MPRI C2-10: Enumerative combinatorics
Lecture 11-13. Markov chains and enumeration

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Pictures are missing, each definition and proposition should be illustrated! The text has been written late in the evening and has not be proofcorrected, take it with a grain of salt... and do not hesitate to contact me by email if you have doubts: it would not be such a surprise if some statements were completely wrong!

1 Finite state Markov chains

1.1 First example and definition of Markov chain

Let us consider a sequence of random permutations $X_0, X_1, \dots$ on the set S_n of permutations of $\{1, \dots, n\}$, constructed as follows:

- The permutation X_0 is drawn according to an initial probability distribution d_0 on S_n: for instance we can start deterministically with the identity by setting $d_0(\sigma) = \delta_{\sigma=\mathrm{id}_n}$.
- The permutation X_i is obtained from X_{i-1} by multiplication by a transposition taken at uniformly random among the $\binom{n}{2}$ transpositions of S_n.

The X_i form a sequence of random variables on S_n. Let d_i be the distribution of X_i. By definition the conditional probabilities that $X_i = \sigma$ knowing that $X_{i-1} = \rho$ is

$$\tau_{\sigma,\rho} := \mathrm{Prob}(X_i = \sigma \mid X_{i-1} = \rho) = \begin{cases} \frac{1}{\binom{n}{2}} & \text{if } \sigma\rho^{-1} \text{ is a transposition,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular this conditional probability does not depend on the sequence $X_0, \dots, X_{i-2}$. This property of a sequence of r.v. is called the *Markov property*, and the triple $\mathcal{X} = (S_n, T, d_0)$ where T is the $n! \times n!$ transition matrix with entries $(\tau_{\sigma,\rho})_{\sigma,\rho \in S_n}$ is called a *finite state Markov chain.*

Definition 1 *A finite state Markov chain (Ω, T, d_0) consists of a finite set Ω of cardinal N (the universe), a $N \times N$ transition matrix $T = (\tau_{\omega,\omega'})_{\omega,\omega' \in \Omega}$ with $\sum_{\omega' \in \Omega} \tau_{\omega,\omega'} = 1$, and an initial probability distribution d_0 on Ω.*

The probability distribution d_i of X_i can be expressed nicely in terms of d_0 and the matrix T:

$$d_i = T^i d_0$$

where the d_i are viewed as column vectors in $\mathbb{R}^N$. However this expression does not provide much explicit information about $d_i(\sigma)$ for large i.

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1.2 Irreducibility, aperiodicity

The Markov chain $\mathcal{X} = (S_n, T, d)$ can visit all permutations of S_n (how many steps are needed ?). In other terms in the *transition graph* with vertex set S_n and with an arc (ρ, σ) if $\tau_{\sigma,\rho} > 0$, there is path between any two vertices.

Definition 2 *A finite state Markov chain is* irreducible *if its transition graph is connected.*

When $i \to \infty$, we expect X_i to grow "ever more random" and to "forget" the starting point X_0. How can we describe X_i when i goes to infinity? The simplest situation would be that for each σ, $d_i(\sigma)$ converges to a limit distribution $d_\infty(\sigma)$. Taking the limit in the relation $d_i = Td_{i-1}$, we see that the limit distribution, if it exists, must satisfy $d_\infty = Td_\infty$.

Definition 3 *A distribution π is* stationary *for the chain (Ω, T, d_0) (or more precisely for the transition matrix T) if*

$$\pi = T\pi$$

If convergence is to hold for $\mathcal{X}$, we should also, in view of the symmetry of our chain, expect the limit distribution d_∞ to be uniform among all permutations. (Propose a way to state formally that the chain is symmetric.) A weaker statement which we can immediately check is that the uniform distribution on S_n is indeed stationary for $\mathcal{X}$ (prove it!).

Observe however more precisely that if X_0 is an even permutation then so is X_{2i} while X_{2i+1} is an odd permutation: the parity of the initial permutation is never forgotten. So the previous statement is not correct: more generally in order to get proper limit behavior we need to exclude periodic behaviors.

