A positive perspective on term representation

Jui-Hsuan (Ray) Wu and Dale Miller

Inria Saclay & LIX, Institut Polytechnique de Paris

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Outline

Introduction

Focusing and synthetic inference rules

Proofs as terms
• Terms (or expressions) exist in several different settings.
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  \[(1 + 2) + (1 + (1 + 2))\]
  \[\text{let } x = 1 + 2 \text{ in let } y = (1 + (1 + 2)) \text{ in } x + y\]
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• What to do with terms? Equality, substitution, evaluation, etc.
Proof theory for term representations

• A framework for describing (unifying) different term representations.
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  - Highly principled and mathematically sound means for describing syntactic structures.
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• Sequent calculus: too little structure, too much non-essential information.
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- Sequent calculus: too little structure, too much non-essential information.
- Focused proof system $LJF$: large-scale rules, flexibility of polarization.
Focusing

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  - invertible rules ↔ non-invertible rules
  - non-essential information ↔ essential information
  - don’t care ↔ don’t know
  - negative phase ↔ positive phase

- Applied to LJ and LK: LJT, LJQ, LKT, LKQ, etc.

- LJF and LKF by Liang and Miller (2009).

- Large-scale rules (not phases!): synthetic inference rules and bipoles.
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• \textit{LJF} and \textit{LKF} by Liang and Miller (2009).

• \textbf{Large-scale} rules (not phases!): \textit{synthetic inference rules} and bipoles.
Two phases: an example in LL

\[
\begin{array}{c}
\vdash A \perp, A \\
\vdash B, B \perp \\
\vdash A \perp, B \perp, A \otimes B \\
\vdash A \perp, B \perp \oplus (C \perp \otimes D \perp), A \otimes B \\
\vdash A \perp, B \perp \oplus (C \perp \otimes D \perp), (A \otimes B) \& (A \otimes (C \Rightarrow D)) \\
\vdash A \perp \Rightarrow (B \perp \oplus (C \perp \otimes D \perp)), (A \otimes B) \& (A \otimes (C \Rightarrow D)) \\
\end{array}
\]
The \textit{LJF} system with only implication

- Formulas are built using atomic formulas and implications.
The *LJF* system with only implication

- Formulas are built using atomic formulas and implications.
- We work with **polarized** formulas.
  - Implications are negative.
  - Atomic formulas are either **positive** or **negative**.
    (forward-chaining / backchaining)
The *LJF* system with only implication

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- A polarized theory is a theory together with an **atomic bias assignment**.
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- Formulas are built using atomic formulas and implications.
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- A polarized theory is a theory together with an **atomic bias assignment**.
- Different polarizations do not affect provability in *LJF*, but give different forms of proofs.
  ▶ If a sequent is provable in *LJF* for some polarization, then it is provable for all such polarizations.
Two kinds of sequents:

- $\uparrow$-sequents, used with invertible rules
  \[
  \Gamma \uparrow \Theta \vdash \Delta \uparrow \Delta'
  \]

- $\downarrow$-sequents, used to specify the formula under focus
  \[
  \Gamma \downarrow B \vdash \Delta' \quad \text{left focus}
  \]
  \[
  \Gamma \vdash B \downarrow \quad \text{right focus}
  \]
**Two kinds of sequents:**

- **⇑-sequents**, used with invertible rules
  \[ \Gamma \uparrow \Theta \vdash \Delta \uparrow \Delta' \]

- **⇓-sequents**, used to specify the formula under focus
  \[ \Gamma \downarrow B \vdash \Delta' \] left focus
  \[ \Gamma \vdash B \downarrow \] right focus

**Border sequents:**

\[ \Gamma \uparrow \cdot \vdash \cdot \uparrow \Delta \sim \Gamma \vdash \Delta \]

▶ Inference rules are collected into large-scale rules (synthetic inference rules) by looking at border sequents in a proof.
The \textit{LJF} system with only implication

\begin{align*}
\text{Decide, Release, and Store Rules} \\
&\frac{N, \Gamma \downarrow N \vdash A}{N, \Gamma \vdash A} D_I \\
&\frac{\Gamma \vdash P \downarrow}{\Gamma \vdash P} D_r \\
&\frac{\Gamma \uparrow P \vdash A}{\Gamma \downarrow P \vdash A} R_I \\
&\frac{\Gamma \vdash N \uparrow}{\Gamma \vdash N \downarrow} R_r \\
&\frac{\Gamma, C \uparrow \Theta \vdash \Delta' \uparrow \Delta}{\Gamma \uparrow \Theta, C \vdash \Delta' \uparrow \Delta} S_I \\
&\frac{\Gamma \uparrow \Theta \vdash A}{\Gamma \uparrow \Theta \vdash A} S_r
\end{align*}

