Multi-Row Cuts
in
Integer Programming

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Mixed Integer Linear Programming

$$\begin{array}{lll} \min & cx & \\ \text{s.t.} & Ax = b & \\ & x_j \in \mathbb{Z} & \text{for } j = 1, \ldots, p \\ & x_j \geq 0 & \text{for } j = 1, \ldots, n. \end{array}$$

Common approach to solving MILP:

- First solve the LP relaxation. Basic optimal solution:

$$x_i \;=\; f_i + \sum_{j \in N} r^j x_j \quad \text{for } i \in B.$$

- If $f_i \notin \mathbb{Z}$ for some $i \in B \cap \{1, \ldots, p\}$, add cutting planes:

Gomory 1963 Mixed Integer Cuts, Marchand and Wolsey 2001 MIR inequalities, Balas, Ceria and Cornuéjols 1993 lift-and-project cuts, for instance, are used in commercial codes.

References

This talk

Borozan and Cornuéjols MOR 2009
Basu, Conforti, Cornuéjols and Zambelli SIDMA 2010
Basu, Conforti, Campelo, Cornuéjols and Zambelli IPCO 2010
Basu, Cornuéjols and Margot working paper 2010
Basu, Cornuéjols and Köppe working paper 2011

Related work

Corner polyhedron

Gomory LAA 1969
Gomory and Johnson MP 1972

Intersection cuts

Balas OR 1971

The work that motivated me

Andersen, Louveaux, Weismantel and Wolsey IPCO 2007
Dey and Richard MOR 2008
Dey and Wolsey IPCO 2008, SIOPT 2010

Corner Polyhedron

Gomory 1969

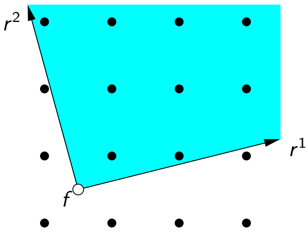
Relax nonnegativity on basic variables x_j.

In our work, we make a further relaxation, as suggested by Andersen, Louveaux, Weismantel and Wolsey 2007

Relax integrality on nonbasic variables.

$$
\begin{aligned}
x &= f + \sum_{j=1}^{k} r^j s_j \\
x &\in \mathbb{Z}^q \\
s &\geq 0
\end{aligned}
$$

Example



Feasible set $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Z}^2 : \right.$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = f + r^1 s_1 + r^2 s_2$$

where $\left. s_1 \geq 0, s_2 \geq 0 \right\}$

Formulas for Deriving Cutting Planes

$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j \\ x & \in & \mathbb{Z}^q \\ s & \geq & 0 \end{array}$$

Every inequality cutting off the point $(\bar{x}, \bar{s}) = (f, 0)$ can be expressed in terms of the nonbasic variables s only, in the form $\sum_{j=1}^{k} \alpha_j s_j \geq 1$.

We are interested in "formulas" for deriving such inequalities.

More formally, we are interested in functions $\psi : \mathbb{R}^q \to \mathbb{R}$ such that the inequality

$$\sum_{j=1}^{k} \psi(r^j) s_j \geq 1$$

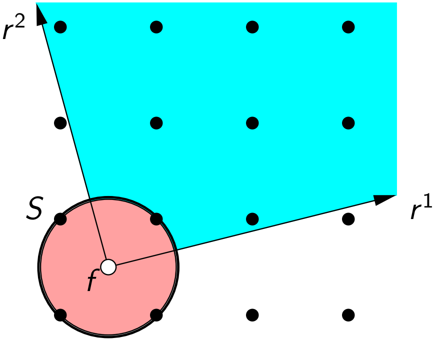
is valid for every choice of k and vectors $r^1, \ldots, r^k \in \mathbb{R}^q$.

Such functions ψ will be called valid functions with respect to f.

Intersection Cuts

Balas 1971

Assume $f \notin \mathbb{Z}^q$. Want to cut off the basic solution $s = 0, x = f$.



Any convex set S with $f \in int(S)$ with no integer point in $int(S)$.

