Topics in the theory of uniqueness of trigonometrical series

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Introduction

The theory of uniqueness of trigonometrical series can be regarded as arising from the question of deciding in what sense the Fourier series of a function may be considered as the legitimate expansion of the function in an infinite trigonometrical series. We know, of course, that if the series converges boundedly to the function, then indeed the coefficients of the series must be given by the Euler–Fourier Formulas. However, in the absence of such a condition, we may ask ourselves whether two trigonometrical series may converge to the same function everywhere. The answer to this question is in the negative and was essentially proved so by Riemann, the proof being completed by Cantor.

It is with the replacement of the condition of convergence everywhere with that of convergence almost everywhere, that the theory of sets of uniqueness is concerned. The situation here is not quite as simple. It was shown by Mensov, that there exist trigonometrical series which converge to zero almost everywhere, but which are not identically zero. By the theorem of Riemann–Cantor mentioned above, this series does not converge to zero almost everywhere. The set at which it fails to converge to zero is an example of what is called a set of multiplicity. To be more precise, we say that a set E of measure zero, is a set of multiplicity or an M–set, if there exists a trigonometrical series converging to zero everywhere outside E, but not identically zero. A set of measure zero is called a set of uniqueness or a U–set, if it is not an M–set.

Young [13] showed that every denumerable set is a U–set. Rajchman and Bary [13] independently discovered the existence of non–denumerable perfect U–sets. The main problem in the subject might be considered to be that of finding necessary and sufficient conditions for a set to be a U–set. In practice, attention has been largely restricted to closed sets. It seems that metrical properties such as capacity have little to do with the question [7]. In general one can say that the problem is related to the arithmetical structure of the set.

Our knowledge of U–sets may be summed up in the following way. On the one hand a theorem of Bary says that a countable union of closed U–sets is a U–set. On the other, certain sets designated as H^r sets are known to be closed U–sets. Here r may be any positive integer. H^1 sets were discovered by Rajchman and Bary while H^r sets, which are generalizations of the H^1 sets, were found by Pyatetskii–Shapiro. In his paper [5], Pyatetskii–Shapiro proved that there were H^2 sets which were not countable unions of H^1 sets. He also stated that for each r, there exists a set of type H^r, which is not a countable union of H^{r-1} sets. One of our theorems will be a complete proof of this fact. It depends upon a certain arithmetical property of H^r sets, which is fairly obvious for H^1 sets, but rather involved for $r > 1$.

Pyatetskii–Shapiro also introduced Banach space methods to give necessary and sufficient conditions for a closed set to be a U–set. This condition cannot be applied to any known sets, however, because it is stated in terms of certain Banach space notions and not in terms of the set itself. Because of the unavailability of the paper in English, I have included a rather brief outline of the proof of his results. Also I give sufficient criteria for U–sets which are more arithmetical in nature and are in terms of the set itself. Pyatetskii–Shapiro's criterion may be used to reprove a result of Zygmund and Marcinkiewicz [4] in the case of closed sets. In the first section is also included a theorem which is a necessary condition for U–sets to be of a certain special type. As no necessary conditions, to my knowledge, have been given for U–sets up to now, I think this may be of some interest.

In the theory of several variables, no analogue of Riemann's uniqueness theorem has ever been proved, without certain strong additional hypotheses. These hypotheses concern the rate at which the coefficients are allowed to grow. In particular the results of [9], imply that if the coefficients of a double series tend to zero, and the series is circularly summable to zero everywhere, then it is identically zero. It seems that no results concerning the rate of growth of the coefficients of a convergent trigonometrical series have been published. In the second chapter I prove a result which implies that the rate of growth is smaller than exponential. This result applies to a general type of method of summation which includes both circular and square summation.

There is a notion of M–sets of restricted type which will be discussed in the third chapter. This is a set on which there exists a measure, whose Fourier–Stieltjes series converges to zero outside the set. The classical M–sets were of this type and the trigonometrical series given were the series associated with the measures. In this section I will give a constructive example of a series which converges to zero almost everywhere, but which is not the Fourier series of a measure. Also I have included a generalization of a theorem of Wiener concerning the Fourier series of measures, to several dimensions.

In the last chapter, I consider Green's theorem for the plane in what might be considered as a best possible form. Bochner in [3], and Shapiro in [10], considered this question, but the theorem that I prove is stronger than Bochner's, and has fewer hypotheses than the one in [10], but it does not allow the same type of exceptional sets as in this latter work.

Chapter 1

1. We first make several definitions which will occur in the course of this chapter. We denote by W the ring of absolutely convergent Fourier series on the interval $[0,1]$. We may think of W as the set of all sequence c_n, $-\infty < n < \infty$, such that $\sum_n |c_n| < \infty$. Then W forms a Banach space under the norm $\sum_n |c_n|$. Let $\mathcal{S}$ denote the Banach space of all sequences a_n, $-\infty < n < \infty$, such that $\operatorname*{Lim}_{|n|\to\infty} a_n = 0$. The norm of the sequence $\{a_n\}$ is defined to be $\mathrm{Max}_n |a_n|$. Then W is the dual space of $\mathcal{S}$ where the sequence $\{c_n\}$ represents a linear functional on $\mathcal{S}$ which, when applied to an element $\{a_n\}$ of $\mathcal{S}$, yields $\sum_n a_n c_{-n}$. In W we have the weak star topology induced by $\mathcal{S}$. In this topology a sequence of elements of W $\{c_n^{(m)}\}$, $1 \leq m < \infty$, converges to the element $\{c'_n\}$ if and only if (a) $\sum_n |c_n^{(m)}|$ is uniformly bounded in m, and (b) $c_n^{(m)}$ approaches c'_n for each n as m tends to infinity. In [5], Pyatetskii–Shapiro proves the following theorem:

Theorem 1. Let E be a closed set in the interval $[0,1]$. Let L be the space of all absolutely converging Fourier series vanishing on open neighborhoods of E. Then E is a U–set if and only if the weak closure of L is all of W.

Proof. We assume familiarity with the concept of formal multiplication as expounded for example in [13]. We further need a little known lemma of Zygmund which is essentially reproved in [5] by means of ideal–theoretic considerations.

Lemma 1. Let $P = \sum_n c_n e^{2\pi inx}$ and $Q = \sum_n d_n e^{2\pi inx}$ be two trigonometrical series where P belongs to W and Q belongs to $\mathcal{S}$. Assume that either P or Q vanishes on some open set U. Then the formal product $R = \sum_n \gamma_n e^{2\pi inx}$ where $\gamma_n = \sum_p c_p d_{n-p}$ vanishes on U.

Postponing the proof of the lemma, we proceed to prove the theorem. Let $f = \sum_n c_n e^{2\pi inx}$ belong to $\mathcal{S}$. A necessary and sufficient condition for f

to converge to zero outside E is that f annihilate the subspace L. For, if f does converge to zero outside E, and $g = \sum_n d_n e^{2\pi inx}$ belongs to L, then by the lemma the formal product of f and g converges to zero outside E since f converges to zero there, and converges to zero on some neighborhood of E, since g does so. Hence this formal product is identically zero, and in particular the constant term $\sum_n c_n d_{-n}$ equals zero. Thus f annihilates the subspace L. Conversely, assume that f does annihilate L. This means then that the constant term of the formal product of f and g is zero. Now every term of this formal product is the constant term of the product of f and $e^{2\pi ikx}\, g$, for some k. Since $e^{2\pi ikx}\, g$ lies in L whenever g does, it follows that the formal product of f and g is zero. Let y not be in E. We can then choose g to be rapidly convergent in the sense of [13], and $g(y) \neq 0$. The theorem on formal multiplication tells us that since this product is zero, f converges to zero at y. Hence E is a U–set if and only if no f annihilates L. By the Hahn–Banach theorem applied to W, this is equivalent to saying that the weak closure of L is all of W.

It remains to prove the lemma. Assume for example that Q vanishes on U. For any x_0 in U, let $T(x) = \sum_n t_n e^{2\pi inx}$ be a rapidly converging Fourier series such that $T(x_0) \neq 0$ and vanishing outside U. Then the triple product PQT, by the associative law, may be evaluated in two different ways. On the one hand QT is identically zero since the formal product vanishes everywhere. On the other hand, since $T(x_0) \neq 0$, it follows that PQ vanishes at x_0. A similar proof holds if P vanishes on U.

The subspace L is of course an ideal in W. Its weak closure $\bar{L}$ is also an ideal. This may be seen as follows. If f belongs to L, then so does $e^{2\pi ikx}\, f$, for any k. Hence $\bar{L}$ is closed under multiplication by finite trigonometric polynomials and hence by approximating an arbitrary element of W by trigonometric polynomials we see that $\bar{L}$ is an ideal. Thus we may rephrase the condition of Theorem 1, namely that $\bar{L}$ is all of W, and say that $\bar{L}$ contains 1.

A theorem of Banach [1] states that if W is the dual space of a separable Banach space, and if L is a convex set, closed under sequential limits in the weak star topology, then it is closed in that topology. If E is a set of uniqueness we know that $\bar{L} = W$. For each ordinal number α set $L_\alpha = \cup_{\beta<\alpha} L_\beta$ ifα is a limit ordinal and set $L_{\alpha+1}$ equal to the closure with respect to sequential limits of L_α. For some α then, we have $L_\alpha = W$. The least such α is an invariant of the set. The H^r sets mentioned in the introduction have the ordinal $\alpha = 1$ associated to them, and there are no sets known which have any other value of α associated to them. It would seem reasonable that a countable union of H^r sets for increasing values of r would be an example of such a set.

2. Let E be a U–set situated in the interval $[0,1]$. Let λ be a positive real number. Consider the set λE consisting of all points of the form λx where x belongs to E. A theorem of Zygmund and Marcinkiewicz says that λE considered modulo 1 is also a U–set. In this section we shall apply the results of the previous section to obtain a new proof of this fact in the case of closed U–sets. First we need the following lemma.

Lemma 2. Assume that the intervals $[\alpha,\beta]$ and $[\lambda\alpha,\lambda\beta]$ are both contained in $[0,1]$. There exists an absolute constant A, such that if $f(x) = \sum_n c_n e^{2\pi inx}$ is a member of W vanishing outside of $[\alpha,\beta]$, then $f\left(\frac{x}{\lambda}\right) = \sum_n c'_n e^{2\pi inx}$ also belongs to W and $\sum_n |c'_n| \leq A \sum_n |c_n|$.

Proof. Let $D(x) = \sum_n d_n e^{2\pi inx}$ belong to W, have two continuous derivatives, and be equal to 1 on $[\alpha,\beta]$. Further, let $D(x)$ vanish outside a small enough neighborhood of $[\alpha,\beta]$ so that $D\left(\frac{x}{\lambda}\right)$ is unambiguously defined. Now we have

$$c'_n = \lambda \int_0^1 f(x) e^{-2\pi inx\lambda} dx\,, \tag{1}$$

so

$$c'_n = \lambda \int_0^1 f(x)\, D(x) e^{-2\pi inx\lambda} dx\,. \tag{2}$$

Also

$$D(x)\, e^{-2\pi in\lambda x} = \sum_m d_m^{(n)} e^{2\pi imx} \tag{3}$$

where

$$d_m^{(n)} = \int_0^1 D(x) e^{-2\pi i(n\lambda+m)x} dx\,. \tag{4}$$

Remembering that $D(x)$ has two continuous derivatives, we get after integrating by parts twice that

$$|d_m^{(n)}| \leq \frac{C_1}{C_2 + (n\lambda+m)^2} \tag{5}$$

where C_1 and C_2 are suitable positive constants. Applying Parseval's formula to (2), we obtain

$$|c_n'| \leq \lambda \sum_m |c_m| \frac{C_1}{C_2 + (n\lambda + m)^2} \,. \tag{6}$$

Hence,

$$\sum_n |c_n'| \leq \sum_m |c_m| \sum_n \frac{\lambda C_1}{C_2 + (n\lambda + m)^2} \leq A \sum_m |c_m| \tag{7}$$

where A is some absolute constant.

If E is a U–set, then we know by §1, that the function 1 belongs to $\bar{L}$. Let $D(x)$ denote the same function as in the proof of the lemma. For any $f(x)$ in W, we have that $f(x)$ belongs to the weak closure of $f(x)\,L$. This we see as follows. If $f(x) = e^{2\pi ikx}$, then it is clear. It then follows easily for a finite linear combination of exponential functions, and finally by approximation in the norm, for arbitrary elements of W. In particular $D(x)$ belongs to the closure of $D(x)\,L$. The elements of $D(x)\,L$ all satisfy the hypothesis of Lemma 2. Since the elements of $D(x)\,L$ vanish outside a neighborhood of $[\alpha, \beta]$ it follows that any function in the closure of $D(x)\,L$ also vanishes there. By the theorem of Banach quoted above, $D(x)$ is the iterated sequential limit of suitable elements of $D(x)\,L$. For each f vanishing outside a neighborhood of $[\alpha, \beta]$ let $\tilde{f}$ denote $f\left(\frac{x}{\lambda}\right)$. If we can verify that whenever f_n tends to g, then $\tilde{f}_n$ tends to $\tilde{g}$, we will have proved that $D\left(\frac{x}{\lambda}\right)$ is in the weak closure of the space consisting of all $\tilde{f}$, where $\tilde{f}$ lies in $D(x)\,L$. In turn this implies that $D\left(\frac{x}{\lambda}\right)$ is in the closure of the ideal of all functions vanishing on some neighborhood of the set λE. Now, with the aid of Lemma 2, we can easily prove this assertion. For, if f_n tends to g, this means precisely two things. First, the norms of the f_n are bounded, and second, each coefficient of f_n approaches the corresponding coefficient of g. Lemma 2 tells us that if the first condition is satisfied for f_n, it is still satisfied for $\tilde{f}_n$. Since the functions f_n are bounded in L^1 norm, it follows that the second condition implies that each Fourier coefficient of $\tilde{f}_n$ approaches the corresponding coefficient of $\tilde{g}$.

