Some Open Problems in Differential Algebra

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This note conatins a discussion of several open problems in differential algebra. The notations for differential polynomials, rings, and ideals follow the lecture notes [10]. I have a code for all the computational experiments described below and will be happy to share it, just email gleb.pogudin@polytechnique.edu.

Contents

1 Differential elimination: support

Background

Differential elimination is a differential analogue of elimination for polynomial systems and Gaussian elimination from linear algebra. It can be stated (and solved) in full generality but here we will focus on an important special case. Consider a system of differential equations of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ is a tuple of differential indeterminates and $\mathbf{f} = (f_1, \ldots, f_n)$ is a tuple of polynomials from $\mathbb{C}[\mathbf{x}]$. Systems of these form describe dynamical systems with polynomial dynamics and appear often in the literature. One natural elimination task is to eliminate all the variables except one, say x_1, that is, describe a differential ideal

$$\langle x_1' - f_1(\mathbf{x}), \ldots, x_n' - f_n(\mathbf{x})\rangle^{(\infty)} \cap \mathbb{C}[x_1^{(\infty)}]. \tag{1}$$

The ring of univariate differential polynomials is "nearly Euclidean" [10, Section 1.3] and, in particular, the ideal (1) is unqiuely determined by its minimal polynomial (polynomials are compared first w.r.t. the order and then w.r.t. total degree).

Question 1 (General)**.** *Describe and/or compute the minimal polynomial of the elimination ideal* (1)*.*

Problem statement

One interesting specific version of Question 1 is to ask about *the support (that is, the set of monomials with nonzero coefficients) of the minimal polynomial of the ideal* (1). If one knows the support, this can be used in practice, for example, as follows. There are very efficient algorithms for finding truncated power series solutions of $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ (e.g. [2, 11]). If one can estimate the support of the minimal polynomial for x_1, then one can write the minimal polynomial with undetermined coefficients and, plugging these solutions, obtain a linear system on these coefficients.

Let us formulate two specific questions about the support at the minimal polynomial of (1).

Question 2 (Generic systems)**.** *Consider a system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ of dimension n, where each f_i is a generic polynomial of degree d. What is the support of the minimal polynomial for x_1 (that is, the minimal polynomial of the corresponding elimination ideal)?*

Question 3 (2D case)**.** *Let $n = 2$, so the supports of f_1 and f_2 are subsets in $\mathbb{Z}^2$. Then the order of the minimal polynomial for x_1 will not exceed two, so its support will be a subset of $\mathbb{Z}^3$. How is this set (or its convex hull) related to the two original sets in $\mathbb{Z}^2$ (or their convex hulls)?*

Experimental results

We start with a series of examples connected to both Questions 2 and 3: we will consider the case of system

$$x_1' = f_1(x_1, x_2), \quad x_2' = f_2(x_1, x_2),$$

where f_1 and f_2 are generic polynomials of degrees d_1 and d_2, respectively. Table 1 below summarizes the results, and one can observe several interesting patterns.

	Newton polytope of the minimal polynomial	
$(d_1,\ d_2)$	Vertices in (x_1, x_1', x_1'')-coordinates	Type
(1, d) (for $d \leqslant 6$)	$(0,0,0),\ (d,0,0),\ (0,d,0),\ (0,0,1)$	tetrahedron
(2, 2)	$(0,0,0),\ (6,0,0),\ (0,3,0),\ (0,0,2)$	tetrahedron
(2, 3)	$(0,0,0),\ (8,0,0),\ (0,4,0),\ (0,0,2)$	tetrahedron
(2, 4)	$(0,0,0),\ (10,0,0),\ (0,5,0),\ (0,0,2)$	tetrahedron
(2, 5)	$(0,0,0),\ (12,0,0),\ (0,6,0),\ (0,0,2)$	tetrahedron
(3, 1)	$(0,0,0),\ (9,0,0),\ (6,3,0),\ (0,5,0),\ (0,0,3)$	pyramid
(3, 2)	$(0,0,0),\ (12,0,0),\ (6,3,0),\ (0,5,0),\ (0,0,3)$	pyramid
(3, 3)	$(0,0,0),\ (15,0,0),\ (6,3,0),\ (0,5,0),\ (0,0,3)$	pyramid
(4, 1)	$(0,0,0),\ (16,0,0),\ (12,4,0),\ (0,7,0),\ (0,0,4)$	pyramid
(4, 2)	$(0,0,0),\ (20,0,0),\ (12,4,0),\ (0,7,0),\ (0,0,4)$	pyramid

