Chromatic simplicial complexes are models for epistemic logic

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Main idea: A close correspondence between two structures.

The chromatic simplicial complexes used in fault-tolerant distributed computability.

The Kripke models used in epistemic logic.
Epistemic logic
Multi-agent epistemic logic

Epistemic logic is the logic of knowledge.

Let \( A \) be a finite set of agents and \( AP \) a set of atomic propositions. The syntax of formulas is:

\[
\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid K_a \varphi \quad \quad p \in AP, \ a \in A
\]

\( K_a \varphi \) is read “\( a \) knows \( \varphi \)”. 
Multi-agent epistemic logic

Epistemic logic is the logic of knowledge.

Let $\mathcal{A}$ be a finite set of agents and $AP$ a set of atomic propositions. The syntax of formulas is:

$$\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid K_a \varphi \mid C_B \varphi \quad p \in AP, a \in \mathcal{A}, B \subseteq \mathcal{A}$$

$K_a \varphi$ is read “$a$ knows $\varphi$”.

Common knowledge:

$$C_B \varphi \equiv \bigwedge_{n \in \mathbb{N}} K_{a_1} \ldots K_{a_n} \varphi$$

$$a_1, \ldots, a_n \in B$$
Example: the two generals problem

Two divisions of the same army, commanded by general $A$ and general $B$, are surrounding an enemy fortress.
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Fortunately, on this particular night, everything goes smooth. How long will it take to coordinate the attack?
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Kripke models

A Kripke frame is a tuple $M = \langle W, (\sim_a)_{a \in \mathcal{A}} \rangle$, where:

- $W$ is a set of worlds
- For every $a \in \mathcal{A}$, $\sim_a \subseteq W \times W$ is an equivalence relation on $W$
Kripke models

A Kripke model is a tuple $M = \langle W, (\sim_a)_{a \in A}, L \rangle$, where:

- $W$ is a set of worlds
- For every $a \in A$, $\sim_a \subseteq W \times W$ is an equivalence relation on $W$
- $L : W \rightarrow \mathcal{P}(AP)$

Example: three agents with binary inputs.

- $a, b, c$ are agents.
- $w \sim_a w'$ is represented as an $a$-labeled edge between $w$ and $w'$.
- 101 : input values of $a, b, c$, in that order.
Semantics of epistemic logic formulas

Let $M = \langle W, \sim, L \rangle$ be a Kripke model and $x \in W$ a world of $M$. We define the truth of a formula $\varphi$ in $x$, written $M, x \models \varphi$, by induction on $\varphi$:

- $M, x \models p$ iff $p \in L(x)$
- $M, x \models \neg \varphi$ iff $M, x \not\models \varphi$
- $M, x \models \varphi \land \psi$ iff $M, x \models \varphi$ and $M, x \models \psi$
- $M, x \models K_a \varphi$ iff for all $y \in W$, $x \sim_a y$ implies $M, y \models \varphi$
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- $M, x \models Ka \varphi$ iff for all $y \in W, x \sim_a y$ implies $M, y \models \varphi$
- $M, x \models CB \varphi$ iff for all $y$ in the $B$-connected component of $x$, $M, y \models \varphi$
Definition

An (abstract) **simplicial complex** is a pair \(\langle V, S \rangle\) where \(V\) is a set of **vertices** and \(S\) is a downward-closed family of subsets of \(V\) called **simplices** (i.e., \(X \in S\) and \(Y \subseteq X\) implies \(Y \in S\)).
Chromatic simplicial complexes

Fix a finite set $\mathcal{A}$ of agents, represented as colors.

**Definition**

A **chromatic simplicial complex** is given by $\langle V, S, \chi \rangle$ where:

- $\langle V, S \rangle$ is a simplicial complex,
- $\chi : V \rightarrow \mathcal{A}$ is a coloring map,

such that every simplex $X \in S$ has vertices of distinct colors.
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The **dimension** of a simplex $X$ is $|X| - 1$. A simplicial complex is **pure** if all the maximal simplices are of the same dimension.
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**Example:** a pure chromatic simplicial complex of dimension $2$. 

![Diagram of a chromatic simplicial complex](image-url)
Equivalence

Assume we have \( n + 1 \) agents \( \mathcal{A} = \{ a_0, \ldots, a_n \} \).