Now, if the U–set E is contained in the interval $[\alpha, \beta]$ it follows that λE is a U–set. For, $D\left(\frac{x}{\lambda}\right)$ is a function which is equal to one on a neighborhood of λE, and is in the closure of L', where L' denotes the ideal of all functions of W vanishing on some neighborhood of λE. If $h(x)$ is an element of L' equal to one on an open set containing all the points where $D\left(\frac{x}{\lambda}\right)$ is not one, we have then that $D\left(\frac{x}{\lambda}\right) + h(x) - D\left(\frac{x}{\lambda}\right) h(x) = 1$ belongs to the closure of L'. Hence λE is a U–set. Since we can always subdivide the set E into sufficiently many portions E_i, such that each is contained in an interval $[\alpha, \beta]$ as above, and since a finite union of closed U–sets is a U–set, the theorem of Zygmund and Marcinkiewicz follows.

3. As the simplest examples of U–sets, we shall consider those sets which have the property that, as explained in §1, the ordinal 1 is invariantly associated with them. Recalling the definition, this means that there exists a sequence of functions in W, each vanishing on some neighborhoodof the set, and such that f_n weakly approaches 1. Such a set we shall call a U_1 set.

Theorem 2. A necessary and sufficient condition for a closed set E to be a U_1 set, is the following. There exists $\epsilon > 0$, such that for no integer N and real number $\delta > 0$ is it true that for every open set O containing E, and every sequence $c_n > 0$, $n \neq 0$, with $c_j < \delta$ for $|j| = 1, \dots, N$ and $\sum c_n = 1$ there exists a finite number of points in O, $x_1, x_2, \dots, x_k$ and constants $\lambda_1, \dots, \lambda_k$ such that $\sum \lambda_j = 1$ and

$$\sum_{n \neq 0} c_n \left| \sum_{j=1}^{k} \lambda_j e^{2\pi i n x_j} \right|^2 < \epsilon \,. \tag{8}$$

Remark: Since the statement of the theorem is rather involved, something should be said concerning the point of the theorem. Equation (8) expresses the fact that the points x_j are such that nx_j are evenly distributed for many values of n. More specifically, the left side of (8) is an average of quantities which will be small if nx_j are well distributed. Thus our theorem might be paraphrased by saying that a set E is a U_1 set if one cannot find points arbitrarily close to E having more or less random distributions modulo $\frac{1}{n}$ for many values of n. The significance of the number δ is that one may ignore what happens for small values of n.

Proof. The condition is necessary. If E is a U_1 set this means that there exists a sequence f_m belonging to W, each function vanishing on an open neighborhood O_m of E, and tending weakly to one. We may assume that

$$f_m = 1 + \sum_{n \neq 0} c_n^{(m)} e^{2\pi i n x} \tag{9}$$

while

$$\sum_n |c_n^{(m)}| < A$$

where A is an absolute constant. Also, we have that $c_n^{(m)}$ tends to zero for fixed n as m tends to infinity. In the following we will drop the superscript m when it is convenient. Let $\mu = \sum_n |c_n|$ and set $\bar{c}_n = \frac{|c_n|}{\mu}$, so that $\sum \bar{c}_n = 1$. Set

$$g_1 = 1 + \sum_n \bar{c}_n e^{2\pi i n x}\,, \quad g_2 = 1 + \sum_n \frac{c_n}{\sqrt{\bar{c}_n}}\, e^{2\pi i n x}\,, \tag{10}$$

and $\epsilon = \frac{1}{2A^2}$. If the theorem were false there would exist N and δ as stated in the theorem. Furthermore for some value of m sufficiently large, the sequence c_n would satisfy the hypothesis $|\bar{c}_n| < \delta$ for $|n| \le N$. Hence there would exist $x_1, \ldots, x_k$, lying in O_m, and $\lambda_1, \ldots, \lambda_k$ such that $\sum \lambda_j = 1$ and

$$\sum_{n \neq 0} \bar{c}_n \left| \sum_{j=1}^{k} \lambda_j e^{2\pi i n x_j} \right|^2 < \epsilon\,. \tag{11}$$

Now we put

$$h(x) = \sum \lambda_j\, g_1(x + x_j)\,.$$

Then we will have

$$\int_0^1 g_2(x)\, h(x) dx = 0$$

by Parseval's formula, remembering that f_m vanishes at x_j since they lie on O_m. On the other hand

$$\int_0^1 g_2(x)\, h(x) dx = 1 + \sum_{n \neq 0} \left(\sum_{j=1}^{k} \sqrt{\bar{c}_n}\, \lambda_j e^{2\pi i n x_j} \right) \frac{c_n}{\sqrt{\bar{c}_n}}\,.$$

By Schwarz's inequality this is greater in absolute value than

$$1 - \left[\epsilon\, \mu \sum |c_n| \right]^{1/2} = 1 - \mu \sqrt{\epsilon} > 0$$

and hence we have a contradiction. Thus the necessity is proved.

The condition is sufficient. The condition means that there exists an $\epsilon > 0$, and a sequence $c_n^{(m)}$, such that $\sum_n c_n^{(m)} = 1$, and $c_n^{(m)}$ tends to zero for a fixed m as n tends to infinity, and open sets O_m containing E, such that

$$\sum_n c_n^{(m)} \left| \sum_{j=1}^{k} \lambda_j e^{2\pi i n x_j} \right|^2 > \epsilon \tag{12}$$

for all x_j in O_m where λ_j are numbers such that $\sum \lambda_j = 1$. Let

$$f_m = 1 + \sum_n \sqrt{c_n^{(m)}}\, e^{2\pi i n x}\,. \tag{13}$$

Clearly f_m belongs to L^2 of the interval $[0,1]$. Furthermore $\int_0^1 |f_m^2(x)|dx = 2$. Let M be the subspace of L^2 consisting of all functions with constant term 1 in their Fourier expansions. Again we drop the index m where convenient. Let M_0 be the subspace of M generated by the functions $f_m(x+t)$ where t belongs to O_m. Let g be that element of M_0 such that $\int_0^1 |g|^2 dx$ is minimum. Set $h = \alpha + \beta\, g(x)$ where α, β are so chosen that

$$\int_0^1 h\,\bar{g} dx = 0\,, \quad \int_0^1 h dx = 1\,. \tag{14}$$

This will be possible provided $\int_0^1 |g|^2 dx > 1$, for in that case we take

$$\alpha = \frac{-\int_0^1 |g|^2 dx}{1 - \int_0^1 |g|^2 dx}\,, \quad \beta = \frac{1}{1 - \int_0^1 |g|^2 dx}\,. \tag{15}$$

Now the hypotheses of the theorem tell us that

$$\int_0^1 |g|^2 dx > 1 + \varepsilon\,.$$

Hence we see that

$$\int_0^1 |h|^2 dx < A$$

where A depends only on ϵ. On the other hand,

$$\int_0^1 h(x)\,\bar{j}(x) dx = 0$$

for any $j(x)$ belonging to M_0. For we have

$$\int_0^1 (\alpha + \beta\, g)\,\bar{j} dx = \int_0^1 (\alpha + \beta\, g)\,\bar{g} dx + \int_0^1 (\alpha + \beta\, g)\,(\overline{j - g}) dx\,. \tag{16}$$

The first integral on the right is zero because of (14), whereas the second integral equals $\beta \int_0^1 g\,(\overline{j - g}) dx$, which is zero by the minimal property of g. Hence if

$$h_m(x) = 1 + \sum_n d_n^{(m)}\, e^{2\pi i n x}\,,$$

the function

$$k_m(x) = 1 + \sum_n d_n^{(m)} \sqrt{c_n^{(m)}}\, e^{2\pi i n x} \tag{17}$$

belongs to W, and has its norm in that ring, by Schwarz's inequality, less than $1+\sqrt{A}$. Furthermore since $\int_0^1 h(x)\,\bar{j}(x)dx = 0$ for all j in M_0 and in particular for $f_m(x+t)$ where t lies in O_m, it follows from Parseval's formula that $k_m(x) = 0$ for x in O_m. Thus it is clear that the sequencesk_m tend weakly to one and vanish on neighborhoods of E, so that E is a U_1 set. The fact that the coefficients of k_m tend to zero as m tends to infinity follows from the fact that the $c_n^{(m)}$ do so and $d_n^{(m)}$ are all bounded by $\sqrt{A}$.

Now we shall turn our attention to the H^r sets which were discovered by Pyatetskii–Shapiro. We need a preliminary definition.

Definition. A sequence of r–tuples of integers, n_i^j, $1 \le j \le r$, $1 \le i < \infty$, is said to be normal if whenever a_j, $1 \le j \le r$, are integers not all zero,

$$\left| \sum_j a_j n_i^j \right| \tag{18}$$

tends to infinity with i.

Then we have:

Definition. A set S is an H^r set if and only if there exist r intervals $L_1, \ldots, L_r$ on the interval $[0,1]$ and a normal sequence of r–tuples of integers n_i^j, such that for all x in S, and all i, there exists j such that $n_i^j x$ does not lie in L_j modulo 1.

We now prove that the H^r sets are sets of uniqueness. For each L_j, let $\lambda_j(x)$ be a function in W vanishing everywhere except in an interval interior to L_j, and such that the constant term of $\lambda_j(x)$ is 1. Let

$$f_i(x) = \prod_{j=1}^r \lambda_j(n_i^j x)\,.$$

These functions then vanish on a neighborhood of E, and their norms as elements of W are uniformly bounded by the product of the norms of λ_j. Let

$$\lambda_j(x) = \sum c_{\alpha_j}^{(j)}\, e^{2\pi i \alpha_j x} \tag{19}$$

where $\alpha_1, \dots, \alpha_r$ range from $-\infty$ to ∞. Then the constant term of $f_i(x)$ is[1]

$$\sum_{n_i^{(1)}\alpha_1+\dots+n_i^{(r)}\alpha_r=0} c_{\alpha_1}^{(1)} \cdot \dots \cdot c_{\alpha_r}^{(r)} \,. \tag{20}$$

This last sum contains the term $c_0^{(1)} \cdot \dots \cdot c_0^{(r)} = 1$. For any fixed choice of $\alpha_1, \dots, \alpha_r$ not all zero there will be an i such that no term corresponding to these values of α_j occurs in the sum (20) because of the normality condition (18), for that value of i or thereafter. Hence it is clear that this constant term (20) approaches one, and a similar argument shows that the non–constant terms approach zero. Thus $f_i(x)$ tend weakly to one, and since they all vanish on open sets containing E by definition of an H^r set, it follows that E is a set of uniqueness. Moreover E is obviously a U_1 set. The next theorem will have as its object to show that in some sense H^r sets are the simplest U_1 sets.

Theorem 3. Let $f_m = \sum_n c_n^{(m)} e^{2\pi inx}$, $c_0^{(m)} = 1$, be functions in W, vanishing on a set E, such that f_m tends weakly to one. Assume that there exists $\epsilon < 1$, and an integer N such that for all m sufficiently large we can find a set J_m of N indices $n_1^{(m)}, \dots, n_N^{(m)}$, such that

$$\sum_{n\neq 0,\ n\notin J_m} |c_n^{(m)}| < \epsilon \,. \tag{21}$$

Then E is a finite union of H^r sets.

Remark: In the proof that H^r sets are sets of uniqueness, we constructed exactly such a sequence of elements of W. On the other hand it is possible by taking appropriate linear combinations of the functions in this sequence, to construct a sequence vanishing on neighborhoods of a H^r set, tending weakly to one, but not satisfying the hypotheses of Theorem 3.