Table 1: Results for 2D systems

Question 4. *Prove the characterization of Newton polytopes for the degree pairs $(2, d), (3, d), (4, d)$ for any d continuing the patterns from Table 1.*

Some basic results for dimension three are collected in Table 2.

(d_1, d_2, d_3)	Vertices in $(x_1, x_1', x_1'', x_1^{(3)})$-coordinates
$(2,2,2)$	$(0,0,0,0),\ (24,0,0,0),\ (0,12,0,0),\ (0,0,8,0),\ (0,0,0,6)$
$(2,2,3)$, $(2,3,2)$	$(0,0,0,0),\ (42,0,0,0),\ (24,0,6,0),\ (12,0,0,6),\ (0,21,0,0),$ $(0,12,6,0)$, $(0,6,0,6),\ (0,0,12,0),\ (0,0,0,8)$

Table 2: 3D models

2 Degree of the prolongation variety

Background

Consider a differential equation or a system of differential equations, say:

$$x' - x^2 = 0.$$

We can transform it into an infinite polynomial system in variables $x, x', x'', \ldots$ by taking all the derivatives:

$$0 = x' - x^2 = x'' - 2x'x = x^{(3)} - 2(x')^2 - 2xx'' = \ldots$$

An important property of this infinite polynomial system is that its solutions are in a bijective correspondence with the power series solutions of the original differential system [10, Proposition 2.3]. However, working constructively with the whole infinite-dimensional system is problematic, so one often works with its "truncations", that is, polynomial systems defined by only first several derivatives, see, e.g., [5, 8]. In other words, one considers a sequence of varieties corresponding to this truncation

$$\mathbb{V}(x' - x^2) \subset \mathbb{A}^2, \quad \mathbb{V}(x' - x^2, x'' - 2xx') \subset \mathbb{A}^3, \quad \mathbb{V}(x' - x^2, x'' - 2xx', x'' - 2(x')^2 - 2xx'') \subset \mathbb{A}^4, \ldots$$

Understanding the geometry of these varieties may be a key to refining the results which use such "truncations" and also design new methods (including homotopy-based ones) to deal with the systems of differential equations.

Problem statement

We consider a system F of differential equations $f_1 = \ldots = f_\ell = 0$ and introduce the following varieties

$$\begin{aligned} X_0(F) &:= \mathbb{V}(f_1, \ldots, f_\ell), \\ X_1(F) &:= \mathbb{V}(f_1, \ldots, f_\ell, f_1', \ldots, f_\ell'), \\ X_2(F) &:= \mathbb{V}(f_1, \ldots, f_\ell, f_1', \ldots, f_\ell', f_1'', \ldots, f_\ell''), \\ &\vdots \end{aligned}$$

living in appropriate finite-dimensional affine spaces.

Question 5 (General)**.** *For an arbitrary differential system, what are the dimension, components, and degree of the varieties $X_i(F)$ for $i \geqslant 0$.*

There is an important special case in which the dimension and the number of components are known. This is the case of polynomial dynamical systems, that is, systems of the form

$$\mathbf{x}' = \mathbf{p}(\mathbf{x}),$$

where $\mathbf{x} = (x_1, \ldots, x_\ell)$ is a tuple of differential indeterminates and $\mathbf{p} = (p_1, \ldots, p_\ell)$ is a tuple of polynomials. It turns out that the system and any number of its derivatives form a Gröbner basis of the ideal they generate in the appropriate polynomial ring [10, Proposition 1.24]. This implies that, for every $i \geqslant 0$, the variety $X_i(F)$ is irreducible and $\operatorname{codim} X_i(F) = \ell$ for every i. So the remaining question is the one about the degree.

