Combinatorial Games

Competitive Economic Games

Cooperative Economic Games

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Planar Slink

Draw a planar map, for example a subset of the squares on a graph paper.
The 1st player chooses a region of it.
Then the players take turns walking across an edge from the last reached region to an unreached region.
A player loses when he can't do that.

Record the moves by arrows crossing the crossed edges
or by a new higher number in each reached square.

Please try the game with a partner, and see if you think of a strategy.

Hint: It uses some well-known combinatorial optimization theory.

Bimatrix Games

A ***bimatrix game*** is given by an mxn payoff matrix A and
an nxm payoff matrix B,
each with positive entries.

Let R = {1,…,m} be the index-set of the rows of A and of the columns of B.
Let S = {m+1,…,m+n} be the index-set of the rows of B and the columns of A.
Player 1 chooses i in R and Player 2 chooses j in S.
The payoff to Player 1 is A_{ij} and the payoff to Player 2 is $B_{ji.}$
If Player 1 chooses i in R with probability x_i and Player 2 chooses j in S with probability y_j , then the expected payoff to Player 1 is $x^R A y^S$ and the expected payoff to Player 2 is $y^s B x^R$.

The pair (x,y) is called a ***Nash equilibrium*** if neither player can improve his expected payoff by unilaterally changing his probability distribution.

Nash's Theorem. ***Every bimatrix game has an equilibrium***.
A Nash equilibrium can be found using the Lemke-Howson Algorithm.

Scarf's Co-operative Game

This is a modified version of Scarf's theory of non-transferable utility (NTU) co-operative n-person games.

Stable allocation of coalitions:
Given a list of subsets of the n persons, called coalitions. (Different coalitions can be the same subset of persons.)
For each person, given a total preference ordering of the coalitions he is in.
A stable allocation, S, is a subset of mutually disjoint coalitions such that for every coalition, c, not in S there is a person in c who prefers the coalition of S that he is in more than c.
Where the persons are the vertices of a bipartite graph G and the coalitions are the edges of G, the Stable Marriage Theorem says there exists a stable allocation. It is proved by an easy algorithm for finding one.
More generally there may not be a stable allocation. However **there always exists a 'fractional stable allocation of coalitions',** proved by a very interesting but exponentially growing algorithm for finding one.

An ***Euler complex (oik)*** (V,M) consists of a finite set V of elements called the ***vertices*** and a set M of (d+1)-element subsets of V called the ***rooms*** such that

every d-element subset of V is in an even number of rooms.

A ***wall*** is a d-element subset obtained by removing one vertex from a room.

If each wall is in exactly 2 rooms, the oik is called a ***manifold oik*** (or a simplicial pseudomanifold).

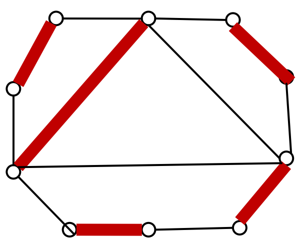
A ***room-partition*** is a partition of the vertices into rooms.

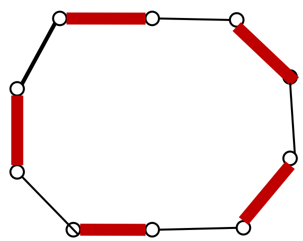
Euler graph (1-d oik).

Vertices of oik = vertices of the graph

Rooms of the oik = edges of the graph

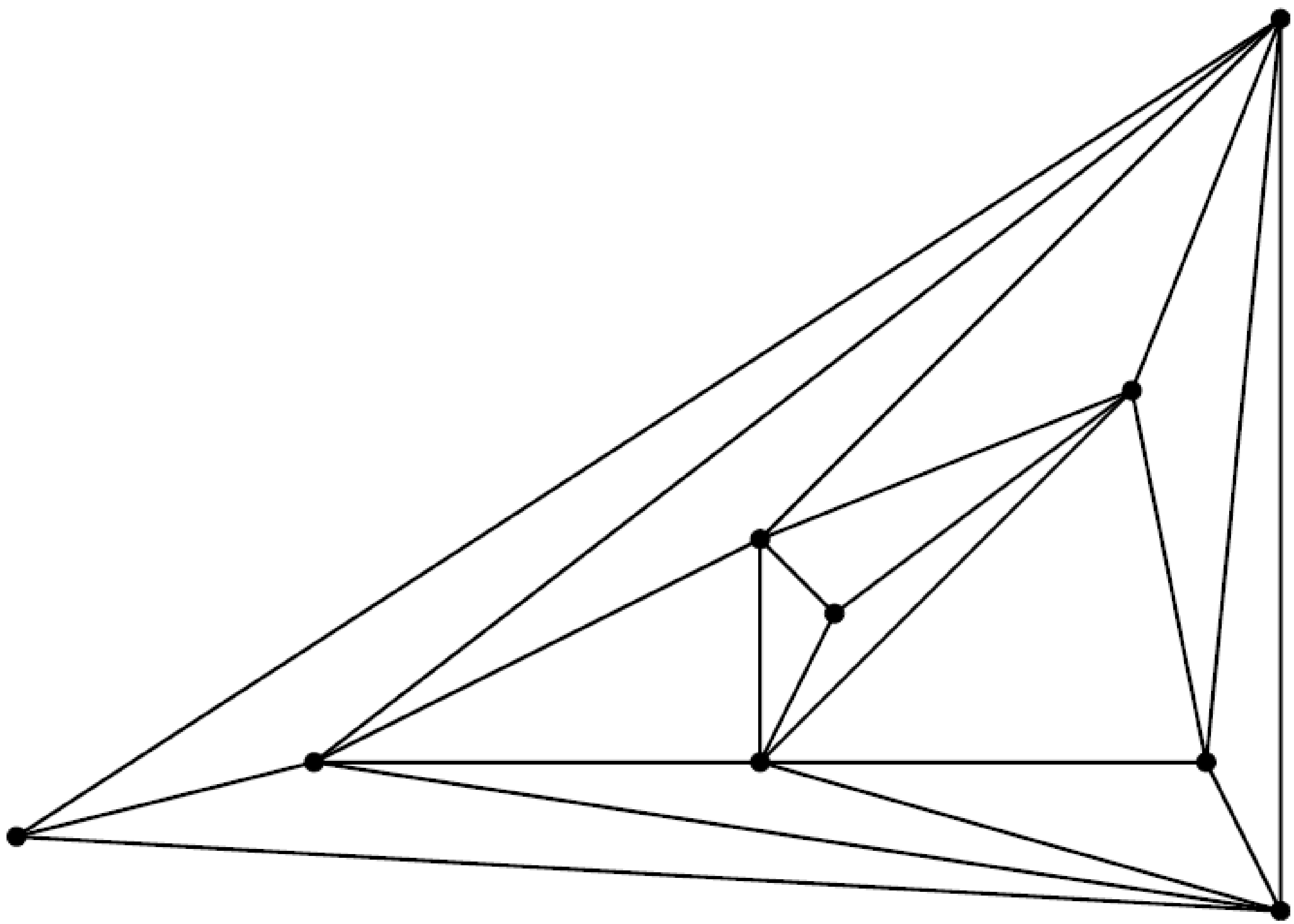
Room partition of the oik = partition of the vertices into rooms (edges) = perfect matching



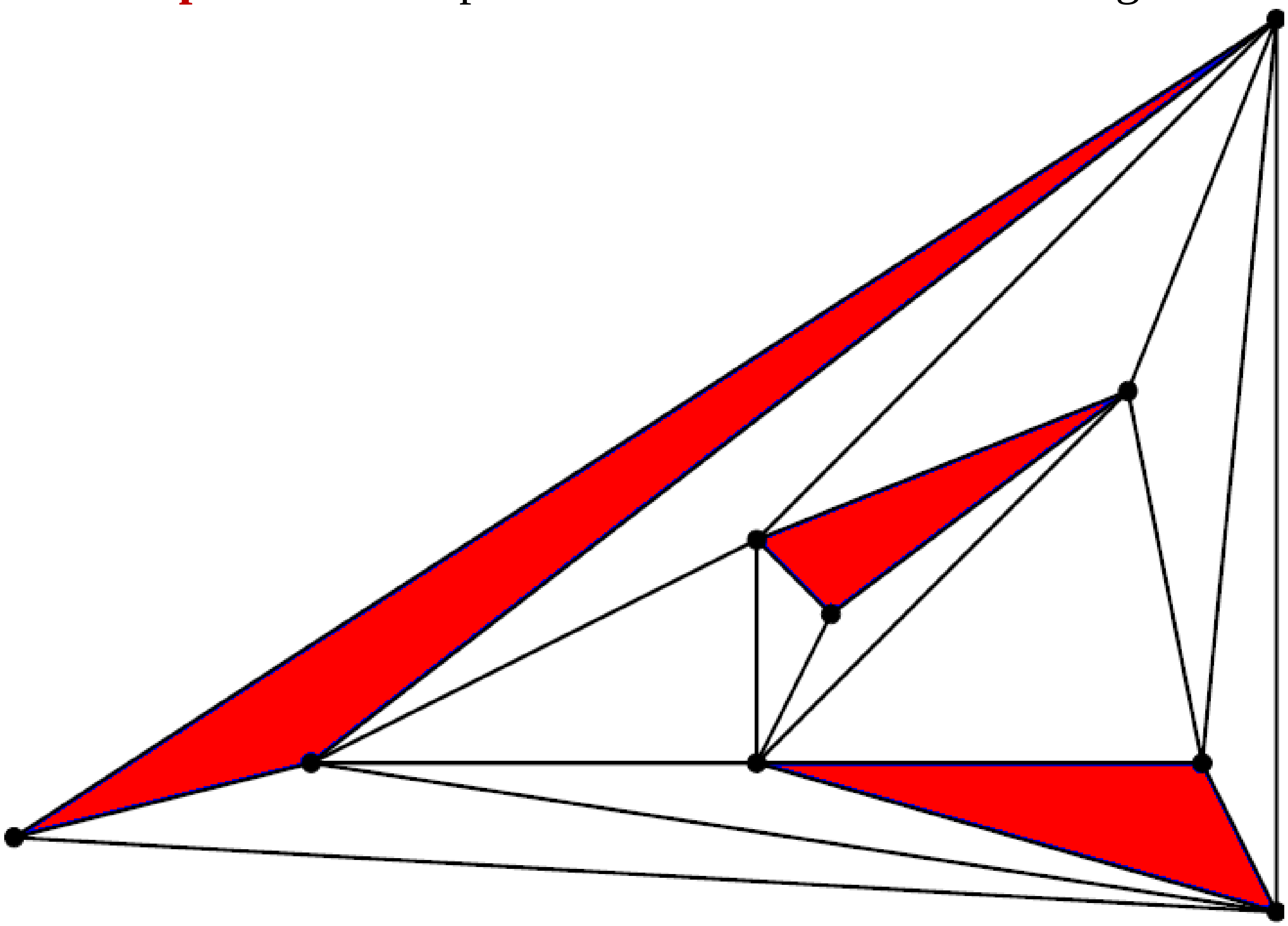


Special case: Even cycle is a 1**-d manifold oik**

A triangulation of a surface is a 2-d manifold oik.
The vertices of the oik are the vertices of the triangulation.
The edges of the oik are the triangular regions.



A triangulation of a surface is a 2-d manifold oik.
The vertices of the oik are the vertices of the triangulation.
The edges of the oik are the triangular regions.
A **room-partition** is a partition of the vertices into triangles.



Summary: An ***Euler complex (oik)*** (V,M) consists of a finite set V of elements called the ***vertices*** and a set M of (d+1)-element subsets of V called the ***rooms*** such that

every d-element subset of V is in an even number of rooms.

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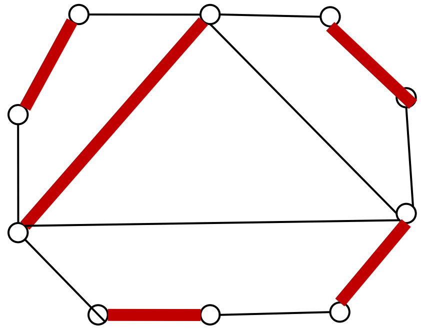
If each wall is in exactly 2 rooms, the oik is called a ***manifold oik*** (or a simplicial pseudomanifold).