Proof. The elements of the sets J_m are not uniformly bounded, because, since f_m tend weakly to one, this would imply that the coefficients corresponding to the indices in J_m tend to zero, and hence by (21) that $\sum_{n\neq 0} |c_n^{(m)}| < 1$ for m sufficiently large. This in turn implies that the functions f_m are never zero. Thus we see that after possible rearranging $n_j^{(m)}$, we have that $n_1^{(m)}$ forms a normal sequence of 1–tuples in the sense of our definition. Assume that after further rearrangement $n_1^{(m)}, \dots, n_r^{(m)}$ form a normal sequence of

[1] Original contained $\sum_{n_i^{(1)}\alpha_1 \dots n_i^{(r)}\alpha_r=0} c_1^{(1)} \cdot \dots \cdot c_r^{(r)}$.

r–tuples, while $n_1^{(m)}, \ldots, n_r^{(m)}, n_{r+1}^{(m)}$ do not form a normal sequence of $r+1$–tuples. By restricting ourselves to a suitable subsequence, this implies that there exist integers $a_{11}, a_{12}, \ldots, a_{1r}$, $b_1 \neq 0$, c_1, such that[2]

$$\sum_{j=1}^{r} a_{1j} n_j^{(m)} + b_1 n_{r+1}^{(m)} = c_1 \,.$$

Now with this new sequence of $n_j^{(m)}$, we consider the sequence of $r+1$–tuples $n_1^{(m)}, \ldots, n_r^{(m)}, n_{r+2}^{(m)}$. If this sequence is not a normal sequence of $r+1$–tuples we proceed exactly as before. If it is a normal sequence we adjoin $n_{r+2}^{(m)}$ to $n_j^{(m)}$, $1 \leq j \leq r$. Thus eventually we will find that for a suitable subsequence of our original sequence and suitable rearrangements, there will be a number r and integers a_{kj} where $1 \leq j \leq r$, $1 \leq k \leq N-r$, $b_k \neq 0$, c_k, $1 \leq k \leq N-r$, such that

$$\sum_{j=1}^{r} a_{kj} n_j^{(m)} + b_k n_{r+k}^{(m)} = c_k \,. \tag{23}$$

Now set

$$B = b_1 \cdot \ldots \cdot b_{N-r}\,, \quad \bar{n}_j^{(m)} = \left[\frac{n_j^{(m)}}{B} \right]\,, \quad \bar{a}_{kj} = \frac{a_{kj}\, B}{b_k}\,. \tag{24}$$

We have then

$$\sum_{j=1}^{r} \bar{a}_{kj} \bar{n}_j^{(m)} + \bar{n}_{r+k}^{(m)} = \bar{c}_k^{(m)} \tag{25}$$

where $\bar{c}_k^{(m)}$ depends on m but remains bounded, $|\bar{c}_k^{(m)}| \leq K$. The sequence $\bar{n}_j^{(m)}$, $1 \leq j \leq r$, is clearly still a normal sequence. We shall show now that the set E is a finite union of H^r sets, each defined relative to some subsequence of this sequence. For each integer d, let $I_1, \ldots, I_d$ be the d consecutive intervals of length $\frac{1}{d}$ which cover the unit interval. If $E \cap I_1$ were not an H^r set, then for each choice of $\alpha_1, \ldots, \alpha_r$ from among the set of positive integers less than or equal to d, the relations

$$\bar{n}_1^{(m)} x \in I_{\alpha_1}, \ldots, \bar{n}_r^{(m)} \in I_{\alpha_r} \tag{26}$$

[2] Typo "$n_r^{(m)}{}_1$" in manuscript.

hold for some x contained in $E \cap I_1$, for some m sufficiently large, since otherwise $E \cap I_1$ would be an H^r set. There are d^r possible choices of the numbers $\alpha_1, \dots, \alpha_r$, and so for each m sufficiently large we have a set of d^r points which we denote by S_m, belonging to $I_1 \cap E$ and satisfying (26). Our next object will be to show that the sums

$$\frac{1}{d^r} \sum_{x \in S_m} e^{2\pi i n_j^{(m)} x}, \quad 1 \le j \le N,$$

are quite small.

First we have

$$\left| \frac{1}{d^r} \sum_{x \in S_m} e^{2\pi i n_j^{(m)} x} - \frac{1}{d^r} \sum_{x \in S_m} e^{2\pi i \bar{n}_j^{(m)} B x} \right| < \left| 1 - e^{\frac{2\pi i B}{d}} \right|, \tag{27}$$

remembering that I_1 consists of the interval $\left[0, \frac{1}{d}\right]$ and x belongs to I_1. For each $\bar{n}_j^{(m)} x$ where x belongs to S_m, has d^{r-1} points in each of the intervals $I_1, \dots, I_d$. Therefore we have that the average value of the function $e^{2\pi i B x}$ taken over these points differs from the integral extended over the interval $[0, 1]$ by at most

$$\operatorname*{Max}_{|x-y| \le \frac{1}{d}} \left| e^{2\pi i B x} - e^{2\pi i B y} \right| .$$

Hence, since the integral of $e^{2\pi i B x}$ is zero, we obtain that

$$\left| \frac{1}{d^r} \sum_{x \in S_m} e^{2\pi i B \bar{n}_j^{(m)} x} \right| \le \left| 1 - e^{\frac{2\pi i B}{d}} \right|, \tag{28}$$

or, from (27)

$$\left| \frac{1}{d^r} \sum_{x \in S_m} e^{2\pi i n_j^{(m)} x} \right| \le 2 \left| 1 - e^{\frac{2\pi i B}{d}} \right| \le 2 \frac{2\pi B}{d}. \tag{29}$$

Now, if $j = r + k$, we have[3] $\bar{n}_{r+k}^{(m)} = \bar{c}_k^{(m)} - (\bar{a}_{k1} \bar{n}_1^{(m)} + \cdots + \bar{a}_{kr} \bar{n}_r^{(m)})$. Assume that $a_{k1} \ne 0$. Consider any set of indices $\alpha_2, \dots, \alpha_r$. Let T denote the set of d points in S_m for which

$$\bar{n}_2^{(m)} x \in I_{\alpha_2}, \dots, \bar{n}_r^{(m)} x \in I_{\alpha_r} . \tag{30}$$

[3] Typo "$\bar{n}_r^{(m)}{}_k$" in manuscript.

Then the points

$$\bar{c}_k^{(m)} x - (\bar{a}_{k2}\bar{n}_2^{(m)} + \ldots + \bar{a}_{kr}\bar{n}_r^{(m)})\, x$$

all lie within distance

$$\frac{|\bar{c}_k^{(m)}| + |\bar{a}_{k2}| + \cdots + |\bar{a}_{kr}|}{d} \le \frac{A}{d}$$

of each other and hence of a single point x_0. Here A denotes an absolute constant. Hence

$$\left| \frac{1}{d} \sum_{x \in T} e^{2\pi i \bar{n}_{r+k}^{(m)} x} - \frac{1}{d} \sum_{x \in T} e^{-2\pi i (\bar{a}_{k1}\bar{n}_1^{(m)} x - x_0)} \right| \le \frac{2\pi A}{d} . \tag{31}$$

But, exactly as above, since the points $\bar{n}_1^{(m)} x$ lie successively in each of the intervals $I_1, \ldots, I_d$, we have

$$\left| \frac{1}{d} \sum_{x \in T} e^{-2\pi i \bar{a}_{k1}\bar{n}_1^{(m)} x} \right| \le \frac{2\pi |\bar{a}_{k1}|}{d} . \tag{32}$$

Since S_m consists of d^{r-1} sets each exactly like our set T, it follows that by combining (31) and (32) we obtain

$$\left| \frac{1}{d^r} \sum_{x \in S_m} e^{2\pi i n_j^{(m)} x} \right| \le \frac{K}{d} \tag{33}$$

where K is some constant, independent of m and d.

Now the theorem follows easily. Since for all points of the set S_m, $f_m(x)$ equals zero, we have

$$\begin{aligned} 0 = \frac{1}{d^r} \sum_{x \in S_m} \sum_n c_n^{(m)} e^{2\pi i n x} &= 1 + \sum_{n \notin J_m} \frac{1}{d^r} \sum_{x \in S_m} c_n^{(m)} e^{2\pi i n x} \\ &+ \sum_{n \in J_m} \sum_{x \in S_m} \frac{1}{d^r} c_n^{(m)} e^{2\pi i n x} . \end{aligned} \tag{34}$$

The first sum on the right is bounded by $1 - \epsilon$ in absolute value, while the second sum does not exceed $\sum_n |c_n^{(m)}| \, \frac{K}{d}$. Hence we have

$$1 \le (1 - \epsilon) + \frac{K}{d} \sum_n |c_n^{(m)}| \tag{35}$$

which is clearly impossible if d is chosen large enough.

In the next theorem the symbol $|E|$ will be used to denote the measure of E.

Theorem 4. Let E_n be a sequence of closed sets in $[0,1]$, such that $|E_n| > \delta > 0$. Assume that for each fixed interval I,

$$\frac{I \cap E_n}{|I|} - |E_n| \tag{36}$$

tends to zero as n tends to infinity. Let F_n denote the set of all x such that $E_n + x$, which means E_n translated by x, does not not intersect E_n. Then any closed set contained in the intersection of all the F_n is a U_1 set.

Proof. Let $\chi_n(x)$ be the characteristic function of the set E_n. Set

$$f_n(x) = \frac{\chi_n(x)}{|E_n|}\,.$$

f_n then has constant term one in its Fourier expansion. Since each $f_n(x)$ belongs to L^2, and $\|f_n\|_2 = \frac{1}{\sqrt{\delta}}$, it follows that $g_n(x) = f_n(x) \star f_n(x)$, where $\star$ denotes convolution, lies in W, and its norm in that ring does not exceed $\frac{1}{\delta}$. For any $k \neq 0$, and $\epsilon > 0$, let d be so large that $|e^{2\pi ikx} - e^{2\pi iky}| < \epsilon$ if $|x-y| < \frac{1}{d}$. Let $I_1, \ldots, I_d$ be as in the proof of Theorem 3. Pick n so large that

$$\left|\frac{|I_\alpha \cap E_n|}{|I_\alpha|} - |E_n|\right| < \epsilon, \quad 1 \leq \alpha \leq d\,. \tag{37}$$

Then we have

$$\left|\int_{I_\alpha \cap E_n} e^{2\pi ikxdx} dx - \frac{|I_\alpha \cap E_n|}{|I_\alpha|}\int_{I_\alpha} e^{2\pi ikx} dx\right| < \epsilon\,|I_\alpha|\,, \tag{38}$$

or by (37),

$$\left|\int_{I_\alpha \cap E_n} e^{2\pi ikxdx} dx - |E_n|\int_{I_\alpha} e^{2\pi ikx} dx\right| < 2\epsilon\,|I_\alpha|\,.$$

Summing over all α, we obtain

$$\left|\int_{E_n} e^{2\pi ikx}\right| < 2\epsilon\,.$$

It follows that the functions $g_n(x)$ tend weakly to one. On the other hand we have

$$g_n(x) = \int f_n(t)\, f_n(x-t)dt\,,$$

so it is clear that $g_n(x)$ vanishes on F_n. Hence the theorem is proved.

We now remark that those H^r sets which are defined with respect to normal sequences where the ratio of successive terms in each r–tuple tends to infinity, obviously satisfy the hypotheses of our theorem 4. The choice of the set E_n is rather obvious.

4. Now we shall consider the question of whether there exist H^r sets which are not countable unions of H^{r-1} sets. We will first prove that the complement of an H^{r-1} set has a certain metric property, and then we will find an H^r set whose complement does not have this property. We need some preliminary definitions and lemmas.

Definition: Let S be any set in $[0,1]$. For $c, \epsilon > 0$, we say that an interval I is of type $(1, c, \epsilon)$ relative to S, if to every point x belonging to I, there exists an interval contained in S of length δ, with its midpoint at m, such that

$$|x-m| < \epsilon, \quad \frac{\delta}{|x-m|} > c \tag{39}$$

both hold. Similarly, we say that an interval I is of type (r, c, ϵ) relative to S, if to every point x belonging to I, there exists an interval of type $(r-1, c, \epsilon)$ relative to S, of length δ, with its midpoint at m, such that

$$|x-m| < \epsilon, \quad \frac{\delta}{|x-m|} > c\,.$$

Finally, if the interval $[0,1]$ itself is of type (r, c, ϵ) relative to S, we say that S is of type (r, c, ϵ).

Roughly speaking, this definition says that a set is of type 1, if every point is close to a relatively long interval of the set. It is of type 2, if every point is close to a relatively long interval of points, each one of which is close to a long interval of the set, etc. Such sets as those in the definition occur in the definition of H^r sets, and the purpose of Theorem 5, is precisely to prove that a certain set if of type (r, c, ϵ).

We first have a lemma.

Lemma. Let L_i be the intervals $(h_i, h_i + d)$ where $0 \leq h_i < 1$, $0 < d < 1$, $1 \leq i \leq r$. There exist constants $B(d)$ and $C(d)$ depending only on d, as follows: For any $\epsilon > 0$, there exists an integer N such that if

$$s_1 = \frac{p_1}{n}\,,\ s_2 = \frac{p_2}{n}\ \ldots,\ s_r = \frac{p_r}{n}$$

are r rational numbers, with $|s_i| \leq 1$, and having the property that for all integers a_j, $0 \leq j \leq r$, not all zero, satisfying $|a_j| \leq N$, we have

$$\left| a_0 n + \sum_{j=1}^{r} a_j p_j \right| > N\,, \tag{40}$$

then there exists an integer $q \leq B(d)$ such that the set of all intervals of the form $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ where k runs through all the integers satisfying $kqs_i \in L_i$ (modulo 1), $1 \leq i \leq r$, forms a set of type $(r, C(d), \epsilon)$.