Definition 4 *A Markov chain is k-*periodic *if there exists a partition of the states in k colors such that the transition graph cycles through colors (*i.e. *each arc joins a vertex of color c to a vertex of color $c+1 \mod k$.*

A Markov chain is aperiodic *if there is no $k > 1$ such that it is k-periodic.*

A simple way to make a Markov chain aperiodic is introducing a positive standby probability at each state: given a Markov chain (Ω, T, d_0) let $\widetilde{T} = \frac{1}{2}T + \frac{1}{2}\mathrm{Id}_N$, then the Markov chain $(\Omega, \widetilde{T}, d_0)$ is aperiodic.

Irreducibility and aperiodicity are fundamental notions for Markov chains because they form compact sufficient conditions for the convergence to a limit.

Theorem 1 *If the finite state Markov chain (Ω, T, d_0) is irreducible then it admits a unique stationary distribution π. If moreover it is aperiodic then the distribution d_n converges to π: for all σ, $d_n(\sigma) \to \pi(\sigma)$. (Since Ω is finite, this convergence is equivalent to convergence in distribution or in the total variation sense.)*

This theorem can be understood in various ways: the most classical proof is based on Perron Frobenius analysis of the dominant eigenvalue of the transition matrix; an alternative, more probabilistic proof can be devised via coupling (see Häggström); from the enumerative point of view finite state Markov chains are closely related to weighted path enumeration in finite directed graphs. The analogous theorem for paths reads:

Theorem 2 (Thm V.8 of Flajolet-Sedgewick (Profile of paths in graphs)) *Let T be a non-negative matrix associated to a weighted digraph G, assumed to be irreducible and aperiodic. Let ℓ, r be, respectively, the left and right eigenvectors corresponding to the dominant (Perron– Frobenius) eigenvalue λ_1. Consider the collection $F^{(a,b)}$ of (weighted) paths in G with fixed origin a and final destination b. Then, the number of traversals of edge (s,t) in a random element of $F^{(a,b)}$ of length i has mean*

$$\tau_{s,t} i + O(1) \text{ where } \tau_{s,t} = \frac{\ell_s g_{s,t} r_t}{\langle \ell, r \rangle}$$

and the probability that a random path with origin fixed at some point a will end up after a large number of steps at point b is asymptotically

$$\frac{\ell_b}{\sum_j \ell_j}$$

Returning to our chain $\mathcal{X}$ on S_n, Theorem 1 implies that the Markov chain $\widetilde{\mathcal{X}}$ above indeed converges to its unique stationary distribution, which we have already identified as the uniform distribution on S_n.

1.3 Identifying limit distributions, symmetry arguments

Consider now another Markov chain $\mathcal{Y} = (S_n, T', d_0)$ on permutations of S_n where only adjacent transpositions (modulo n) are used. In other terms the transition matrix T' has entries:

$$\tau'_{\sigma,\rho} := \text{Prob}(Y_i = \sigma \mid Y_{i-1} = \rho) = \begin{cases} \frac{1}{n} & \text{if } \sigma\rho^{-1} \text{ is one of } \{(1,2),(2,3),\ldots,(n-1,n),(n,1)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Again the symmetry of the chain implies that the uniform distribution is stationary and the theorem yields the convergence of $\widetilde{Y}$. Can we get away from that?

Let us try to break the symmetry by remove the transposition $(1,n)$ and consider only standard adjacent transposition $(k,k+1)$ for $k = 1,\ldots,n-1$:

$$\tau''_{\sigma,\rho} := \text{Prob}(Z_i = \sigma \mid Z_{i-1} = \rho) = \begin{cases} \frac{1}{n-1} & \text{if } \sigma\rho^{-1} \text{ is one of } \{(1,2),(2,3),\ldots,(n-1,n)\}, \\ 0 & \text{otherwise.} \end{cases}$$

In the resulting Markov chain $\mathcal{Z} = (S_n, T'', d_0)$ the entry 1 and n appear to play a special role since there is only one adjacent transposition involving 1 or n. What distribution do we get for $\widetilde{\mathcal{Z}}$?

What about multiplication by a random element of $\{(1,2,3),(2,3,4),(3,4,5),\ldots,(n,1,2)\}$? There are two orbits so we should concentrate on A_n the group of odd permutations. Has the resulting chain $\mathcal{W}$ a uniform stationary distribution on A_n?

