Perspective cuts for the ACOPF with generators

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Abstract The alternating current optimal power flow problem is a fundamental problem in the management of smart grids. In this paper we consider a variant which includes activation/deactivation of generators at some of the grid sites. We formulate the problem as a mathematical program, prove its **NP**-hardness w.r.t. activation/deactivation, and derive two perspective reformulations.

1 Introduction

The Alternating Current Optimal Power Flow (ACOPF) problem is one of the most important problems arising in the energy industry. It models the propagation of power flows in electrical grids. It is often used as second-level subproblem in bilevel problems modelling the decision of electricity prices subject to production and demands [14]. Multilevel problems with ACOPF at different time-scales are also considered [1]. The ACOPF received a lot of attention over the years, and specifically after smart grids were introduced [2].

The ACOPF asks for the best power flow over an electrical network modelled by a digraph $\mathscr{D} = (\mathscr{N}, \mathscr{L})$, where $\mathscr{N}$ is the set of *buses* and $\mathscr{L}$ the set of *lines*. It is well known that the natural formulation can be simplified using only voltage variables [10]. The ACOPF is usually cast as a Mathematical Programming (MP) problem

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over the complex numbers (which make their appearance due to the cyclic nature of alternating currents). The standard ACOPF can be reformulated as a (larger) MP over the reals, by separating real and complex parts [17].

While the standard version of the ACOPF only has continuous variables, more realistic variants include binary variables which activate/deactivate various electrical components. In this paper we consider the possibility of activating/deactivating electrical generation at some of the buses. This defines an ACOPF variant which we call ACOPF with Generators (ACOPFG) [15].

Note that the ACOPF is **NP**-hard even without binary variables, as shown in [13]. Experimentally, however, it was found that many standard benchmarks, as well as randomly generated instances, can be solved efficiently. It is shown in [12] that this happens whenever the duality gap is zero. One might then question whether the ACOPFG is **NP**-hard simply because of the addition of the binary activation variables. The first contribution of this paper is to prove that this is indeed the case.

While ACOPF objective functions vary in the literature, it is common to consider quadratic objectives with respect to voltage. In this paper, however, we focus on a more general objective function, quadratic with respect to active power and quartic (without cubic terms) w.r.t. voltage [10]. The second contribution of this paper is the application of two perspective reformulations (PR) to the ACOPFG with the more general (quartic) objective [8, 6].

2 MP formulation

We consider the network digraph $\mathscr{D}$ mentioned in Sect. 1. Let $n = |\mathscr{N}|$ and $\ell = |\mathscr{L}|$. We identify a subset $\mathscr{G}$ of *generator buses*, and let $n' = |\mathscr{G}|$. We note that, in modern "smart grids", generators may produce *and* consume electricity. Because we are dealing with alternating currents, power is represented by a complex number. The real part is the *active power* while the complex part is *reactive*.

Notationwise, we use $[\underline{\alpha}, \overline{\alpha}]$ to denote lower/upper bounds to a quantity, and α^* to denote complex conjugate.

The parameters of our problem are as follows:

- $\forall b \in \mathscr{N}$ $S_b \in \mathbb{C}$ is the power demand at bus b;
- $\forall g \in \mathscr{G}$ $\mathfrak{S}_g = [\underline{S}_g, \overline{S}_g]$ is the (complex) interval where g can generate power if active;
- $\forall b \in \mathscr{N}$ $\mathfrak{v}_b = [\underline{v}_b, \overline{v}_b]$ is the (real) interval where the voltage magnitude at bus b can range;
- $\forall (a,b) \in \mathscr{L}$ $\bar{\imath}_{ab}$ is the maximum current which can flow through the line (a,b);
- Y is a complex $n \times n$ *bus admittance matrix* (it plays a role analoguous to the reciprocal of resistance in Ohm's law);
- Y^0, Y^1 are complex $\ell \times n$ *line admittance matrices* (they "encode" some electrical properties of the lines).

The decision variables are:

- $\forall g \in \mathscr{G}$ $s_g \in \mathbb{C}$ is the power generated at g;
- $\forall g \in \mathscr{G}$ $z_g \in \{0,1\}$ denotes the deactivation (0) or activation (1) of generator g;
- $\forall b \in \mathscr{N}$ $v_b \in \mathbb{C}$ is the voltage at bus b;
- $\forall (a,b) \in \mathscr{L}$ $i_{ab} \in \mathbb{C}$ is the current on the line (a,b).