Experimental results

Consider our example system $x' - x^2 = 0$, which is a polynomial dynamical system. One can prove that $\deg X_m(x' - x^2) = m + 2$ for every $m \geqslant 0$. Furthermore, the same argument shows that, for an arbitrary polynomial $p(x)$ of degree d, one has $\deg X_m(x' - p(x)) = m + d$.

Situation becomes more interesting if one goes to dimension 2. Consider the following system F_1

$$F_1 \colon x_1' = x_2^2, \quad x_2' = x_1^2.$$

For $m \leqslant 10$, the following has been verified:

$$\deg X_m(F_1) = \begin{cases} (m+2)^2, \text{ if } m \equiv 0, 2 \pmod 3, \\ (m+2)^2 - 2, \text{ if } m \equiv 1 \pmod 3. \end{cases} \tag{2}$$

By taking the degree to be larger by one, that is, considering F_2:

$$F_2 \colon x_1' = x_2^3, \quad x_2' = x_1^3,$$

we find (for $m \leqslant 10$)

$$\deg X_m(F_2) = \begin{cases} (2m+3)^2, \text{ if } m \equiv 0, 3 \pmod 4, \\ (2m+3)^2 - 8, \text{ if } m \equiv 1, 2 \pmod 4. \end{cases} \tag{3}$$

Question 6. *Prove* (2) *and* (3).

Gabriela Jeronimo and independently Sebastian Falkensteiner and Rafael Sendra kindly shared with me the proofs that $\deg X_m(F_1)$ does not exceed the number from (2) but the problem of showing the equality is still open to the best of my knowledge. We also observe that in all the cases above, the sequence $\deg X_m(F)$ is a *quasi-polynomial* (a polynomial with the coefficients being periodic functions in m with integer period) in m.

Question 7. *Is it true that, for a polynomial dynamical system F, $\deg X_m(F)$ is always a quasi-polynomial in m? How does this quasi-polynomial depend on the system?*

Here are several first numbers $\deg X_m(F_3)$ for $F_3 = \{x_1' = x_2^2, x_2' = x_1^3\}$:

$$6, 11, 26, 46, 66, 91, 121, 156, 196, \ldots$$

What would be the quasi-polynomial?

Similar results can be obtained for more general differential-algebraic equations. For example, here are the values of $\deg X_m((x')^2 + x^2 - 1)$:

$$2,\ 4,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18, \ldots$$

Note the jump from 4 to 8!

Question 8. *For a univariate differential polynomial $p \in \mathbb{C}[x^{(\infty)}]$, what does $\deg X_m(p = 0)$ look like? For example, can we prove that the sequence above (for $p = (x')^2 + x^2 - 1$) continues as an arithmetic progression?*

3 Homogeneous Lagrangians

Background

A number of problems in the calculus of variations can be stated as finding functions $x(t)$ and $y(t)$ (subject to some constraint) maximizing/minimizing an integral of the form

$$\int_{t_0}^{t_1} L(x(t), y(t))\,\mathrm{d}t, \tag{4}$$

where $L \in \mathbb{C}(x^{(\infty)}, y^{(\infty)})$ is a differential rational function. The function L is then referred to as *the Lagrangian.* An important special case is the case when the value of the integral (4) is invariant under the reparametrization of the time axis. This corresponds to the situation in which one is looking for a curve $\{(x(t), y(t)) \mid t \in [t_0, t_1]\}$ with certain extremal property rather than for particular functions $x(t)$ and $y(t)$. In this case the Lagrangian L is called *homogeneous* [9, §8.1].