One can easily check experimentally that all these chains converge to the uniform distribution... Why?

Reversible distributions, reversible Markov chains

Definition 5 *A distribution π is* reversible *for a transition matrix T if*

$$T_{i,j}\pi_j = T_{j,i}\pi_i.$$

A Markov chain is reversible *if its transition matrix admits a reversible distribution.*

Observe that the reversibility equation can be interpreted as a equality of flow: the "flow of proba" going from j to i starting from distribution π is equal to the "flow of proba" going from i to j.

Proposition 1 *Reversible $\Rightarrow$ stationary.*

(Check it using $\pi = T\pi$ and the definition).

In the case of our chain $\mathcal{Z}$ on transpositions, the uniform distribution is clearly reversible because the probability to go from σ to ρ is equal to the probability to go from τ to σ.

More generally we can interpret a Markov chain as a random walk on the transition graph. If this oriented graph is symmetric, and if the transition probabilities at a given vertex are all equal (all ways to leave a configuration are equiprobable) then the degree distribution (*ie* each vertex gets a probability equal to its degree in the graph) is easily checked to be reversible. As a consequence it is stationary. If the graph is irreducible it is the unique stationary distribution and if it is aperiodic convergence holds.

This result is however not sufficient to explain the convergence of $\mathcal{W}$ to the uniform distribution since the transition graph is in this case non symmetric.

Uniform distribution without reversibility: using bijections Reversibility is a local balance condition for each edge of the transition graph. However to say that the uniform distribution is stationary we only need to show that the total flow of proba arriving at each vertex of the transition graph is the same. Another simple sufficient condition for this for a simple random walk on a oriented graph is that the graph has regular in- and out-degrees.

A "combinatorial" way to construct such a simple random walk on a set Ω of configuration is to use a k-to-k application:

Lemma 1 *Assume that ϕ is a bijection from $\Omega \times [1, k]$ into itself. Then the Markov chain with transition matrix T given by*

$$T_{y,x} = \frac{1}{k} |\{(i,j) \mid \phi(x,i) = (y,j)\}|$$

admit the uniform distribution as stationary distribution.

Apply this lemma to prove that the uniform distribution is stationary for the chain $\mathcal{W}$. Complete then the proof of the convergence of $\mathcal{W}$ to the uniform distribution on A_n?

2 TASEP, ASEP, multicolor TASEP; lien avec Catalan et les permutations

2.1 Definition of the models

Let $\Omega_n = \{0,1\}^n$. We interpret[1] $\omega \in \Omega_n$ as a linear configuration of n sites with 0 representing an empty site $\circ$ and 1 representing an occupied site $\bullet$. For instance the following configurations of Ω_{10} has 5 particles distributed among 10 sites:

$$1001001110 = \bullet\circ\circ\bullet\circ\circ\bullet\bullet\bullet\circ$$

Let $\Omega_{n,k}$ denote the subset of configurations with k particles.

The *asymmetric simple exclusion process (ASEP)* is a random process on Ω_n in which particles jump at random from their current position to a empty neighboring site one according to some fixed rates. In the model with periodic boundaries the ith site is adjacent to the $i \pm 1 \mod n$ sites, and the number of particles never changes. In the model with open boundaries the ith site is adjacent to the $i \pm 1$ site for $i \in [1, n-1]$ while the first site is adjacent only to the second and the nth site only to the $(n-1)$th but particles can enter or exit the system at the borders.

Equivalently the evolution of the particle configurations in the ASEP can be described by Markov chains on Ω_n.

Definition 6 (ASEP with periodic boundaries) *The ASEP with periodic boundary is a Markov chain $\mathcal{X}_{n,k} = (\Omega_{n,k}, T(p,q), d_0)$ with*

$$\tau_{\omega,\omega'} = \begin{cases} p/n & \text{if } \omega = u\bullet\circ v, \text{ and } \omega' = u\circ\bullet v \\ & \text{or if } \omega = \circ u\bullet, \text{ and } \omega' = \bullet u\circ \\ q/n & \text{if } \omega = u\circ\bullet v, \text{ and } \omega' = u\bullet\circ v \\ & \text{or if } \omega = \bullet u\circ, \text{ and } \omega' = \circ u\bullet \\ * & \text{if } \omega' = \omega \\ 0 & \text{otherwise.} \end{cases}$$

where $*$ denotes the complement to 1 of the other cases.