At each generator $g \in \mathscr{G}$, the injected complex power $s_g - S_g = v_g \sum_{(g,a)\in\mathscr{L}} i^*_{ga}$, and at each non-generator bus $b \in \mathscr{N} \smallsetminus \mathscr{G}$, we have $-S_b = v_b \sum_{(b,a)\in\mathscr{L}} i^*_{ba}$. Kirchoff's law and a generalized form of Ohm's law allow us to derive $i = Yv$, which implies that the RHS of the above equations can be reformulated to $v_b(Y^*v^*)_b = \sum_{(a,b)\in\mathscr{L}} v_b v^*_a Y^*_{ab}$ for each $b \in \mathscr{N}$ [17]. This allows us to express current in function of voltage and power. We obtain the following constraints:

$$\forall g \in \mathscr{G} \quad \sum_{(g,a)\in\mathscr{L}} Y^*_{ga} v_g v^*_a = s_g z_g - S_g \tag{1}$$

$$\forall b \in \mathscr{N} \smallsetminus \mathscr{G} \quad \sum_{(b,a)\in\mathscr{L}} Y^*_{ba} v_b v^*_a = -S_b \tag{2}$$

$$\forall (a,b) \in \mathscr{L}, \omega \in \{0,1\} \quad \sum_{h\neq k\in\mathscr{N}} (Y^{\omega}_{abh})^*(Y^{\omega}_{abk})^* v^*_h v_k \leq \bar{\imath}_{ab} \tag{3}$$

$$\forall g \in \mathscr{G} \quad s_g \in \mathfrak{S}_g \tag{4}$$

$$\forall b \in \mathscr{N} \quad |v_b| \in \mathfrak{v}_b \tag{5}$$

$$\forall g \in \mathscr{G} \quad z_g \in \{0,1\}. \tag{6}$$

We remark that complex power variables s only appear in Eq. (1) and (4). We can eliminate them by replacing Eq. (1) and (4) with the following inequalities:

$$\forall g \in \mathscr{G} \quad \underline{S}_g z_g \leq \sum_{(g,a)\in\mathscr{L}} Y^*_{ga} v_g v^*_a + S_g \leq \overline{S}_g z_g. \tag{7}$$

Moreover, if we define z over all of $\mathscr{N}$ and fix $z_b = 0$ for all $b \notin \mathscr{G}$, Eq. (7) quantified on $\mathscr{N}$ can also replace Eq. (2).

In the ACOPF literature [12, 11, 4, 15] we consider the following generation cost function, to be minimized:

$$f(s,z) = \sum_{g\in\mathscr{G}} z_g(c_{g2}(\mathsf{Re}(s_g))^2 + c_{g1}\mathsf{Re}(s_g) + c_{g0}). \tag{8}$$

Again we can replace s by v using Eq. (1) and removing constant terms in order to express Eq. (8) as a function of voltage: essentially, we obtain $f(v,z)$ from Eq. (8) by replacing s_g with $\sum_{(g,a)\in\mathscr{L}} Y^*_{ga} v_g v^*_a + S_g$.

Let F be the feasible subset of $\mathbb{C}^n$ defined by Eq. (2)-(6) and (7). We call $\mathsf{ACOPFG}_{\mathbb{C}}$ the formulation $\min_{(v,z)\in F} f(v,z)$.

Finally, we can obtain a real formulation as follows:

1. replace each quadratic constraint $v^H M v \; \Diamond \; \alpha + j\beta$ (where $\Diamond \in \{=,\leq,\geq\}$ and $j = \sqrt{-1}$) by the pair of constraints

$$v^H M^+ v \; \Diamond \; \alpha + j\beta \wedge v^H M^- v \; \Diamond \; \alpha + j\beta,$$

where $M^+ = \frac{1}{2}(M + M^H)$ and $M^- = \frac{1}{2}(M - M^H)$;

2. replace each complex matrix M by the real matrix $\begin{pmatrix} \mathsf{Re}(M) & -\mathsf{Im}(M) \\ \mathsf{Im}(M) & \mathsf{Re}(M) \end{pmatrix}$;
3. replace each complex vector v by the real vector $(\mathsf{Re}(v)\ \mathsf{Im}(v))^\top$.