\begin{align*}
\text{Initial Rules} \\
A \text{ positive} &\quad & A \text{ negative} \\
&\frac{A, \Gamma \vdash A \downarrow}{A \vdash A \downarrow} I_r & &\frac{\Gamma \downarrow A \vdash A}{I_l}
\end{align*}

\begin{align*}
\text{Introduction Rules for Implication} \\
&\frac{\Gamma \vdash B \downarrow \Gamma \downarrow B' \vdash A}{\Gamma \downarrow B \supset B' \vdash A} \supset L \\
&\frac{\Gamma \uparrow \Theta, B \vdash B' \uparrow}{\Gamma \uparrow \Theta \vdash B \supset B' \uparrow} \supset R
\end{align*}
Synthetic inference rules

**Synthetic inference rule** = large-scale rule = $\downarrow$-phase + $\uparrow$-phase

**Definition**

A *left synthetic inference rule* for $B$ is an inference rule of the form

$$
\frac{\Gamma_1 \vdash A_1 \quad \ldots \quad \Gamma_n \vdash A_n}{\Gamma \vdash A} \quad B
$$

justified by a derivation (in $LJF$) of the form

$$
\frac{\Gamma_1 \vdash A_1 \quad \ldots \quad \Gamma_n \vdash A_n}{\Downarrow \text{phase} \quad \Uparrow \text{phase}} \quad \frac{\Gamma \Downarrow B \vdash A}{\Gamma \vdash A} \quad D_I
$$
Bipoles:

A (left) bipele for a formula \( B \) is a (left) synthetic inference rule such that only atomic formulas are stored in its corresponding derivation (in \( LJF \)).

Order of a formula:

- \( \text{ord}(A) = 0 \) for \( A \) atomic.
- \( \text{ord}(B_1 \supset B_2) = \max(\text{ord}(B_1) + 1, \text{ord}(B_2)) \).

Theorem

Let \( B \) be a negative polarized formula. If \( \text{ord}(B) \leq 2 \), then the left synthetic rule for \( B \) is a bipele.
Axioms as rules

Definition

Let $\mathcal{T}$ be a finite polarized theory of order 2 or less, We define $LJ\langle \mathcal{T} \rangle$ to be the extension of $LJ$ with the left synthetic inference rules for the formulas in $\mathcal{T}$. More precisely, for every left synthetic inference rule

$$
\frac{B, \Gamma_1 \vdash A_1 \ldots B, \Gamma_n \vdash A_n}{B, \Gamma \vdash A}
$$

with $B \in \mathcal{T}$, the inference rule

$$
\frac{\Gamma_1 \vdash A_1 \ldots \Gamma_n \vdash A_n}{\Gamma \vdash A}
$$

is added to $LJ\langle \mathcal{T} \rangle$. 
Definition
Let $\mathcal{T}$ be a finite polarized theory of order 2 or less. We define $LJ\langle \mathcal{T} \rangle$ to be the extension of $LJ$ with the left synthetic inference rules for the formulas in $\mathcal{T}$. More precisely, for every left synthetic inference rule

$$
B, \Gamma_1 \vdash A_1 \quad \ldots \quad B, \Gamma_n \vdash A_n \quad B, \Gamma \vdash A
$$

with $B \in \mathcal{T}$, the inference rule

$$
\Gamma_1 \vdash A_1 \quad \ldots \quad \Gamma_n \vdash A_n \quad \Gamma \vdash A
$$

is added to $LJ\langle \mathcal{T} \rangle$.

Theorem
$\mathcal{T}, \Gamma \vdash B$ provable in $LJ \iff \Gamma \vdash B$ provable in $LJ\langle \mathcal{T} \rangle$. 
An example

Let $\mathcal{T}$ be the collection of formulas
\[ D_1 = a_0 \supset a_1, \cdots, D_n = a_0 \supset \cdots \supset a_n, \cdots \] where $a_i$ are atomic.
An example

Let $\mathcal{T}$ be the collection of formulas

$D_1 = a_0 \supset a_1, \cdots, D_n = a_0 \supset \cdots \supset a_n, \cdots$ where $a_i$ are atomic.

If $a_i$ are given the **negative** bias,
An example

Let \( T \) be the collection of formulas
\[ D_1 = a_0 \supset a_1, \cdots, D_n = a_0 \supset \cdots \supset a_n, \cdots \] where \( a_i \) are atomic.