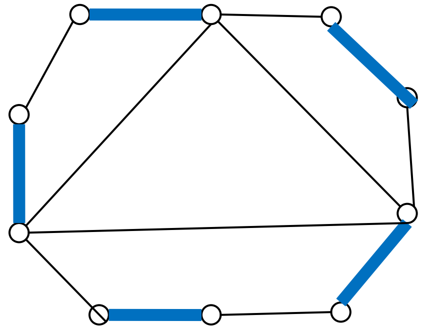
A ***room-partition*** is a partition of the vertices into rooms.

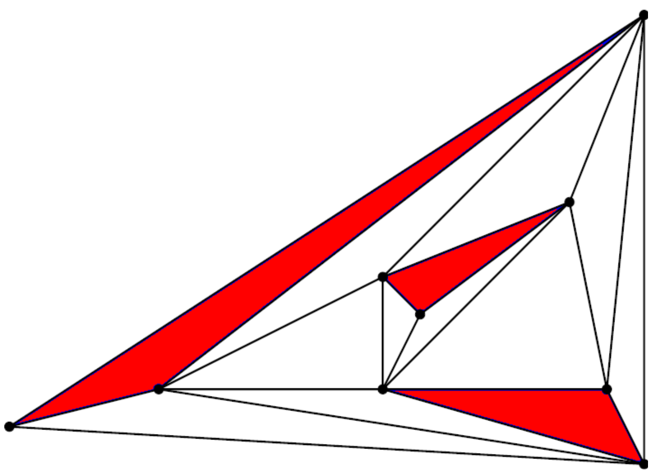
Oik	**Vertices**	**Rooms**	**Walls**	**Room partition**
Euler graph (1-d oik)	vertices	edges	vertices	partition of vertices into edges i.e. perfect matching
Even cycle (1-d manifold oik)	vertices	edges	vertices	perfect matching
Triangulation of a surface (is a 2-d manifold oik)	vertices	triangular regions	edges	partition into triangular regions

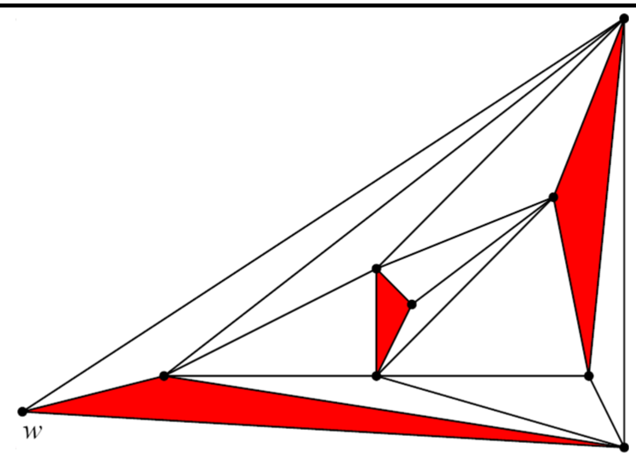
Room-Partition Theorem.

For every oik, there is an even number of partitions of the vertices into rooms. Thus, if there is one room-partition, there is another room-partition.







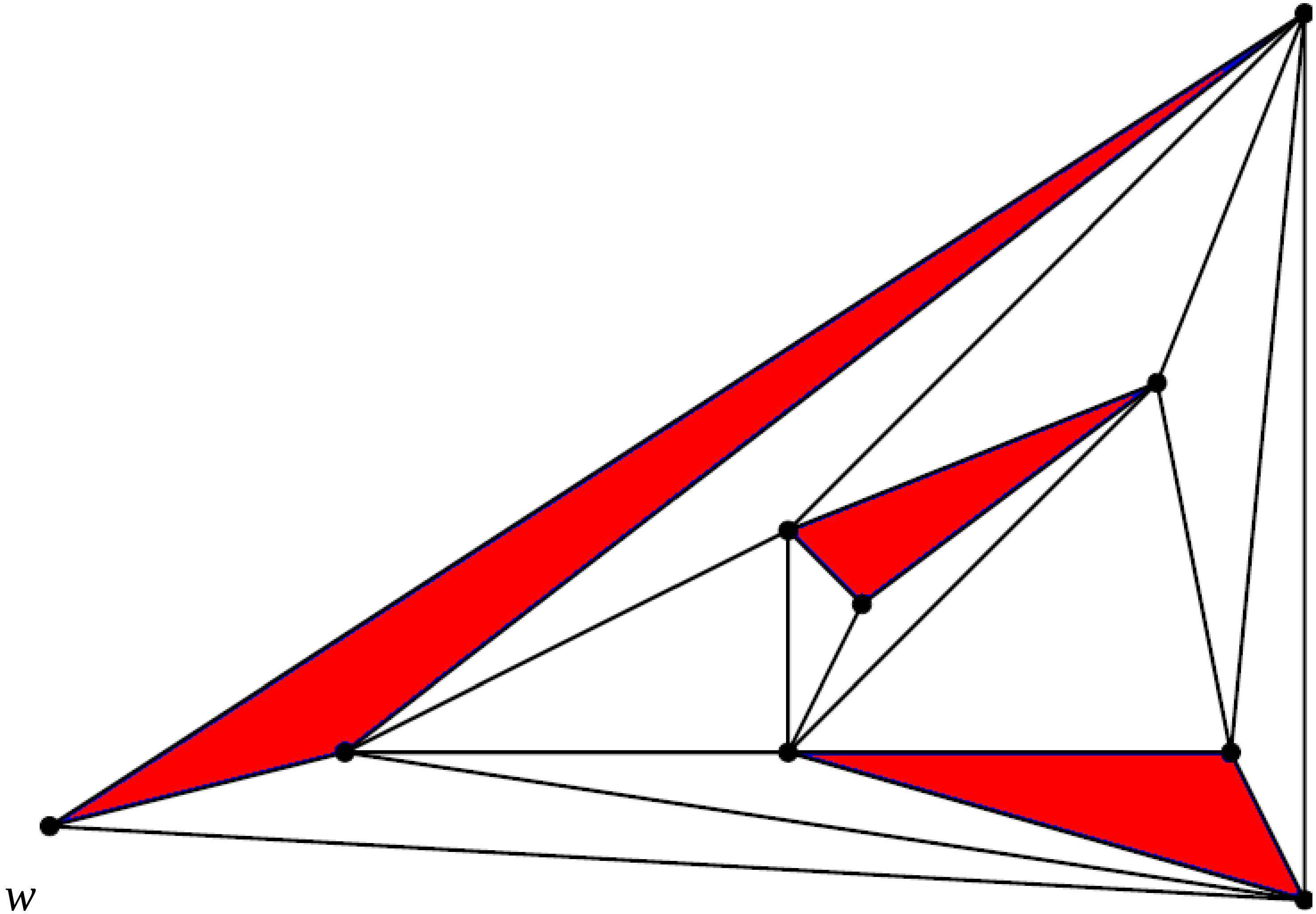


Manifold Oik Slink
Given a subset of the rooms of a manifold oik,
The 1st player chooses a room.
Then the players take turns walking across a wall from the last reached room to an unreached room.
A player loses when he can't do that.
Record the moves by arrows crossing the crossed walls.

How to Find Another Room Partition of the Vertices

A ***room-family*** means a set $R = \{R_i : i=1,\ldots,h\}$ of rooms.
Let w be a vertex. A ***w-skew room-family*** is a room-family $R = \{R_i : i=1,\ldots,h\}$ such that w is not in any of the rooms R_i, some vertex v is in exactly two of the R_i, and every other vertex is in exactly one of the R_i.