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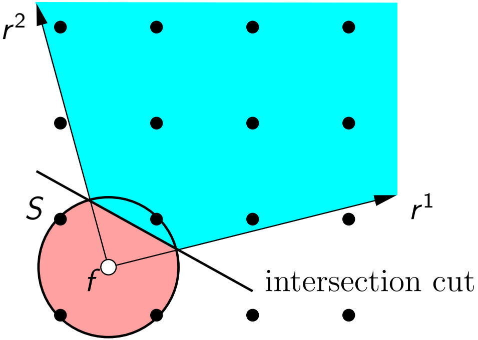


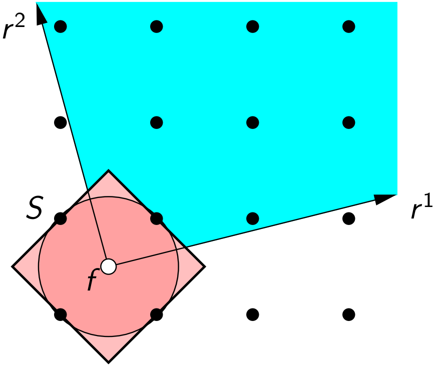
Any convex set S with $f \in int(S)$ with no integer point in $int(S)$.

The gauge of $S - f$, i.e. $\psi(r) = inf\{\lambda > 0 : \frac{1}{\lambda} r \in S - f\}$ is a valid function.

Intersection cut: $\psi(r^1)s_1 + \psi(r^2)s_2 \geq 1$.

Minimal Valid Functions

Our main interest is in minimal valid functions $\psi : \mathbb{R}^q \to \mathbb{R}$, i.e. there is no valid function $\psi' \leq \psi$ where $\psi'(r) < \psi(r)$ for at least one $r \in \mathbb{R}^q$.



Bigger convex set

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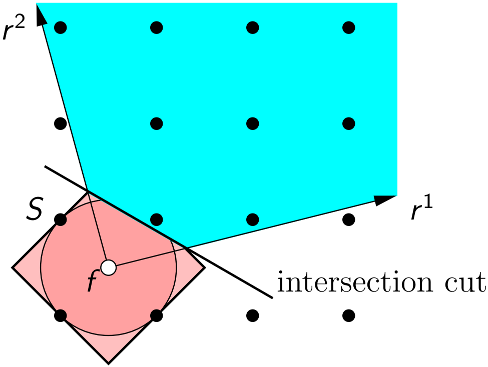


Bigger convex set

Better cut: $\psi(r^1)s_1 + \psi(r^2)s_2 \geq 1$.

Theorem

Borozan and Cornuéjols MOR 2009

On $\mathbb{Q}^q$ (extension to $\mathbb{R}^q$ due to Basu, Conforti, Cornuéjols, Zambelli)

Let $f \in \mathbb{R}^q \setminus \mathbb{Z}^q$.

If $\psi : \mathbb{R}^q \to \mathbb{R}$ is a minimal valid function, then ψ is

- nonnegative
- piecewise linear
- positively homogeneous
- and convex.

Furthermore $B_\psi := \{x \in \mathbb{R}^q : \ \psi(x - f) \leq 1\}$ is a maximal $\mathbb{Z}^q$-free convex set containing f in its interior.

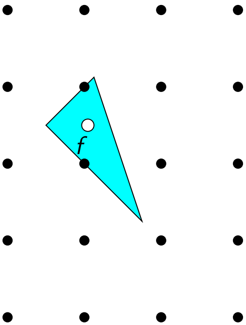
Conversely, for any maximal $\mathbb{Z}^q$-free convex set B containing f in its interior, the gauge of $B - f$ is a minimal valid function ψ.

DEFINITION A convex set is $\mathbb{Z}^q$-free if it does not have any integral point in its interior. However, it may have integral points on its boundary.

Maximal $\mathbb{Z}^q$-Free Convex Sets

...are polyhedra Lovász 1989

- $\mathbb{Z}^q$-free convex set contains no integral point in its interior



Maximal: each edge contains an integral point in its relative interior.

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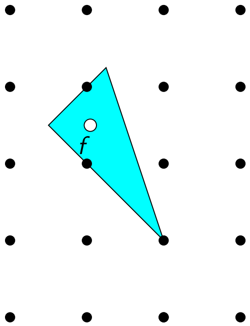

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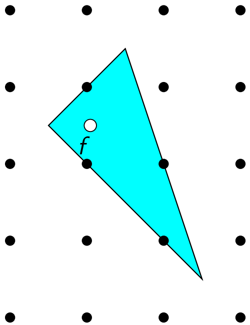
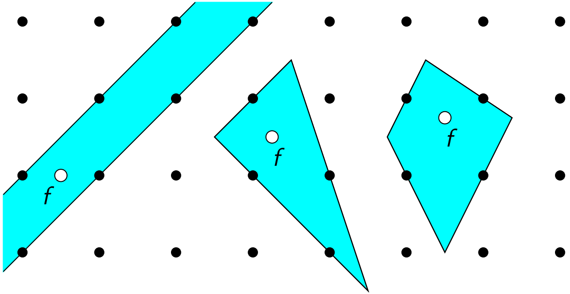


Maximal: each edge contains an integral point in its relative interior.