[1] An equivalent formulation is sometimes used in the literature in which empty sites are viewed as white particles and occupied sites as black particles.

Compute the probability $*$. Check that the above formal definition is equivalent to the following more intuitive definition: X_i is obtained from X_{i-1} by choosing a wall in $\{0, \dots, n-1\}$ uniformly at random and, if the configuration around the ith wall is $\circ \,|_i\, \bullet$ let the particle jump to the left with probability q, and if the configuration around the ith wall is $\bullet \,|_i\, \circ$ let the particle jump to the right with probability p, and do nothing otherwise.

Definition 7 (ASEP with open boundaries) *The ASEP with periodic boundary is a Markov chain* $\mathcal{Y}_n = (\Omega_n, T(p, q, \alpha, \beta, \gamma, \delta), d_0)$ *with*

$$\tau_{\omega,\omega'} = \begin{cases} p/(n+1) & \text{if } \omega = u \bullet \circ v, \text{ and } \omega' = u \circ \bullet v \\ q/(n+1) & \text{if } \omega = u \circ \bullet v, \text{ and } \omega' = u \bullet \circ v \\ \alpha/(n+1) & \text{if } \omega = \circ v, \text{ and } \omega' = \bullet v \\ \beta/(n+1) & \text{if } \omega = u \bullet, \text{ and } \omega' = u \circ \\ \gamma/(n+1) & \text{if } \omega = \bullet v, \text{ and } \omega' = \circ v \\ \delta/(n+1) & \text{if } \omega = u \circ, \text{ and } \omega' = u \bullet \\ * & \text{if } \omega' = \omega \\ 0 & \text{otherwise.} \end{cases}$$

where $*$ denotes the complement to 1 of the other cases.

Again one should compute the quantity $*$ and check that this formal definition is equivalent to the intuitive definition: X_i is obtained from X_{i-1} by choosing a wall in $\{0, \dots, n\}$ uniformly at random and, letting a particle jump across the ith wall with probabilities p if $\bullet|_i\circ$, q if $\circ|_i\bullet$, α if $|_0\circ$, β if $\bullet|_n$, γ if $|_0\bullet$, δ if $\circ|_n$.

The *asymmetry* in ASEP refers to the fact that in general $p \neq q$. Accordingly, the special cases $q = \gamma = \delta = 0$ of these two Markov chains are called TASEP for *totally* asymmetric simple exclusion process: in the TASEP, particles only travel forward (*ie* to the right).

Observe that if we multiply all parameters by a common factor then the $*$ term adjusts resulting in a slower or faster evolution of the chain but the relative probabilities are unchanged and so are the stationary distributions. In other terms, if π is a stationary distribution for T it is also for $T' = x \cdot \mathrm{id} + (1 - x)T$. As a consequence we always set $p = 1$.

In the TASEP case $q = \gamma = \delta = 0$, the further special case $\alpha = \beta = 1$ is called the *maximal current TASEP* (show that the expected number of particles that leave the system is indeed an increasing function of α and β).

Finally we call *partially* asymmetric exclusion process (PASEP) the special case $\gamma = \delta = 0$ but $q > 0$ of the ASEP with open boundary: particles can jump backward but can only enter from the left hand side and exit from right hand side.

3 Analysis of the TASEP

3.1 Periodic boundaries and the covering Markov chain method

The TASEP is clearly not reversible: particles keep jumping to the right... However, one can experimentally check that the TASEP with periodic boundary indeed converges to the uniform distribution.

Let us therefore try to apply Lemma 1 to prove this result. For the transition matrix of the TASEP to fit in this lemma we need a bijection ϕ such that for any configuration ω of the TASEP and any choice $i \in [1, n]$ of a wall $\phi(\omega, i) = (\omega', j)$ with ω' the configuration obtained by making a TASEP transition at position i in ω.