Algorithm.
Start with a room-partition R.
Remove w from the room, say R_1, it is in.
R_1-w is a wall, so it is contained in a room, say R_2, different from R_1
Replace R by $R' = R - R_1 + R_2$.
If R' is a room-partition, stop.
Otherwise, R' is a w-skew room-family, with say v in two rooms of the family, R' and R_3. Remove v from R_3, to get a wall, ….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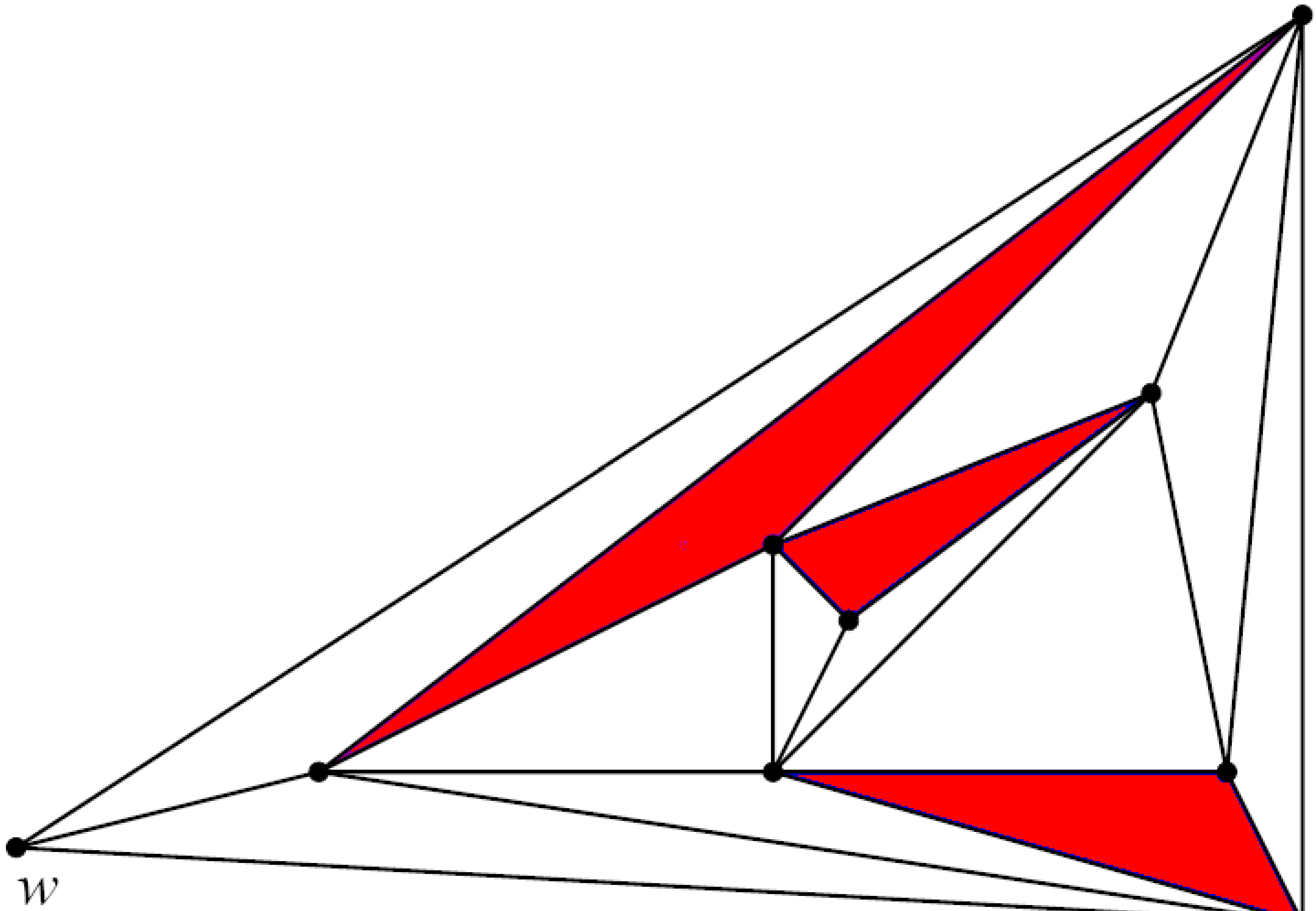
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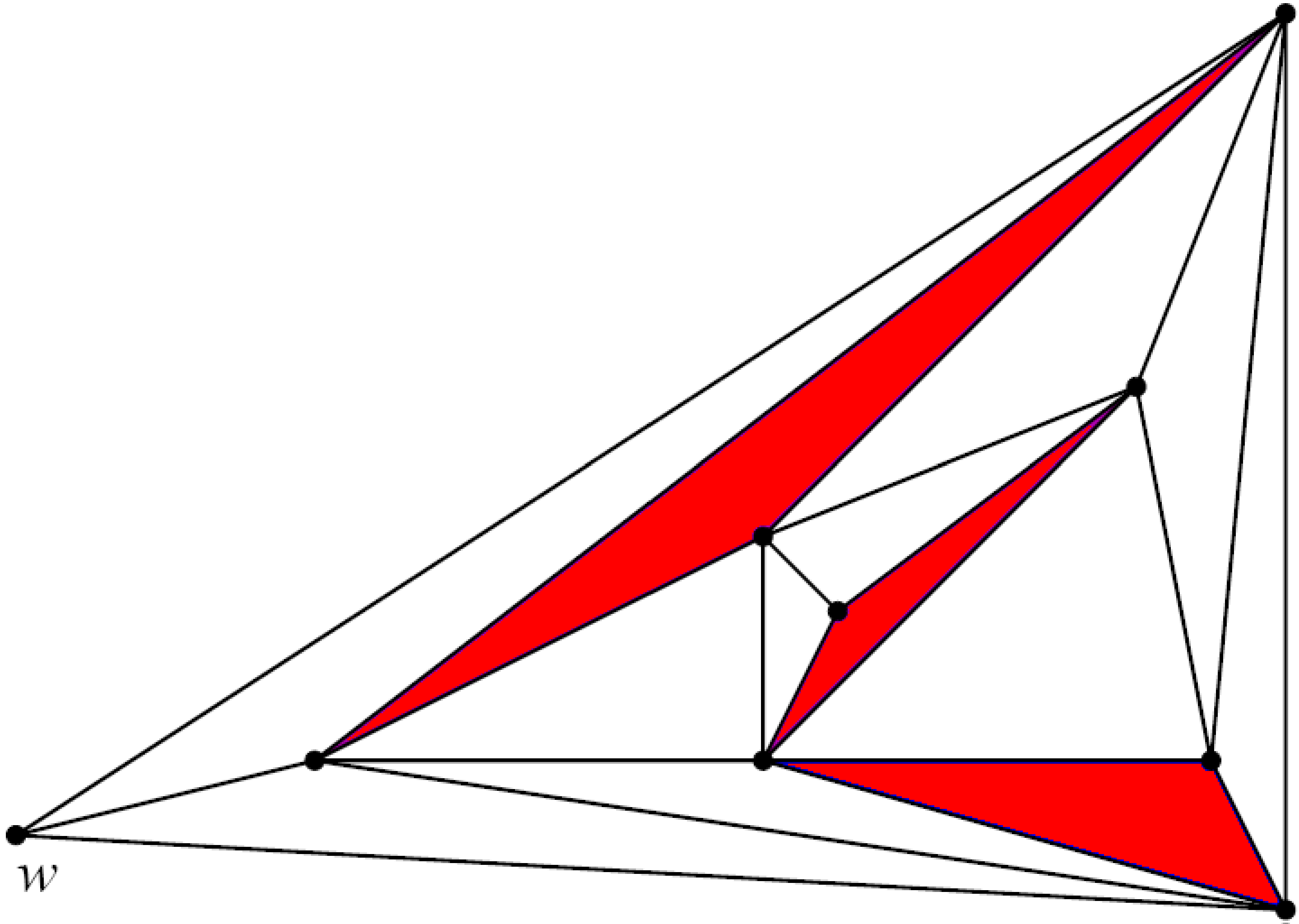
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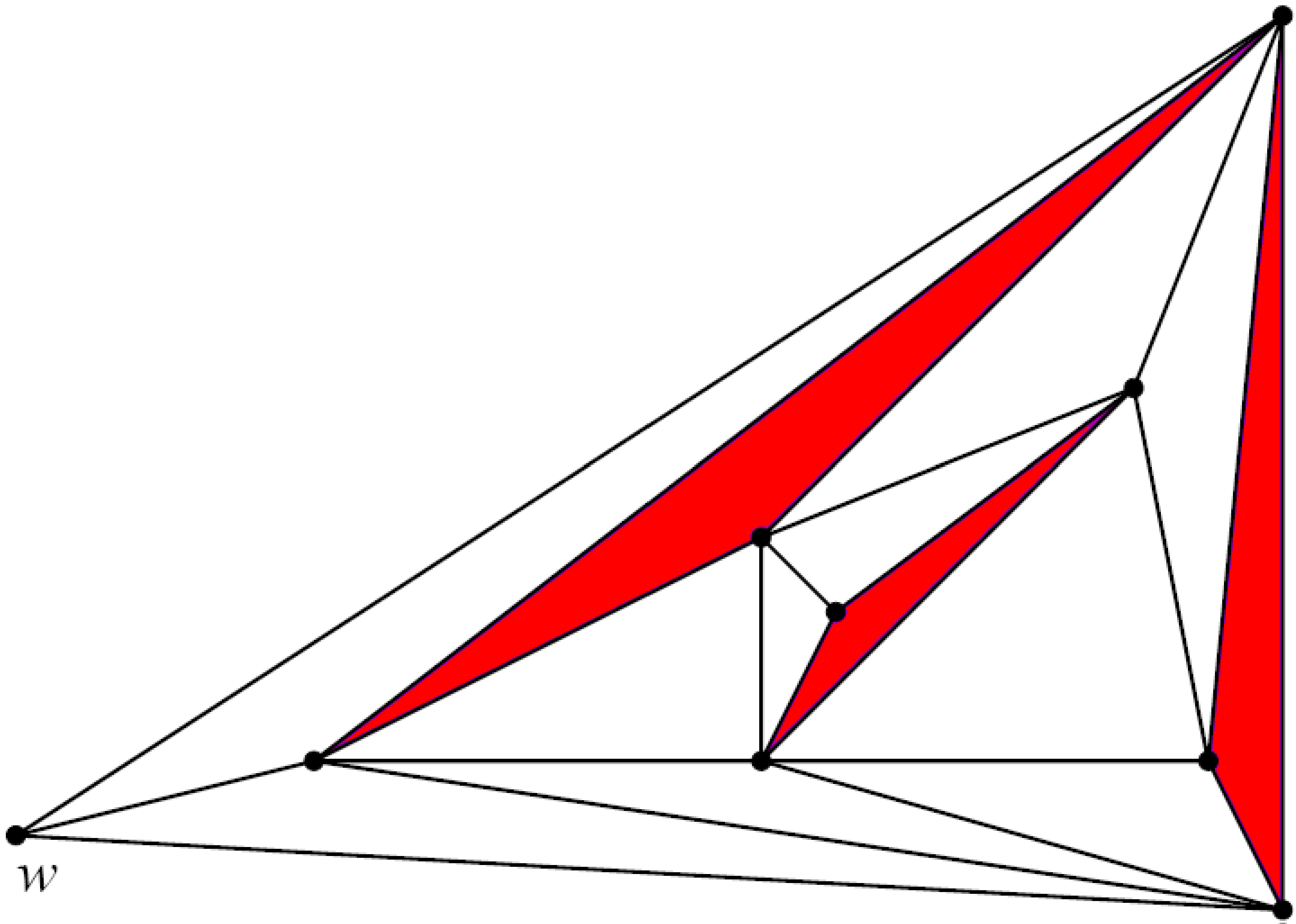
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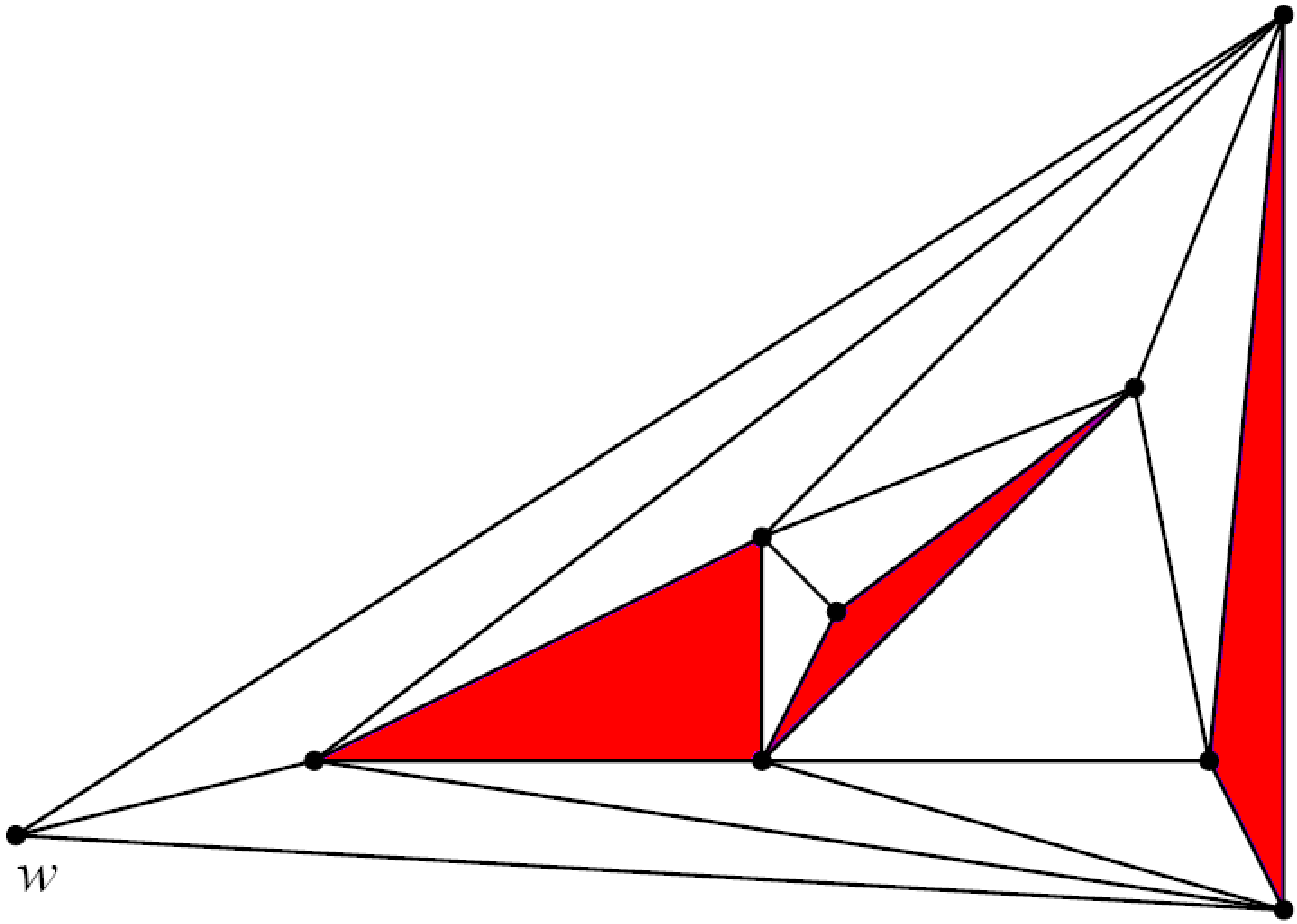
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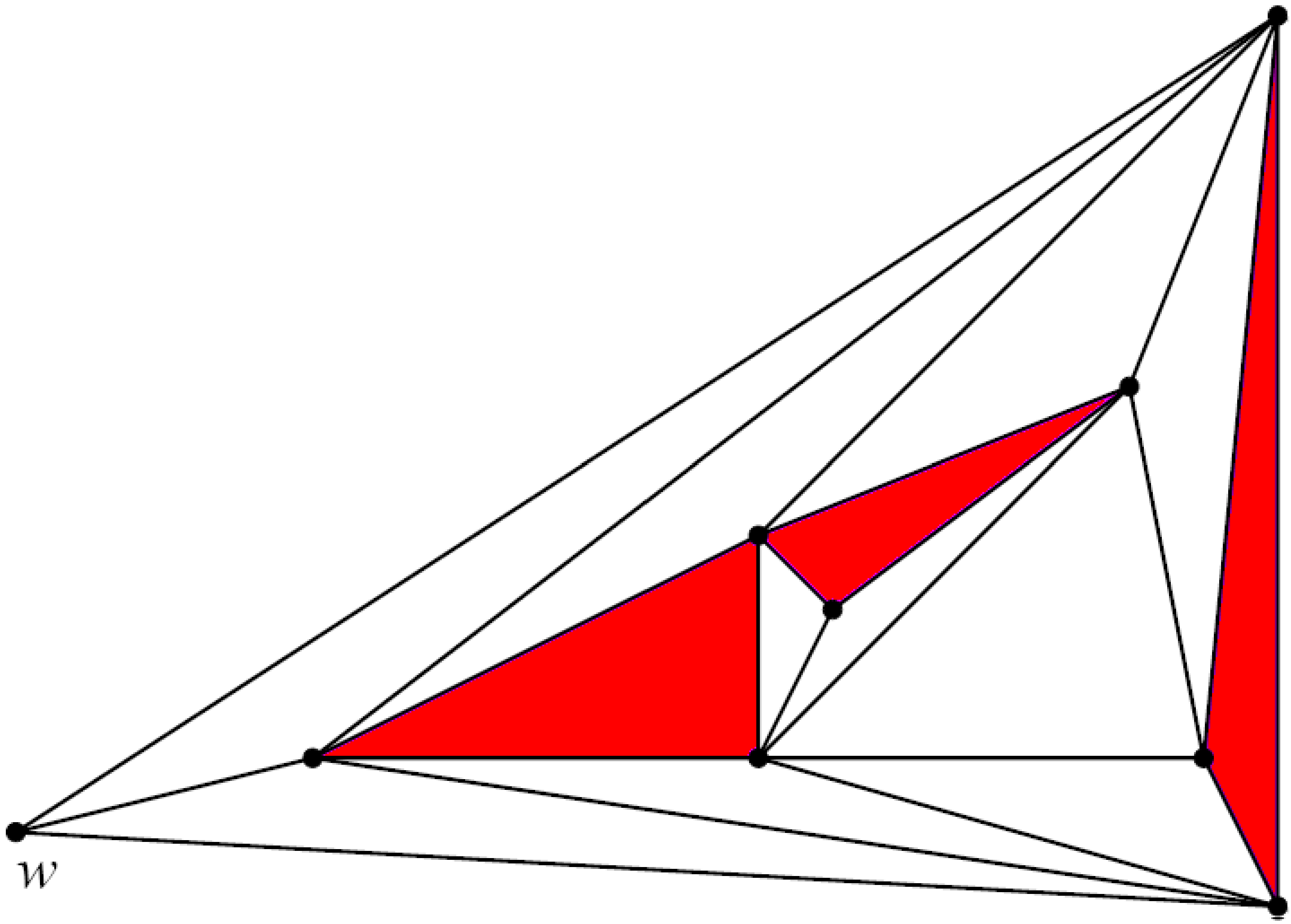