In the plane: it is a strip, a triangle or a quadrilateral.

Maximal $\mathbb{Z}^q$-Free Sets in the Plane

Split, triangles and quadrilaterals



generate split, triangle and quadrilateral inequalities $\sum \psi(r) s_r \geq 1$, where the function ψ is the gauge of $S - f$.

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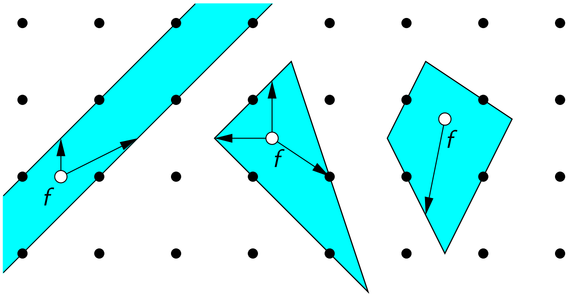
Maximal $\mathbb{Z}^q$-Free Sets in the Plane

Split, triangles and quadrilaterals



generate split, triangle and quadrilateral inequalities $\sum \psi(r) s_r \geq 1$,
where the function ψ is the gauge of $S - f$.

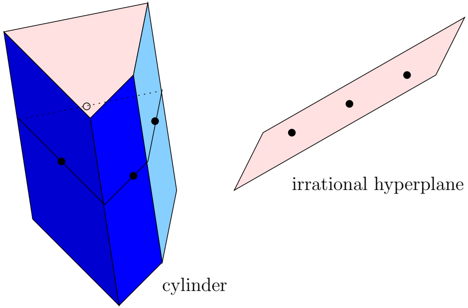
If $S = \{x \in \mathbb{R}^q : a_i(x - f) \leq 1, i = 1, \ldots, t\}$,
then $\psi = max_{i=1,\ldots,t} a_i r$.

Lovász' Theorem

THEOREM A set $K \subset \mathbb{R}^q$ is a maximal $\mathbb{Z}^q$-free convex set if and only if

- either K is a polyhedron of the form $K = P + L$ where P is a polytope, L is a rational linear space, $\dim(P) + \dim(L) = p$, K does not contain any point of $\mathbb{Z}^q$ in its interior and there is a point of $\mathbb{Z}^q$ in the relative interior of each facet of K.

- or K is an irrational hyperplane.



Generalization of the Lovász and Borozan-Cornuéjols theorems

Here, we consider a system of the form

$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j \\ x & \in & S \\ s & \geq & 0 \end{array}$$

where $S = P \cap \mathbb{Z}^q$ for some rational polyhedron $P \subseteq \mathbb{R}^q$.

This model has been studied in the 70s by Glover 1974, Balas 1972 and Johnson 1981, and recently by Dey and Wolsey 2009, and Günlük and Fukusawa 2009.

Basu, Conforti, Cornuéjols and Zambelli generalize the Lovász and Borozan-Cornuéjols theorems to such systems SIDMA 2010.

Integer Lifting   Dey-Wolsey 2010

QUESTION: How should we deal with INTEGER nonbasic variables?

Integer Lifting

Here, we consider a system of the form

$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j + \sum_{i=1}^{\ell} \rho^i y_i \\ x & \in & \mathbb{Z}^q \\ s & \geq & 0 \\ y & \in & \mathbb{Z}^\ell. \end{array}$$

We are interested in functions $\psi : \mathbb{R}^q \to \mathbb{R}$ and $\phi : \mathbb{R}^q \to \mathbb{R}$ such that the inequality

$$\sum_{j=1}^{k} \psi(r^j) s_j + \sum_{i=1}^{\ell} \phi(\rho^i) y_i \geq 1$$

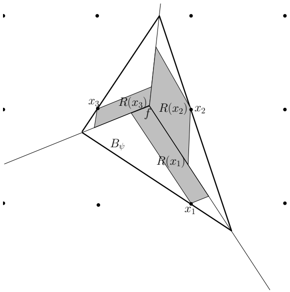
is valid for every choice of integers k, ℓ and vectors $r^1, \ldots, r^k \in \mathbb{R}^q$ and $\rho^1, \ldots, \rho^\ell \in \mathbb{R}^q$.