More precisely, if around wall i, $\omega = u| \bullet |_i \circ |v$, then we must have $\omega' = u| \circ |_i \bullet |v$, while otherwise $\omega' = \omega$. This suggest to take $\phi(\omega, i) = (\omega, i)$ unless $\omega = u| \bullet |_i \circ |v$, but in the other case we cannot take $\phi(\omega, i) = (\omega', i)$ since in ω', the wall i is of the form $\circ|\bullet$ (so that it should be a fix point of ϕ). Instead we take j to be the first wall of the form $\bullet|\circ$ on the righthand side of i in ω'.

More precisely, if

$$\omega = u|\bullet|_i \circ |\underbrace{\bullet|\cdots|\bullet}_{k}|_j \circ |v$$

with $j = i + k$ then let $\phi(\omega, i) = (\omega', j)$ with

$$\omega' = u|\circ|_i \bullet |\underbrace{\bullet|\cdots|\bullet}_{k}|_j \circ |v$$

(the transformation should be understood with cyclic boundaries: if $i + k > n$ then $j = i + k - n$ and the picture should be "wrapped around"). Then (ω, i) can be recovered uniquely from (ω', j) as the first wall of the form $\circ|\bullet$ on the left hand side of j. This implies that the application ϕ from $\Omega_{n,k} \times [1, n]$ into itself is a bijection and we can apply Lemma 1 to conclude:

Theorem 3 *The TASEP with periodic boundaries converges to the uniform distribution on $\Omega_{n,k}$.*

3.2 TASEP with open boundaries, binary trees, canopy and alternative tableaux

The TASEP with open boundary does not converge to the uniform distribution, even in the maximal current case. This can be checked already for $n = 2$, there are four configurations: make the picture! The stationary probability for $n = 2$ are found to be:

$$d(\circ\circ) = \frac{1}{5}, \quad d(\circ\bullet) = \frac{1}{5}, \quad d(\bullet\circ) = \frac{2}{5}, \quad d(\bullet\bullet) = \frac{1}{5}.$$

The stationary probability for $n = 3$ are found to be:

$$d(\circ\circ\circ) = \frac{1}{14}, \quad d(\circ\circ\bullet) = \frac{1}{14}, \quad d(\circ\bullet\circ) = \frac{2}{14}, \quad d(\circ\bullet\bullet) = \frac{1}{14},$$
$$d(\bullet\circ\circ) = \frac{3}{14}, \quad d(\bullet\circ\bullet) = \frac{2}{14}, \quad d(\bullet\bullet\circ) = \frac{3}{14}, \quad d(\bullet\bullet\bullet) = \frac{1}{14},$$

Derrida, Evans, Hakim and Pasquier obtained in 1993 the following expression of the stationary distribution.

Theorem 4 *Suppose that D and E are matrices and w and v are row and column vectors (not necessarily finite-dimensional), respectively, such that:*

$$DE = D + E, \quad w \cdot E = \alpha^{-1} w, \quad D \cdot v = \beta^{-1} v, \quad w \cdot v = 1.$$

Then the stationary probability that the TASEP with n sites is in state $\omega = (\tau_1, \ldots, \tau_n)$ is equal to

$$\frac{w \cdot \prod_{i=1}^{n} (\tau_i D + (1 - \tau_i) E) \cdot v}{w \cdot (D + E)^n \cdot v}.$$

For instance to compute the probability of the configuration $\bullet\circ\circ\bullet\circ$ we observe the following relation: $wDEEv = w(D+E)Ev = w(D+E)v + w(EE)v = \beta^{-1} + \alpha^{-1} + \alpha^{-2}$. In the maximal current case, $\alpha = \beta = 1$, we find $d(\bullet\circ\circ) = 3/Z_3$ and doing the same computation for all configurations with $n = 3$ we have $Z_3 = 14$.

Moreover there exists (infinite) matrices D and E satisfying the conditions.

This expression is quite amazing: there is absolutely no reason that such a matrix product form should exist, in particular the matrices here have no direct relation with the transition matrix of the chain.

In particular using this expression one can show that $Z_n(1, 1, 1) = \frac{1}{n+2}\binom{2n+2}{n+1}$.