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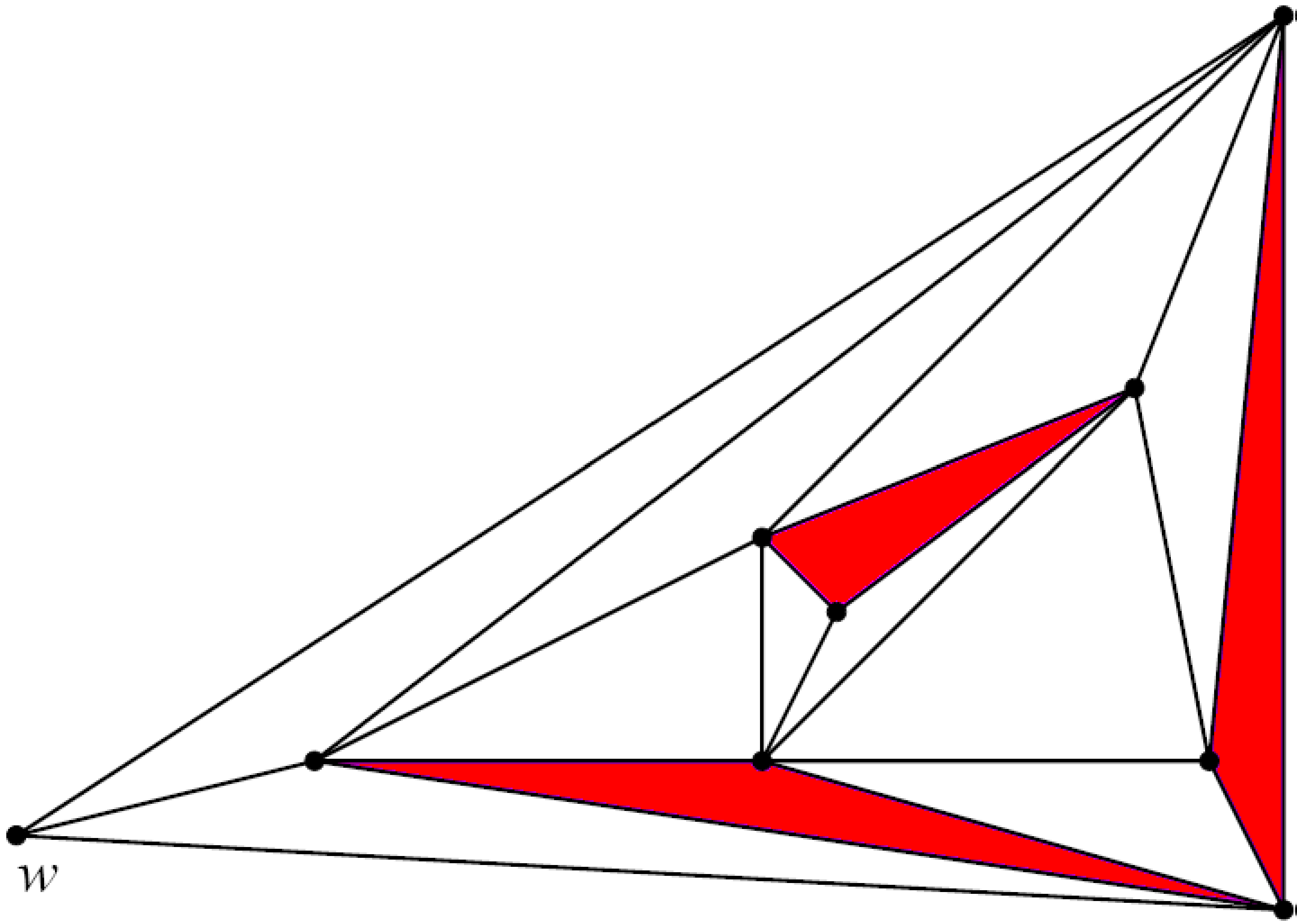
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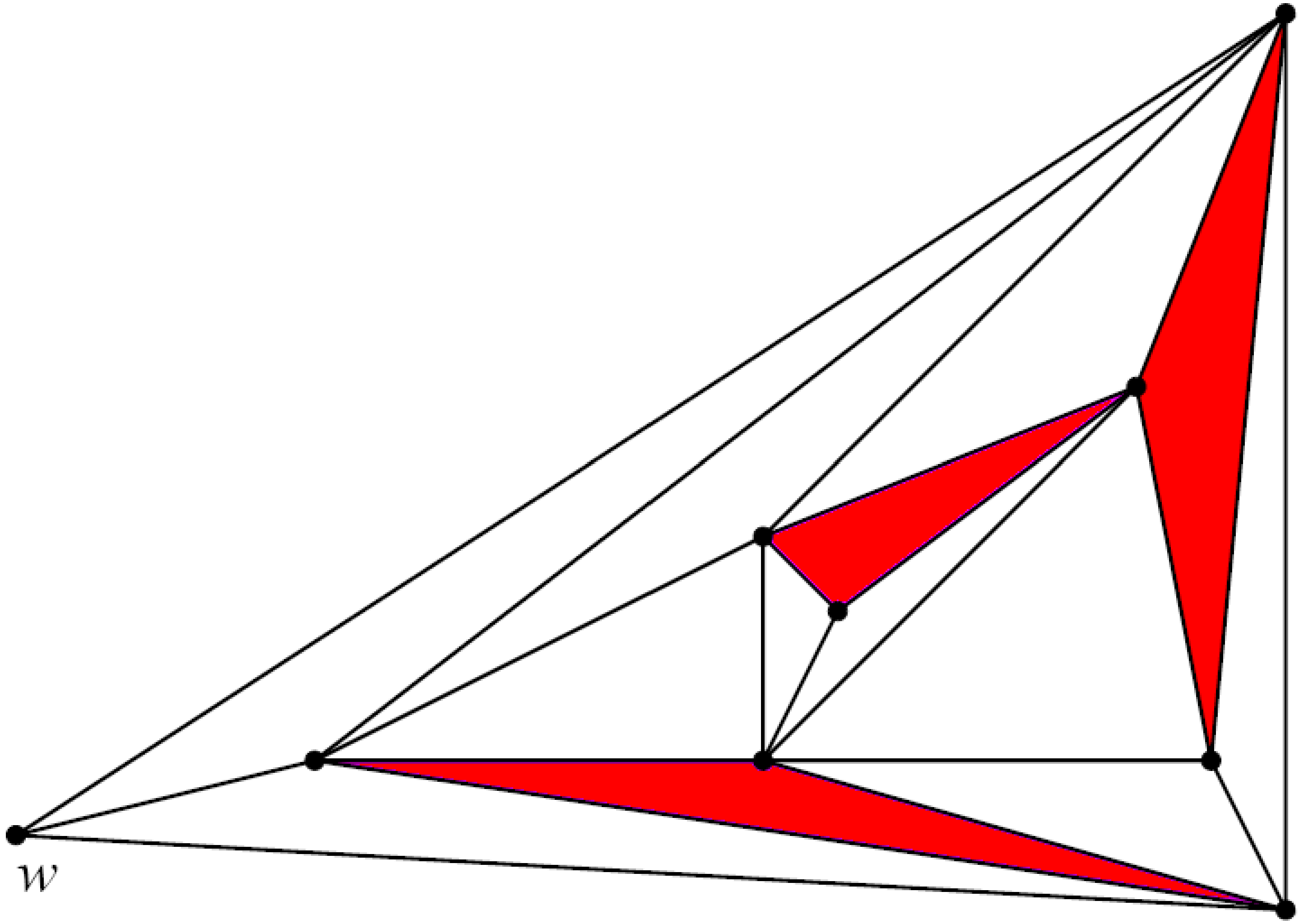
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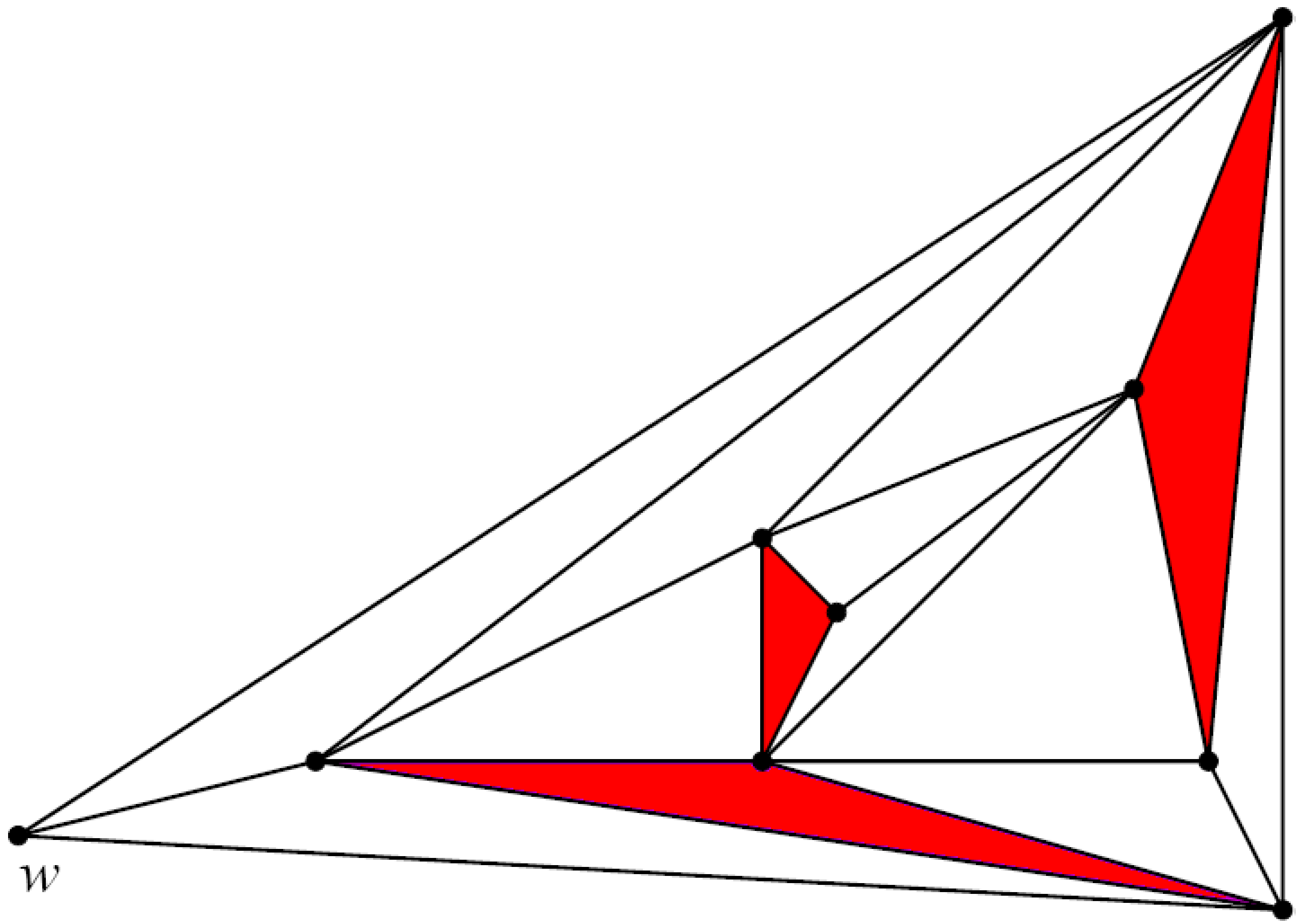
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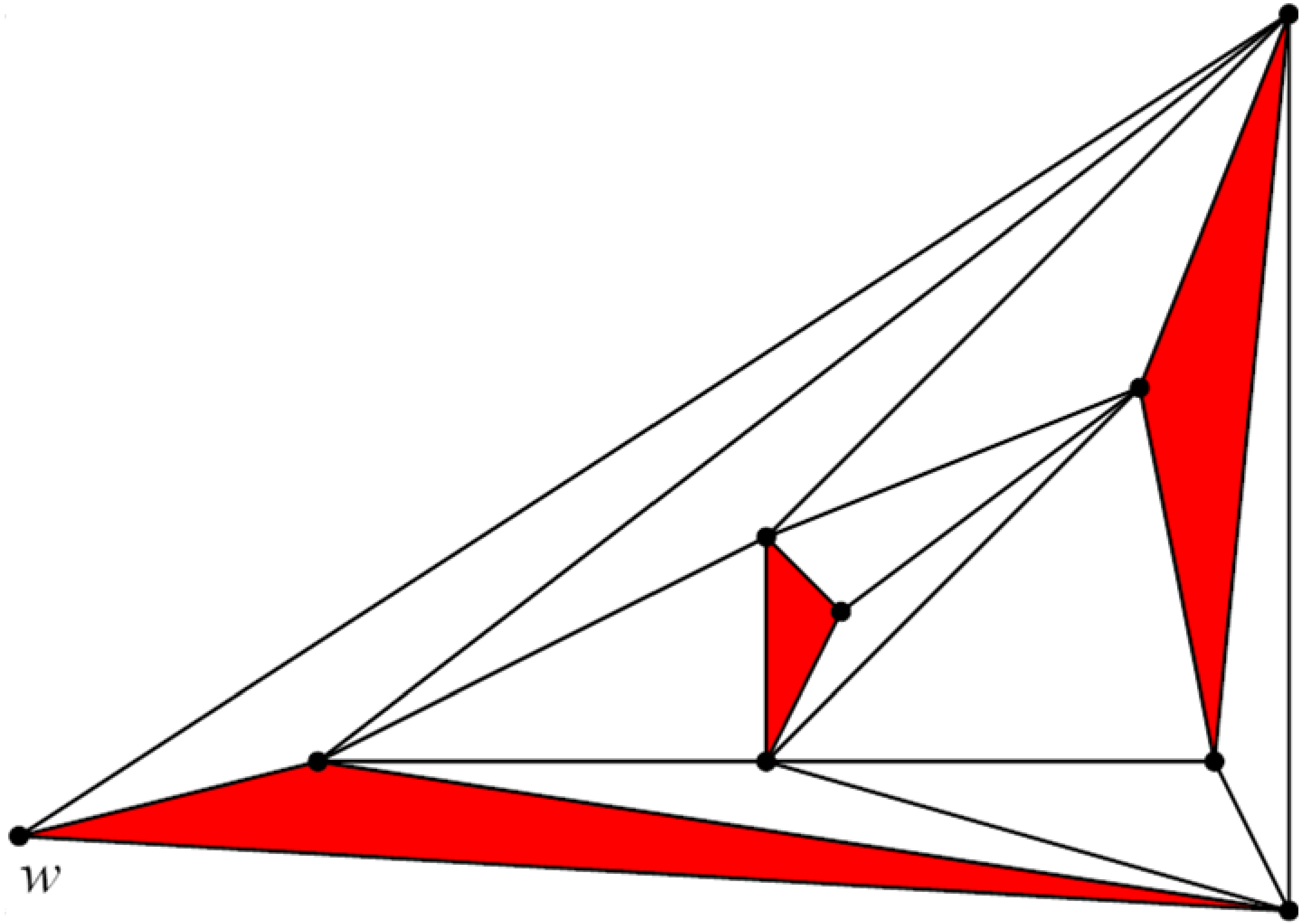
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Polytopal Oiks