Integer Lifting Basu, Campelo, Conforti, Cornuéjols, Zambelli IPCO 2010

Starting from a minimal valid function $\psi : \mathbb{R}^q \to \mathbb{R}$, what can we say about a minimal lifting function ϕ?

Clearly, $\phi \leq \psi$. Are there regions R where we can guarantee that $\phi(r) = \psi(r)$ for all $r \in R$?

THEOREM Let ψ be minimal.
$\phi(r) = \psi(r)$ for $r \in R = \bigcup_t R(x_t)$ where the union is taken over all integral points x_t on the boundary of the maximal $\mathbb{Z}^q$-free convex set B_ψ defining ψ.
Conversely, if $r \notin R$, there exists a minimal lifting ϕ where $\phi(r) < \psi(r)$.



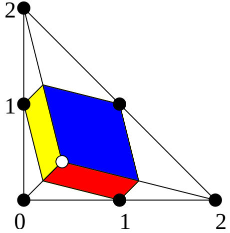
THEOREM A minimal function ψ has a unique minimal lifting ϕ if and only if $R + \mathbb{Z}^q$ covers $\mathbb{R}^q$.

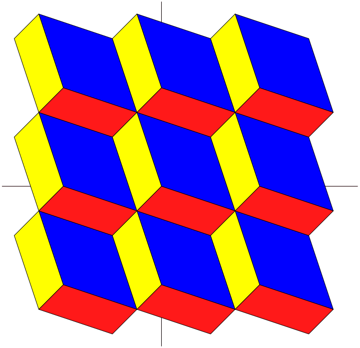
Body with a Unique Lifting

Characterizing when the integer lifting is unique.

Example: Split inequalities, Gomory Mixed Integer Cuts.

Another example:

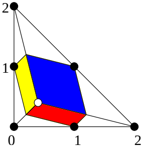


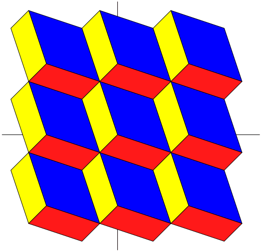


Bodies with a Unique Lifting Basu, Cornuéjols, Köppe 2011

THEOREM Let B be a maximal lattice-free simplicial polytope in $\mathbb{R}^n$. Then B is either a body with a unique lifting for all $f \in \mathrm{int}(B)$, or a body with multiple liftings for all $f \in \mathrm{int}(B)$.

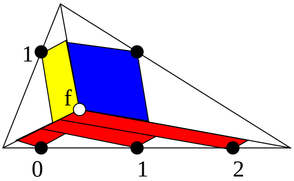
THEOREM Let Δ be a simplex in $\mathbb{R}^n$ such that it is a maximal lattice-free convex body and each facet of Δ has exactly one integer point in its relative interior. Then Δ is a body with a unique lifting for all $f \in \mathrm{int}(B)$ if and only if all the vertices of Δ are integral, i.e., Δ is an affine unimodular transformation of $\mathrm{conv}\{0, ne^1, \ldots, ne^n\}$.





Bodies with a Unique Lifting Basu, Cornuéjols, Köppe 2011

THEOREM
Let $\Delta \subset \mathbb{R}^{n+1}$ be a maximal lattice-free 2-partitionable simplex with hyperplanes H_1, H_2 such that H_1 defines a facet of Δ and this is the only facet of Δ with more than one lattice point in its relative interior. Then Δ is a body with a unique lifting for all $f \in \mathrm{int}(B)$ if and only if $\Delta \cap H_2$ is an affine unimodular transformation of $\mathrm{conv}\{0, ne^1, \ldots, ne^n\}$.



THEOREM Let $B \subset \mathbb{R}^n$ be a maximal lattice-free simplicial polytope and let $f \in \mathrm{int}(B)$. Then the volume of the region R where the lifting is unique is an affine function of the coordinates of f.

Split Inequalities   Cook-Kannan-Schrijver 1990

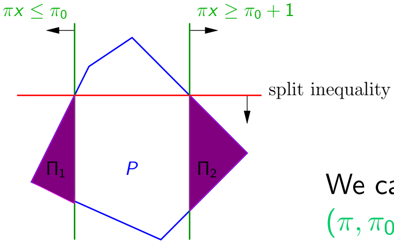
Widely used in commercial solvers.

QUESTION: Can we generate any intersection cut using a sequence of split inequalities?