Corollary 1 *In the maximal current case, the stationary probability that the TASEP on n sites is in a configuration with k black particles is*

$$\frac{\frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1}}{\frac{1}{n+2}\binom{2n+2}{n+1}}$$

Several combinatorial equivalent reformulations of the stationary distribution have been proposed using different Catalan counted structures. A very elegant one can be given in terms of *rooted binary trees.*

Definition 8 *A rooted binary tree of size n is a rooted plane tree with n vertices of arity 2 (called* inner nodes*) and $n+1$ leaves. The* canopy *of a rooted binary tree is obtained along a prefix order traversal writing* 0 *for left leaves and* 1 *for right leaves.*

The canopy of a binary tree of size n is a sequence of length $n+1$ starting with 0 and ending with 1. Let us denote $\mathcal{B}_n$ the set of binary trees with n inner nodes, and canopy(t) the canopy of tree t.

Theorem 5 *The stationary distribution of the maximal current TASEP is*

$$d(\omega) = \frac{|\text{canopy}^{-1}(0\omega 1)|}{|B_{n+1}|}.$$

This theorem can be deduced from the matrix expression of Derrida et al (Angel 2001). We shall instead give a direct combinatorial proof and extend the result to the PASEP. For this task we need alternative Catalan structures...

3.3 Alternative tableaux, tree-like tableaux and Catalan tableaux

Definition 9 *An* alternative tableau *of size n is a diagram consisting of a set of lozange unit cells between an arbitrary path of length n with steps in $\{u = (1,1), d = (1,-1)\}$ and a path of the form $d^i u^{n-i}$, tiled coherently with the following six types of cells:*

- *arrows or lines, which can be of type SW (south-west) or SE (south-east)*
- *crossings, corresponding to the superposition of two perpendicular lines,*
- *and empty cells.*

(A tiling is coherent if each arrow points to a sequence of tiles that extend the arrow in a straight line reaching the bottom path.)

In particular an arrow cannot point toward another arrow or an empty cell. Equivalently, alternative tableaux can be defined using only empty and arrow tiles, with the condition that an arrow cannot point toward another arrow.

Definition 10 *The* shape *$sh(T)$ of an alternative tableau T is given by the upper path, upon replacing u steps by 1 and d steps by 0. Let also $e(T)$ denote its number of empty cells, $\ell(T)$ the number of d steps of the bottom path that are not pointed to by an arrow, and $r(T)$ the number of u steps of the bottom path that are not pointed to by an arrow.*

Theorem 6 (Viennot 2007, Nadeau 2009) *The number of alternative tableau of size n is $n!$. The number of alternative tableau of size n without empty cells, also called* Catalan tableaux*, is $\frac{1}{(n+1)}\binom{2n}{n}$.*

In order to prove the theorem we shall use yet another representation of these tableaux: start with an alternative tableau T with bottom path $d^i u^{n-i}$ and add below ribbon of $n+1$ cell (one cell on the south-west of each d step, one cell on the south-east of each u step and one bottom cell connecting these two sequences to form a V shape). Put a dot in all the new cells that are not pointed to by an arrow and replace all arrows of T by dots. The result $T' = tlt(T)$ is a *tree-like tableau*, *i.e.* a (rotated) ferrer diagram with upper boundary $u \cdot sh(T) \cdot v$ and lower boundary $d^i + 1u^{n-i+1}$, filled with dots with the property that, apart from the bottom dot, each dot has dots below him in exactly one of the two diagonals.

Proposition 2 *The tlt construction is a bijection between alternative tableaux of size $n-1$ and tree-like tableaux of size $n+1$.*

The name tree-like tableau refers to the fact that upon connecting each dot to the closest dot below him in diagonal we obtain a rooted partial binary tree. Adding a leaf on the middle of each step of the upper path and connecting these in the same way we complete this tree into a rooted binary tree $tree(T')$ drawn inside the tree like tableau T'. In general there are edge crossings in the drawing of the tree. However if the original alternative tableau T is a Catalan tableau then the rooted binary tree $tree(T')$ drawn in the associated tree-like tableau $T' = tlt(T)$ is non-crossing.

Proposition 3 *Given a rooted binary tree t there is a unique tree-like tableau T' without crossings such that $tree(T') = t$. In particular Catalan tableaux of size n are in bijection with rooted binary trees with $n+2$ leaves. Moreover the upper border path of the tableau corresponds to the canopy of the tree.*

This completes the proof of the second assertion of Theorem 6.

