Multi-focusing on extensional rewriting with sums (introduction)

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$$\forall (K[A_1 + A_2] : B), \quad K[t] =_\eta \delta(t,\, x_1.K[\sigma_1\;x_1],\, x_2.K[\sigma_2\;x_2])$$

- Sum equivalence looks hard. Can we implement it?
- Are there representations of programs (proofs) that quotient over those equivalences?

My paper in one slide

The equivalence algorithm of

Sam Lindley.
Extensional rewriting with sums.
In *TLCA*, pages 255–271, 2007.

and the normalization of proof representations in

Kaustuv Chaudhuri, Dale Miller, and Alexis Saurin.
Canonical sequent proofs via multi-focusing.
In *IFIP TCS*, pages 383–396, 2008.

are doing (almost) the same thing
– and we had not noticed.

In this talk

Sam Lindley's rewriting-based algorithm is the first **simple** solution (first solution: Neil Ghani, 1995) to deciding sum equivalences.

It's easy to understand and follow. But to me it felt a bit arbitrary.

On the other hand, (multi-)focusing is beautiful, but requires some background knowledge.

Providing it is the purpose of this talk.

Sequent calculus

(Can be done in natural deduction, but less regular)

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C} -$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

$$\frac{\Gamma, A_i \vdash C}{\Gamma, A_1 * A_2 \vdash C} -$$

$$\frac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\Gamma \vdash A_1 * A_2}$$

$$\frac{\Gamma, A_1 \vdash C \qquad \Gamma, A_2 \vdash C}{\Gamma, A_1 + A_2 \vdash C}$$

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 + A_2} +$$

Inversible vs. non-inversible rules.

Negatives (interesting on the left): products, arrow, atoms.

Positives (interesting on the right): sum, atoms (or products).

Inversible phase

Focusing restriction 1: inversible phases

Inversible rules must be applied as soon and as long as possible
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– and their order does not matter.

Imposing this restriction gives a single proof of $(X \to Y) \to (X \to Y)$ instead of two ($\lambda(f)\, f$ and $\lambda(f)\, \lambda(x)\, f\ x$).

Non-inversible phases

After all inversible rules, $\Gamma_n \vdash A_p$

Only step forward: select a formula, apply some non-inversible rules on it.

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Focusing restriction 2: non-inversible phase

When a principal formula is selected for non-inversible rule, they should be applied as long as possible – until its polarity changes.

Completeness: this restriction preserves provability. **Non-trivial !**
Example of removed redundancy:

$$\dfrac{\dfrac{\dfrac{X_2, \qquad Y_1 \vdash A}{X_2 * X_3, \qquad Y_1 \vdash A}}{X_2 * X_3, \quad Y_1 * Y_2 \vdash A}}{X_1 * X_2 * X_3, Y_1 * Y_2 \vdash A}$$

This was focusing

Focused proofs are structured in alternating phases,
inversible (boring) and non-inversible (focus).

Phases are forced to be as long as possible – to eliminate duplicate proofs.

The idea is independent from the proof system.
Applies to sequent calculus or natural deduction;
intuitionistic, classical, linear, you-name-it logic.

On proof terms, these restrictions correspond to $\beta\eta$-normal forms (for products and arrows only). But the fun is in the search.

Restrictive syntax

So far we've defined **focused proofs** as a subset of proofs in our system. We can give them a syntax that enforces their structure.

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A + B \vdash C} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \times B} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

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$$\frac{X \text{ atomic}}{\Gamma_n, X \vdash X} \qquad \frac{\Gamma_{na}, [A_n] \vdash B_{pa}}{\Gamma_{na}, A_n \vdash B_{pa}} \qquad \frac{\Gamma_{na} \vdash [B_{pa}]}{\Gamma_{na} \vdash B_{pa}}$$

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$$\frac{\Gamma \vdash [A_i]}{\Gamma \vdash [A_1 + A_2]} \qquad \frac{\Gamma, [A_i] \vdash B}{\Gamma, [A_1 \times A_2] \vdash B} \qquad \frac{\Gamma \vdash [A] \quad \Gamma, [B] \vdash C}{\Gamma, [A \to B] \vdash C}$$

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$$\frac{\Gamma, A_{pa} \vdash B}{\Gamma, [A_{pa}] \vdash B} \qquad \frac{\Gamma \vdash B_{na}}{\Gamma \vdash [B_{na}]}$$

Success stories

Focusing was introduced by Andreoli in 1992.
Revolution in logic programming.

