figure 2(c)) or $s'_y \neq 2$ and $s'_x < t'_x - 1$ (Figure 2(d)).

A Hamiltonian path problem $P(R(m,n),s,t)$ is *acceptable* if s and t are color-compatible and (R,s,t) does not satisfy any of the conditions $(F1)$,

($F2$) and ($F3$).
The following theorem has been proved in [4].

Theorem 2.1. *Let $R(m,n)$ be a rectangular graph and s and t be two distinct vertices. Then $(R(m,n),s,t)$ is Hamiltonian if and only if $P(R(m,n),s,t)$ is acceptable.*

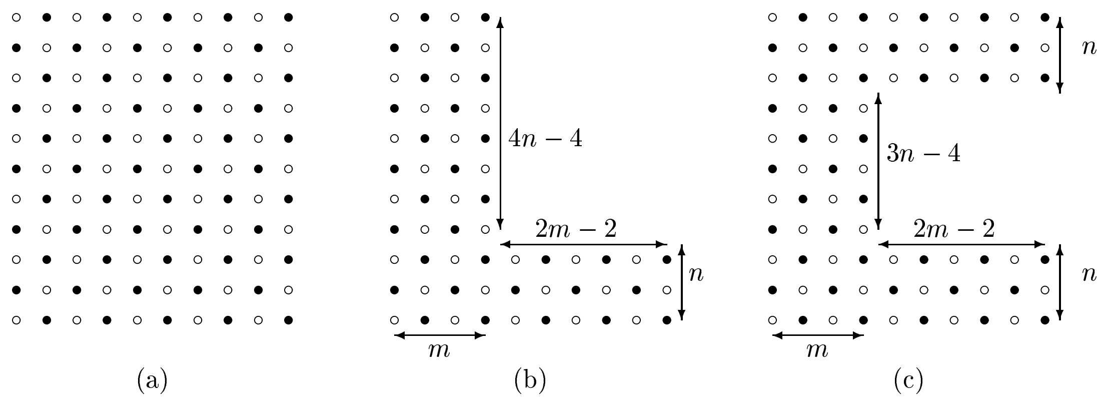


Figure 1: (a) A rectangular grid graph $R(10,11)$, (b) a $L-$alphabet grid graph $L(4,3)$, (c) a $C-$alphabet grid graph $C(4,3)$.

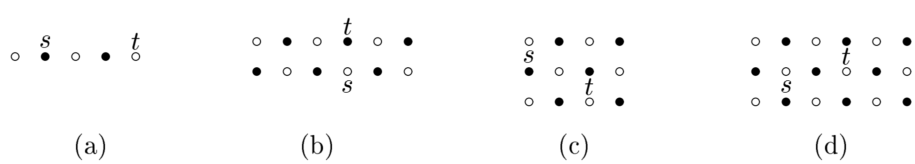


Figure 2: Rectangular grid graphs in which there is no Hamiltonian path between s and t.

3. The Hamiltonian path in alphabet graphs

In this section, we give sufficient and necessary conditions for the existence of a Hamiltonian path in $L-$alphabet and $C-$alphabet graphs. We also present an algorithm for finding a Hamiltonian path between two given vertices of these graphs.
A *separation* of $L-$alphabet graph is a partition of L into two vertex disjoint rectangular grid graphs R_1 and R_2, i.e. $V(L) = V(R_1) \cup V(R_2)$, and

$V(R_1) \cap V(R_2) = \emptyset$. A *separation* of $C-$alphabet graph is a partition of C into a $L-$alphabet graph and a rectangular grid graph.

Lemma 3.1. *Let $A(m,n)$ be a $L-$alphabet or $C-$alphabet grid graph and let R be the smallest rectangular grid graph includes A. If $(A(m,n),s,t)$ has a Hamiltonian path between s and t, then $(R(m,n),s,t)$ also has a Hamiltonian path.*

Proof. Since $R - A$ is a rectangular grid graph with even size of $(2m-2) \times (4n-4)$ or $(2m-2) \times (3n-4)$, then by Lemma 2.1 it has a Hamiltonian cycle. By combining the Hamiltonian cycle and the Hamiltonian path of $(A(m,n),s,t)$ of [1], a Hamiltonian path between s and t for (R,s,t) is obtained. $\square$

Combining Lemma 3.1 and Theorem 2.1 the following corollary is trivial.