We call this reformulation $\mathsf{ACOPFG}_{\mathbb{R}}$.

3 Complexity

Assume $c_{g2} = 0$ for all $g \in \mathscr{G}$ in Eq. (8). By ignoring activation variables we obtain the ACOPF, which is a Quadratically Constrained Quadratic Program (QCQP). Since the ACOPF is **NP**-hard [3], it follows by inclusion that the ACOPFG is also **NP**-hard. On the other hand, it was shown in [12] that many practical ACOPF instances turn out to be easy rather than hard. We remark that "easy", in this setting, does not necessarily mean "in **P**", since the decision version of the QCQP is not known to be in **NP** (unless there are no quadratic constraints, in which case the problem class is known to be in **NP** [19]). The meaning of "easy" in this context is that global optima can be obtained by means of a local, rather than global, optimization procedure.

The question we answer in this section is whether the addition of the binary activation variables constitute an actual additional difficulty. To show that this need not necessarily be the case, we consider a linear system

$$\left.\begin{array}{r} Ax \leq b \\ x \leq 1 \\ x \in \mathbb{R}^n_+, \end{array}\right\} \tag{9}$$

where A is totally unimodular. Finding a feasible solution can obviously be done in polynomial time by the, say, interior point algorithm (irrespective of total unimodularity), and so this formulation is in **P**. If we add n additional binary activation variables $y_1, \ldots, y_n \in \{0,1\}$ and n additional activation/deactivation constraints

$$\forall j \leq n \quad x_j \leq y_j, \tag{10}$$

then the new system has a constraint matrix:

$$\begin{pmatrix} A & 0 \\ I_n & 0 \\ I_n & -I_n \end{pmatrix},$$

which is easily seen to also be totally unimodular [20]. Therefore this new Mixed-Integer Linear Programming (MILP) formulation is in **P**. This provides an example where adding boolean activation variables does not make the underlying problem more difficult.

Having established that the question makes sense, we present a reduction of the weakly **NP**-complete SUBSET-SUM problem to a subclass of ACOPFG. Given an instance $(\sigma_1,\ldots,\sigma_n,S_0)$ of SUBSET-SUM, we must decide whether there is a subset $I\subseteq\{1,\ldots,n\}$ such that

$$\sum_{i\in I}\sigma_i = S_0. \tag{11}$$

This is equivalent to asking whether the following linear diophantine equation has a solution $x\in\{0,1\}^n$:

$$\sum_{i\le n}\sigma_i x_i = S_0. \tag{12}$$

We now show that we can naturally express Eq. (12) using the ACOPFG formulation of Sect. 2. We consider a simple network $\mathscr{D}$ with $\mathscr{G}=\{1,\ldots,n\}$ generators with demand $S_g=0$ for $g\le n$ and a single non-generator bus (indexed by 0) with demand S_0 (so that $\mathscr{N}=\{0,\ldots,n\}$). The set $\mathscr{L}$ of lines is $\{(g,0)\mid 1\le g\le n\}$, namely each generator is linked to the only non-generator bus. Each generator $g\in\mathscr{G}$ has generation interval $\mathfrak{S}_G=[\sigma_g,\sigma_g]$, i.e. each generator can either be inactive, or else, if active, must produce exactly σ_g. Then Eq. (7) becomes:

$$\forall g\le n\quad Y^*_{g0}v_g v^*_0=\sigma_g z_g. \tag{13}$$

Since we know σ_g is real and positive, we arrange Y^* so that the complex part of the LHS of Eq. (13) is zero; in particular, we arrange $Y^*_{g0}v^*_0$ to yield a $j^2=-1$ coefficient (this can be easily done when we derive $\text{ACOPFG}_{\mathbb{R}}$). So we get:

$$\forall g\le n\quad \mathsf{Re}(Y^*_{g0}v_g v^*_0)=-\sigma_g z_g. \tag{14}$$

Furthermore, Eq. (2) is:

$$\sum_{(g0)\in\mathscr{L}} Y^*_{g0}v_0 v^*_g=-S_0,$$

whence, by Eq. (14), we have:

$$\sum_{g\le n}(-\sigma_g z_g)=-S_0,$$

which is exactly Eq. (12).