Consider a feasible tableau T of a non-degenerate bounded linear system
$Ax=b$, $x \geq \underline{0}$

A ***polytopal oik*** has vertex-set the indices of columns of A
and the rooms are the complements of feasible bases of A.

Proof that this is a manifold oik – the ratio test of the Simplex Algorithm.
For any feasible basis B and any column j not in B,
There is a unique column i in B such that $B' = B + j - i$ is a feasible basis.

Sperner Oiks

The vertices are a set of coloured points.
The rooms are sets of vertices whose complement consists of exactly one vertex of each colour.
A wall W is a set of vertices whose complement consists of two vertices of some colour and one vertex of every other colour.
Clearly, a wall W is in exactly two rooms: Where a and be are the two vertices of the same colour in the complement of W, W+a and W+b are the two rooms containing W.

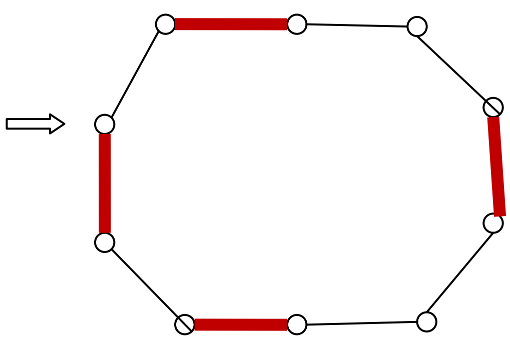
A version of Sperner's Lemma
For any manifold oik,
If the vertices are coloured,
There is an even number of rooms containing exactly one vertex of each colour.

Gale Oiks

Consider an even simple cycle C_n with n vertices.
The vertices of the oik are the vertices of the graph.
The rooms are the sets of vertices V(M) in a matching M of size k, where $2k < n$.

Thus a room induces a set of even paths in the cycle, each starting and ending with a matching edge.
A wall is the set of vertices of such a matching M with one vertex removed.

It is easy to see that a wall W is in exactly two rooms. So Gale oiks are manifold oiks.



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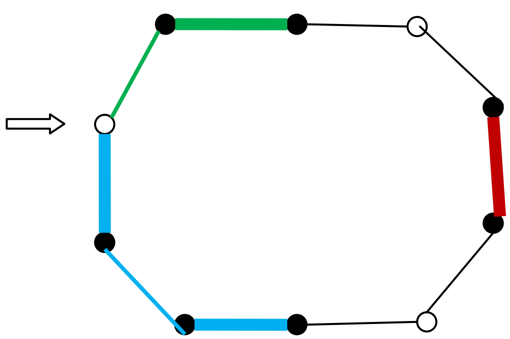

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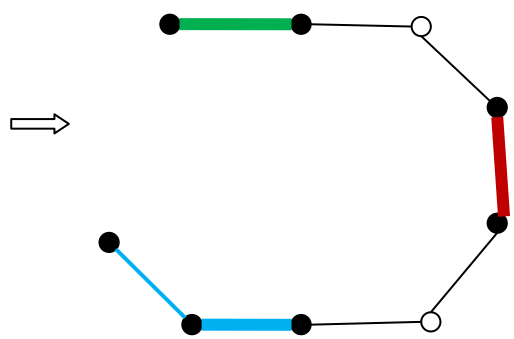

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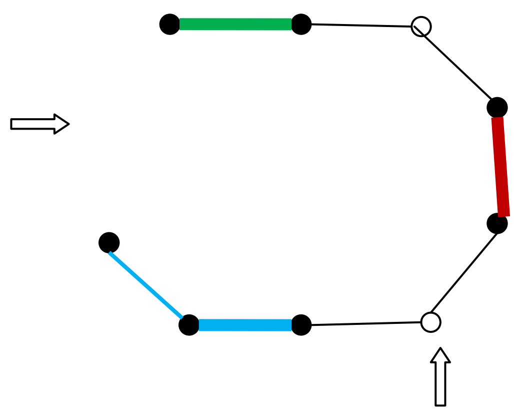

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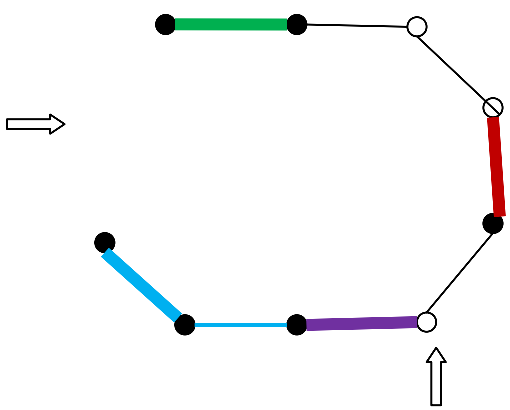

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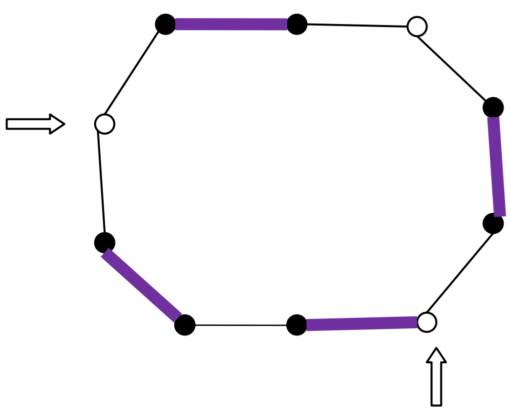
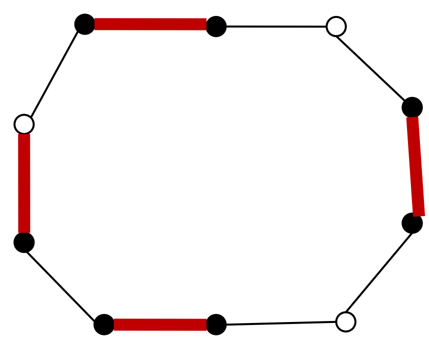




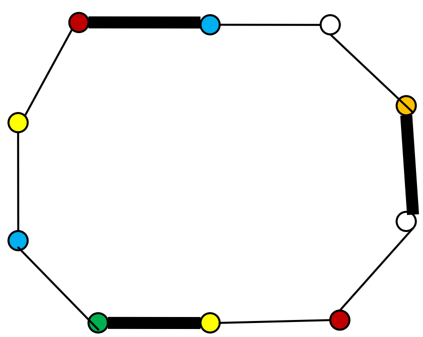
Summary

Oik	**Vertices**	**Rooms**
Euler graph	vertices	edges
Even cycle	vertices	edges
Triangulation of a surface	vertices	triangular regions
Polytopal oiks $Ax=b, \ x \geq \underline{0}$	columns	complements of bases
Sperner oiks	coloured points	complement consists of a vertex of each colour
Gale oiks From even cycle C_n	vertices	matchings of size $2k < n$

Rainbow Matchings in Gale Oiks

Recall: The vertices of a Gale oik are the vertices of the simple cycle C_n.
The rooms are the sets of vertices V(M) in a matching M of size k, where $2k < n$.

Where the vertices are coloured, a ***rainbow matching*** is a matching such that the vertices it contains have distinct colours.



For every colouring of the vertices with 2k colours,
There exists an even number of rainbow matchings.
In particular, if there is one rainbow matching, then there is another.

Obvious:
A rainbow matching in a graph
is the same as
a perfect matching in the graph obtained by gluing together all vertices of the same colour.