Split Inequalities Cook-Kannan-Schrijver 1990

$P := \{x \in \mathbb{R}^n : \; Ax \geq b\}$
$S := P \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.

For $\pi \in \mathbb{Z}^n$ such that $\pi_{p+1} = \ldots = \pi_n = 0$ and $\pi_0 \in \mathbb{Z}$, define



$$\Pi_1 := P \cap \{x : \; \pi x \leq \pi_0\}$$

$$\Pi_2 := P \cap \{x : \; \pi x \geq \pi_0 + 1\}$$

We call $cx \leq c_0$ a split inequality if there exists $(\pi, \pi_0) \in \mathbb{Z}^p \times \mathbb{Z}$ such that $cx \leq c_0$ is valid for $\Pi_1 \cup \Pi_2$.

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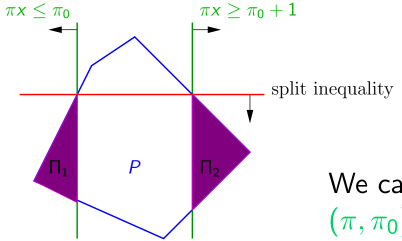

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The split closure is the intersection of all split inequalities.

THEOREM Cook, Kannan, Schrijver 1990
The split closure is a polyhedron.

Split Rank Cook-Kannan-Schrijver 1990

$P := \{x \in \mathbb{R}^n : \; Ax \geq b\}$
$S := P \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.

Let $P^0 = P$. For $k \geq 1$, let P^k denote the split closure of P^{k-1}.

Let $\alpha x \leq \beta$ be a valid inequality for $\text{conv}(S)$. The smallest k such that $\alpha x \leq \beta$ is valid for P^k is called the split rank of $\alpha x \leq \beta$, if such an integer k exists.

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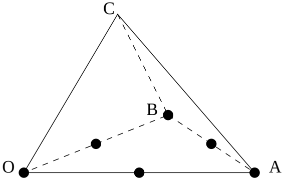
In the mixed integer case, inequalities may have infinite split rank, i.e. there is no finite k such that $\alpha x \leq \beta$ is valid for P^k, as shown by the following example.



P is a simplex with vertices $O = (0,0,0)$, $A = (2,0,0)$, $B = (0,2,0)$ and $C = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
$S := P \cap (\mathbb{Z}^2 \times \mathbb{R})$.
Thus $\text{conv}(S) = P \cap \{y \leq 0\}$.

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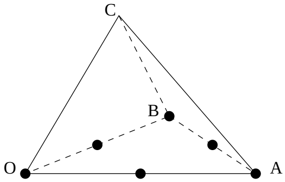

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Thus $\mathrm{conv}(S) = P \cap \{y \leq 0\}$.

Consider a simplex P with vertices O, A, B and $C = (\frac{1}{2}, \frac{1}{2}, t)$ with $t > 0$. Let $C_1 = C$, let C_2 be the point on the edge AC with coordinate $x_1 = 1$ and C_3 the point on BC with $x_2 = 1$. Observe that no split disjunction removes all three points $C_1,\ C_2,\ C_3$. Thus $(\frac{1}{2}, \frac{1}{2}, \frac{t}{3}) \in P^1$. By induction, $(\frac{1}{2}, \frac{1}{2}, \frac{t}{3^k}) \in P^k$. Therefore $y \leq 0$ has infinite split rank.

The Andersen-Louveaux-Weismantel-Wolsey Model

The Cook-Kannan-Schrijver example can be written as
$x_1 \geq y, \quad x_2 \geq y, \quad x_1 + x_2 + 2y \leq 2$.

Introducing nonnegative slack variables, and eliminating y, we get

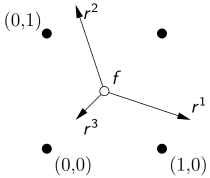
$$\begin{cases} x_1 & = & \frac{1}{2} + \frac{3}{4}s_1 - \frac{1}{4}s_2 - \frac{1}{4}s_3 \\ x_2 & = & \frac{1}{2} - \frac{1}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{4}s_3 \\ x_i & \in \mathbb{Z} & \text{for } i = 1, 2 \\ s_j & \geq 0 & \text{for } j = 1, 2, 3. \end{cases}$$

Note that $\quad y \leq 0 \quad \Longleftrightarrow \quad s_1 + s_2 + s_3 \geq 2$