The homogeneity can be expressed as an algebraic condition on the differential rational function L. Consider a reparametrization of the time axis $t = t(\tau)$. Let use denote by $L^{(t)}(x, y)$ and $L^{(\tau)}(x, y)$ the value of the differential rational function L on $x(t(\tau)), y(t(\tau))$ with the derivatives taken with repect to t and τ (so, using the chain rule), respectively. Then the homogeneity can be expressed as the equality of (4) after the substitution $t = t(\tau)$ and the same functional applied to $x(t(\tau)), y(t(\tau))$ as functions of τ. This yields

$$\int_{\tau_0}^{\tau_1} L^{(t)}(x, y) t'(\tau)\,\mathrm{d}\,\tau = \int_{\tau_0}^{\tau_1} L^{(\tau)}(x, y)\,\mathrm{d}\,\tau.$$

Since the equality above is expected to hold for all functions $x(t), y(t), t(\tau)$ and intervals $[\tau_0, \tau_1]$, we obtain the following condition on L:

$$L^{(\tau)}(x, y) = L^{(t)}(x, y) t'(\tau) \quad \text{for every } t(\tau). \tag{5}$$

Problem statement

We will now transform the condition (5) to a formal algebraic definition.

Definition 1 (homogeneous Lagrangian)**.** *Let $L \in \mathbb{C}(x^{(\infty)}, y^{(\infty)})$ be a differential rational function. We will write it $L(\partial, x, y)$ to make the dependence on the derivation explicit. We consider one more differential indeterminate w and introduce the following derivation ∂_w on $\mathbb{C}(x^{(\infty)}, y^{(\infty)}, w^{(\infty)})$:*

$$\partial_w(w^{(i)}) = w^{(i+1)}, \quad \partial_w(z^{(i)}) = w' z^{(i+1)} \qquad \text{for } i \geqslant 0,\ z \in \{x, y\},$$

which encodes the chain rule if one considers x and y as "functions in w".

Then L is called homogeneous Lagrangian *if*

$$L(\partial_w, x, y) = w' L(\partial, x, y),$$

where by $L(\partial_w, x, y)$ we mean $L(x, y)$, in which every occurence of $x^{(i)}$ or $y^{(i)}$ is replaced by $\partial_w^i(x)$ or $\partial_w^i(y)$, respectively.

Example 1. *Consider $L = \frac{x''y' - x'y''}{(x')^2}$, that is, $L(\partial, x, y) = \frac{(\partial^2 x)(\partial y) - (\partial x)(\partial^2 y)}{(\partial x)^2}$. For every $z \in \{x, y\}$, we have*

$$\partial_w z = w' z' \qquad \text{and } \partial_w^2 z = w'' z' + (w')^2 z''.$$

So we obtain

$$L(\partial_w, x, y) = \frac{(w''x' + (w')^2 x'')w'y' - w'x'(w''y' + (w')^2 y'')}{(w')^2 (x')^2} = w' L(\partial, x, y),$$

so L is a homogeneous Lagrangian.

A natural general question then is:

Question 9 (General)**.** *Describe the set of all homogeneous Lagrangians.*

In order to state more precise questions in this direction, we will extend Definition 1:

Definition 2 (semi-homogeneous Lagrangian)**.** *In the notation of Definition 1, L will be called* semi-homogeneous Lagrangian *of order h if*

$$L(\partial_w, x, y) = (w')^h L(\partial, x, y).$$

Then one can study Question 9 using the following lemma.

Lemma 1. *Let $L \in \mathbb{C}(x^{(\infty)}, y^{(\infty)})$ be a homogeneous Lagrangian. Then there exist $P, Q \in \mathbb{C}[x^{(\infty)}, y^{(\infty)}]$ such that $L = \frac{P}{Q}$ and P and Q are semi-homogeneous Lagrangians of orders $h+1$ and h for some integer h.*

Notation 1. *For an integer $h \geqslant 0$, we define $\mathcal{L}_h$ to be the space of all semi-homogeneous Lagrangians of order h in $\mathbb{C}[x^{(\infty)}, y^{(\infty)}]$.*

We observe that, for every (nondifferential) polynomial $P(x, y) \in \mathbb{C}[x, y]$, we have

$$Q \in \mathcal{L}_h \implies PQ \in \mathcal{L}_h$$

meaning that $\mathcal{L}_h$ is a module over $\mathbb{C}[x, y]$. It is not hard to show that this is a free module.