Proof. In the statement of the lemma, the number N in reality depends upon both d and r. Therefore, at times we shall denote it by $N(r, \epsilon, d)$. The proof proceeds by induction on r. We assume the lemma true for $r - 1$. Set $C' = C\left(\frac{d}{4}\right)$ and $B' = B\left(\frac{d}{4}\right)$, where we consider these quantities defined for the case $r - 1$. Let ϵ be an arbitrary positive quantity. Then if we set $N' = N\left(\epsilon, \frac{d}{4}\right)$, again for the case $r - 1$, we shall prove the lemma for the case r with the following determination of constants,

$$C = \mathrm{Min}\left(\frac{d}{32B'}, C'\right) \qquad B = \left(\frac{8B'}{d} + 1\right) B'\,, \tag{41}$$

$$N = \mathrm{Max}\left(\frac{5}{\epsilon}, r\,(B+1)N'\right)\,.$$

Therefore, assume that $s_i = \frac{p_i}{n}$ are r rational numbers which satisfy the condition (40) of the lemma where N is given by (41). Thus in particular for no k satisfying

$$k \leq \left(\frac{8B'}{d} + 1\right)^r \tag{42}$$

do we have

$$k\,(s_1, s_2, \ldots, s_r) = 0 \text{ (modulo 1)}\,, \tag{43}$$

by which we mean that the corresponding vector does not have all its components integral. This is so since otherwise $k\,s_1 = t$, where t is an integer and both k and t are smaller than N, so that we have $kp_1 - tn = 0$ which violates (40). Now we apply the well known box principle of Dirichlet. We divide the unit interval up into $\frac{8B'}{d}+1$ equal intervals. If we do this for each axis in r–dimensional space, we will have subdivided the r–dimensional cube into at most $\left(\frac{8B'}{d}+1\right)^r$ cubes. Now, consider all the vectors $k\,(s_1,\ldots,s_r)$, where k satisfies (42). There must exist two values of k such that the corresponding vectors lie in the same cube. Their difference then will be an integer k_1 such that

$$k_1 \le \left(\frac{8B'}{d}+1\right)^r , \quad k_1 s_i = \alpha_i + E_i\,, \quad |\alpha_i| \le \frac{d}{8B'}\,, \tag{44}$$

and E_i are integers. Now let α_1 be the largest of the α_i in absolute value, and assume that α_1 is positive.

Let C denote the cube in $r-1$ dimensional space defined by the following inequalities,

$$\begin{aligned} &\text{if } \alpha_i > 0\,, \quad h_i - \frac{h_1}{\alpha_1}\,\alpha_i + \frac{d}{4} \le y_i \le y_i - \frac{h_1}{\alpha_1}\,\alpha_i + \frac{d}{2}\,,\\ &\text{if } \alpha_i < 0\,, \quad h_i - \frac{h_1}{\alpha_1}\,\alpha_i + \frac{d}{2} \le y_i \le y_i - \frac{h_1}{\alpha_1}\,\alpha_i + \frac{3d}{4}\,, \end{aligned} \tag{45}$$

where $y_2,\ldots,y_r$ are the variables. If $(y_2,\ldots,y_r)$ lies in the cube C, and λ is an integer satisfying

$$h_1 + \frac{d}{8} \le \alpha_1\lambda \le h_1 + \frac{3d}{8}\,, \tag{46}$$

then it will follow that

$$\lambda\,(\alpha_1,\ldots,\alpha_r) + (0, y_2,\ldots,y_r) \tag{47}$$

lies in the cube L' which is defined as the direct product of the intervals $L'_i = \left(h_i + \frac{d}{8}, h_i + \frac{7d}{8}\right)$. This is so since the first component of the vector (47) lies in the interval L'_i by (46), while if α_i is greater than zero we have

$$h_1\,\frac{\alpha_i}{\alpha_1} \le \lambda\,\alpha_i \le h_1\,\frac{\alpha_i}{\alpha_1} + \frac{3d}{8} \tag{48}$$

since $|\alpha_i| \leq \alpha_1$ and for $\alpha_i < 0$ we have

$$h_1 \, \frac{\alpha_i}{\alpha_1} - \frac{3d}{8} \leq \lambda \, \alpha_i \leq h_1 \, \frac{\alpha_i}{\alpha_1} \, . \tag{49}$$

Thus comparing (48), (49), and (46) we see that (47) lies in the cube C. Now set

$$\beta_i = \frac{\alpha_i}{\alpha_1} = \frac{k_1 p_i - E_i n}{k_1 p_1 - E_1 n} = \frac{p_i'}{n'} \, , \quad 2 \leq i \leq r \, , \tag{50}$$

where

$$n' = k_1 p_1 - E_1 n \, , \quad p_i' = k_1 p_i - E_i n \, .$$

Now if a_i, $1 \leq i \leq r$, are r integers not all zero and such that $|a_i| \leq N'$, where we recall that N' was defined as $N'\left(r-1, \epsilon, \frac{d}{4}\right)$, we have

$$\left| \sum_{i=2}^{r} a_i p_i' + a_1 n' \right| = \left| \sum_{i=1}^{r} a_i k_1 p_i - n \left(\sum_{i=1}^{r} E_i a_i \right) \right| \, . \tag{51}$$

Using the inequality $|E_i| \leq B+1$, we see that we have a linear combination of p_i and n with coefficients less than or equal to $r(B+1)\,N'$ which does not exceed N. Hence the quantity in (51) exceeds N and therefore N'. We also note that $|\beta_i| \leq 1$ and that the length of each side of the cube C is $\frac{d}{4}$. Thus applying our lemma to these numbers and the cube C for the case $r-1$, we deduce that there exists a number $q' \leq B'$ such that the set of intervals of the form $\left[\frac{k'}{n'}, \frac{k'+1}{n'}\right]$ where k' ranges over all integers such that $q'k'\left(\frac{\alpha_i}{\alpha_1}\right)$ lies in the cube C modulo 1 for $2 \leq i \leq r$, forms a set of type $(r-1, C', \epsilon)$. Remembering that $C \leq C'$, it follows that to prove the lemma we must merely show that each one of the intervals $\left[\frac{k'}{n'}, \frac{k'+1}{n'}\right]$ is of type $(1, C, \epsilon)$ relative to the original set mentioned in the lemma. Let us then consider a particular k' such that $q'k'\left(\frac{\alpha_i}{\alpha_1}\right)$ lies in C modulo 1. Set $m = \left[\frac{k'}{\alpha_1}\right]$. By what was said above concerning the vector in (47), it follows that if an integer λ satisfies

$$h_1 + \frac{d}{8} \leq \alpha_1 q' \, \lambda \leq h_1 + \frac{3d}{8} \, , \tag{52}$$

then the vector

$$q' \left(\lambda + \frac{k'}{\alpha_1} \right) \, (\alpha_1, \ldots, \alpha_r) \tag{53}$$

lies in the cube L' modulo 1. The vector (53), however, differs in every component from the vector

$$q'(\lambda+m)(\alpha_1,\ldots,\alpha_r) \tag{54}$$

by at most $|q'\alpha_i| \le \frac{d}{8}$. Hence the vector (54) lies in the cube L modulo 1, where L is defined as the direct product of the intervals L_i. Now set $q = q'k$, so that we have $q \le B$, and all intervals of the form $\left[\frac{\lambda+m}{n}, \frac{\lambda+m+1}{n}\right]$ belong to the original set described in the lemma. The condition (52) on λ can be rewritten as follows:

$$\frac{1}{\alpha_1 q'}\left(h_1+\frac{d}{8}\right) \le \lambda \le \frac{1}{\alpha_1 q'}\left(h_1+\frac{3d}{8}\right).$$

Obviously then, there is a string of consecutive such λ numbering at least[4]

$$\frac{d}{4\alpha_1 q'} - 1 \ge \frac{d}{8\alpha_1 q'}$$

since

$$\frac{d}{4\alpha_1 q'} \ge 2\,.$$

The corresponding intervals $\left[\frac{\lambda+m}{n}, \frac{\lambda+m+1}{n}\right]$ which we know belong to the original set of the lemma, form on large interval of length at least $\frac{d}{8\alpha_1 q' n} = \frac{d}{8q'n'}$. This large interval is contained in $\left[\frac{m}{n}, \frac{m+1}{n}+\frac{2}{n'q'}\right]$ since $h_1+\frac{3d}{8} < 2$. In turn, this interval is of length at most $\frac{3}{n'}$. We also have $\frac{k'}{n'} \ge \frac{m}{n} \ge \frac{k'}{n'} - \frac{1}{n}$. Thus we conclude that the distance from any point in $\left[\frac{k'}{n'}, \frac{k'+1}{n'}\right]$ to the midpoint of our block of intervals is at most $\frac{4}{n'}$. Now $n' \ge N$ so that this distance is smaller than ϵ, and the ratio of the length of the block of intervals to this distance is at least $\frac{d}{32q'} \ge \frac{d}{32B'}$. Thus the lemma is proved for the case r under the assumption that it holds for the case $r-1$. Notice that at this last step if we took each point in the interval $\left[\frac{k'}{n'}, \frac{k'+1}{n'}\right]$ and examined its distance to the block of intervals of the original set which we associated with those values of λ such that $1+h_1+\frac{d}{8} \le \alpha_1 q'\lambda \le h_1+\frac{3d}{8}+1$, then we could modify the argument very slightly to show that with a different choice of $C(d)$ we can assume that all the intervals which occur in the definition of a set of type (r,c,ϵ) occur to the right. We will need this remark later. Thus, it only remains to prove the lemma in the case $r=1$.

[4] Original: "α" missing on right hand side.

This is very simple. If L_1 is the interval $(h, h+d)$, we set $C(d)$ equal to $\frac{d}{4}$, and $B(d)$ equal to $\frac{20}{d}+1$. Finally for $\epsilon > 0$ we set N equal to Max $\left(\frac{2}{\epsilon}, B+1\right)$. Then if $s = \frac{p}{n}$, $|s| < 1$, we have as above for no $k \leq B(d)$ is $ks \equiv 0$ (modulo 1), so that there exists an integer $k \leq B(d)$, such that $ks = \alpha + E$, where E is an integer and $|\alpha| < \frac{d}{20}$. Also $|E| < B+1$. We have $\alpha = \frac{kp - En}{n} = \frac{n'}{n}$, and $n' \geq N$ since k and E do not exceed N. Let v be any integer. There exists a consecutive sequence of integers λ satisfying both $\frac{v}{\alpha} \leq \lambda \leq \frac{v+1}{\alpha}$ and $h \leq \lambda\alpha \leq h+d$ (modulo 1) numbering at least $\frac{d}{2\alpha}$ terms. If we set $q = k$, intervals of the form $\left[\frac{\lambda}{n}, \frac{\lambda+1}{n}\right]$ will be of the desired type, and the total length of these intervals will e at least as great as $\frac{d}{2\alpha n} = \frac{d}{2n'}$. The distance from the midpoint of this block of intervals to any point in the interval $\left[\frac{v}{\alpha n}, \frac{v+1}{\alpha n}\right]$ is at most $\frac{2}{n'}$, using again the fact that $n > n'$. Thus we see that our choice of C and B satisfy the conditions of the lemma, since for some v every point is contained in an interval of the form $\left[\frac{v}{\alpha n}, \frac{v+1}{\alpha n}\right]$.

Theorem 5. Let n_j^i, $j = 1, \ldots, r$ be a normal sequence of r–tuples. Let $L_1, \ldots, L_r$ be the intervals $[h_j, h_j + d]$. Then there exists a constant $C(d)$ depending only on d, such that if $\epsilon > 0$, there exists an integer N such that for $i > N$ the set $S_i = \{x \mid n_j^i x \in L_j \bmod 1, \text{ all } j\}$ is a set of type $(r, C(d), \epsilon)$.

Proof. We shall drop the superscript in n_j^i without risk of confusion. Let $n_1 > n_2 > \cdots > n_r$. If for $j = 2, \ldots, r$ and for some integer k, we have

$$h_j - \frac{3}{4}\, d\, \frac{n_j}{n_1} \leq \frac{h_1 + k}{n_1}\, n_j \leq h_j + d - \frac{n_j}{n_1}\, d \text{ (modulo 1)}\,, \tag{55}$$

then the interval $A_k = \left[\frac{k+h_1}{n_1} + \frac{3}{4}\frac{d}{n_1}, \frac{k+h_1}{n_1} + \frac{d}{n_1}\right]$ will belong to the set S. We see this because if x lies in this interval then $n_1 x$ lies in $\left[h_1 + \frac{3}{4}\, d, h_1 + d\right]$ (modulo 1), and $n_j x$ for $j \neq 1$ lies in $[h_j, h_j + d]$ modulo 1. We notice that the intervals A_k are of length at least $\frac{d}{4n_1}$, and that every point of the interval $\left[\frac{k}{n_1}, \frac{k+1}{n_1}\right]$ is no more than $\frac{2}{n_1}$ distant from its midpoint. This means that the interval $\left[\frac{k}{n_1}, \frac{k+1}{n_1}\right]$ is of type $\left(1, \frac{d}{8}, \frac{2}{n_1}\right)$ relative to S. The condition on k in (55) says that $k\,\frac{n_j}{n_1}$ lies in a certain interval modulo 1 of length at least $\frac{3}{4}\, d$, remembering that $n_1 > n_j$. We may thus apply our lemma to the numbers $\frac{n_j}{n_1}$. We then conclude that there exists an integer $q \leq B(d)$, such that the intervals of the form $\left[\frac{k}{n_1}, \frac{k+1}{n_1}\right]$ where qk satisfies (55), is of type $(r-1, C, \epsilon)$. This follows because the hypotheses of the lemma will be satisfied for any

N if we go out far enough in the sequence of r–tuples, since the sequence is normal. Therefore we may say that the intervals $\left[\frac{qk}{n_1}, \frac{qk+1}{n_1}\right]$ are of type $\left(1, \frac{d}{8}, \frac{2}{n_1}\right)$ relative to S. Now, if a set S is of type (r, C, ϵ) then the set qS is easily seen to be of type $(r, C, q\epsilon)$. Thus the intervals $\left[\frac{qk}{n_1}, \frac{qk+1}{n_1}\right]$ are of type $(r-1, C, q\epsilon)$. Each such interval is of type $\left(1, \frac{d}{8q}, \frac{q+2}{n_1}\right)$ relative to S since the interval $\left[\frac{qk}{n_1}, \frac{qk+1}{n_1}\right]$ is of type $\left(1, \frac{d}{8q}, \frac{2}{n_1}\right)$. Thus if we insure that n_1 is sufficiently large we will have that S is of type (r, C', ϵ), where C' is a new constant which depends only on d. Thus the theorem is proved. Now we have the main theorem of this section:

Theorem 6. There exists an H^r set which is not contained in a countable union of H^{r-1} sets.