Forward-chaining and backward-chaining expressed in a single system by assigning polarities to atoms.

Syntethic connectives: state-of-the-art automated theorem proving for non-classical logics
(+ Jumbo connectives, Paul Blain Levy, 2006)

Lazy vs. strict evaluation (Zeilberger 2008)

A sequent calculus with cut-free search bisimilar to DPLL (Lengrand, 2013).

This is **not** the end

$$(X + X) \to X$$

$$(1 \to (X + X)) \to X$$

$$\lambda(f)\,\delta(f\ 1,\ x_1.x_1,\ x_1.x_1)$$

$$\lambda(f)\,\delta(f\ 1,\ x_1.\delta(f\ 1,\ x_2.x_2,\ x_2.x_2),\ x_1.x_1)$$

$$\lambda(f)\,\delta(f\ 1,\ x_1.x_1,\ x_1.\delta(f\ 1,\ x_2.x_1,\ x_2.x_2))$$

$$\ldots$$

Multi-focusing

Sometimes several independent foci are possible to make progress in a proof.

Multi-focusing (Miller and Saurin, 2007): do them all at once, in parallel.

$$\dfrac{\dfrac{\dfrac{X_2, \qquad Y_1 \vdash A}{\times X_2 X_3, \qquad Y_1 \vdash A}}{\times X_2 X_3, \quad \times Y_1 Y_2 \vdash A}}{\times X_1 \times X_2 X_3, \times Y_1 Y_2 \vdash A} \qquad \Rightarrow \qquad \dfrac{\dfrac{\dfrac{X_2, \qquad Y_1 \vdash A}{\times X_2 X_3, \qquad Y_1 \vdash A}}{\times X_2 X_3, \quad \times Y_1 Y_2 \vdash A}}{\times X_1 \times X_2 X_3, \times Y_1 Y_2 \vdash A}$$

$$\dfrac{\Gamma_{na}, [\Delta_n] \vdash B^?{}_{pa} \mid [C^?{}_{pa}]}{\Gamma_{na}, \Delta_n \vdash B^?{}_{pa} \mid C^?{}_{pa}}$$

Maximal multi-focusing

Given a focused proof, it is possible to put focused sequences in parallel to exhibit some parallelism – without changing the operational meaning of the proof, seen as a pure program.

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Maximally multi-focusing is a powerful notion of canonical structure for proof.

- linear logic: proof nets (Chaudhuri, Miller, Saurin, 2008)
- first-order classical logic: expansion proofs (Chaudhuri, Hetzl, Miller, 2013)

"Evolution rather than revolution" (Dale Miller)

Computing a maximal proof

Preemptive rewriting temporarily breaks the focused structure to move foci as far down as possible.

$$\left\{\begin{array}{cc} & \mathrm{I}_3 \\ & \mathrm{NI}_3 \\ \mathrm{I}_2 & \\ \mathrm{NI}_2 & \\ \mathrm{I}_1 & \\ \mathrm{NI}_1 & \end{array}\right\}$$

$$?\left\{\begin{array}{cc} \mathrm{I}_2 & \\ \mathrm{NI}_2 & \\ \mathrm{I}_1 & \mathrm{I}_3 \\ \mathrm{NI}_1 & \mathrm{NI}_3 \end{array}\right\}$$

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\end{array}\right\}
\to^*
\left\{\begin{array}{cc}
 & \mathrm{I}_3 \\
\mathrm{I}_2 & \\
 & \mathrm{NI}_3; (\mathrm{I}_2) \\
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\mathrm{I}_1 & \\
\mathrm{NI}_1 &
\end{array}\right\}
\to^*
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\rhd
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This is the heart of the correspondence with Sam Lindley's work

Contribution conclusion