Question 10. *What is the rank of $\mathcal{L}_h$ as a free module over $\mathbb{C}[x, y]$?*

Experimental results

We computed the rank of $\mathcal{L}_h$ for h from 0 to 16, and got the following sequence:

$$1, 2, 3, 4, 6, 8, 10, 14, 18, 22, 29, 36, 44, 56, 68, 82, \ldots$$

There are two sequences in OEIS[1] containing this: A053253 and A095913. Based on this, we can state the following conjecture: *the rank of $\mathcal{L}$ is a number of Young diagrams of size $2h+1$ such that each row and each column consists of an odd number of cells.*

In the trivariate case, we get the sequence $1, 3, 6, 10, 18, 30, 45, 69$ which does not match any entry in OEIS.

[1] http://oeis.org

4 Integrating differential polynomials

Background

Consider a nonlinear differential equation $p(x) = 0$, where $p(x) \in \mathbb{C}[x^{(\infty)}]$. If we take $p(x) = x''x + (x')^2$, then one can use integration to lower the order of the equation:

$$0 = x''x + (x')^2 = (xx')' \implies xx' = c \text{ for some } c \in \mathbb{C}.$$

Such integration can be used to simplify the analysis of the equation both in theory and in practice. An algorithm for determining if a differential polynomial is integrable (and even a differential rational function) has been designed in [3].

There are, however, differential polynomials which do not have antiderivatives but allow similar "integration". For example, consider $p(x) = xx'' - (x')^2$. One can show that it is not a derivative of any other differential polynomial. However, if we divide it by $(x')^2$, we get

$$\frac{x''x - (x')^2}{(x')^2} = -\left(\frac{x}{x'}\right)' \implies \frac{x}{x'} = c \text{ for some } c \in \mathbb{C}.$$

So we managed to integrate the differential polynomial by using an "integrating factor" $\frac{1}{(x')^2}$. If we consider $p(x) = xx'' + \alpha(x')^2$ for $\alpha \in \mathbb{C}$, we can see that it may be beneficial to use non-rational expressions as integrating factors:

$$(xx'' + \alpha(x')^2)\frac{(x')^{1/\alpha - 1}}{\alpha} = \left(x(x')^{1/\alpha}\right)' \implies x(x')^{1/\alpha} = c \text{ for some } c \in \mathbb{C}. \tag{6}$$

Problem statement

In general, if we have an equation $p(x) = 0$, where $p(x) \in \mathbb{C}[x^{(\infty)}]$ is a differential polynomial of order $h > 0$, we can multiply the equation by some "functions" in $x, x', \ldots, x^{(h-1)}$ (a generic solution of the original equation will not annihilate a function of order $h-1$) aiming at making the resulting product to be someone's derivative. We can formulate a two-part general question.

Question 11 (General)**.**

- *Which differential polynomials $p(x) \in \mathbb{C}[x^{(\infty)}]$ can be integrated by using an appropriate integrating factor?*
- *What is the class of functions in $x, x', \ldots, x^{(h-1)}$ where these integrating factors should be sought (e.g., polynomials with arbitrary complex exponents as in the example above)?*

Experimental results

In order to use computational methods for experiments, let us restrict ourselves to a finite dimensional space of polynomials

$$V_2 := \{p(x) \in \mathbb{C}[x^{(\infty)}] \mid \operatorname{ord} p \leqslant 2,\ \deg p \leqslant 2\}.$$