Proof. We remark first that every H^{r-1} set is trivially also an H^r set. For if a give H^{r-1} set is defined relative to the intervals $L_1, \ldots, L_{r-1}$ and the sequence n_j^i, $1 \le j \le r-1$, then it is also defined relative to the sequence n_j^i, $1 \le j \le r$ where n_r^i is taken to be a sufficiently rapidly increasing sequence so that the sequence of r–tuples is still normal, and L_r is taken to be the interval $[0, 1]$.

We shall now define a set S, which satisfies the statement of the theorem. Let $a_1^k < a_2^k \cdots < a_r^k$ be a sequence of increasing positive integers such that

$$a_2^k - a_1^k \to \infty, \ldots, a_r^k - a_{r-1}^k \to \infty \tag{56}$$

as k tends to infinity. We define the set S as the set of all x such that for each k, not all the points

$$3^{a_1^k} x, 3^{a_2^k} x, \ldots, 3^{a_r^k} x$$

lie in the open interval $\left[\frac{1}{3}, \frac{2}{3}\right]$. Clearly S is an H^{r-1} set, so that by an application of the Baire category theorem we need only prove the following:

If $x \in S$, and I is an interval containing x, no H^{r-1} set can contain $S \cap I$.

Let x and I be such a point and interval. Every number z can be expanded in the ternary system,[5]

$$z = \sum_{k=1}^{\infty} \frac{\theta(k)}{3^k} \tag{57}$$

[5] In original "∞" missing.

where $\theta(k)$ takes only the values $0, 1, 2$. Unless the expansion of z ends in all zeroes or all twos, then saying that $3^n z$ lies in the interval $\left[\frac{1}{3}, \frac{2}{3}\right]$, is equivalent to asserting that $\theta(n+1) = 1$. So excluding the exceptional numbers mentioned of which there are only a countable number, to say that z lies in S means exactly that for no k do we have

$$\theta(a_1^k + 1) = \theta(a_2^k + 1) = \ldots = \theta(a_r^k + 1) = 1\,. \tag{58}$$

Even if the number z does terminate in zeroes then the above condition implies that z is in the set S. Therefore if we modify our original x by setting all $\theta(k) = 0$ for k sufficiently large, we may still assume that it lies in $S \cap I$, and that it terminates in zeroes. Assume further that x is the midpoint of I, and that I is of length 2δ. Let T be an H^{r-1} set which contains $S \cap I$. Then by our theorem, we know that there exists a constant C, such that the complement of T, which we denote by U, is a set of type $(r-1, C, \epsilon)$ for arbitrary ϵ. Let n be an integer having certain properties which we shall specify later on, and also having the property that for $k \geq a_1^n$, $\theta(k) = 0$. Since U is of type $(r-1, C, \epsilon)$, there exists an interval I_1 with midpoint m_1 and length δ_1 such that

$$|x - m_1| < \epsilon\,, \quad \frac{\delta_1}{|x - m_1|} > C\,, \tag{59}$$

and I_1 is an interval of type $(r-2, C, \epsilon)$ with respect to U. By the remark which was made in the course of the proof of the lemma, we can assume that I_1 lies to the right of x. By choosing ϵ small enough we can insure that the interval I_1 lies completely within our original interval I. As a matter of fact, we can choose ϵ so small that each one of the $r-1$ intervals which we shall choose all lie in the original interval I. Now since $m_1 > x$, it must agree with x in its ternary expansion up to the place

$$\left[\frac{-\log|x - m_1|}{\log 3}\right] - 1\,. \tag{60}$$

Now if we assume that

$$r \in < 3^{-a_1^n - 2}\,, \tag{61}$$

this place will be one beyond which x terminates in zeroes. Define z_1 to be the number which agrees with m_1 up to the place

$$\left[\frac{-\log\frac{1}{2}}{\log 3}\right] + 1\,,$$

beyond which z_1 terminates in zeroes. Let us call the range of places between

$$\left[\frac{-\log|x-m_1|}{\log 3}\right]-1 \qquad \left[\frac{-\log\frac{1}{2}}{\log 3}\right]+1$$

R_1. Now because I_1 is of length δ_1, z_1 lies in the interval I_1. Also because of (59), the number of places in R_1 is at most some bounded quantity which depends only on C. Now there must exist an interval I_2 with midpoint m_2 and of length δ_2, which is of type $(r-2, C, \epsilon)$ with respect to U and such that

$$\frac{\delta_2}{|z_1-m_2|} > C, \quad |z_1-m_2| < \epsilon\,. \tag{62}$$

Again we may assume that[6] z_1 and m_2 will then agree in their ternary expansions up to the place

$$\left[\frac{-\log|z_1-m_2|}{\log 3}\right]-1\,.$$

Define z_2 to agree with m_2 up to the place

$$\left[\frac{-\log\frac{2}{2}}{\log 3}\right]+1$$

and be zero beyond that. Then again, the number z_2 lies in the interval I_2, and this number agrees with the number x in all but possibly two ranges of places R_1 and R_2, the second being defined as all places lying between

$$\left[\frac{-\log|z_1-m_2|}{\log 3}\right]-1 \quad \text{and} \quad \left[\frac{-\log\frac{2}{2}}{\log 3}\right]+1\,.$$

Because of (62), both ranges have at most a bounded number of places in them. Proceeding in this manner, we eventually obtain a number z, which lies in U, and which agrees with x in all but $r-1$ ranges of places each of length bounded by a number depending only on C. Now if n were chosen so large that the difference[7] $a^n_{j+1}-a^n_j$ was always greater than this constant, for all j, it would follow that this number z would lie in S, since the conditions (58) could only occur for at most $r-1$ of the a^n_j. Hence z is in S and

[6] Original contains here "$m_2\ z_1 \cdot z_1$ and m_2".

[7] Original contains typo "$a^n_j\ _1$".

simultaneously in the complement of T, so that we have a contradiction. Therefore the theorem is proved.

5. It is not a priori obvious from the definition that an H^r set for $r > 1$ is a set of measure zero. Of course, once we know that such a set is a set of uniqueness then it follows that it must be a set of measure zero. For $r = 1$ it is clear that an H^1 set is a set of measure zero, for we constantly remove from the unit interval a set of fixed measure which is more and more evenly dispersed, so that we decimate any fixed in the set. For $r > 1$ the situation is clarified by the following theorem:

Theorem 7. Let n_j^k, $1 \leq j \leq r$ be a sequence of r–tuples such that for any r integers a_j, not all zero, the sum $\sum_{j=1}^r a_j n_j^k$ is zero for only a finite number of values of k. Then for f_j, $1 \leq j \leq r$, bounded measurable functions, we have

$$\int_0^1 f_1(n_1^k x)\, f_2(n_2^k x) \ldots f(n_r^k x) dx \to \prod_{j=1}^r \int_0^1 f_j(x) dx \,. \tag{63}$$

Proof. Assume first that f_j are finite exponential polynomials. Then the expression on the left of (63) will consist of the product of the constant terms of f_j if k is sufficiently large. This is so because of the condition on n_j^k. This is exactly what the right side is, so that the theorem is true in this case. In the general case, we may approximate each f_j by an exponential polynomial in the L^1 norm, such that the maximum of each polynomial is not more than the maximum of the corresponding f_j. Then it follows that the right and left hand sides of (63) approach the corresponding expressions for the f_j as the approximation becomes closer. Here we must use the fact that nx is a measure–preserving transformation for all n. Hence the theorem is proved in this case also.

Now, assume that n_j^k are a normal sequence of r–tuples. This means precisely that the $r+1$–tuples $1, n_1^k, \ldots, n_r^k$ satisfy the condition of our theorem. Thus if I is an arbitrary interval and $L_1, \ldots, L_r$ are intervals of length δ each, then the set $S_k = \{x \mid x \in I,\ n_j^k x \in L_j\,,\ \text{all } j\}$ has measure approaching $\delta^r\,|I|$. This we see by applying the theorem to the characteristic functions of these intervals. Thus if E is an H^r set defined by the normal sequence n_j^k and the intervals L_j, then if U is any open set containing E, for some k each interval in U is decimated by the corresponding S_k to the extent that δ^r times the measure of U lies in the complement of E. This clearly implies that E is of measure zero.

Chapter 2

In this section we shall prove a theorem concerning multiple trigonometric series. In the theory of trigonometric series in one variable, the famous Cantor–Lebesgue theorem states that if a trigonometric series converges on a set of positive measure, then the coefficients must tend to zero. No complete analogue of this theorem can exist, since we must first specify a particular method of summation when we are dealing with the case of several variables. Shapiro in [9] has obtained results which would give an almost complete answer to the question of the uniqueness of multiple series, if some kind of analogue could be proved. It appears that no results along this line have been published. The theorem which we shall prove is much weaker than the corresponding Cantor–Lebesgue theorem. The terms of a multiple series are in one to one correspondence with the set of all r–tuples of integers. A method of summation is described by a sequence E_n, of finite sets of r–tuples of integers, such that E_n is contained in E_{n+1} and the union of E_n consists of all r–tuples. At the n^{th} stage of the summation we consider the sum of all terms corresponding to r–tuples contained in E_n. Then we have the following definition.

Definition. A method of summation E_n, for a trigonometric series in r variables, is said to be regular if and only if there exists a constant K such that for every lattice point $a = (a_1, \ldots, a_r)$, there exists n_0 such that a belongs to E_{n_0} and the maximum of the absolute value of all coordinates of all lattice points in E_{n_0} is smaller than $K \operatorname*{Max}_{1 \le i \le r} |a_i|$.

It is clear that the usual methods of spherical and cubical summation are regular. Now we have our theorem.

Theorem. Let

$$(1) \qquad \sum_{j_1,\ldots,j_r=-\infty}^{\infty} c_{j_1\ldots j_r}\, e^{i\,(j_1x_1+\ldots+j_rx_r)}$$

be a trigonometric series in r variables, which converges almost everywhere by a regular method of summation. Then for all $\gamma > 1$, there exists $b > 0$, such that

$$(2) \qquad |c_{j_1\ldots j_r}| \le b\,\gamma^{|j|} \quad \text{where } \ |j| = \underset{1\le i\le r}{\mathrm{Max}}\,|j_i|\,.$$

Proof. The variables $x_1,\ldots,x_r$ range over the reals modulo 2π, which we consider as being canonically identified with the reals modulo 1.

By applying Egoroff's theorem to the partial sums

$$s_n = \sum_{j\in E_n} C_j\, e^{i\,j\cdot x}\,,$$

where $j = (j_1,\ldots,j_r)$, $x = (x_1,\ldots,x_r)$, and $j\cdot x = j_1x_1 + \ldots + j_rx_r$, we find that for any $\epsilon > 0$, there exists a closed set G, of measure greater than $1-\epsilon$, such that s_n are uniformly bounded in absolute value by some constant B, for all x in G. Let $a = (a_1,\ldots,a_r)$ be a lattice point. Let E_{n_0} be the set which corresponds to it in the definition of regularity. Set $N = 2 \underset{\substack{j\in E_n\\ 1\le i\le r}}{\mathrm{Max}} |j_i| + 2$. Then $N \le K \underset{1\le i\le r}{\mathrm{Max}} |a_i|$. The set G can be visualized as being contained in the r–fold direct product T^r of the unit circle. Because the measure of G is more than $1-\epsilon$, it follows that there is a set G', in the space T^{r-1} of measure greater than $1-\epsilon^{1/2}$ such that if $(x_2,\ldots,x_r)$ belongs to G', the set of all x_1 such that $(x_1,x_2,\ldots,x_r)$ belongs to G is of measure also greater than $1-\epsilon^{1/2}$. We now need the following simple lemma.

Lemma. Let S be a set on the unit circle of measure greater than δ. Then for any integer $k > 0$, there exist k points in S, $z_1,\ldots,z_k$, such that the distance between any two of them is greater than $\frac{\delta}{k}$.