In particular,
the rainbow rooms of a Gale manifold oik with coloured vertices
are the same as
the perfect matchings in the Euler graph you get by gluing together the vertices of C_n which have the same colour.

There is an even number of perfect matchings in an Euler graph,
so there is an even number of rainbow rooms in a Gale manifold oik with coloured vertices.

Room Partitioning Oik Families

An **oik-family** is a family of oiks, all with the same vertex-set V.
A ***room-partition of the oik family*** is a partition of V into sets, such that the ith set is a room in the ith oik.

Room-partitions where all oiks same corresponds to room-partitions of a single oik.

Room Partitioning Theorem for Oik Families.
For any oik family, there is an even number of room-partitions.

In particular, if there is one room-partition of a family of oiks, there exists a different room-partition.

Room Partitioning Theorem for Oik Families (again)
For any oik family, there is an even number of room-partitions.

Proof. Choose a vertex, say w, to be special. A ***w-skew room-family*** for oik family M mean a room-family, R = $\{R_i : I – 1,\ldots,h\}$ such that w is not in any of the rooms R_i, some vertex v is in exactly two of the R_i, and every other vertex is in exactly one of the R_i.

Consider the so-called exchange graph X, determined by M and w, where the nodes of X are the room-partitions for M and all the w-skew room-families for M.
Two nodes of X are joined by an edge in X when each is obtained from the other by replacing one room by another.
It’s easy to see the room-partitions for M are the odd-degree nodes of X.
The result follows since X, like every graph, must have an even number of odd-degree nodes. □

Consider an oik-family where one oik is a Sperner oik.

Where the other oik polytopal (that is, the oik whose rooms are the complements of feasible bases of a simplex tableau T):
A room-partition for this pair of oiks is a partition of the columns of T into the complement N of a basis of T and a set B of columns whose complement is rainbow.
Thus N is actually a rainbow complement of a feasible basis.
It follows that the number of rainbow complements of feasible bases is even.
Geometrically interpreted, the complements of feasible bases are the facets of a simplicial polytope, and this gives the classical Sperner Lemma: the number of rainbow facets of a simplicial polytope is even.

Where the other oik is a Gale oik:
A room partition for this pair of oiks is a partition of the vertices of C_n into a the vertices in a matching of size k, and a set whose complement is rainbow. Thus the number of rainbow matchings is even (as said before).

Theorem. The Lemke-Howson Algorithm for a Nash-equilibrium of a bimatrix game is the exchange algorithm for an oik family consisting of two oiks, where each oik is polytopal.

Theorem. Scarf's Algorithm for a stable solution for his NTU game is the exchange algorithm applied to an oik family consisting of two oiks where one is polytopal and the second is slightly more complicated and remains to be described.

Recall:

Bimatrix Games

A ***bimatrix game*** is given by an mxn payoff matrix A and
an nxm payoff matrix B,
each with positive entries.

Let R = {1,…,m} be the index-set of the rows of A and of the columns of B.
Let S = {m+1,…,m+n} be the index-set of the rows of B and the columns of A.
Player 1 chooses i in R and Player 2 chooses j in S.
The payoff to Player 1 is A_{ij} and the payoff to Player 2 is B_{ji}.
If Player 1 chooses i in R with probability x_i and Player 2 chooses j in S with probability y_j , then the expected payoff to Player 1 is $x^R A y^S$ and the expected payoff to Player 2 is $y^s B x^R$.

The pair (x,y) is called a ***Nash equilibrium*** if neither player can improve his expected payoff by unilaterally changing his probability distribution.

Nash's Theorem. ***Every bimatrix game has an equilibrium***.

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Finding a Nash equilibrium is equivalent to finding a pair of complementary feasible bases of two bounded non-degenerate linear feasibility problems:

$[\ A \mid I_m\ ]\ \mathbf{y''} = \underline{1}$ $\qquad\qquad$ $[\ I_n \mid B]\ \mathbf{x''} = \underline{1}$

$\mathbf{y''} \geq \underline{0}$ $\qquad\qquad$ $\mathbf{x''} \geq \underline{0}$

Theorem. The equilibria of the bimatrix game are given by the basic solutions corresponding to complementary pairs of feasible bases other than the starting pair (I_m , I_n) (that is, the basis given by (R,T)).

Precisely, the Nash equilibrium (x^R, y^S) is

$x^R = (\mathbf{x''})^{\mathbf{R}} \,/\, \mathbf{\Sigma}\ \{\mathbf{x''}_{\mathbf{j}} : j \in R\}$, $\qquad$ $y^R = (\mathbf{y''})^{\mathbf{S}} \,/\, \mathbf{\Sigma}\ \{\mathbf{y''}_{\mathbf{j}} : j \in S\}$

A crucial part of an algorithmic proof of Nash's Theorem:
Theorem. There is another complimentary pair of feasible bases different from the starting pair (R,T).

If Player 1 chooses i in R with probability x_i and Player 2 chooses j in S with probability y_j , then the expected payoff to Player 1 is $x^R A y^S$ and to Player 2 is $y^s B x^R$.

A ***best response*** to Player 2's mixed strategy y^s is a mixed strategy x^R of Player 1 which maximizes $x^R A y^s$. Analogously, a ***best response*** to Player 1's mixed strategy x^R is a mixed strategy y of Player 2 which maximizes $y^s B x^R$.
A Nash equilibrium is a pair (x^R, y^s) of best responses to eachother.

Lemma. Let x, y be mixed strategies for Players 1 and 2, respectively.
x is a best response to y if and only if
$x_i > 0$ => the ith component of Ay is the largest component of Ay

Proof. Note that the ith component of Ay^S, denoted $(Ay^S)_i$, is the expected payoff to Player i when she chooses row i.
Let u denote the largest component of Ay^S.

$$x^R A y^S = \Sigma\, x_i\, (Ay^S)_i = \Sigma\, x_i\, (u - (\, u - (Ay^S)_i\,)) = u - \Sigma\, x_i\, (u - (Ay^S)_i\,) \leq u$$

$$x^R A y^S = u \;\Leftrightarrow\; [\; x_i > 0 \Rightarrow (Ay^S)_i = u\;]$$

Note that $Ay^S \leq \underline{u}$ means that u is at least as large as the expected payoff for each of Player 1's pure strategies. [$\underline{u}$ is a vector of all u's]

A strategy y for Player 2 can be formulated as finding a vector y and number u such that

(1) $\quad Ay^S \leq \underline{u}\,, \qquad y^S \geq \underline{0}, \quad \underline{1}\cdot y^S = 1.$

Analogously, a strategy x for Player 1 can be formulated as finding a vector x and number v such that

(2) $\quad Bx^R \leq \underline{v}\,, \qquad x^R \geq \underline{0}, \quad \underline{1}\cdot x^R = 1.$

u and v are unknown, but since A and B have positive entries, u and v are positive.

Replace variable y_j by y_j/u variable x_i by x_i/v in (1) and (2) respectively to get equivalents systems

(1') $\quad A\mathbf{y'} \leq \underline{1}\,, \qquad \mathbf{y'} \geq \underline{0}$

and

(2') $\quad B\mathbf{x'} \leq \underline{v}\,, \qquad \mathbf{x'} \geq \underline{0}$

$Ay^S \leq \underline{u}$	became	$A\mathbf{y'} \leq \underline{1}$	$Bx^R \leq \underline{v}$	became	$B\mathbf{x'} \leq \underline{1}$
$y^S \geq \underline{0}$		$\mathbf{y'} \geq \underline{0}$	$x^R \geq \underline{0}$		$\mathbf{x'} \geq \underline{0}$
$\underline{1}\cdot y^S = 1$			$\underline{1}\cdot x^R = 1$		

The equilibrium conditions:

$x_i > 0 \Rightarrow (Ay^S)_i = u$ $\qquad$ $y_j > 0 \Rightarrow (Bx^R)_j = v$

becomes $\mathbf{x'_i} > 0 \Rightarrow (A\mathbf{y'})_i = 1$ $\qquad$ becomes $\mathbf{y'_j} > 0 \Rightarrow (B\mathbf{x'})_i = 1$

Inserting slack variables:

$[\,A \mid I_m\,]\,\mathbf{y''} = \underline{1}$ $\qquad$ $[\,I_n \mid B\,]\,\mathbf{x''} = \underline{1}$

$\mathbf{y''} \geq \underline{0}$ $\qquad$ $\mathbf{x''} \geq \underline{0}$

Let y” and x” be basic feasible solutions. Remember: the systems are non-degenerate.

i in basis of $[\,I_n \mid B\,]$ <=> $x''_i > 0$ <=> $(A\mathbf{y'})_i = 1$ <=> slack variable i of $[\,A \mid I_m\,]$ is 0 <=>
i is not in basis of $[\,A \mid I_m\,]$

j in basis of $[\,A \mid I_m\,]$ <=> $y''_j > 0$ <=> $(B\mathbf{x})_j = 1$ <=> slack variable i of $[\,I_n \mid B\,]$ is 0 <=>
i is not in basis of $[\,I_n \mid B\,]$

The Nash equilibrium (x^R, y^S) is

$x^R = (\mathbf{x''})^{\mathbf{R}} \,/\, \boldsymbol{\Sigma}\, \{\mathbf{x''_j} : j \in R\},$ $\qquad$ $y^R = (\mathbf{y''})^{\mathbf{S}} \,/\, \boldsymbol{\Sigma}\, \{\mathbf{y''_j} : j \in S\}$

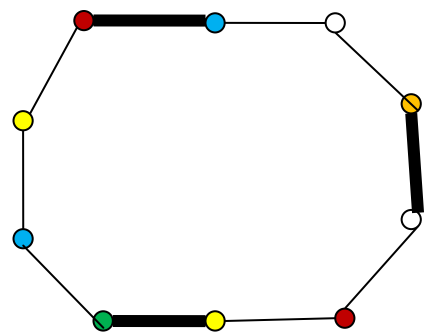
So: **Finding a Nash equilibrium is equivalent to finding a pair of complementary feasible bases of two bounded non-degenerate linear feasibility problems (at top of page) different from the starting pair of complementary feasible bases.**

Finding a Nash equilibrium is equivalent to finding a pair of complementary bases of two bounded non-degenerate linear feasibility problems different from the starting pair of complementary feasible bases.

This is an instance of
Given two polytopal (manifold) oiks, and a partition of the vertices into a room in each, find a different partition of the vertices into a room in each oik.

Rainbow Matchings in Gale Oiks

Where the vertices are coloured, a ***rainbow matching*** is a matching such that the vertices it contains have distinct colours.



A rainbow matching in a graph is the same as a perfect matching in the graph obtained by gluing together all vertices of the same colour.

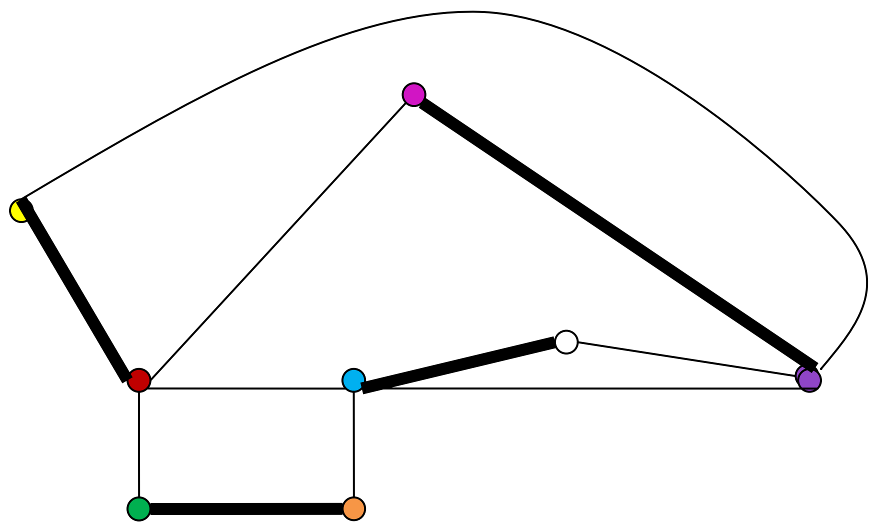
Thus, the rainbow rooms of a Gale manifold oik with coloured vertices are the same as the perfect matchings in the Euler graph you get by gluing together the vertices of C_n which have the same colour.