4 Perspective reformulation

The objective function Eq. (8) can be restated using additional variables $p_g=\mathsf{Re}(v_g\sum_{(g,a)\in\mathscr{L}}v^*_aY^*_{ga}+S_g)$. In practice use the convex constraints

$$\forall g\in\mathscr{G}\text{ s.t. }c_{g2}>0\quad p_g\ge\mathsf{Re}(v_g\sum_{(g,a)\in\mathscr{L}}v^*_aY^*_{ga}+S_g), \tag{15}$$

which are justified by the objective function direction. We now reformulate Eq. (8) using these new variables:

$$f(p,z) = \sum_{g\in\mathscr{G}} \left(c_{g2}p_g^2 + c_{g1}p_g + c_{g0}z_g\right). \tag{16}$$

The reformulations proposed below can all be carried out on a per-generator basis. In the rest of this paper, we assume they are only applied to generators $g \in \mathscr{G}$ for which $c_{g2} > 0$.

The power p_g is subject to the following activation constraints:

$$p_g \le \overline{P}_g z_g \quad \wedge \quad p_g \ge \underline{P}_g z_g \tag{17}$$

where $\overline{P}_g = \mathsf{Re}(\overline{S}_g)$ and $\underline{P}_g = \mathsf{Re}(\underline{S}_g)$. The PR reformulation [8] can be applied to (16) as follows:

$$\hat{f}(p,z) = \sum_{g\in\mathscr{G}} \left(c_{g2}\frac{p_g^2}{z_g} + c_{g1}p_g + c_{g0}z_g\right). \tag{18}$$

The function (18) can be optimized using the perspective cuts (PC) method [8], which works as follows: (i) first we add new variables t_g representing the nonlinear part of the cost in (18) by considering the following constraints

$$t_g \ge c_{g2}\frac{p_g^2}{z_g}, \tag{19}$$

and replacing (18) with $\tilde{f}(t,p,z) = \sum_{g\in\mathscr{G}}(t_g + c_{g1}p_g + c_{g0}z_g)$; (ii) then constraints (19) can be replaced by PCs:

$$t_g \ge c_{g2}(2\check{p}_g p_g - \check{p}_g^2 z_g), \tag{20}$$

where $\check{p}_g$ are fixed values of the real power p_g varying in the feasible interval $\underline{P}_g \le \check{p}_g \le \check{P}_g$ when $z_g = 1$. The addition of PCs does not add further difficulties in the problem formulation except for the condition that they should be generated iteratively as their number is not finite.

We can alternatively apply the AP2R technique [6, 7], which works in two phases. The first phase is a projection where the optimal value of z_g for the continuous relaxation of Eq. (18) subject to Eq. (17) is found depending on p_g. The second phase is a lifting where the variables z_g are lifted back. The resulting problem can be solved using an off-the-shelf MIP solver. This is equivalent to replacing (18) and (17) with:

$$\left.\begin{array}{lc} \min \sum_{g\in\mathscr{G}} \left(z_g(p_g^{\mathsf{int}})^2 + \check{f}_g(\pi_g + p_g^{\mathsf{int}}) - \check{f}_g(p_g^{\mathsf{int}}) + c_{g1}p_g + c_{g0}z_g\right) & \\ \forall g \in \mathscr{G} \qquad (\underline{P}_g - p_g^{\mathsf{int}})z_g \le \pi_g \le (\overline{P}_g - p_g^{\mathsf{int}})z_g & \\ \forall g \in \mathscr{G} \qquad \pi_g = p_g - p_g^{\mathsf{int}} z_g, & \end{array}\right\} \tag{21}$$

where $\check{f}_g(x) = c_{g2}x^2$ and p_g^{int} is

$$p_g^{\mathsf{int}} = \max(\underline{P}_g, \min(\sqrt{c_{g0}/c_{g2}}, \overline{P}_g)). \qquad (22)$$

The final AP2R reformulation consists in Eq. (21), Eq. (2)-(6), the complex part of Eq. (7), and Eq. (15).

5 Computational results

We tested PRs with 4 cuts and AP2R (implemented using AMPL [5]): both on the ACOPFG formulation $\mathsf{ACOPFG}_{\mathbb{R}}$ in Sect. 2 (Table 1) and on the dual Diagonally Dominant Programming (DDP) outer-approximation proposed in [18] (Table 2) solved using CPLEX [9]. We compared these results with local optima of the ACOPF (all active generators) obtained by MatPower [21] and by solving the $\mathsf{ACOPFG}_{\mathbb{R}}$ using Baron [16] to global optimality (within a limited CPU time of 1h). The test set includes small to medium scale instances taken from MatPower; results on one larger-scale instance are reported in Table 2. All results were obtained on an Intel i7 dual-core CPU at 2.1GHz with 16GB RAM.