Corollary 3.1. *Let $A(m,n)$ be a $L-$alphabet or $C-$alphabet grid graph and R be its smallest including rectangular grid graph. If $(A(m,n),s,t)$ has a Hamiltonian path between s and t, then s and t must be color-compatible in $R(m,n)$.*

Therefore, the color-compatibility of s and t is a necessary condition for $(L(m,n),s,t)$ and $(C(m,n),s,t)$ to have Hamiltonian paths.
The length of a path in a grid graph means the number of vertices of the path. In any grid graph, the length of any path between two same-colored vertices is odd and the length of any path between two different-colored vertices is even.

Lemma 3.2. *Let $L(m,n)$ be a $L-$alphabet grid graph and s and t be two given vertices of L. Let $R(2m-2,n)$ and $R(m,5n-4)$ be a separation of $L(m,n)$. If $t_x > m+1$ and $R(2m-2,n)$ satisfies condition $(F3)$, then $L(m,n)$ does not have any Hamiltonian path between s and t.*

Proof. Without loss of generality, let s and t be color-compatible. Assume that $R(2m-2,n)$ satisfies condition $(F3)$. We show that there is no Hamiltonian path in $L(m,n)$ between s and t. Assume to the contrary that $L(m,n)$ has a Hamiltonian path P. The following cases are possible:

Case 1. In this case, t is in $R(2m-2,n)$ and s is not in $R(2m-2,n)$. Since $n=3$ there are exactly three vertices v, w and u in $R(2m-2,n)$ which are connected to $R(m,5n-4)$, as shown in Figure 3. The following sub-cases

are possible for the Hamiltonian path P.

Case 1.1. The Hamiltonian path P of $L(m,n)$ that starts from s may enter to $R(2m-2,n)$ for the first time through one of the vertices v, w or u and passes through all the vertices of $R(2m-2,n)$ and end at t. This case is not possible because we assumed that $R(2m-2,n)$ satisfies $(F3)$ ($t^{'} = t$ and $s^{'} = w$).

Case 1.2. The Hamiltonian path P of $L(m,n)$ may enter to $R(2m-2,n)$, passes through some vertices of it, then leaves it and enter it again and passes through all the remaining vertices of it and finally ends at t. In this case, two sub-paths of P which are in $R(2m-2,n)$ are called P_1 and P_2, P_1 from v to u (v to w or u to w) and P_2 from w to t (u to t or v to t). This case is not also possible because the size of P_1 is odd (even) and the size of P_2 is even (odd), then $|P_1 + P_2|$ is odd while $R(2m-2,n)$ is even, which is a contradiction.

Case 2. In this case, s and t are in $R(2m-2,n)$. The following cases may be considered:

Case 2.1. The Hamiltonian path P of $L(m,n)$ starts from s passes through some vertices of $R(2m-2,n)$, leaves $R(2m-2,n)$ at v (u), then passes through all the vertices of $R(m,5n-4)$ and reenter to $R(2m-2,n)$ at w goes to $u(v)$ and passes through all the remaining vertices of $R(2m-2,n)$ and ends at t. In this case by connecting $v(u)$ to w we obtain a Hamiltonian path from s to t in $R(2m-2,n)$, which contradicts to the assumption that $R(2m-2,n)$ satisfies $(F3)$, see Figure 4(a).

Case 2.2. The Hamiltonian path P of $L(m,n)$ starts from s leaves $R(2m-2,n)$ at $v(u)$, then passes through all the vertices of $R(m,5n-4)$ and reenter to $R(2m-2,n)$ at $u(v)$ goes to w and passes through all the remaining vertices of $R(2m-2,n)$ and ends at t. In this case, two parts of P resides in $R(2m-2,n)$. The part P_1 starts from s ends at $v(u)$, and the part P_2 starts from $u(v)$ ends at t. The size of P_1 is even and the size of P_2 is odd while the size of $R(2m-2,n)$ is even, which is a contradiction, see Figure 4(b).