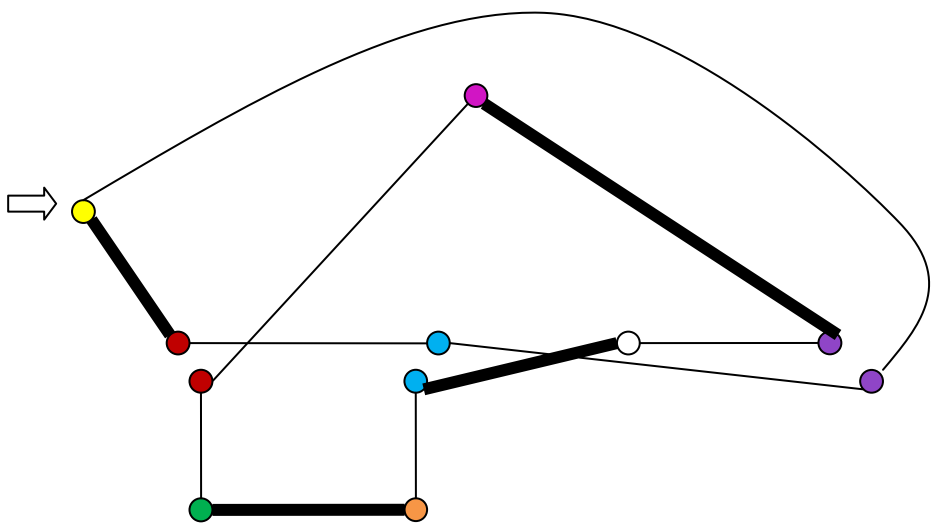
Rainbow rooms of a Gale manifold oik with coloured vertices corresponds to partitioning the vertices into a room in the Gale manifold and a room in a Sperner manifold.

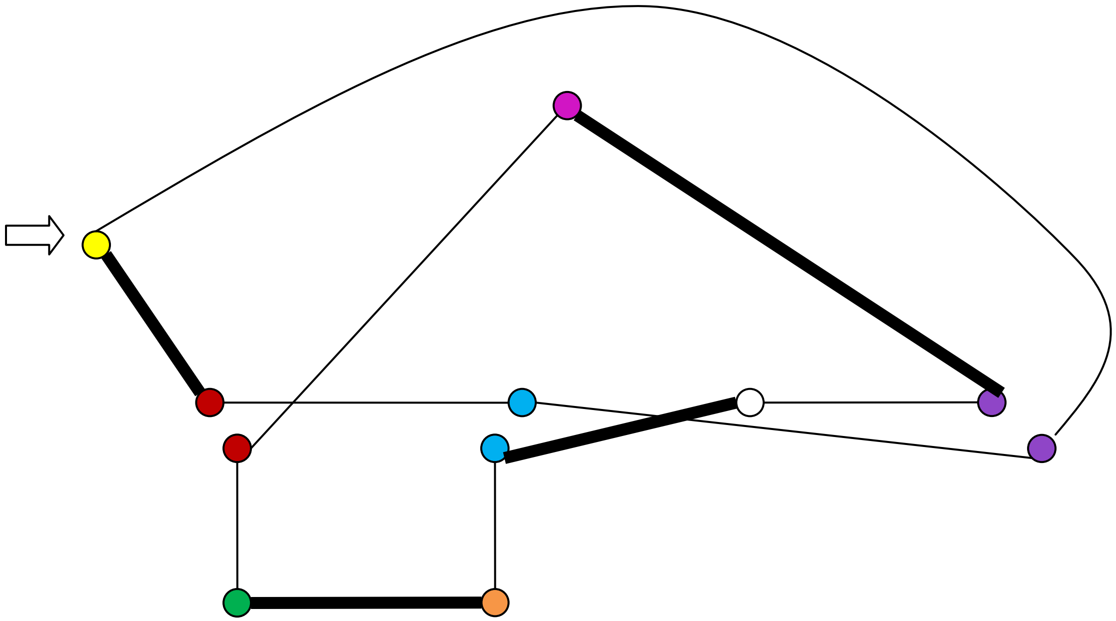
Sperner oiks are polytopal (trivially).
Gale oiks are polytopal (not so trivially – they come from cyclic polytopes)

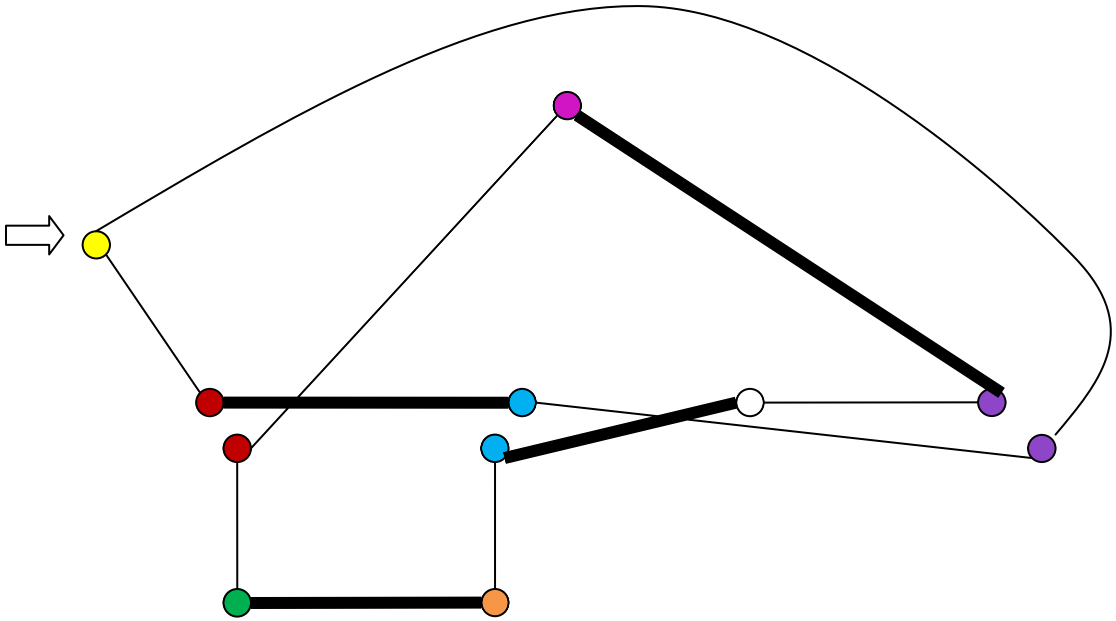
We don't need to know that in order to apply the exchange algorithm for finding a room partition of an oik family consisting on one Sperner oik and one Gale oik, to find another rainbow matching in an even cycle with coloured vertices
(or a another perfect matching in an Euler graph).

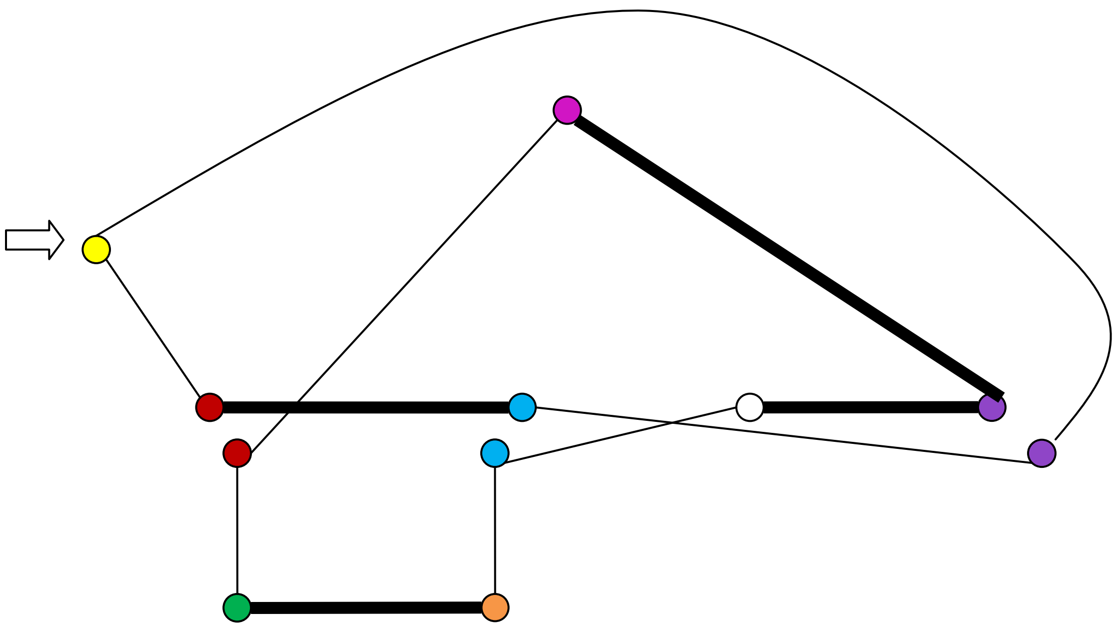
Our purpose is only to conclude that the exchange algorithm on this oik family is an instance of the Lemke-Howson algorithm.