The first assertion immediately follows from the following point insertion procedure due to Aval, Boussicault and Nadeau (2010).

Proposition 4 *There is a bijection between tree-like tableaux of size n with a marked edge on the upper boundary path and tree-like tableaux of size $n+1$.*

Let T be a tree-like tableau of size n and $i \in [0, n]$ the index of a step on the upper boundary of T. Let x be the leftmost dot of T without a cell at its north-east. The insertion is in two steps:

- first insert a pointed cell followed by an empty diagonal above the ith step: if it is a d step a south-west diagonal should be inserted, otherwise a south-east diagonal;
- if the inserted point y is on the left hand side of x, this new tree-like tableau is T'; otherwise T' is obtained by adding a ribbon of empty cells between x and y on the upper border of the tableau: this ensures that y is the leftmost point of T'.

The resulting tree-like tableau T' has size $n+1$ and the reverse construction consists in locating and deleting y together with its diagonal and the longest possible ribbon of empty cells on its left hand side.

3.4 Construction of a Markov chain on alternative tableaux

Let us define a Markov chain on the set of Catalan tableaux that projects on the TASEP and has uniform distribution.

Definition 11 *The chain $\mathcal{C} = (\Omega, T, d_0)$ is a* covering *of the chain $\mathcal{P} = (\Omega', T', d_0')$ (and $\mathcal{P}$ is a* projection *of $\mathcal{C}$) if there exists a mapping $\pi : \Omega \to \Omega'$ such that*

$$\tau'_{\omega_1', \omega_2'} = \sum_{\omega_2 \in \pi^{-1}(\omega_2')} \tau_{\omega_1, \omega_2} \quad \textit{for all } \omega_1 \in \pi^{-1}(\omega_1').$$

and $d_0'(\omega') = \sum_{\omega \in \pi^{-1}(\omega')} d_0(\omega)$. The mapping π is then called a projection *of $\mathcal{C}$ onto $\mathcal{P}$.*

Given a projection $\pi : \Omega \to \Omega'$ and a distribution d on Ω, let πd be the distribution defined as above by $\pi d(\omega') = \sum_{\omega \in \pi^{-1}(\omega')} d(\omega)$.

Theorem 7 *If d is a stationary distribution of a chain $\mathcal{C}$ which is a covering of a chain $\mathcal{P}$ via a projection π, then πd is a stationary distribution of the chain $\mathcal{P}$.*

Prove the theorem!

In order to apply this theorem to prove Theorem 5, we need to construct the markov chain on Catalan tableaux. The bijection Φ takes a Catalan tableaux T and an index i and maps (T, i) on a pair (T', j) obtained as follows:

- if the TASEP does nothing on $(sh(T), i)$ do nothing
- otherwise i is the index of a pair ud of steps of the upper border path of T'; the correponding cells contains an arrow pointing to a diagonal; we move this diagonal as illustrated by the MISSING PICTURE.

Proposition 5 Φ *is a bijection.*

3.5 Extension to the PASEP

The above Markov chain can be extended to deal with the PASEP in a very simple way: just replace Catalan tableaux by alternative tableaux. The only extra case to describe is when the selected index points to a pic covering an empty cell: in this case we remove the cell with probability $1 - q$ (and do nothing otherwise); conversely, when the selected index point to a valley we add an empty cell with probability q.

Theorem 8 (Corteel, Williams, 2005) *The stationary probability that the PASEP with parameter q, α, β is in configuration ω*

$$\frac{1}{Z_n} \sum_{T \in \mathrm{sh}^{-1}(\omega)} q^{e(T)} \alpha^{-\ell(T)} \beta^{-r(T)}$$

where

$$Z_n = \sum_{T} q^{e(T)} \alpha^{-\ell(T)} \beta^{-r(T)}$$

is the partition function of the model, i.e. *the generating polynomial of alternative tableaux of size n.*

3.6 Staircase tableaux and the general ASEP

A combinatorial description of the stationary distribution of the general case of the ASEP was only very recently obtained by Corteel and Williams using *staircase tableaux*. However its formulation is non symetric and the covering chain is not known: the proof is based on the matrix ansatz.