**Theorem**

There is an equivalence of categories between the category of (proper) Kripke frames and the category of pure chromatic simplicial complexes of dimension \( n \).
Proof of the theorem
Proof of the theorem

From simplicial complexes to Kripke frames.
Let $C$ be a chromatic simplicial complex. We associate the Kripke frame $F(C) = \langle W, \sim \rangle$, where:

- $W$ is the set of maximal simplices
- For $X, Y \in W$, $X \sim_a Y$ if $X \cap Y$ has an $a$-colored vertex.
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From Kripke frames to simplicial complexes.
Let $M = \langle W, \sim \rangle$ be a Kripke frame and $\mathcal{A} = \{a_0, \ldots, a_n\}$ the set of agents, then:

\[
G(M) = \left( \bigsqcup_{x \in W} \{v_0^x, \ldots, v_n^x\} \right) / \equiv
\]

where $v_i^x \equiv v_i^y$ iff $x \sim_{a_i} y$. 

Proof of the theorem
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From Kripke frames to simplicial complexes.

Let $M = \langle W, \sim \rangle$ be a Kripke frame and $A = \{a_0, \ldots, a_n\}$ the set of agents, then:

$$G(M) = \left( \bigcup_{x \in W} \{v_{x0}, \ldots, v_{xn}\} \right) / \equiv$$

where $v_{xi} \equiv v_{yi}$ iff $x \sim a_i y$.

The set $v_{xi}$
Proof of the theorem

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where $v_i^x \equiv v_i^y$ iff $x \sim_{a_i} y$. 
Simplicial models

Assume we have \( n + 1 \) agents \( \mathcal{A} = \{a_0, \ldots, a_n\} \).

**Definition**

A simplicial model is given by \( \langle V, S, \chi, \ell \rangle \) where:

- \( \langle V, S, \chi \rangle \) is a pure chromatic simplicial complex of dimension \( n \).
- \( \ell : V \to \mathcal{P}(AP) \)
Example: binary input complex for 3 agents

- Every agent has input value either 0 or 1.
- Every agent knows its value, but not the other values.
Example: binary input complex for 3 agents

▶ Every agent has input value either 0 or 1.
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In the picture below, the three agents are represented as the colors black, grey, white:
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Theorem

The previous theorem still holds for models!
Defining truth in simplicial models

Let $M = \langle V, S, \chi, \ell \rangle$ be a simplicial model and $X \in \mathcal{F}(S)$ a maximal simplex of $M$.

- $M, X \models p$ iff $p \in \ell(X)$
- $M, X \models \neg \varphi$ iff $M, X \not\models \varphi$
- $M, X \models \varphi \land \psi$ iff $M, X \models \varphi$ and $M, X \models \psi$
- $M, X \models K_a \varphi$ iff for all $Y \in \mathcal{F}(S)$, if $a \in \chi(X \cap Y)$, then $M, Y \models \varphi$
Defining truth in simplicial models

Let $M = \langle V, S, \chi, \ell \rangle$ be a simplicial model and $X \in \mathcal{F}(S)$ a maximal simplex of $M$.

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$M, X \models K_a \varphi \iff$ for all $Y \in \mathcal{F}(S)$, if $a \in \chi(X \cap Y)$, then $M, Y \models \varphi$

$M, x \models C_B \varphi \iff \ldots$
Defining truth in simplicial models

Let $M = \langle V, S, \chi, \ell \rangle$ be a simplicial model and $X \in \mathcal{F}(S)$ a maximal simplex of $M$.

\[
\begin{align*}
M, X &\models p \quad \text{iff} \quad p \in \ell(X) \\
M, X &\models \neg \varphi \quad \text{iff} \quad M, X \not\models \varphi \\
M, X &\models \varphi \land \psi \quad \text{iff} \quad M, X \models \varphi \quad \text{and} \quad M, X \models \psi \\
M, X &\models K_a \varphi \quad \text{iff} \quad \text{for all } Y \in \mathcal{F}(S), \text{ if } a \in \chi(X \cap Y), \text{ then } M, Y \models \varphi \\
M, x &\models C_B \varphi \quad \text{iff} \quad \ldots
\end{align*}
\]

**Theorem**

This definition agrees with the usual one:

\[
\begin{align*}
M, X &\models_S \varphi \quad \text{iff} \quad F(M), X \models_\kappa \varphi \\
N, x &\models_\kappa \varphi \quad \text{iff} \quad G(N), G(x) \models_S \varphi
\end{align*}
\]
Example: card dealing

Consider the following situation: *there are three agents and a deck of four cards* \{0, 1, 2, 3\}. *Each agent is given a card at random, and the remaining card is kept hidden.*
Example: card dealing

Consider the following situation: **there are three agents and a deck of four cards** \( \{0, 1, 2, 3\} \). **Each agent is given a card at random, and the remaining card is kept hidden.**
So what?