Proof. Let z_1 be any point in S. Then for any $\eta > 0$, let z_2 be a point in S, if it exists, such that $z_2 > z_1 + \frac{\delta}{k}$ and the measure of the set of all points in S lying in the interval $[z_1,z_2]$ is smaller than $\frac{\delta}{k}+\eta$. Here $[z_1,z_2]$ denotes the set of all points lying to the right of z_1 and to the left of z_2. Choose z_3 to be a point in S, if it exists, which has the same relationship to z_2, as z_2 had to z_1. Now assume that the process terminates after m steps, that is, no z_{m+1}

can be found having the appropriate property. This means that the only elements of S lying in the interval $[z_m, z_1]$ are contained in $\left[z_m, z_m + \frac{\delta}{k}\right]$. Thus it follows that the measure of S is smaller than $m\left(\frac{\delta}{k} + \eta\right)$, unless $m = k$. This shows that $z_1, \ldots, z_k$ can be so chosen in S that they are placed cyclically around the circle, and the distance between two consecutive points is greater than $\frac{\delta}{k}$. This prove the lemma.

Now consider $s_{n_0}(x_1, \ldots, x_r)$ as a function of x_1 alone, holding $x_2, \ldots, x_r$ fixed. By multiplying $s_{n_0}(x_1, \ldots, x_n)$ by a suitable power of e^{ix}, we obtain a trigonometric polynomial $\bar{s}_{n_0}(x_1, \ldots, x_r)$ which contains only positive powers of e^{ix_1} and is of degree smaller than N. Set $\zeta = e^{x_1}$, so that $\bar{s}_{n_0}$ can be regarded as a polynomial of degree smaller than N in the complex variable ζ. For each $(x_2, \ldots, x_n)$ in G', $\bar{s}_{n_0}$ is still bounded in absolute value by B on the points $(x_1, \ldots, x_n)$ where x_1 belongs to a set S of measure greater than $1 - \epsilon^{1/2}$. Thus by the lemma there are N points $z_1, \ldots, z_N$ in S, such that the distance between any two of them is greater than $\lambda = \frac{1-\epsilon^{1/2}}{N}$. Set $\zeta_k = e^{iz_k}$. Now, if we apply the Lagrange interpolation formula to $\bar{s}_{n_0}$, it follows that

$$\bar{s}_{n_0}(\zeta) = \sum_{k=1}^{N} \bar{s}_{n_0}(\zeta_k) \frac{\prod_{j\neq k}(\zeta - \zeta_j)}{\prod_{j\neq k}(\zeta_k - \zeta_j)}. \tag{3}$$

The denominator of each term in (3) obviously exceeds the corresponding product where the ζ_j are m^{th} roots of unity, and $m = \left[\frac{1}{\lambda}\right] + 1$. Now, the product $\prod_\theta (1 - \theta)$, where the product is extended over all the m^{th} roots of unity different from 1, equals m. On the other hand, in the denominator which occurs in (3), only $N - 1$ terms occur in the product. Thus there are $m - N$ terms which are additional, and each one of these terms is bounded in absolute value by 2. Hence we obtain

$$\prod_{j\neq k}(\zeta_k - \zeta_j) \geq \frac{m}{2^{m-N}} \geq \frac{N}{2^{\delta N}} \tag{4}$$

where δ is a quantity which goes to zero with ϵ. Now we estimate the numerators which occur in (3). We have

$$\left|\prod_{j\neq k}(\zeta - \zeta_j)\right| \leq \left|\prod_\theta (1 - \theta)\right|, \tag{5}$$

where the product is now extended over those $N-1$ m^{th} roots of unity which are furthest from 1. The same product extended over the remaining roots is bounded from below by

$$\left|\prod_{k=1}^{\delta m}(1-e^{\frac{2\pi ik}{m}})^2\right| \geq \frac{1}{2^{2\,\delta m}}\left(\frac{1}{m}\cdot\frac{2}{m}\cdots\frac{[\delta m]}{m}\right)^2 , \tag{6}$$

where δ as before tends to zero with ϵ. The right hand side of (6) can be estimated by Stirling's formula. It follows easily that the right side of (6) is greater than $2^{-\delta N}$, where δ here is again a quantity which tends to zero with ϵ. Hence it follows from (5) that

$$\left|\prod_{j\neq k}(\zeta-\zeta_j)\right| \leq N\,2^{\delta N} . \tag{7}$$

Now $|\bar{s}_{n_0}(\zeta_j)| \leq B$, so that from (3) it follows that $\bar{s}_{n_0}(\zeta)$ is bounded by $b\,\gamma^N$, where b depends only on B, and γ is a number which tends to one as ϵ tends to zero. Thus it follows that the coefficients of $\bar{s}_{n_0}(\zeta)$ are bounded by $b\,\gamma^N$. These coefficients are functions of $x_2,\ldots,x_n$, and are bounded by $b\,\gamma^N$ whenever $(x_2,\ldots,x_n)$ lies in G', which is a set of measure greater than $1-\epsilon^{1/2}$. Now if we apply the same argument as above to each of these functions we find that they in turn are bounded by $b_1\,\gamma_1^N$ where γ_1 also tends to one as ϵ tends to zero. If we apply this process r times we eventually obtain the result that the coefficient $|c_a| \leq b\,\gamma^N$ for suitable choices of b and γ. Since $N \leq K \operatorname*{Max}_{1\leq i\leq r}|a_i|$, it follows that the theorem holds after a suitable redefinition of γ and b.

2. Because our theorem allows the coefficients to grow quite rapidly, it may be of some interest to construct an example of a series where the coefficients grow at a reasonably rapid pace. However, we cannot prove that our theorem is a best possible result. Given any function $\omega(n)$ which tends to zero as n tends to infinity, we shall now show that there exists a double trigonometric series with the property that it converges almost everywhere, and yet for some sequence of coefficients m_k and n_k we have $|C_{m_k,n_k}| > \omega(N_k)\,N_k$ where $N_k = \operatorname{Max}(m_k,n_k)$. Let $K_n(t)$ denote the Fejer kernel, that is,

$$K_n(t) = \frac{1}{2\,(n+1)}\left(\frac{\sin(n+1)\frac{1}{2}\,t}{\sin\frac{1}{2}\,t}\right)^2 . \tag{8}$$

Then constant coefficient of $K_n(t)$ is 1, and $K_n(t)$ is a trigonometric polynomial of degree n. Let n_k be an increasing sequence of integers, and d_k positive numbers such that $\sum_{k=1}^{\infty} \frac{d_k}{n_k}$ converges and $d_k > \omega(n_k)\, n_k$. Then it is clear that the series $\sum_{k=1}^{\infty} d_k\, K_{n_k}(x)$ converges absolutely at all points except for $x = 0$. Hence it follows that the corresponding double series,

$$\sum_{k=1}^{\infty} d_k\, e^{in_k y}\, K_{n_k}(x) \tag{9}$$

converges at all points where $x \neq 0$, if we sum the series by the method of square summation. More precisely, we take as the n^{th} partial sum all those terms of degree not greater than n in either x or y. On the other hand it is quite clear that the coefficients tend to infinity rapidly as was indicated.

Chapter 3

1. The first example of a trigonometric series which converges to zero almost everywhere, and which is not identically zero, was given by Mensov. He constructed a measure on a perfect set of Lebesgue measure zero and considered its Fourier–Stieltjes series. Riemann's localization theorem then tells us that such a series converges to zero on the complement of the support of the measure provided only that the coefficients of this series tend to zero. An exposition of Mensov's result can be found either in [2] or [13]. The particular set of multiplicity, or M–set, which is thus constructed is very similar to the usual Cantor set. The Cantor set is constructed by removing from the unit interval the middle third. From each of the remaining segments one removes the middle third and so on. What remains is precisely the Cantor set. If at each stage, instead of using the fraction one third, we use the number α_i, where $0 < \alpha_i < 1$, we obtain a more general class of sets. Mensov's example is exactly when the α_i tend to zero, while $\sum \alpha_i = \infty$. Salem [8], investigated the case of equal α_i and proved the remarkable theorem that unless this common ratio belongs to a certain denumerable class of algebraic numbers, the set is an M–set, whereas in the contrary case it is a U–set. In his proof he used a formula for the Fourier–Stieltjes coefficients of a certain measure which originated with Carleman. This formula is valid in the case of Mensov's example, but the proofs of Mensov's result referred to above do not use this explicit formula. Therefore, it seems of interest to give a proof which unifies the treatment of these two cases, and seems to be conceptually much clearer. We begin by deriving the formula which was referred to.

Let ξ_i be a sequence of real numbers such that $0 < \xi_i < \frac{1}{2}$. Let μ_i be the measure which assigns mass $\frac{1}{2}$ to the point 0 and mass $\frac{1}{2}$ to the point $\xi_1 \ldots \xi_{i-1}(1-\xi_i)$. Now set $\nu_j = \mu_1 \star \mu_2 \ldots \star \mu_j$ where $\star$ denotes convolution. Set $\alpha_i = 1 - 2\xi_i$, and let S_j be the collection of intervals which remain after j dissections as described above, using the ratios α_i. Then clearly S_j consists of 2^j intervals and the measure ν_j assigns mass $\frac{1}{2^j}$ to the left hand endpoint of each of these intervals. As j tends to infinity, ν_j tends weakly to a measure ν which has its support on the set $S = \cap_{j=1}^{\infty} S_j$. This measure has been constructed entirely analogously to the ordinary Cantor function. Now let C_n be the Fourier coefficients of ν, that is,

$$C_n = \int_0^1 e^{-2\pi inx}\, d\nu(x)\,. \tag{1}$$

Since ν is an infinite convolution, C_n is the infinite product of the corresponding coefficients for each of the measures μ_k. So[1]

$$C_n = \prod_{k=1}^{\infty} \left(\frac{1 + e^{-2\pi in\xi_1 \ldots \xi_{k-1}(1-\xi_k)}}{2} \right) \tag{2}$$

$$= \prod_{k=1}^{\infty} e^{-\pi in\xi_1 \ldots \xi_{k-1}(1-\xi_k)} \prod_{k=1}^{\infty} \cos \pi n\xi_1 \ldots \xi_{k-1}(1-\xi_k)$$

$$= (-1)^n \prod_{k=1}^{\infty} \cos \pi n\xi_1 \ldots \xi_{k-1}(1-\xi_k)\,, \tag{3}$$

Since the first factor is obviously $(-1)^n$. Now under the assumption that ξ_k tends to $\frac{1}{2}$, we shall prove that the product (3) tends to zero as n tends to infinity. Set $\theta_k = \xi_1 \ldots \xi_{k-1}(1-\xi_k)$. Then clearly $\frac{\theta_{k+1}}{\theta_k}$ tends to $1/2$. In the case where $\xi_i = \frac{1}{2}$, then clearly the measure ν_j consists of 2^j equally spaced masses of mass $\frac{1}{2^j}$ each. The Fourier coefficients of this measure are easily computed and we may express the result in the following equation.

$$\cos x \, \cos \frac{x}{2} \ldots \cos \frac{x}{2^k} = \frac{\sin 2x}{2^{k+1} \, \sin \frac{x}{2^k}}\,. \tag{4}$$

This formula may also be easily verified by induction on k. Now if we assume that $\frac{x}{2^k} < \frac{1}{10}$, it follows that

$$\left| \cos x \, \cos \frac{x}{2} \ldots \cos \frac{x}{2^k} \right| \leq \frac{1}{x}\,. \tag{5}$$

Let $A > 0$ and m an integer such that $\frac{3A}{2^m} < \frac{1}{10}$. Then the expression

$$\left| \prod_{\ell=0}^{m} \cos \frac{x}{2^\ell} - \prod_{\ell=0}^{m} \cos \frac{\theta_{k+\ell}}{\theta_k}\, x \right| \tag{6}$$

[1] In original "$(-1)^n$" missing.

where x lies between A and $3A$ approaches 0 uniformly in x as k tends to infinity and m is kept fixed. This is true because the quantity $\frac{\theta_{k+\ell}}{\theta_k}$ approaches $\frac{1}{2}$ for fixed ℓ as k tends to infinity. Now if n is large enough there is a value of k such that $A < n\,\theta_k < 3A$ and such that (6) is smaller than a positive number ϵ, with $n\theta_k$ substituted for x. By (5) it follows that

$$\left|\prod_{\ell=0}^{m} \cos \frac{n\,\theta_k}{2^\ell}\right| \leq \frac{1}{A}\,. \tag{7}$$

Thus a partial product of (3) is smaller than $\frac{1}{A} + \epsilon$, and since all the terms of (3) are bounded by 1 in absolute value (3) itself is bounded by $\frac{1}{A} + \epsilon$. Since A can be taken larger and larger as n tends to infinity, it follows that (3) tends to zero.

2. A closed M–set such that there exists a measure with support contained in that set, whose Fourier coefficients tend to zero is called an M–set of restricted type. Pyatetskii–Shapiro has shown the existence of M–sets which are not of restricted type. His proof is non–constructive and does not yield an example of a trigonometrical series which converges to zero almost everywhere and yet which is not a Fourier–Stieltjes series. In this section we shall give an example of such a series using a modification of the construction given by Salem in [7]. We first need the following result of Wiener, (see Zygmund, [13], p. 221).