	Perspective reformulation ($\mathsf{ACOPFG}_{\mathbb{R}}$)						AP2R ($\mathsf{ACOPFG}_{\mathbb{R}}$)				MATPOWER	Mi-Quartic			Solution's distances	
Instance	it	time	first value	last value	% active	real value	time	%active	value	real value	value	time	%active	best value	Persp/Mi-Quartic	AP2R/Mi-Quartic
WB2	2	24	878.182	878.182	100	878.182	13	100	878.18	878.182	877.78	13	100	878.182	0	0
WB3	2	169	417.244	417.244	100	417.244	109	100	417.244	417.244	417.25	108	100	417.244	0	0
WB5	2	3600	947.056	947.056	100	947.056	2269	100	947.056	947.056	1082.33	2454	100	947.056	0	0
6ww	1	3600	2913.58	2913.58	x	2881.28	3600	100	10948.7	3135.18	3134.35	3600	100	3135.18	0.1584	0
case9	2	3600	2062.65	5115.72	100	5430.38	3600	66.7	10948.74	7335.42	5296.69	3600	100	5296.69	0.2561	0.6234
case14	1	3600	5250.22	5250.22	x	5375.94	3600	80	6589.72	5287.72	8081.53	3600	60	5476.90	0.1370	0.8304
case30	1	3600	430.906	430.906	x	536.307	3600	66.7	503.508	503.508	576.89	3600	83.3	515.807	0.4234	0.3503

Table 1 Results on $\mathsf{ACOPFG}_{\mathbb{R}}$ ('x': solution not found within time limit).

In the "Perspective reformulation" columns we show: number of iterations, CPU seconds (limited to 1h), PR objective value obtained on 1st iteration and final value, original objective function value at optimum, percentage of active generators at optimum. In the "AP2R" and "Mi-Quartic" columns we show CPU time, objective function value and percentage of active generators. In "Solution's distances" we report a scaled distance of the optima found by PR/AP2R w.r.t. Mi-Quartic, namely $\frac{\|p^{\omega}-p^{\mathsf{MI\text{-}Quartic}}\|_1}{\|p^{\omega}\|_1}$, $\omega \in \{\mathsf{PR},\ \mathsf{AP2R}\}$.

	Perspective reformulation (dual DDP salgado3)						AP2R (dual DDP salgado3)				MATPOWER	Mi-Quartic			Solution's distances	
Instance	it	time	first value	last value	% active	real value	time	%active	value	real value	value	time	%active	best value	Persp/Mi-Quartic	AP2R/Mi-Quartic
WB2	2	0.001	876.923	876.923	100	876.923	0.001	100	876.923	876.923	877.78	13	100	878.182	0.0014	0.0014
WB3	2	0.002	398.443	398.443	100	398.443	0.001	100	398.443	398.443	417.25	108	100	417.244	0.0472	0.0472
WB5	2	0.005	677.688	677.688	100	677.688	0.008	100	677.688	677.688	1082.33	1200	100	947.056	0.3563	0.3992
6ww	4	0.02	2760.751	2838.693	66.7	2844.44	0.008	66.7	57287.6	2841.61	3134.35	1200	100	3135.177	0.6108	0.6108
case9	2	0.03	2012.135	5034.028	100	5430.38	0.031	66.7	10810.8	7197.45	5296.69	1200	100	5296.686	0.2422	0.8423
case14	3	0.15	5091.340	5390.800	60	5406.53	1200	80	4746.99	4746.99	8081.53	1200	60	5476.905	0.0572	0.0325
case30	2	3.28	398.427	509.554	83.3	518.42	21	66.7	492.232	492.232	576.89	1200	83.3	515.807	0.6031	0.4068
case89pegase	2	297.33	5730.152	5730.152	100	5730.15	286.484	100	5730.15	5730.15	5817.60	x	x	x	x	x

Table 2 Results on dual DDP [18] ('x': solution not found within time limit).

While it is clear that the tests with $\mathsf{ACOPFG}_{\mathbb{R}}$ are inconclusive, those on the dual DDP approximation give very tight bounds in relatively little time.

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