If \( a_i \) are given the negative bias, we have the derivation

\[
\frac{\Gamma \vdash a_0}{\Gamma \vdash a_0 \downarrow} R_r/S_r \quad \cdots \quad \frac{\Gamma \vdash a_{n-1}}{\Gamma \vdash a_{n-1} \downarrow} R_r/S_r \quad \frac{\Gamma \vdash a_n \vdash a_n}{\Gamma \vdash a_n} I_l
\]

\[
\frac{\Gamma \downarrow a_0 \supset \cdots \supset a_n \vdash a_n}{\Gamma \vdash a_n} D_l
\]

and the inference rules in \( LJ\langle T \rangle \) include

\[
\frac{\Gamma \vdash a_0 \quad \cdots \quad \Gamma \vdash a_{n-1}}{\Gamma \vdash a_n}
\]
An example

Let $\mathcal{T}$ be the collection of formulas $D_1 = a_0 \supset a_1, \cdots, D_n = a_0 \supset \cdots \supset a_n, \cdots$ where $a_i$ are atomic.

If $a_i$ are given the negative bias, we have the derivation

\[ \frac{\Gamma \vdash a_0}{\Gamma \vdash a_0 \ \downarrow} R_r/S_r \] \[ \frac{\Gamma \vdash a_{n-1}}{\Gamma \downarrow a_{n-1} \ \downarrow} R_r/S_r \] \[ \frac{\Gamma \vdash a_n \vdash a_0 \supset \cdots \supset a_n \vdash a_n}{\Gamma \vdash a_n} D_l \]

and the inference rules in $LJ\langle \mathcal{T} \rangle$ include

\[ \frac{\Gamma \vdash a_0 \ \cdots \ \Gamma \vdash a_{n-1}}{\Gamma \vdash a_n} \]

"backchaining"
Let $\mathcal{T}$ be the collection of formulas

$D_1 = a_0 \supset a_1, \ldots, D_n = a_0 \supset \cdots \supset a_n, \ldots$ where $a_i$ are atomic.

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An example

Let $\mathcal{T}$ be the collection of formulas
$D_1 = a_0 ⊃ a_1, \cdots, D_n = a_0 ⊃ \cdots ⊃ a_n, \cdots$ where $a_i$ are atomic.

If $a_i$ are given the positive bias, we have the derivation

\[
\begin{align*}
\Gamma \vdash a_0 \Downarrow & \quad \cdots \quad \Gamma \vdash a_{n-1} \Downarrow \\
\Gamma \Downarrow a_0 \supset \cdots \supset a_n \vdash A & \quad \Gamma \Downarrow a_n \vdash A \\
\Gamma \Downarrow a_0 \supset \cdots \supset a_n \vdash A & \quad \Gamma \vdash A \\
\end{align*}
\]

and the inference rules in $LJ(\mathcal{T})$ include

\[
\begin{align*}
\Gamma, a_0, \cdots, a_{n-1}, a_n \vdash A & \\
\Gamma, a_0, \cdots, a_{n-1} \vdash A \\
\end{align*}
\]
An example

Let $\mathcal{T}$ be the collection of formulas
\[ D_1 = a_0 \supset a_1, \ldots, D_n = a_0 \supset \cdots \supset a_n, \ldots \] where $a_i$ are atomic.

If $a_i$ are given the positive bias, we have the derivation

\[
\begin{array}{c}
\Gamma \vdash a_0 \Downarrow \\
\vdots \\
\Gamma \vdash a_{n-1} \Downarrow \\
\Gamma \Downarrow a_0 \supset \cdots \supset a_n \vdash A \\
\Gamma \vdash A
\end{array}
\]

and the inference rules in $LJ\langle \mathcal{T} \rangle$ include

\[
\begin{array}{c}
\Gamma, a_0, \cdots, a_{n-1}, a_n \vdash A \\
\Gamma, a_0, \cdots, a_{n-1} \vdash A
\end{array}
\]

"forward-chaining"
Backchaining and Forward-chaining

What are the proofs of $a_0 \vdash a_n$?
What are the proofs of $a_0 \vdash a_n$?

When $a_i$ are all given the negative bias, we have:

$$
\begin{array}{c}
\Gamma \vdash a_0 \\
\Gamma \vdash a_1 \\
\Gamma \vdash a_2 \\
\Gamma \vdash a_{n-1} \\
\Gamma \vdash a_n \\
\end{array}
$$

▷ a unique proof of exponential size
Backchaining and Forward-chaining

What are the proofs of $a_0 \vdash a_n$?

When $a_i$ are all given the **negative** bias, we have:

$$\frac{\Gamma \vdash a_0}{\Gamma \vdash a_1} \quad \frac{\Gamma \vdash a_0}{\Gamma \vdash a_1} \quad \frac{\Gamma \vdash a_1}{\Gamma \vdash a_2} \quad \cdots \quad \frac{\Gamma \vdash a_0}{\Gamma \vdash a_{n-1}} \quad \frac{\Gamma \vdash a_{n-1}}{\Gamma \vdash a_n}$$

▷ a unique proof of exponential size

When $a_i$ are all given the **positive** bias, we have:

$$\frac{\Gamma, a_0, a_1 \vdash A}{\Gamma, a_0 \vdash A} \quad \frac{\Gamma, a_0, a_1, a_2 \vdash A}{\Gamma, a_0, a_1 \vdash A} \quad \cdots \quad \frac{\Gamma, a_0, \ldots, a_{n-1}, a_n \vdash A}{\Gamma, a_0, \ldots, a_{n-1} \vdash A}$$

▷ a shortest proof of linear size
Proofs as terms

We want to use terms to annotate proofs.
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How much information do we need?
Proofs as terms

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How much information do we need?