We have uncovered higher-dimensional topological information which is hidden in Kripke models.
So what?

→ We have uncovered higher-dimensional topological information which is hidden in Kripke models.

Does it allow us to say anything new about logic?
Yes: examples from distributed computability!

Herlihy, Kozlov, Rajsbaum, 2013
Distributed computability (Herlihy et. al.)
Distributed computability (Herlihy et. al.)

Output complex

Task specification

Input complex
Distributed computability (Herlihy et. al.)

Protocol complex

Computation

Input complex

Task specification

Output complex
Distributed computability (Herlihy et. al.)

Input complex

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Decision

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∃ ?
Dynamic epistemic logic
Dynamic Epistemic Logic (DEL)

Syntax:
Let $\mathcal{A}$ be a finite set of agents and $AP$ a set of atomic propositions. The syntax of formulas is:

$$
\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid K_a \varphi \mid C_B \varphi \mid [\alpha] \varphi
$$

$$
\alpha ::= (\text{see next slide})
$$

$[\alpha] \varphi$ intuitively means “$\varphi$ will be true after the action $\alpha$ occurs”.
Dynamic Epistemic Logic (DEL)

**Syntax:**
Let $\mathcal{A}$ be a finite set of *agents* and $AP$ a set of *atomic propositions*. The syntax of formulas is:

$$\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid K_a \varphi \mid C_B \varphi \mid [\alpha] \varphi$$

$$\alpha ::= \text{(see next slide)}$$

$[\alpha] \varphi$ intuitively means “$\varphi$ will be true after the action $\alpha$ occurs”.

**Semantics:**

| $M, x \models p$ | iff $p \in L(x)$ |
| $M, x \models \neg \varphi$ | iff $M, x \not\models \varphi$ |
| $M, x \models \varphi \land \psi$ | iff $M, x \models \varphi$ and $M, x \models \psi$ |
| $M, x \models K_a \varphi$ | iff for all $y \in W, x \sim_a y$ implies $M, y \models \varphi$ |
| $M, x \models C_B \varphi$ | iff ... |
| $M, x \models [\alpha] \varphi$ | iff $M[\alpha], x[\alpha] \models \varphi$ |
Action models

Three agents, three cards \{1, 2, 3\}.

Black announces publicly: "I do not have card 2."

Black says privately to White: "I do not have card 2."

→ this does not work.
Three agents, three cards \( \{1, 2, 3\} \).

Black announces publicly: “I do not have card 2”.

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Action models

Three agents, three cards \( \{1, 2, 3\} \).

Black announces publicly: 

“I do not have card 2”.

\[ \begin{array}{ccc}
1 & 2 & 1 \\
3 & 2 & 3 \\
1 & 1 & 1
\end{array} \]
Three agents, three cards \( \{1, 2, 3\} \).

Black announces publicly: 
"I do not have card 2".

Black says privately to White: 
"I do not have card 2".

→ this does not work.
An **action model** is a tuple \( \langle T, \left( \sim_a \right)_{a \in A}, \text{pre} \rangle \) where:

- \( T \) is a set of *actions*,
- for each \( a \in A \), \( \sim_a \) is an equivalence relation on \( T \),
- for each \( t \in T \), \( \text{pre}(t) \in \mathcal{L}_{A,AP} \) is a *precondition*. 

**Example:**

Public announcement

Black: \( \neg 2 \)

Black: \( \neg 1 \)

Black: \( \neg 3 \)
Definition

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Example:

Public announcement

\begin{align*}
\text{Black: } & \text{“¬2”} \\
\text{Black: } & \text{“¬1”} \\
\text{Black: } & \text{“¬3”}
\end{align*}
Definition

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Example:

Private announcement of Black to White

\[
\begin{align*}
\text{Black: } \neg 2 \\
\text{Black: } \neg 1 & \quad \text{g} \\
\text{Black: } \neg 3 & \quad \text{g}
\end{align*}
\]
**Product update**

