We define binary decision variables $x_{ij} \in \{0,1\}$ to mean that the order visits city i immediately before city j if and only if $x_{ij} = 1$. We also introduce continuous decision variables $u_0, \ldots, u_n \in \mathbb{R}$. Consider the following MILP formulation:

$$\min_{x,u} \sum_{i \neq j}^{n} c_{ij} x_{ij} \tag{2.3}$$

$$\forall i \leq n \quad \sum_{j=0}^{n} x_{ij} = 1 \tag{2.4}$$

$$\forall j \leq n \quad \sum_{i=0}^{n} x_{ij} = 1 \tag{2.5}$$

$$\forall i \neq j \neq 0 \quad u_i - u_j + n x_{ij} \leq n - 1 \tag{2.6}$$

$$\forall i, j \leq n \quad x_{ij} \in \{0,1\}. \tag{2.7}$$

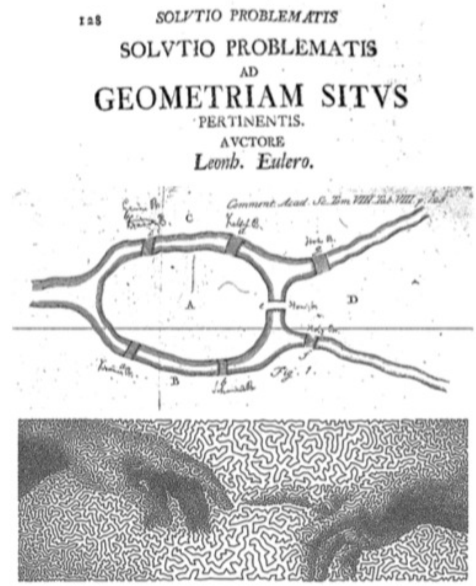

Figure 2.2: Euler, and TSP art.

The objective function Eq. (2.3) aims at minimizing the total cost of the selected legs of the traveling salesman tour. By Eq. (2.4)-(2.5),[9] we know that the feasible region consists of permutations of $\{0, \ldots, n\}$: if, by contradiction, there were two integers j, ℓ such that $x_{ij} = x_{i\ell} = 1$, then this would violate Eq. (2.4); and, conversely, if there were two integers i, ℓ such that $x_{ij} = x_{\ell j} = 1$, then this would violate Eq. (2.5). Therefore all ordered pairs (i, j) with $x_{ij} = 1$ define a bijection $\{0, \ldots, n\} \to \{0, \ldots, n\}$, in other words a permutation. Permutations can be decomposed in products of disjoint cycles. This, however, would not yield a tour but many subtours. We have to show that having more than one tour would violate Eq. (2.6). Suppose, to get a contradiction, that there are at least two tours. Then one cannot contain city 0 (since the tours have to be disjoint by definition of bijection): suppose this is the tour $i_1, \ldots, i_h$. Then from Eq. (2.6), by setting the x variables to one along the relevant legs, for each $\ell < h$ we obtain $u_{i_\ell} - u_{i_{\ell+1}} + n \leq n - 1$ as well as $u_{i_h} - u_{i_1} + n \leq n - 1$. Now we sum all these inequalities and observe that all of the u variables cancel out, since they all occur with changed sign in exactly two inequalities. Thus we obtain $n \leq n - 1$, a contradiction.[10] Therefore the above is a valid MILP formulation for the TSP.

Eq. (2.6) is not the only possible way to eliminate the subtours.

[9] Constraints (2.4)-(2.5) define the *assignment constraints*, which define an incredibly important substructure in MP formulation — these crop up in scheduling, logistics, resource allocation, and many other types of problems. Assignment constraints define a bijection on the set of their indices.

[10] At least one tour (namely the one containing city 0) is safe, since we quantify Eq. (2.6) over i, j both non-zero.