Consider the formula $A \supset A \supset A$ in $LJ$.

How many proofs are there?
Proofs as terms

We want to use terms to annotate proofs.

How much information do we need?

Consider the formula $A \supset A \supset A$ in LJ.

How many proofs are there?

\[
\frac{\frac{\text{init}}{A, A \vdash A}}{A \vdash A \supset A} \supset R
\]

\[
\frac{\frac{\frac{\frac{\text{init}}{A, A \vdash A}}{A \vdash A \supset A}}{A \vdash A \supset A}}{A \vdash A \supset A} \supset R
\]
Proofs as terms

We want to use terms to annotate proofs.

How much information do we need?

Consider the formula $A \supset A \supset A$ in $LJ$.

How many proofs are there?

\[
\begin{align*}
A, A & \vdash A & \text{init} \\
A & \vdash A \supset A & \supset R \\
\vdash A \supset A \supset A & \supset R
\end{align*}
\]
Proofs as terms

We want to use terms to annotate proofs.

How much information do we need?

Consider the formula $\forall A \supset A \supset A$ in $LJ$.

How many proofs are there?

\[
\frac{A, A \vdash A}{\vdash A \supset A \supset A} \supset R
\]
\[
\frac{A \vdash A \supset A}{\vdash A \supset A \supset A} \supset R
\]
\[
\Rightarrow \lambda x. \lambda y. x
\]
Proofs as terms

We want to use terms to annotate proofs.

How much information do we need?

Consider the formula $A \supset A \supset A$ in $LJ$.

How many proofs are there?

$\begin{align*}
\text{init} & \quad A, A \vdash A \\
\text{R} & \quad A \vdash A \supset A \\
\text{R} & \quad \vdash A \supset A \supset A \\
\Rightarrow & \quad \lambda x.\lambda y.y
\end{align*}$
Proofs as terms

We want to use terms to annotate proofs.

How much information do we need?

Consider the formula $A \supset A \supset A$ in $LJ$.

How many proofs are there?

$$\frac{\text{init}}{A, A \vdash A}$$

$$\frac{\vdash A \supset A \supset A}{\vdash A \supset A \supset A}$$

$\Rightarrow \lambda x.\lambda y.y$

$\triangleright$ each formula on the left hand side is given a label.
Untyped \( \lambda \)-terms

We fix a theory \( \mathcal{T} = \{ \Phi : D \supset D \supset D, \Psi : (D \supset D) \supset D \} \) with \( D \) atomic and consider proofs of sequents of the form \( \mathcal{T}, x_1 : D, \cdots, x_k : D \vdash t : D \)
Untyped $\lambda$-terms

We fix a theory $\mathcal{T} = \{ \Phi : D \supset D \supset D, \Psi : (D \supset D) \supset D \}$ with $D$ atomic and consider proofs of sequents of the form $\mathcal{T}, x_1 : D, \cdots, x_k : D \vdash t : D$

When $D$ is given the negative bias, we have

\[
\begin{align*}
\Gamma \vdash D & \quad R_r/S_r \\
\Gamma \vdash D & \quad R_r/S_r \\
\Gamma \vdash D & \quad L^2
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash D & \quad \downdownarrows \quad D_l \\
\Gamma, D \vdash D & \quad R_r/S_l/S_r \\
\Gamma \vdash D & \quad I_l \\
\Gamma \vdash D & \quad L
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash (D \supset D) \supset D & \quad \downdownarrows \quad D_l \\
\Gamma \vdash D & \quad I_l
\end{align*}
\]
Untyped \( \lambda \)-terms

We fix a theory \( \mathcal{T} = \{ \Phi : D \supset D \supset D, \Psi : (D \supset D) \supset D \} \) with \( D \) atomic and consider proofs of sequents of the form \( \mathcal{T}, x_1 : D, \cdots , x_k : D \vdash t : D \)

When \( D \) is given the negative bias, we have

\[
\begin{align*}
\Gamma & \vdash t : D & & \Gamma \vdash u : D & & \Gamma \Downarrow D \vdash D \\
\Gamma & \Downarrow D & & \Gamma \Downarrow D & & \Gamma \Downarrow D \vdash D \\
\Gamma & \Downarrow D \supset D \supset D \vdash D & & \Gamma \vdash D \\
\Gamma & \Downarrow (D \supset D) \supset D \vdash D & & \Gamma \vdash D
\end{align*}
\]

Here we use the \( \lambda \kappa \)-calculus\(^1\) to annotate terms.