Let $\mathcal{M} = \langle V, S, \chi, \ell \rangle$ be a simplicial model and $\mathcal{T} = \langle T, \sim, \text{pre} \rangle$ an action model. The **product update model** $\mathcal{M}[\mathcal{T}]$ is the following simplicial model:

- its vertices are of the form $(v, t) \in V \times T$,
- $\chi(v, t) = \chi(v)$ and $\ell(v, t) = \ell(v)$,
- the maximal simplices are the $(X, t)$ such that $\mathcal{M}, X \models \text{pre}(t)$.
Product update

Let $\mathcal{M} = \langle V, S, \chi, \ell \rangle$ be a simplicial model and $\mathcal{T} = \langle T, \sim, \text{pre} \rangle$ an action model. The **product update model** $\mathcal{M}[\mathcal{T}]$ is the following simplicial model:

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An **action** is $\alpha := (\mathcal{T}, t)$.

The truth of a DEL formula is defined as:

\[ \mathcal{M}, X \models [(\mathcal{T}, t)] \varphi \iff \mathcal{M}[\mathcal{T}], (X, t) \models \varphi \]
Example: Public announcement

\[ \mathcal{M} = \]  

\[ \mathcal{T} = \]  

\[ \mathcal{M} \times \mathcal{T} = \]
Example: Public announcement

\[ \mathcal{M} = \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
  \node (a) at (0,0) [circle, fill=black] {1};
  \node (b) at (1,1.732) [circle, fill=white] {2};
  \node (c) at (2,0) [circle, fill=black] {3};
  \node (d) at (1,-1.732) [circle, fill=white] {1};
  \node (e) at (-1,0) [circle, fill=black] {2};
  \node (f) at (-1,1.732) [circle, fill=white] {3};
  \draw (a) -- (b) -- (c) -- (d) -- (e) -- (f) -- (a);
\end{tikzpicture}
\end{array} \]

\[ \mathcal{T} = \begin{array}{c}
\begin{tikzpicture}[scale=0.5]
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  \draw (a) -- (b) -- (c) -- (d) -- (e) -- (f) -- (a);
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Example: Private announcement

\[ M = \quad T = \]

\[ M[T] = \]
Distributed computability via logic

Protocol complex

Computation

Input complex

Output complex

Task specification

Decision

∃?
Distributed computability via logic

\[ \mathcal{M}[\mathcal{P}] \quad \text{Protocol model} \quad \exists \, ? \quad \mathcal{M}[\mathcal{T}] \quad \text{Output model} \]

\[ \mathcal{M} \quad \text{Input model} \]

Computation

Task specification
Key Lemma: simplicial maps cannot gain knowledge

Lemma

Consider two simplicial models $M$ and $M'$, and a morphism $f : M \rightarrow M'$. Let $X \in \mathcal{F}(M)$ be a maximal simplex of $M$, $a$ an agent, and $\varphi$ a positive formula ($\varphi$ does not contain negations except, possibly, in front of atomic propositions). Then,

$$M', f(X) \models \varphi \quad \text{implies} \quad M, X \models \varphi$$

Recipe for impossibility proofs:

- Assume $\delta : M[\mathcal{P}] \rightarrow M[\mathcal{T}]$
- Find a suitable formula $\varphi$ such that:
  - $\varphi$ is true everywhere in the output model
  - $\varphi$ is false somewhere in the protocol model
Conclusions and perspectives

Benefits in both areas

- **For computer scientists:** we can now understand the abstract topological proofs of impossibility in terms of *knowledge*.
Conclusions and perspectives

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- **For logicians:** Kripke models contain higher-dimensional topological information, and it is actually useful!
Conclusions and perspectives

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- **For computer scientists:** we can now understand the abstract topological proofs of impossibility in terms of *knowledge*.
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Future work

- Simplicial complexes that are not pure
  - variable number of agents
Conclusions and perspectives

Benefits in both areas

▶ For computer scientists: we can now understand the abstract topological proofs of impossibility in terms of knowledge.

▶ For logicians: Kripke models contain higher-dimensional topological information, and it is actually useful!

Future work

▶ Simplicial complexes that are not pure
  → variable number of agents

▶ New notions of knowledge?

<table>
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<th>Topology</th>
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<tbody>
<tr>
<td>consensus</td>
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<td>common knowledge</td>
</tr>
<tr>
<td>$k$-set agreement</td>
<td>$k$-connectedness</td>
<td>???</td>
</tr>
</tbody>
</table>
Thanks!