Lemma. If μ is a measure on $[0,1]$ of total variation V, let $\eta(\delta)$ denote the maximum value of the total measure contained in an interval of length δ. Then if $d\mu$ has the Fourier series $d\mu \sim \sum C_n e^{2\pi i n x}$ we have

$$\frac{1}{2N}\sum_{n=-N}^{N} |C_n|^2 \leq A \cdot V \cdot \eta\left(\frac{1}{N}\right)\,, \tag{8}$$

where A is an absolute constant.

Also we need a transformation which was first used by Wiener and Wintner in a paper dealing with this subject [12]. Let $\xi(x)$ be a function defined on $\left[-\frac{1}{2}, 0\right]$ which is quadratic, increasing, and $\xi\left(-\frac{1}{2}\right) = -\frac{1}{2}$, $\xi(0) = 0$. Extend ξ to the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ by requiring it to be an odd function. Then as is proved in [12], we have the following properties of $\xi(x)$. Let[2]

$$e^{-2\pi i m\xi(x)} = \sum_{n=-\infty}^{\infty} \lambda_{n,m}\, e^{-2\pi i n x}\,.$$

Then

$$|\lambda_{n,m}| \le C\,|m|^{-1/2} \quad \text{for all } m,$$

$$|\lambda_{n,m}| \le C\,|n|^{-2} \quad \text{for all } |n| > 2\,|m|,$$

where C is some absolute constant.

Assume that the measure μ has Fourier coefficients C_n, and $\sum_{n=-\infty}^{\infty} |C_n| \le B$. Then the measure $\mu(\xi^{-1}(x))$ is a measure whose Fourier coefficients we denote by $\tilde{C}_m$. We have then

$$\tilde{C}_m = \int_0^1 e^{-2\pi i m x}\, d\mu(\xi^{-1}(x)) = \int_0^1 e^{-2\pi i m\xi(x)}\, d\mu(x)\,. \tag{10}$$

Now the Fourier series for $e^{-2\pi i m\xi(x)}$ converges absolutely, so that in (10) we may integrate term by term to obtain

$$\tilde{C}_m = \sum_n \lambda_{n,m} C_m = \sum_{|n|\ge 2\,|m|} \lambda_{n,m} C_n + \sum_{|n|<2\,|m|} \lambda_{n,m} C_n\,. \tag{11}$$

The first sum in (11) is less than $\alpha_1\, B\,|m|^{-1/2}$, while the second is less than $\frac{\alpha_2}{m}\, V$, where V is the total variation of μ and α_1 and α_2 are constants.

Let $\Omega(N)$ be an arbitrary positive function tending monotonically to infinity. We now propose to construct a set of measures μ_k with the following properties:

a) the total variation of μ_k is not greater than k,

b) if μ_k has the Fourier series $\sum_{n=-\infty}^{\infty} d_{n,k}\, e^{2\pi i n x}$, then

$$\sum_{n=-\infty}^{\infty} |d_{n,k}| \le \omega(k)\,, \tag{12}$$

where $\omega(k)$ tends monotonically to infinity.

c) For each k, let $p = p(k)$ be a positive integer such that $\frac{\alpha_2\, k}{q} < \frac{\Omega(q)}{q^{1/2}}$ for all $q \ge p$, and $p(k+1) > p(k)$. We also demand that $\omega(k)$ satisfy $\alpha_1\, \omega(k) < \Omega(p)$. This can be achieved by choosing p large enough. Set

$$\tilde{d}_{m,k} = \sum_n \lambda_{n,m}\, d_{n,k}\,. \tag{13}$$

[2] In original "λ" is missing.

Then (9), (11) and a) imply that there exists M and ϵ, such that

$$|d_{n,k} - d_{n,k+1}| < \epsilon \quad \text{for } \ |n| < M \tag{14}$$

implies

$$|\tilde{d}_{m,k} - \tilde{d}_{m,k+1}| < \frac{\Omega(|m|)}{2^k\,|m|^{1/2}} \quad \text{for } \ |m| \leq p(k+1)\,. \tag{15}$$

This is so because in the second summand of (11), $\sum_{|n|>2\,|m|} \lambda_{n,m}$ convergent sequence while $d_{n,k}$ are bounded by k. We now assume that μ_k satisfy condition (14).

Under these conditions let m be an integer, k_1 such that $p(k_1) \leq |m| \leq p(k_1+1)$. Then by (11),

$$|\tilde{d}_{m,k_1}| \leq \frac{\alpha_1\,\omega(k_1)}{|m|^{1/2}} + \frac{\alpha_2\,k_1}{|m|} < \frac{2\,\Omega(|m|)}{|m|^{1/2}}\,. \tag{16}$$

On the other hand, (15) implies that for all $k > k_1$ we have

$$|\tilde{d}_{m,k_1} - \tilde{d}_{m,k}| < \frac{\Omega(|m|)}{|m|^{1/2}} \tag{17}$$

so that $\tilde{d}_m = \operatorname*{Lim}_{k\to\infty} \tilde{d}_{m,k}$ is such that

$$|\tilde{d}_m| < \frac{3\,\Omega(|m|)}{|m|^{1/2}}\,. \tag{18}$$

We also observe that since (16) holds for $|m| > |p|$, it follows that $|\tilde{d}_{m,k}| \leq \frac{3\,\Omega(|m|)}{|m|^{1/2}}$. In particular $\tilde{d}_{m,k}$ are uniformly bounded if[3] $\Omega(N)$ tends to infinity slowly enough.

Now we shall proceed to construct a sequence of sets S_k. let $d_1, d_2, \ldots$ be a sequence of integers tending to infinity. Let S_1 be the union of all intervals of the form $\left[\frac{r}{d_1}, \frac{r}{d_1} + \frac{1}{4\,d_1}\right]$ for some integer r. Then the measure of S_1 is one–fourth. S_1 has the property that the set $S_1 + S_1$, which means all points of the form $x+y$ where x and y belong to S_1, is the union of all intervals of the form $\left[\frac{r}{d_1}, \frac{r}{d_1} + \frac{1}{2\,d_1}\right]$ and is of measure $\frac{1}{2}$. S_1 thus consists of the unit interval with d_1 intervals removed. Let S_2 consist of all the intervals of S_1, from each one of which d_2 intervals have been removed in precisely

[3] In original "Ω" missing.

the same manner. Then the measure of S_2 will be $\frac{1}{16}$ while that of $S_2 + S_2$ will be $\frac{1}{4}$. We proceed in this manner with $d_3, d_4, \ldots$, to construct the sets S_k. The measure of S_k is $\frac{1}{4^k}$ while that of $S_k + S_k$ is $\frac{1}{2^k}$.

Now measures ν_k will be defined inductively. Each interval of S_k has the property that ν_k assigns to the first half of it a multiple of Lebesgue measure and to the second half another multiple of Lebesgue measure. Now each half interval of S_k contains certain intervals of S_{k+1}. ν_{k+1} assigns to each of these intervals identical distributions of mass so that the sum of the measures of these intervals is the same as that of the original half interval. However this is done in such a manner that the *total* variation is $\frac{k+1}{k}$ times that of the original interval. We first observe that ν_{k+1} will have any finite number of its Fourier coefficients arbitrarily close to the corresponding coefficients of ν_k provided d_{k+1} is chosen large enough. For, as d_{k+1} approaches infinity it is obvious that ν_{k+1} approaches ν_k weakly. In particular it follows that if $F_{k+1}(x)$ and $F_k(x)$, defined as the integrals of ν_{k+1} and ν_k respectively, i.e.,

$$F_{k+1}(x) = \int_0^x d\nu_{k+1}\,, \quad F_k(x) = \int_0^x d\nu_k\,,$$

then $|F_{k+1}(x) - F_k(x)|$ can be made uniformly small. We can thus assume that all the $F_k(x)$ are uniformly bounded. This remark will be needed presently.

The quantity $\operatorname*{Max}_{\delta>0} \delta\,\eta(\delta)$ also does not depend upon the choice of the sequence d_k. This is so because this quantity is the maximum "density" of the measure and only depends upon the Lebesgue measure of S_k. Now set $\mu_k = \nu_k \star \nu_k$. Then by the lemma we have

$$\sum_{n=-N}^{N} |d_{n,k}| \le A \cdot V \cdot N \cdot \eta\left(\frac{1}{N}\right) . \tag{19}$$

Thus it follows that by choosing d_k large enough we can be certain that μ_k satisfy all the properties mentioned above.

The support of μ_k is contained in $S_k + S_k$. The intersection of these sets is a closed set of measure zero. Now I claim that the series $\sum_n \tilde{d}_n e^{2\pi i n x}$ converges to zero everywhere outside the transform of this set by the transformation $\xi(x)$. For it is certainly true that the series $\sum_n \tilde{d}_{n,k} e^{2\pi i n x}$ converges to zero outside the transform of $S_k + S_k$. Now the assertion follows from the following lemma:

Lemma. If the sequence of trigonometrical series $\sum_n \tilde{d}_{n,k} e^{2\pi i n x}$ converges to zero in an open interval, and $d_{n,k}$ are uniformly bounded, if $\tilde{d}_n = \operatorname*{Lim}_{k\to\infty} \tilde{d}_{n,k}$

exist, then $\sum_n \tilde{d}_n e^{2\pi inx}$ converges to zero in that interval, provided $\operatorname*{Lim}_{k\to\infty} \tilde{d}_n = 0$.

Proof. This is a trivial consequence of formal multiplication. If $\gamma(x)$ converges rapidly, is non–zero at a point of the interval, zero outside it, then the formal product of $\gamma(x)$ with each of the series $\sum_n \tilde{d}_{n,k} e^{2\pi inx}$ is identically zero. The hypotheses imply that the product of $\gamma(x)$ and $\sum_n \tilde{d}_n e^{2\pi inx}$ is zero. Therefore the conclusion of the lemma follows from well–known theorems concerning formal multiplication.

Since $\xi(x)$ is piecewise differentiable it follows that $\sum_n \tilde{d}_n e^{2\pi inx}$ converges almost everywhere to zero. It remains to prove that $\sum_n \tilde{d}_n e^{2\pi inx}$ is not a Fourier–Stieltjes series. Let $F_k(x)$ be the integral of the measure $\nu_k(x)$, that is, $\nu_k(x) = dF_k(x)$. All the functions $F_k(x)$ are uniformly bounded provided we take the sequence d_k growing quickly enough. The functions $F_k(x)$ converge boundedly to a function $F(x)$ of infinite variation. The Fourier series of $F(x)$ is the integrated series of $\sum_n \tilde{d}_n e^{2\pi inx}$. Since $F(x)$ is of infinite variation it follows that $\sum_n \tilde{d}_n e^{2\pi inx}$ is not a Fourier–Stieltjes series.

Thus we have shown that there exist series which converge to zero almost everywhere and which are not Fourier–Stieltjes series. Furthermore the coefficients can be made smaller than $\frac{\Omega(|m|)}{|m|^{1/2}}$ where $\Omega(m)$ is any function tending to infinity.

3. In the preceding section we quoted a theorem of Wiener concerning the Fourier series of measures. In this section we shall present a generalization of the theorem to several dimensions by means of a rather simple proof which seems more lucid than the one in [13]. Also a generalization in a different direction will be given.

Theorem. Let μ be a measure on the n–dimensional torus T^n, with Fourier series

$$d\mu \sim \sum_j C_j \, e^{2\pi i\,(j\cdot x)} \tag{20}$$

where j ranges over all n–dimensional lattice points $j = (j_1, \ldots, j_n)$, $x = (x_1, \ldots, x_n)$ and $j \cdot x = j_1 x_1 + \ldots + j_n x_n$. Let S_p be an expanding sequence of rectangles in the space of all lattice points, and $|S_p|$ denote the number of lattice points contained in S_p. Then

$$\operatorname*{Lim}_{p\to\infty} \frac{1}{|S_p|} \sum_{j\in S_p} |C_j|^2 \tag{21}$$

exists and equals $\sum_Q |\mu(Q)|^2$ where Q ranges over all points having non–zero mass.

Proof. We assume S_p is the set of all lattice points j, such that $-m_{p,k} \leq j_k \leq m_{p,k}$, $1 \leq k \leq n$. We assume that $m_{p,k}$ tends to infinity as p tends to infinity. Set

$$
\begin{aligned}
f_p(x_1, \ldots, x_n) &= \frac{1}{|S_p|} \sum_{j \in S_p} e^{2\pi i\,(j \cdot x)} \\
&= \prod_{k=1}^{n} \frac{1}{2\,m_{p,k}+1} \sum_{-m_{p,k} \leq j_k \leq m_{p,k}} e^{2\pi i j_k x_k}\,.
\end{aligned} \tag{22}
$$

It is clear that $|f_p| \leq 1$, and since f_p is a product of Dirichlet kernels, it follows from the formula for the sum of a geometric series that f_p converges to 0 everywhere except at $(0, 0, \ldots, 0)$ where it is 1. It is also clear that

$$
\frac{1}{|S_p|} \sum_{j \in S_p} |C_j|^2 = \iint\limits_{T^n \times T^n} f_p(x-y)\, d\mu(x) d\bar{\mu}(y)\,, \tag{23}
$$

as may be seen by direct substitution. By the Lebesgue monotone convergence theorem it follows that the limit of (21) exists and equals

$$
\iint\limits_{T^n \times T^n} \delta(x-y)\, d\mu(x) d\bar{\mu}(y)\,, \tag{24}
$$

where $\delta(x)$ is zero for $x \neq 0$, $\delta(0) = 1$. This quantity by Fubini's theorem is clearly $\sum_Q |\mu(Q)|^2$.