Remember the Andersen, Louveaux, Weismantel, Wolsey 2007 model:

$$\begin{cases} x_1 & = & f_1 + \sum_{j=1}^{n} r_1^j s_j \\ x_2 & = & f_2 + \sum_{j=1}^{n} r_2^j s_j \\ x_i & \in \mathbb{Z} & \text{for } i = 1, 2 \\ s_j & \geq 0 & \text{for } j = 1, \ldots, n. \end{cases}$$

The Cook-Kannan-Schrijver Example Continued



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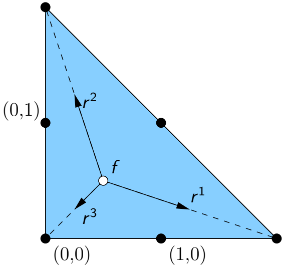
Recall: Inequality with infinite split rank is $s_1 + s_2 + s_3 \geq 2$

This is the intersection cut associated with the triangle

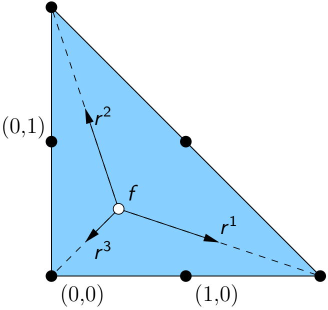


The Dey-Louveaux Theorem 2009

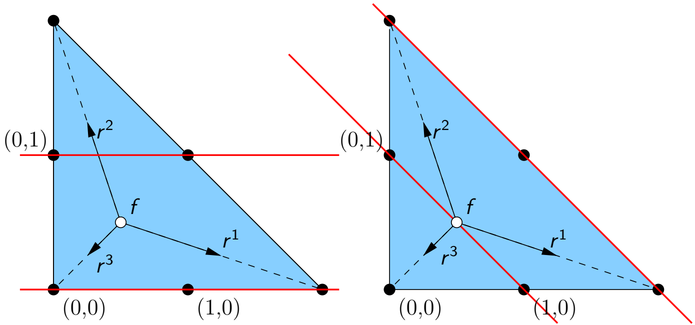
Andersen, Louveaux, Weismantel, Wolsey 2007 model in $\mathbb{R}^2$:

$$\begin{cases} x_1 & = & f_1 + \sum_{j=1}^{n} r_1^j s_j \\ x_2 & = & f_2 + \sum_{j=1}^{n} r_2^j s_j \\ x_i & \in \mathbb{Z} & \text{for } i = 1, 2 \\ s_j & \geq 0 & \text{for } j = 1, \ldots, n. \end{cases}$$

THEOREM Every intersection cut has a finite split rank, except for those generated from a maximal $\mathbb{Z}^2$-free triangle with integral vertices and rays pointing to the corners.



A Property of the Triangles that Generate Intersection Cuts with Infinite Split Rank



Not all integral points can fit on the two parallel lines of a split.

IMPRECISE DEFINITION If every integral point of $K \subset \mathbb{R}^q$ lies on the two parallel hyperplanes of a split, we say that K has the 2-hyperplane property.

Intersection Cuts with Finite Split Rank

THEOREM Basu, Cornuéjols, Margot 2010

Let K be a rational lattice-free polytope in $\mathbb{R}^q$ containing f in its interior and having rays going into its corners.
The intersection cut arising from K has finite split rank if and only if K has the 2-hyperplane property.

PRECISE DEFINITION

A set S of points in $\mathbb{R}^q$ is 2-partitionable if either $|S| \leq 1$ or there exists a partition of S into nonempty sets S_1, S_2 and a split such that S_1 is contained in one of its boundary hyperplanes and S_2 is contained in the other.

A polytope is 2-partitionable if its integer points are 2-partitionable.

Let K_I be the convex hull of the integer points in K. We say that K has the 2-hyperplane property if every face of K_I that is not contained in a facet of K is 2-partitionable.

Idea of Proof

If K does not have the 2-hyperplane property, it is not too hard to show that the intersection cut arising from K has infinite split rank.

The difficult part of the theorem is to show if K has the 2-hyperplane property, then the intersection cut arising from K has finite split rank.

Our proof is by induction on the dimension q.

We define the notions of intersecting split and englobing split, and we show that the theorem holds when there is a sequence of intersecting splits followed by an englobing split.

We use Chvátal cuts to reduce K to K_I. The theorem is proved by replacing each of the Chvátal cuts by a finite collection of intersecting splits for enlarged polytopes, and using the 2-hyperplane property for proving that a final englobing split exists.

Thank you

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