\(^1\)Taus Brock-Nannestad, Nicolas Guenot, and Daniel Gustafsson. Computation in focused intuitionistic logic. PPDP 2017.
Untyped $\lambda$-terms

We fix a theory $\mathcal{T} = \{ \Phi : D \supset D \supset D, \Psi : (D \supset D) \supset D \}$ with $D$ atomic and consider proofs of sequents of the form $\mathcal{T}, x_1 : D, \cdots, x_k : D \vdash t : D$

When $D$ is given the negative bias, we have

\[
\frac{\Gamma \vdash t : D}{\Gamma \vdash \lfloor t \rfloor : D \Downarrow} R_r/S_r \quad \frac{\Gamma \vdash u : D}{\Gamma \vdash \lfloor u \rfloor : D \Downarrow} R_r/S_r \quad \frac{\Gamma \Downarrow D \vdash \epsilon : D}{I_l} \quad \Gamma \Downarrow D \vdash \epsilon : D \supset L^2
\]

\[
\frac{\Gamma \Downarrow D \supset D \supset D \vdash \lfloor t \rfloor :: \lfloor u \rfloor :: \epsilon : D}{D_l} \quad \Gamma \vdash \Phi \bowtie (\lfloor t \rfloor :: \lfloor u \rfloor :: \epsilon) : D
\]

\[
\frac{\Gamma, D \vdash D}{\Gamma \vdash D \supset D \Downarrow} R_r/S_r/S_r \quad \frac{\Gamma \Downarrow D \vdash D}{I_l} \quad \frac{\Gamma \Downarrow D \vdash D}{I_l} \quad \frac{\Gamma \Downarrow (D \supset D) \supset D \vdash D}{D_l} \quad \Gamma \vdash D \supset L
\]

Here we use the $\lambda\kappa$-calculus\(^1\) to annotate terms.

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We fix a theory $\mathcal{T} = \{\Phi : D \supset D \supset D, \Psi : (D \supset D) \supset D\}$ with $D$ atomic and consider proofs of sequents of the form $\mathcal{T}, x_1 : D, \cdots, x_k : D \vdash t : D$

When $D$ is given the negative bias, we have

\[
\frac{\Gamma \vdash t : D}{\Gamma \vdash [t] : D \downarrow} \quad R_r / S_r \quad \frac{\Gamma \vdash u : D}{\Gamma \vdash [u] : D \downarrow} \quad R_r / S_r \quad \frac{\Gamma \Downarrow D \vdash \epsilon : D}{\Gamma \Downarrow D \vdash [u] : D \Downarrow} \quad I_i \quad \frac{\Gamma \vdash \Phi \Downarrow ([t] :: [u] :: \epsilon) : D}{D_i}
\]

\[
\frac{\Gamma, x : D \vdash t : D}{\Gamma \vdash D \supset D \Downarrow} \quad R_r / S_l / S_r \quad \frac{\Gamma \Downarrow D \vdash D}{\Gamma \Downarrow (D \supset D) \supset D \vdash D \Downarrow} \quad D_i \quad \frac{\Gamma \Downarrow (D \supset D) \supset D \vdash D}{\Gamma \vdash D \Downarrow} \quad \supset L
\]

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We fix a theory $T = \{ \Phi : D \supset D \supset D, \Psi : (D \supset D) \supset D \}$ with $D$ atomic and consider proofs of sequents of the form $T, x_1 : D, \cdots, x_k : D \vdash t : D$

When $D$ is given the negative bias, we have

\[
\frac{\Gamma \vdash t : D}{\Gamma \vdash [t] : D \downarrow} \quad R_r/S_r \quad \frac{\Gamma \vdash u : D}{\Gamma \vdash [u] : D \downarrow} \quad R_r/S_r \quad \frac{\Gamma \vdash \epsilon : D}{\Gamma \vdash [\epsilon] : D \downarrow} \quad I_l \quad \Gamma \vdash \Phi \downarrow (\cdot) : D \quad D_l
\]

\[
\frac{\Gamma \vdash \lambda x.t : D \supset D \downarrow}{\Gamma \vdash \lambda x.t : D \supset D \downarrow} \quad R_r/S_r/S_r \quad \frac{\Gamma \vdash \epsilon : D}{\Gamma \vdash [\epsilon] : D \downarrow} \quad I_l \quad \Gamma \vdash \Psi \downarrow (\cdot) : D \quad D_l
\]