Case 2.3. Another case that may imagine is that the Hamiltonian path P of $L(m,n)$ starts from s leaves $R(2m-2,n)$ at w and reenters $R(2m-2,n)$ at $v(u)$ and then goes to t. But in this case vertex $u(v)$ can not be in P, which is a contradiction, see Figure 4(c).

Thus the proof of Lemma 3.2 is completed. $\square$

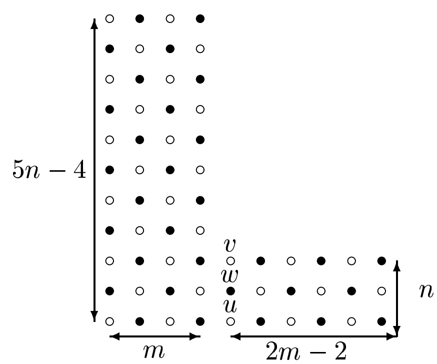


Figure 3:

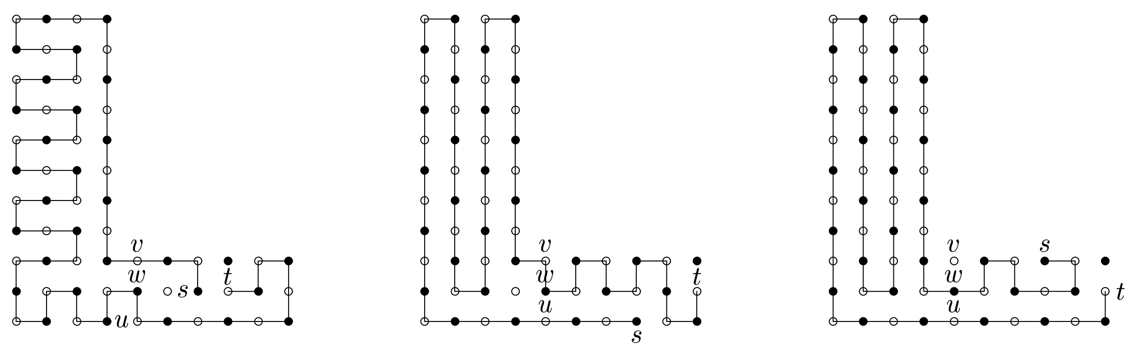
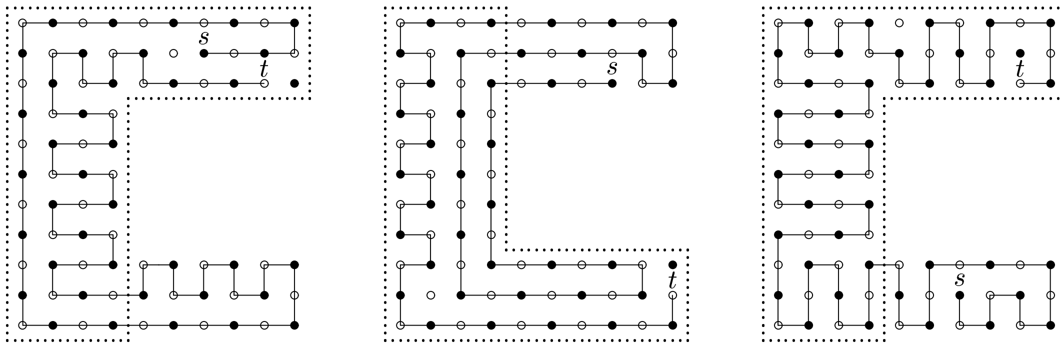


Figure 4: $L-$alphabet grid graphs in which there is no Hamiltonian path between s and t.