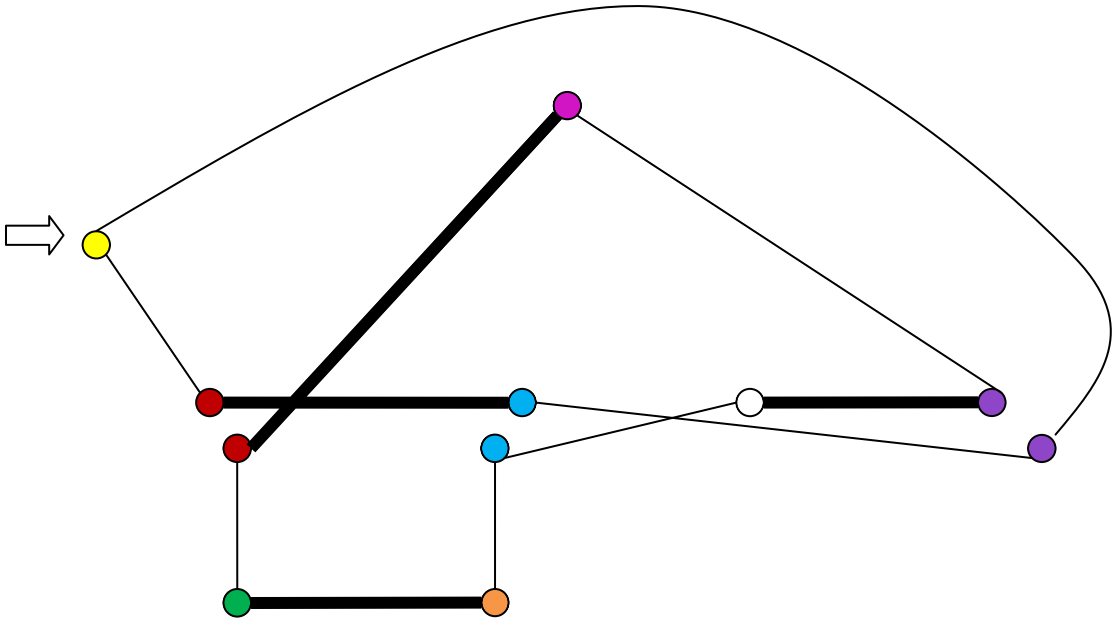


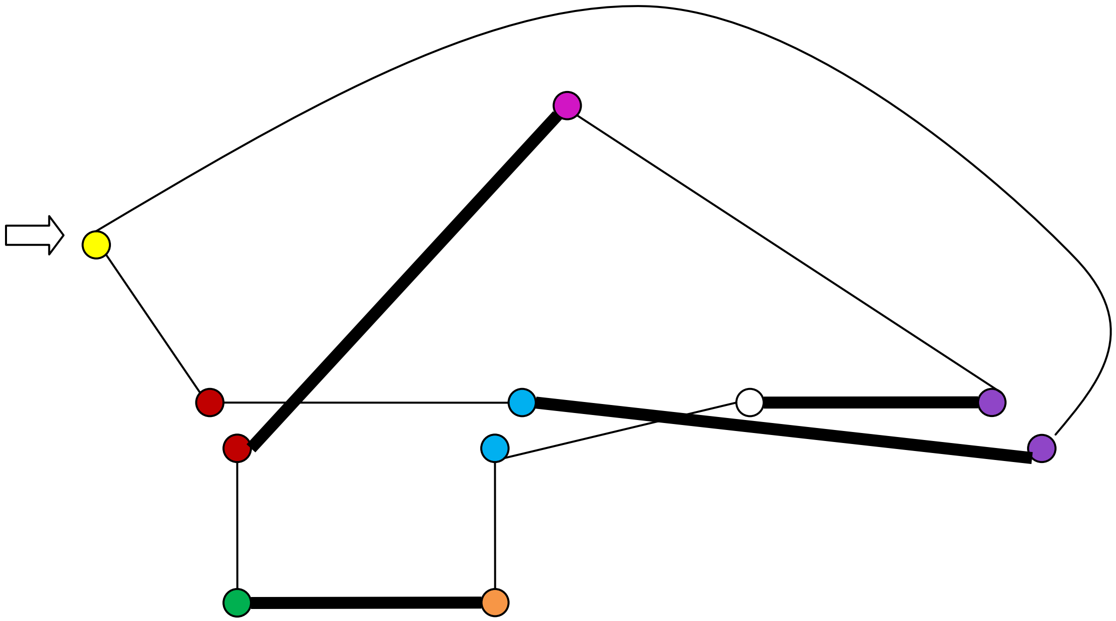


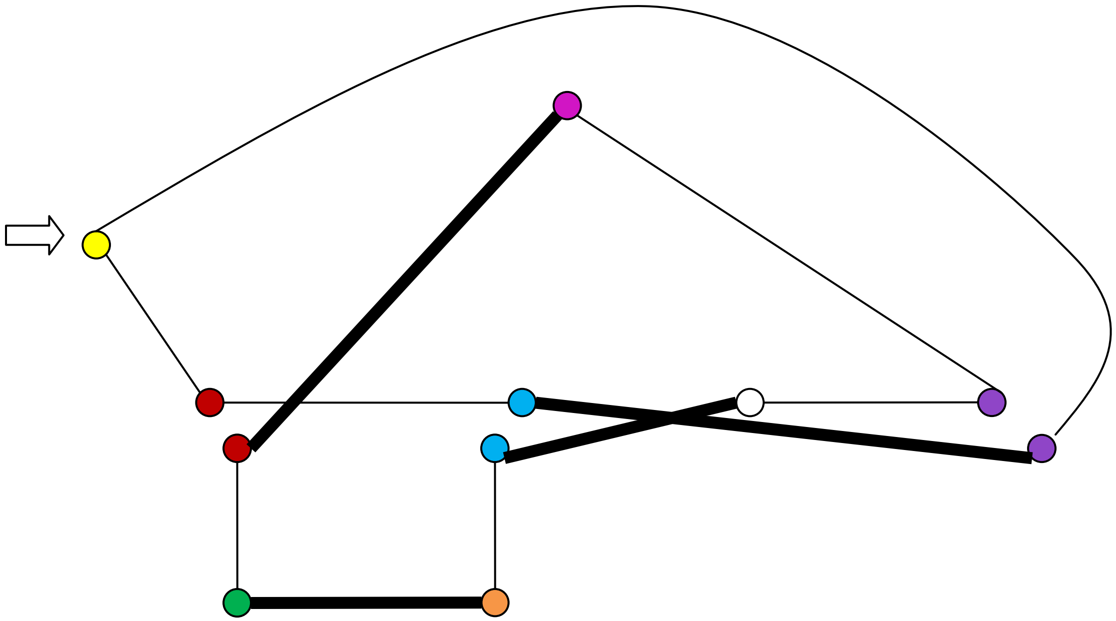


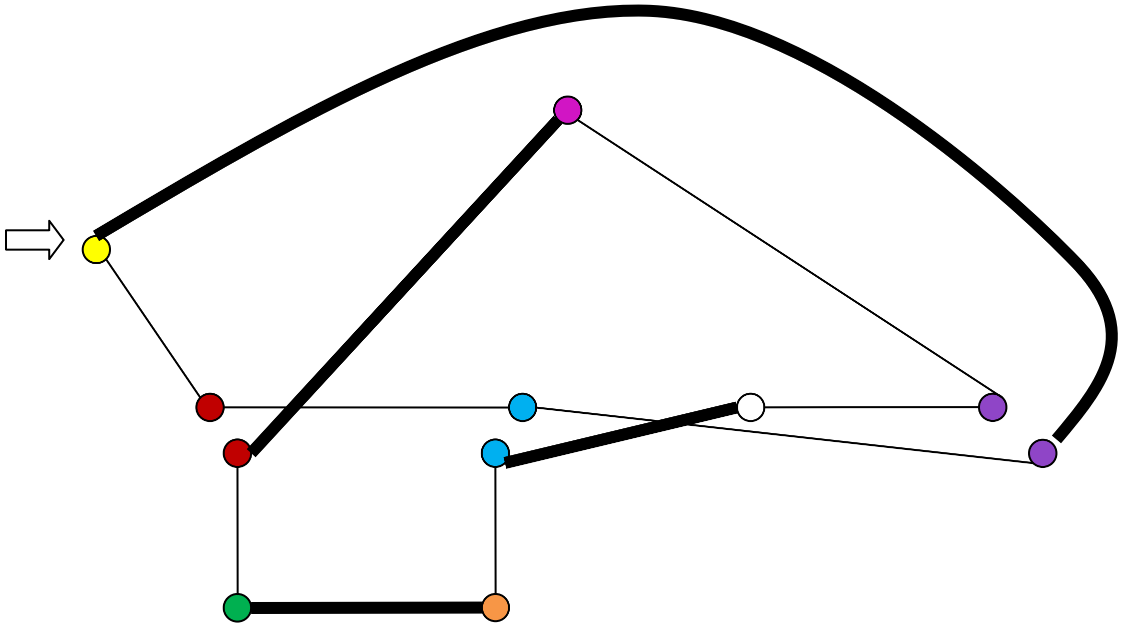


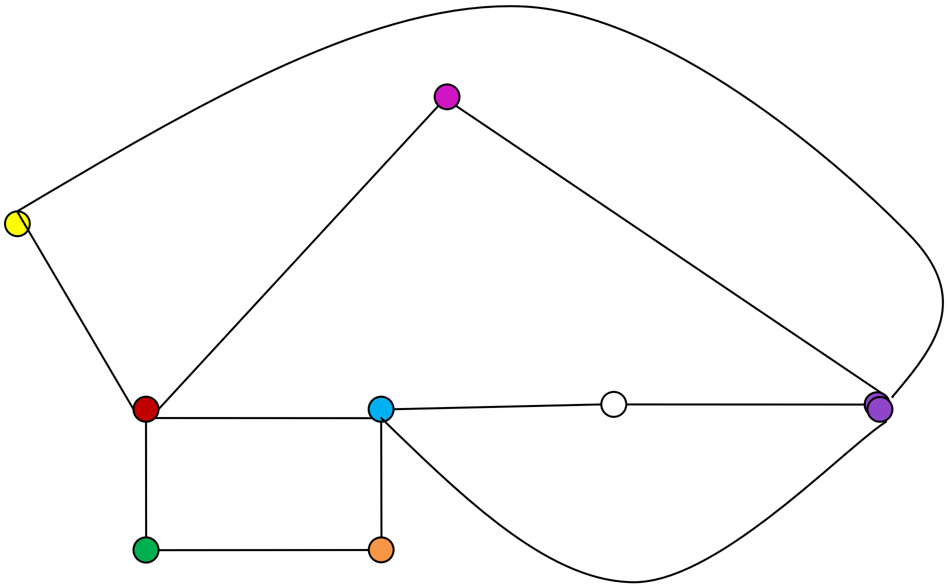


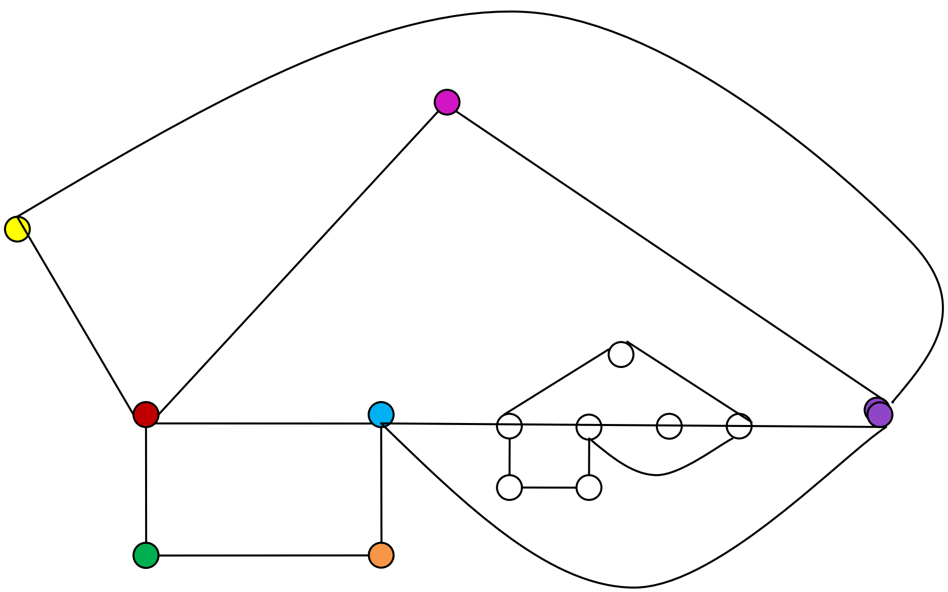












Scarf's Co-operative Game. A modified version of Scarf's theory of Non-transferable Utility (NTU) Co-operative n-person games.
(aka *fractional stable matching in hypergraphs*, Aharoni & Fleiner, 2003, JCT)**:**

Stable allocation of coalitions:
Given a list of subsets of the n persons, called coalitions.
(Different coalitions can be the same subset of persons.)
For each person, given a total preference ordering of the coalitions she is in.
A stable allocation, S, is a subset of mutually disjoint coalitions
such that for every coalition, c, not in S
there is a person in c who prefers the coalition of S that she is in more than c.

Where the persons are the vertices of a bipartite graph G and the coalitions are the edges of G, the Stable Marriage Theorem says there exists a stable allocation. It is proved by an easy algorithm for finding one.

More generally there may not be a stable allocation. However **there always exists a 'fractional stable allocation of coalitions',** proved by a very interesting but exponentially growing algorithm for finding one. In other words:

In any hypergraph preference system there exists a fractional stable matching.

fractional allocation of coalitions (i.e., *fractional matching):*
an assignment of non-negative weights, w(h), to the hypergraph edges, h,
such that for each vertex, v, sum of w(h), where h contains v, is at most 1.

A fractional matching w is called *stable* when
Every edge e contains a vertex v such that sum of w(h)
over edges containing v and having preference at least as great as e
equals 1.

Existence of a fractional stable matching follows immediately from
Scarf's Theorem:
Given an m by n matrix B whose first m columns are identity I;
Given non-neg b such that $\{x: Bx = b, x \geq \underline{0}\}$ is bounded;
And given m by n matrix $C = \{c(i,j): i = 1\ldots m, j = 1\ldots n\}$
of preference ordering numbers for each row i,
such that $c(i,j) \leq c((i,k) \geq c(i,j)$ for row i, row j, and any non-row column k.
Then there exists a size m subset J of columns
such that $Bx = b$ for some $x \geq \underline{0}$ where $x_j = 0$ for j not in J,

and such that J is "dominating",
i.e., for every column k there is a row i such that $c(i,k) \leq c(i,j)$ for all $j \in J$.
For a matrix C where in each row the entries
are the elements of a preference total ordering, say 0, 1, 2, …, n, ∞,
A subset D of columns of C is called *dominating* if
for every column k of C there is a row i such that $c(i,k) \leq c(i,j)$ for all $j \in D$.

Clearly subsets of a dominating set are dominating.
For the C of Scarf's Theorem the first m columns, call it set H, is not dominating.
However every subset of H is dominating.

Scarf's Lemma:
The family consisting set H together with all size m dominating sets is the rooms of a manifold oik.

Important example (showing exponential growth of the exchange algorithm):

$$\begin{matrix} \infty & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & \infty & 0 & 0 & 0 & 5 & 1 & 2 & 3 & 4 \\ 0 & 0 & \infty & 0 & 0 & 5 & 4 & 1 & 2 & 3 & \dots \\ 0 & 0 & 0 & \infty & 0 & 5 & 4 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & \infty & 5 & 4 & 3 & 2 & 1 \end{matrix}$$