Here we use the $\lambda\kappa$-calculus\footnote{Taus Brock-Nannestad, Nicolas Guenot, and Daniel Gustafsson. Computation in focused intuitionistic logic. PPDP 2017.} to annotate terms.
Untyped $\lambda$-terms

We fix a theory $T = \{ \Phi : D \supset D \supset D, \Psi : (D \supset D) \supset D \}$ with $D$ atomic and consider proofs of sequents of the form $T, x_1 : D, \cdots, x_k : D \vdash t : D$

When $D$ is given the positive bias, we have

$$
\begin{align*}
\frac{\Gamma \vdash D \downarrow}{\Gamma \vdash D \downarrow} & \quad \frac{\Gamma \vdash D \downarrow}{\Gamma \vdash D \downarrow} \quad \frac{\Gamma, D \vdash D}{\Gamma \vdash D} \quad \frac{\Gamma, D \vdash D}{\Gamma \vdash D} \quad \frac{\Gamma \downarrow D \vdash D}{\Gamma \downarrow D \vdash D} \\
\frac{\Gamma \downarrow D \vdash D \supset D \vdash D}{\Gamma \vdash D} & \quad \frac{\Gamma \downarrow D \vdash D \supset D \vdash D}{\Gamma \vdash D} \\
\frac{\Gamma, D \vdash D}{\Gamma \vdash D} & \quad \frac{\Gamma, D \vdash D}{\Gamma \vdash D} \\
\frac{\Gamma \vdash D \downarrow}{\Gamma \vdash D \downarrow} & \quad \frac{\Gamma \vdash D \downarrow}{\Gamma \vdash D \downarrow} \quad \frac{\Gamma \downarrow D \vdash D}{\Gamma \downarrow D \vdash D} \\
\frac{\Gamma \downarrow (D \supset D) \vdash D}{\Gamma \vdash D} & \quad \frac{\Gamma \downarrow (D \supset D) \vdash D}{\Gamma \vdash D}
\end{align*}
$$

Here we use the $\lambda\kappa$-calculus\(^1\) to annotate terms.

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Untyped $\lambda$-terms

We fix a theory $\mathcal{T} = \{ \Phi : D \supset D \supset D, \Psi : (D \supset D) \supset D \}$ with $D$ atomic and consider proofs of sequents of the form $\mathcal{T}, x_1 : D, \cdots, x_k : D \vdash t : D$.

When $D$ is given the positive bias, we have

\[
\begin{align*}
& \Gamma \vdash x : D \Downarrow \quad \Gamma \vdash y : D \Downarrow \quad \Gamma, z : D \vdash t : D \\
& \quad \Gamma \Downarrow D \supset D \supset D \vdash D \quad \Gamma \Downarrow D \vdash D \\
& \quad \Gamma \Downarrow D \vdash D \\
& \quad \Gamma, D \vdash D \\
& \quad \Gamma \Downarrow (D \supset D) \supset D \vdash D \\
\end{align*}
\]

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When $D$ is given the positive bias, we have

\[
\begin{align*}
\Gamma &\vdash x : D \downarrow & \text{Ir} \quad \Gamma &\vdash y : D \downarrow & \text{Ir} \\
\Gamma &\downarrow D \supset D \supset D \vdash x :: y :: \kappa z.t : D & \quad \text{L}^2
\end{align*}
\]

Here we use the $\lambda\kappa$-calculus\(^1\) to annotate terms.

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When $D$ is given the positive bias, we have

$$
\begin{align*}
\Gamma \vdash x : D & \quad \Gamma \vdash y : D & \quad \Gamma, z : D \vdash t : D \\
\Gamma \vdash x :: y :: \kappa z.t : D & \quad \Gamma \vdash \Phi (x :: y :: \kappa z.t) : D & \quad R_l/S_l \\
\Gamma \vdash D \supset D \supset D & \quad \Gamma \vdash D \vdash D & \quad R_l/S_l \\
\Gamma \vdash (D \supset D) \supset D & \quad \Gamma \vdash D & \quad R_l/S_l \\
\Gamma \vdash D & \quad \Gamma \vdash \kappa z.t : D & \quad \Gamma \vdash \kappa z.t : D & \quad \Gamma \vdash D
\end{align*}
$$