Now we shall restrict ourselves to the case of one variable. Let μ be a measure which has no point masses, i.e., a continuous measure. If $d\mu$ has the Fourier series $\sum_n C_n\, e^{2\pi i n x}$, let us examine the quantity $\frac{1}{N} \sum_{k=1}^{N} |C_{n_k}|^2$, where n_k is some increasing sequence of integers. Then this expression is equal to

$$
\frac{1}{N} \iint\limits_{T^1 \times T^1} f_N(x-y)\, d\mu(x) d\bar{\mu}(y)\,, \tag{25}
$$

where

$$
F_N(x) = \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i n_k x}\,.
$$

As before the $f_N(x)$ are unformly bounded by 1 in absolute value. If the n_k are a polynomial sequence, i.e., n_k is a polynomial in k, then well–known results of Weyl [11], tell us that $f_N(x)$ approaches zero for all irrational values of x. Hence (25) approaches (24) where $\delta(x)$ is now a function which is zero at all irrational points. Since there are only a countable number of rational points and μ is assumed continuous we have the following result:

Theorem. If μ is a continuous measure, then

$$\frac{1}{N}\sum_{k=1}^{N}|C_{n_k}|^2$$

approaches zero if n_k is a polynomial sequence.

Chapter 4

Green's theorem in two dimensions says that if C is a simple closed curve bounding the region Q, if $A(x,y)$ and $B(x,y)$ are continuous functions having derivatives, then under suitable further conditions we have,

$$\int_C Ad + Bdy = \iint\limits_Q \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dxdy\,, \tag{1}$$

where the line integral is taken in a positive sense around the curve C. In [3], Bochner investigated under which conditions (1) holds. There it was shown that if A and B have certain regularity properties and if the integrand on the right of (1) behaves well, then (1) does hold. Here we shall prove (1) under what may be regarded as the weakest possible hypotheses. This question was also treated by Shapiro in [10], and though he assumes certain regularity of A and B, namely, the existence of the differential, he allows certain exceptional sets which we cannot allow. The proof of our theorem is modeled after the proof of the Looman–Mensov theorem as contained, for example, in [6]. We will not deal with the topological difficulties involved so that our theorem will only treat the case in which Q is a rectangle.

Theorem. Let $A(x,y)$ and $B(x,y)$ be two functions defined on the rectangle Q, and continuous on the closure of Q. Assume further that the partial derivatives

$$\frac{\partial A}{\partial x} \quad \frac{\partial A}{\partial y} \quad \frac{\partial B}{\partial x} \quad \frac{\partial B}{\partial y}$$

exist everywhere in the interior of Q, except perhaps at a countable number of points. If $\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}$ is Lebesgue integrable in the rectangle Q, then (1) holds.

Proof. We first need a lemma which is contained in [6]. In the following, the word "rectangle" means the direct product of two intervals.

Lemma. Let $w(x,y)$ be a function defined in a square Q, such that $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ exist almost everywhere in Q. Let F be a closed, non–empty set in Q, and N a finite constant such that

$$
\begin{aligned}
&w(x, y+k) - w(x,y)| \leq N\,|k|\,, \\
&w(x+h, y) - w(x,y)| \leq N\,|h|\,,
\end{aligned} \tag{2}
$$

whenever (x,y) belongs to F and $(x+h,y)$ and $(x,y+k)$ belong to Q. Let J be the smallest rectangle containing F and assume J is the product of (a_1,b_1) and (a_2,b_2). Then

$$
\begin{aligned}
&\left| \iint\limits_F \frac{\partial w}{\partial x} dxdy - \int_{a_2}^{b_2} [w(b_1,y) - w(a_1,y)]dy \right| \leq 5\,N \cdot |Q-F|\,, \\
&\left| \iint\limits_F \frac{\partial w}{\partial y} dxdy - \int_{a_1}^{b_1} [w(x,b_2) - w(x,a_2)]dx \right| \leq 5\,N \cdot |Q-F|\,.
\end{aligned} \tag{3}
$$

It is clearly enough to prove the theorem for all rectangles Q' properly contained in Q, for, in this case, we can approximate Q from the interior by a sequence of such rectangles, and for each of which (1) holds. Now by the Lebesgue integrability of $\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}$ the right side approaches the corresponding integral over Q, and by the continuity of A and B so does the left side. Hence we may assume in the original statement of the theorem that A and B are actually defined in a neighborhood of Q where they are continuous and have derivatives at all but a countable number of points. Now, let E be the set of all points P in Q, such that (1) holds for integrations taken over all rectangles in a sufficiently small neighborhood of P. We shall show that E is all of Q. Let $F = Q - E$. Since E is obviously open, F is closed. Let H_n be the set of all points (x,y) in Q, such that $\left|\frac{A(x+h,y)-A(x,y)}{h}\right|$, $\left|\frac{A(x,y+k)-A(x,y)}{h}\right|$, $\left|\frac{B(x+h,y)-B(x,y)}{h}\right|$, $\left|\frac{B(x,y+k)-B(x,y)}{h}\right|$ are all bounded by n whenever $|h| \leq \frac{1}{n}$, $|k| \leq \frac{1}{n}$, and all the quantities involved are defined. Clearly Q, with a countable number of exceptions, is the union of all these closed sets H_n. Therefore by the Baire category theorem, since F is also closed, either F contains an isolated point in the interior of Q, or there is some square I, in the interior of Q, such that $I \cap F$ is non–empty and is contained in H_N for some N. If a rectangle lies completely in E, then the Heine–Borel theorem shows that (1) holds for it. Hence we see that isolated points of F cannot occur and so the second alternative holds. Then the conditions of the lemma hold and we have

$$\left| \int_J A dx + B dy - \iint\limits_{J\cap F} \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx dy \right| \leq 10\, N \cdot |J - F| \,. \tag{4}$$

Here J is the smallest rectangle containing $I \cap F$ and ∂J denotes the boundary of J. The set $I - J$ is a finite union of rectangles each of which can be approximated from the interior by rectangles wholly contained in E. Hence (1) holds for the set $I - J$, where the line integral is taken around its boundary in the positive sense. Thus we have

$$\int_{\partial(I-J)} A dx + B dy = \iint\limits_{I-J} \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx dy \,. \tag{5}$$

So by (4) we have

$$\left| \int_{\partial I} A dx + B dy - \iint\limits_{(J\cap F)\cup(I-j)} \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx dy \right| \leq 10\, N \cdot |J - F| \,. \tag{6}$$

From (6) it follows that

$$\left| \int_{\partial I} A dx + B dy \right| \leq 10\, N \cdot |I| + \iint\limits_{I} \left| \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right| dx dy \,. \tag{7}$$

Now (7) holds equally well for any square I' contained in I. Thus the set function which assings to every square I' the quantity[1] $\int_{\partial I'} A dx + B dy$ is dominated by an absolutely continuous measure and hence extends to an absolutely continuous measure defined on all Borel sets. Thus, it is given by the indefinite integral of some function. If we can then show that the derivative of this measure in the sense of averages taken over smaller and smaller squares is equal to $\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}$ almost everywhere, then we will know that (1) holds for all rectangles in I and hence that $I \cap F$ is empty, which is a contradiction. How the derivative of the measure at almost all points not in F is clearly $\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}$. This is merely the theorem concerning the differentiation of indefinite integrals, since at such points (1) does hold. On the other hand, if P is a point of density of F, then for[2] a sufficiently small

[1] In original the $'$ is missing.
[2] In original "a" missing.

square I around P, we have that $\frac{|I-F|}{|I|}$ is arbitrarily small. Thus from (6) it follows that $\frac{1}{|I|}\int_{\partial I} Adx + Bdy$ approaches $\frac{1}{|I|}\iint\limits_{(J\cap F)\cup(I-j)} \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dxdy.$ This last quantity approaches the derivative of the integral taken with respect to the sets $(J\cap F)\cup(I-J)$. These are a regular sequence of sets in the sense of [6] p. 106, since $\frac{|(J\cap F)\cup(I-J)|}{|I|}$ tends to 1, and so this derivative is equal to $\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}$ at almost all points of F. Thus the measure is the desired indefinite integral and so (1) holds at all points P. Now by the Heine–Borel theorem it follows that (1) holds for the rectangle Q itself.

We may easily generalize this result to any number of dimensions.

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Notes

Paul Cohen's thesis has remained unpublished, except for the results of Chapter 4 (the shortest chapter) which appeared in [Coh59]. However, the rest of the thesis also presents interesting results, and in fact, some of these were reproduced by Ash and Welland in [AWe72], see also [AWa97]. Apart from this, it appears that the results of the thesis have largely gone unnoticed. The main references on the uniqueness questions are the books of Bary [Ba64], Salem [Sal63], Meyer [Mey72], Zygmund [Zy79], Kechris and Louveau [KL87], and Kahane and Salem [KS94]. Even though a number of the results stated in these works were proved in Paul Cohen's thesis, it is not cited in the references. Therefore, it seemed appropriate that this work should finally be more widely available. I have included a short list of references in order to indicate the progress made in the last 40 years.

The most substantial reference is the book of Kechris and Louveau [KL87] which also gives a complete update on the progress done on the topics of Chapter 1. As noted above, some of the results of Chapter 2 are reproved in [AWe72]. The general reference for Chapter 3 is the updated version of the book of Kahane and Salem [KS94] which summarizes progress until 1994.

Chapter 1

Section 1. This section simplifies and extends some results previously obtained by Piatetski–Shapiro [PS52]. The notation used here is not standard, and the notations of Kahane and Salem [KS94] have been adopted. The class W is written A. The class of sequences a_n such that $\operatorname{Lim} a_n = 0$ is now referred to as the class of *pseudofunctions.*

At the end of the section it is stated that there are no known sets which have an ordinal > 1 associated to them. This ordinal, introduced by Piatetski–Shapiro, is now known as the *rank* of the set of uniqueness. A survey of results is given by Kechris and Louveau [KL87, Chapter V]. For example, it follows from a theorem of Banach [KL87, p. 156] that the rank is $< \omega_1$ and McGehee [MG68] has shown that the rank is in unbounded in ω_1.

Section 2. This section uses the results of Piatetski–Shapiro to give a spimple proof of a special case of a theorem of Zygmund and Marcinkiewicz. The proof should be compared with the one given by Kahane and Salem [KS94, p. 61], see also [KL87, p. 71]. The general proof, i.e., not limited to closed sets, is given by N. Bary [Ba64, p. 364].

Section 3. The classes H^n studied here are now denoted $H^{(n)}$. Moreover, the notation n_i^j used in this section should be $n_i^{(j)}$ in order to be consistent with the rest of the text.

Section 4. Another explicit construction of an $H^{(n)}$ set which is not a countable union of $H^{(n-1)}$ sets is given by N. Bary [Ba64, p. 382].

Chapter 2

This chapter presents a generalization of the Cantor–Lebesgue theorem to multiple trigonometric series. Recall that this theorem states that if a trigonometric series converges on a set of positive measure then its coefficients must go to zero. In this chapter, Paul Cohen proves that if a multiple trigonometric series converges almost everywhere for a "regular" method of summation, e.g., circular or square, then the coefficients grow more slowly than exponential. Moreover, the other result in this chapter shows that an analogue of the Cantor–Lebesgue theorem cannot generalize directly, as an example is given of a double trigonometric series which converges almost everywhere using square convergence, but whose coefficients do not go to zero.

The first result may appear to be somewhat weak, but in fact it was recently shown by Ash and Wang [AWa97] that Cohen's result is optimal for square summation, i.e., for any function $\varphi(n)$ which goes to infinity slower than exponentially, there is a square convergent trigonometric series which has coefficients which grow like $\varphi(n)$.

In the case of spherical summation, much progress has been made. The Cantor–Lebesgue theorem does generalize in this case as was shown by

Cooke [Co71] and Zygmund [Zy72] in dimension 2, and by B. Connes (in slightly less general form) in dimensions greater than 2 [BC76].

Due to a result of Shapiro [Sh57], Cooke's result immediately proved the uniqueness of double trigonometric results for circular summation. A recent advance was made by Bourgain [Bo76] who extended this uniqueness result to all dimensions.

Similarly, uniqueness results were also obtained for "unrestricted rectangular convergence" by Ash, Freiling, and Rinne [AFR93]. The case of square convergence still remains open.

Chapter 3

In the first section, a simple proof is given that Mensov's original example is a set of multiplicity. The following comment is implicit in the proof, but is added here for clarity: In order to ensure that the resulting set has measure zero, one just have that $\prod(2\xi_k) = 0$, while $\xi_k < 1/2$ for all k, i.e., $\sum \alpha_i = \infty$ as in Mensov's original construction.

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