Lemma 3.3. *Assume that $C(m,n)$ is a $C-$alphabet grid graph and s and t are two given vertices of $C(m,n)$. Let $L(m,n)$ and $R(2m-2,n)$ be a separation of $C(m,n)$. If $L(m,n)$ does not have Hamiltonian path, then $C(m,n)$ does not have Hamiltonian path between s and t.*

Proof. The proof is similar to the proof of Lemma 3.2, for more details see Figure 5. □

A Hamiltonian path problem $P(L(m,n),s,t)$ is *acceptable* if s and t are color-compatible and $R(2m-2,n)$ does not satisfy the condition $(F3)$, and also $P(C(m,n),s,t)$ is *acceptable* if $P(L(m,n),s,t)$ is acceptable, where $L(m,n)$ is a partition of $C(m,n)$.
Now, we are have shown that all acceptable Hamiltonian path problems have



(a) (b) (c)

Figure 5: $C-$alphabet grid graphs in which there is no Hamiltonian path between s and t.

solutions. Our algorithm is described in the following:
If the given graph is $L-$alphabet, then it is divided into two rectangular grid graphs and two Hamiltonian paths in them are found by the algorithm of [1].
If the given graph is $C-$alphabet, then it is divided into a $L-$alphabet graph and a rectangular grid graph, and the Hamiltonian path in $L-$alphabet graph is found as before.
In the following we discuss the details of this dividing and merging.
A rectangular subgraph S of $L-$alphabet or $C-$alphabet graph A *strips* a Hamiltonian path problem $P(A(m,n),s,t)$, if:

1. S is even-sized.
2. S and $A-S$ is a separation of A.
3. $s,t \in A-S$
4. $A-S$ is acceptable.

Lemma 3.4. *Let $P(L(m,n),s,t)$ be an acceptable Hamiltonian path problem, and S strips it. If $L-S$ has a Hamiltonian path between s and t, then $(L(m,n),s,t)$ has a Hamiltonian path between s and t.*

Proof. Assume that $L-S$ has a Hamiltonian path H. S is an even-sized rectangular grid graph and it has Hamiltonian cycle by Lemma 2.1. There exists an edge $ab \in H$ such that ab is on the boundary of $L-S$ facing S. A Hamiltonian path for $(L(m,n),s,t)$ can be obtained by merging H and the Hamiltonian cycle of S as shown in Figure 6(a). $\square$

Let (R_p, R_q) be a separation of $L(m,n)$. If s and t are in different partitions, then we consider two vertices p and q such that $s, p \in R_p$, $q, t \in R_q$ and (R_p, R_q) are acceptable. Therefore, a Hamiltonian path for $(L(m,n), s, t)$ can be obtained by connecting two vertices p and q as shown in Figure 7(a).

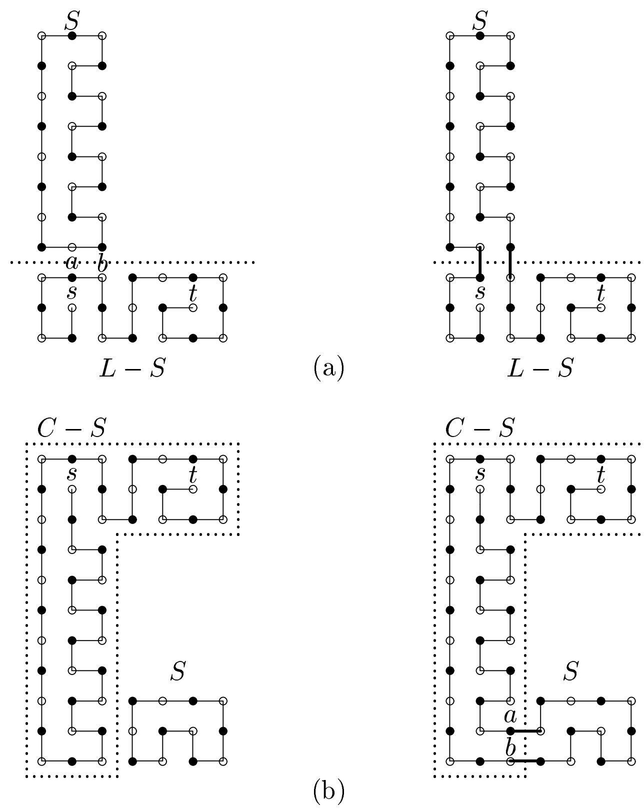


Figure 6: (a) A strip of $L(3,3)$, (b) A strip of $C(3,3)$.