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When $D$ is given the positive bias, we have

\[
\frac{\Gamma \vdash x : D \Downarrow}{\Gamma \vdash \iota_r x : D} \quad \frac{\Gamma \vdash y : D \Downarrow}{\Gamma \vdash \iota_r y : D} \quad \frac{\Gamma, z : D \vdash \iota : t : D}{\Gamma \vdash \iota D \supset D \vdash \iota : t : D} \quad \frac{\Gamma \vdash \iota D \supset D \vdash \iota \iota : t : D}{\Gamma \vdash \iota D \supset \iota \iota : t : D} \quad \frac{\Gamma \vdash \psi \iota (\lambda x. t) : D}{\Gamma \vdash \iota D} \quad \frac{\Gamma \vdash \psi \iota (\lambda x. t) : D}{\Gamma \vdash \iota D} \quad \frac{\Gamma, y : D \vdash s : D}{\Gamma \vdash \iota r s : D} \quad \frac{\Gamma \vdash (\lambda x. t) : D \vdash s : D}{\Gamma \vdash \psi \iota (\lambda x. t) \iota : s : D} \quad \frac{\Gamma \vdash (\lambda x. t) : D \vdash s : D}{\Gamma \vdash \psi \iota (\lambda x. t) \iota : s : D}
\]

Here we use the $\lambda\kappa$-calculus\(^1\) to annotate terms.

---

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Untyped $\lambda$-terms

Two different polarity assignments give **two different term structures**:

- **$D$ is negative:**

  $$x \sim \epsilon \quad \text{nvar } x \quad x$$

  $$\Phi \sim ([t] :: [u] :: \epsilon) \quad \text{napp } t \ u \quad tu$$

  $$\Psi \sim ([\lambda x. t] :: \epsilon) \quad \text{nabs } (x \ \ t) \ \ \lambda x. t$$

  $\rightarrow$ **Top-down / tree-like structure**

- **$D$ is positive:**

  $$\lceil x \rceil \quad \text{pvar } x \quad x$$

  $$\Phi \sim (x :: y :: \kappa z. t) \quad \text{papp } x \ y \ (z \ \ t) \quad \text{name } z = xy \ in \ t$$

  $$\Psi \sim ([\lambda x. t] :: \kappa y. s) \quad \text{pabs } (x \ \ t) \ (y \ \ s) \quad \text{name } y = \lambda x. t \ in \ s$$

  $\rightarrow$ **Bottom-up / DAG structure**
Some examples for the positive-bias syntax

name y = app x x in name z = app y y in z

Arguments of app are all names
Some examples for the positive-bias syntax

name y = app x x in name z = app y y in z

▷ Arguments of app are all names

name y1 = app x x in name y2 = app x x in
name z = app y1 y2 in z

▷ Redundant naming
Some examples for the positive-bias syntax

\[
\text{name } y = \text{app } x \ x \ \text{in name } z = \text{app } y \ y \ \text{in } z
\]

▷ Arguments of app are all names

\[
\text{name } y_1 = \text{app } x \ x \ \text{in name } y_2 = \text{app } x \ x \ \text{in name } z = \text{app } y_1 \ y_2 \ \text{in } z
\]

▷ Redundant naming

\[
\text{name } y_1 = \text{app } x \ x \ \text{in name } y_2 = \text{app } y \ y \ \text{in name } z = \text{app } y_1 \ y_1 \ \text{in } z
\]

▷ Vacuous naming
Some examples for the positive-bias syntax

name y = app x x in name z = app y y in z
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▷ Vacuous naming

name y1 = app x x in name y2 = app y y in
name z = app y1 y2 in z
name z = abs (x \ name y1 = app y y in y1) in z
▷ Parallel naming
Cut-elimination for $LJ\langle\mathcal{T}\rangle$

The following theorem\(^2\) states that cut is admissible for the extensions of $LJ$ with polarized theories based on synthetic inference rules.

**Theorem (Cut admissibility for $LJ\langle\mathcal{T}\rangle$)**

Let $\mathcal{T}$ be a finite polarized theory of order 2 or less. Then the cut rule is admissible for the proof system $LJ\langle\mathcal{T}\rangle$.

The following theorem\(^2\) states that cut is admissible for the extensions of \(LJ\) with polarized theories based on synthetic inference rules.

**Theorem (Cut admissibility for \(LJ\langle T\rangle\))**

Let \(T\) be a finite polarized theory of order 2 or less. Then the cut rule is admissible for the proof system \(LJ\langle T\rangle\).

The proof is based on a cut elimination procedure for \(LJF\)

\[\triangleright\] This defines the notion of substitution for terms.

The following theorem\textsuperscript{2} states that cut is admissible for the extensions of $LJ$ with polarized theories based on synthetic inference rules.

**Theorem (Cut admissibility for $LJ\langle T\rangle$)**

Let $T$ be a finite polarized theory of order 2 or less. Then the cut rule is admissible for the proof system $LJ\langle T\rangle$.

The proof is based on a cut elimination procedure for $LJF$

- This defines the notion of substitution for terms.

When we restrict to atomic cut formulas, the cut elimination procedure can be presented in a big-step style.

- Cuts are permuted with synthetic rules instead of $LJF$ rules.