We will start with extending the idea of the example (6), that is, determining which polynomials $p(x) \in V_2$ can be represented in the form

$$\frac{(A(x) \cdot B(x)^{\alpha})'}{B(x)^{\alpha - 1}}, \tag{7}$$

where $A(x)$ and $B(x)$ are polynomials in x and x' of degree at most one and α is a number. We can take $A(x) = a_1x' + a_0x + a_{-1}$ and $B(x) = b_1x' + b_0x + b_{-1}$ and obtain a polynomial map φ_2 from the space with coordinates $(a_{-1}, a_0, a_1, b_{-1}, b_0, b_1, \alpha)$ to V_2 defined by

$$\varphi_2\colon (a_{-1}, a_0, a_1, b_{-1}, b_0, b_1, \alpha) \mapsto \frac{(A(x) \cdot B(x)^{\alpha})'}{B(x)^{\alpha-1}} \in V_2.$$

Although φ_2 is a polynomial map defined by polynomials of degree three, the Zariski closure of its image turns out to be a *linear subspace* of V_2 (compare with (6)) spanned by

$$x',\ xx',\ xx'',\ (x')^2,\ x'x'',\ x''.$$

Question 12. *Is it possible to explain this linear space structure without carrying out the computation?*

Now we perform the same computation for $p(x)$ being still quadratic but of order at most three, that is, we study the set of polynomials representable as (7) with $A(x)$ and $B(x)$ being still of degree at most one but of order at most two. The Zariski image of the corresponding map

$$\varphi_3 \colon (a_{-1}, a_0, a_1, a_2, b_{-1}, b_0, b_1, b_2, \alpha) \mapsto \frac{(A(x) \cdot B(x)^{\alpha})'}{B(x)^{\alpha-1}} \in V_3 := \{p(x) \in \mathbb{C}[x^{(\infty)}] \mid \operatorname{ord} p \leqslant 3, \deg p \leqslant 2\}$$

is not longer a linear space. However, the equations defining the image have interesting features. We introduce the coordinates on V_3 by writing any element as

$$p(x) = \sum_{0 \leqslant i \leqslant j \leqslant 3} a_{i,j} x^{(i)} x^{(j)} + \sum_{i=0}^{3} a_i x^{(i)} + a.$$

The the defining ideal of the closure of the image of φ_3 contains, for example, the following polynomials:

$$\begin{vmatrix} a_1 & a_{0,1} & a_{0,3} \\ a_2 & a_{0,2} & a_{1,3} \\ a_3 & a_{0,3} & a_{2,3} \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_{0,3} & a_{0,2} - a_{1,1} \\ a_2 & a_{1,3} & 0 \\ a_3 & a_{2,3} & a_{1,3} - a_{2,2} \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} a_1 & a_{0,1} & a_{0,2} - a_{1,1} \\ a_2 & a_{0,2} & 0 \\ a_3 & a_{0,3} & a_{1,3} - a_{2,2} \end{vmatrix}.$$

Question 13.

- *Give a complete description of the defining equations of the image of φ_3 (or even φ_d).*
- *Is it possible to extend the representation (7) so the the image of the analogue of φ_3 will be again a linear space?*

Now let us return to the space V_2. We have shown that any (technically, almost any) element of the subspace spanned by x', xx', xx'', $(x')^2$, $x'x''$, x'' can be integrated. However, the set of integrable elements of V_2 is larger: consider $p(x) = x'' + 1$

$$0 = x'(x'' + 1) = (1/2(x')^2 + x)' \implies 1/2(x')^2 + x = c \text{ for some } c \in \mathbb{C}.$$

In order to generalize this example, we can search for $p(x) \in V_2$ representable as

$$\frac{C(x)'}{A(x)},$$

where $C(x)$ and $A(x)$ are at most cubic and linear polynomials in x, x', x'', respectively. We perform the corresponding polynomial elimination and find that the closure of the set of such $p(x) \in V_2$ consists of the union of the following components:

- Linear subspace $a_{2,2} = a_1 = a_{0,1} = a_{0,2} - 2a_{1,1} = 0$;
- A component defined by linear equations $a = a_{2,2} = a_0 = a_{0,0} = 0$ and one nonlinear

$$a_{1,2}a_{0,1} + a_{1,1}^2 - 5a_{1,1}a_{0,2} + 2a_{0,2}^2 = 0.$$

Question 14. *What is the "meaning" of the last equation? What will the corresponding equations look like for higher orders/degrees of $p(x)$?*

Question 15. *Do the two presented constructions exhaust the list of integrable elements in V_2?*

5 Quadratization

Background

The quadratization problem is, given a system of ordinary differential equations (ODEs) with polynomial right-hand side, transform it into a system with at most quadratic right-hand side. For example, consider a scalar ODE:

$$x' = x^5. \tag{8}$$

The right-hand side has degree larger than two but if we introduce a new variable $y := x^4$, then we can write:

$$x' = xy, \quad \text{and} \quad y' = 4x^3x' = 4x^4y = 4y^2 \implies \begin{cases} x' = xy, \\ y' = 4y^2. \end{cases} \tag{9}$$

The right-hand sides of (9) are of degree at most two, and every solution of (8) is the x-component of some solution of (9). Therefore, we have embedded our original system into a system with at most quadratic right-hand side. This transformation arises in different application areas including model reduction [6] and synthetic biology [7]. It is known (e.g. [6, Theorem 3]) that a quadratization always exists, and moreover the new variables can be taken to be monomials in the original ones. An algorithm for finding a monomial quadratization with the smallest possible number of new variables has been designed in [4].

It is natural to ask whether one could use fewer variables for quadratization if the new variables were arbitrary polynomials in the original ones. The answer is yes: consider a scalar equation:

$$x' = (x+1)^{100}.$$

It can be quadratized with a single new variable $y := (x+1)^{99}$:

$$\begin{cases} x' = y(x+1), \\ y' = 99(x+1)^{198} = 99y^2. \end{cases}$$

On the other hand, a simple combinatorial argument [1, Section 4] shows that one needs at least ten new variables if they must be monomials in x.

Problem statement

The discussion above motivates the following general problem statement:

Question 16 (General). *For a given ODE system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with polynomial right-hand side, determine (or bound) the minimal number of new variables sufficient to embed the system into a system with at most quadratic right-hand side.*

We propose the following specific subproblems.

Question 17. *Given an integer d, what is the minimal number of new variables sufficient to quadratize any scalar equation $x' = p(x)$, where $\deg p \leqslant d$?*

Question 18. *Given integers $d_1, \ldots, d_n$, what is the minimal number of new variables sufficient to quadratize any system*

$$\begin{cases} x_1' = p_1(x_1, \ldots, x_n), \\ \vdots \\ x_n' = p_n(x_1, \ldots, x_n) \end{cases},$$

where $\deg p_i \leqslant d_i$ for every $1 \leqslant i \leqslant n$?

Experimental results

Question 17 has been studied for small d's in [1]. The results are

- for $d = 3, 4$ one new variable is sufficient;
- for $d = 5, 6$ two new variables are sufficient (and, in general, necessary).

For $d = 5$, these new variables can always be taken to be powers of x. But this is not the case for $d = 6$ which is more mysterious. A general formula for an arbitrary degree six polynomial $p(x)$ is given in [1, Theorem 3.2], we will show it on a more special case which still highlights the features of the result. Consider a scalar ODE

$$x' = x^6 + p_4x^4 + p_3x^3 + p_2x^2 + p_1x + p_0.$$

Then introducing the following two new variables allows embed it into a system of ODEs with at most quadratic right-hand side:

$$y_1 := x^5 + \boxed{\frac{5p_3}{8}x^2}, \quad y_2 := x^3.$$

The coefficient 5/8 is essential but we do not have any good high-level explanation where it could come from.

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