Lemma 3.5. *Let $P(C(m,n), s, t)$ be an acceptable Hamiltonian path problem, and S strips it. If $C - S$ has a Hamiltonian path between s and t, then $(C(m,n), s, t)$ has a Hamiltonian path between s and t.*

Proof. The proof is similar to Lemma 3.4. Notice that $C - S$ is a $L-$alphabet grid graph, see Figure 6(b). $\square$

Let (R_p, L_q) be a separation of $C(m,n)$. If s and t are in different partitions, then we consider two vertices p and q such that $s, p \in R_p$, $q, t \in L_q$ and (R_p, L_q) are acceptable. Therefore, a Hamiltonian path for $(L(m,n), s, t)$ can

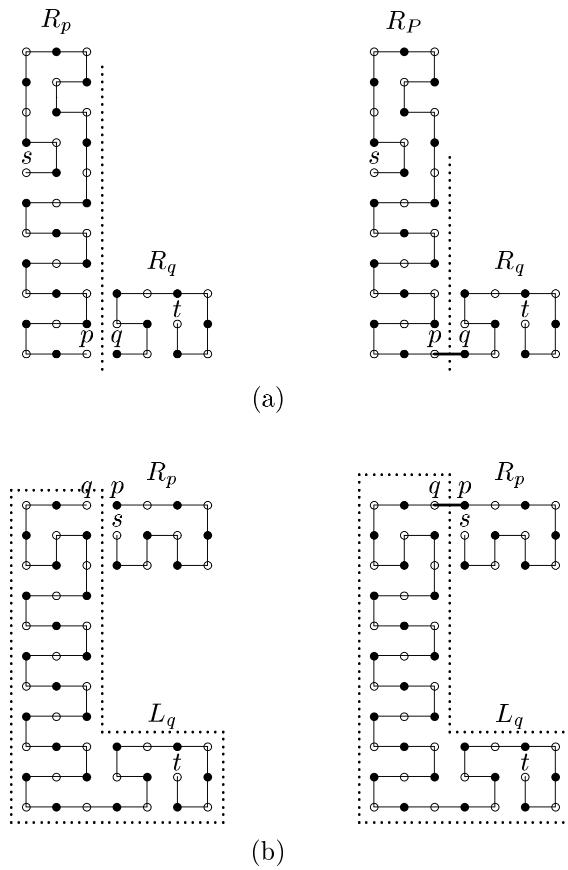


Figure 7: (a) A split of $L(3,3)$, (b) A split of $C(3,3)$.

be obtained by connecting two vertices p and q as shown in Figure 7(b). From Corollary 3.1 and Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5, the following theorem holds.

Theorem 3.1. *Let $A(m,n)$ be a $L-$alphabet or $C-$alphabet graph, and let s and t be two distinct vertices of it. $A(m,n)$ has a Hamiltonian path if and only if $P(A(m,n),s,t)$ is acceptable.*

Theorem 3.1 provides the necessary and sufficient conditions for the existence of Hamiltonian paths in $L-$alphabet and $C-$alphabet grid graphs.

Theorem 3.2. *In $L-$alphabet and $C-$alphabet grid graphs, a Hamiltonian path between any two vertices s and t can be found in linear time.*

Proof. We divide the problem into two (or three) rectangular grid graphs in O(1). Then we solve the subproblems in linear-time and merge the results in O(1) using the method proposed in [1]. □

4. Conclusion and future work

We presented a linear-time algorithm for finding a Hamiltonian path in $L-$alphabet and $C-$alphabet grid graphs between any two given vertices. Since the Hamiltonian path problem is NP-complete in general grid graphs, it remains open if the problem is polynomially solvable in solid grid graphs.

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