Untyped $\lambda$-terms (substitution)

The cut-elimination procedure of $LJF$ gives us the following definitions of substitutions.

\[
\begin{align*}
type \ nsubst, \ psubst \ & \quad tm \to (\text{val} \to tm) \to tm \to o. \\
nsubst \ T \ (x\ \text{nvar} \ x) \ & \quad T.
\end{align*}
\]

\[
\begin{align*}
nsubst \ T \ (x\ \text{nvar} \ Y) \ (nvar \ Y). \\
nsubst \ T \ (x\ \text{napp} \ (R \ x) \ (S \ x)) \ (napp \ R' \ S') \ & \quad :- \\
& \quad nsubst \ T \ R \ R', \ nsubst \ T \ S \ S'. \\
nsubst \ T \ (x\ \text{nabs} \ y\ \text{nvar} \ R \ x \ y) \ (nabs \ y\ \text{nvar} \ R' \ y) \ & \quad :- \\
& \quad \pi y\ \text{nsubst} \ T \ (x\ \text{R} \ x \ y) \ (R' \ y).
\end{align*}
\]

\[
\begin{align*}
\text{psubst} \ (papp \ U \ V \ K) \ R \ (papp \ U \ V \ H) \ & \quad :- \ \pi x\ \text{psubst} \\
& \quad (K \ x) \ R \ (H \ x).
\end{align*}
\]

\[
\begin{align*}
\text{psubst} \ (pabs \ S \ K) \ R \ (pabs \ S \ H) \ & \quad :- \ \pi x\ \text{psubst} \\
& \quad (K \ x) \ R \ (H \ x).
\end{align*}
\]

\[
\begin{align*}
\text{psubst} \ (p\text{var} \ U) \ R \ (R \ U).
\end{align*}
\]
An example

\[
\begin{align*}
\text{output} & \quad \\
\text{z} & \quad \text{app} \quad \\
\text{y} & \quad \text{app} \\
\text{x} & \quad \\
\text{name } y & = \text{app } x \ x \ \text{in} \\
\text{name } z & = \text{app } y \ y \ \text{in} \ z
\end{align*}
\]
An example

name y = app x x in
name z = app y y in z

name y' = app a a in
name z' = app y' y' in z'
An example

```
name y = app x x in
name z = app y y in z

name y' = app a a in
name z' = app y' y' in
name y = app z' z' in
name z = app y y in z

name y' = app a a in
name z' = app y' y' in z'
```
Untyped $\lambda$-terms (equality)

We have now two different formats for untyped $\lambda$-terms.

When should two such expressions be considered the same?
Untyped $\lambda$-terms (equality)

We have now two different formats for untyped $\lambda$-terms.

When should two such expressions be considered the same?

"White box" approach:

- Look at the actual syntax of proof expressions.
  - not working since we have two different sets of synthetic inference rules.
Untyped $\lambda$-terms (equality)

We have now two different formats for untyped $\lambda$-terms.

When should two such expressions be considered the same?

”White box” approach:
- Look at the actual syntax of proof expressions.
  $\Rightarrow$ not working since we have two different sets of synthetic inference rules.

”Black box” approach:
- Describe *paths* by probing a term.
Path equality

We use λProlog programs to illustrate the idea.

\[ \text{npath } T \ P \ (\text{resp. } \text{ppath } T \ P) \text{ if } P \text{ is a path in the } T. \]

```
type npath, ppath tm -> path -> o.

npath (napp M _) (left P) :- npath M P.
npath (napp _ N) (right P) :- npath N P.
npath (nabs R) (bnd P) :- pi x\pi p\ npath (nvar x) p => npath (R x) (P p).

ppath (papp U V K) P :-
  pi x\ (pi P\ ppath (pvar x) (left P) :- ppath (pvar U) P) =>
  (pi P\ ppath (pvar x) (right P) :- ppath (pvar V) P) =>
  ppath (K x) P.
ppath (pabs R K) P :-
  pi x\ (pi Q\ ppath (pvar x) (bnd Q) :-
  pi p\ pi u\ ppath (pvar u) p => ppath (R u) (Q p)) => ppath (K x) P.
```
Related and future work

- Generalize to full \( LJF \).

- Multi-focusing:
  - \( \triangledown \) Parallel actions (parallel name introductions).
  - \( \triangledown \) Maximal multi-focused proofs \( \leftrightarrow \) graphical representations.
  - Conjecture: MMF proofs are isomorphic to \( \lambda \)-graphs in the case for untyped \( \lambda \)-terms.

- Big-step cut-elimination for arbitrary cut formulas
  - \( \triangledown \) At the level of synthetic rules (not phases)!

- Connection with the literature in programming language theory (A-normal form, etc)

- There exist some other frameworks for term structures, such as terms-as-graphs by Grabmayer. Are there some connections or overlaps?

- Proof-theoretic methods for checking term equality.
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