Multi-focusing the λ-calculus

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Plan

1. Simply-typed λ-calculus
2. Focusing
3. Focused λ-calculus
4. (Maximal) Multi-focusing
5. First application: equivalence of (focused) λ-terms
6. Second application: which types have a unique inhabitant?

Section 1

Simply-typed λ-calculus

Simply-typed lambda-calculus

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.\, t : A \to B} \qquad \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B}$$

$$\frac{\begin{array}{c}\Gamma \vdash t_1 : A_1 \\ \Gamma \vdash t_2 : A_2\end{array}}{\Gamma \vdash (t_1, t_2) : A_1 \times A_2} \qquad \frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \pi_i\ t : A_i}$$

$$\frac{\Gamma \vdash t : A_i}{\Gamma \vdash \sigma_i\ t : A_1 + A_2} \qquad \frac{\Gamma \vdash t : A_1 + A_2 \qquad \begin{array}{c}\Gamma, x_1 : A_1 \vdash u_1 : C \\ \Gamma, x_2 : A_2 \vdash u_2 : C\end{array}}{\Gamma \vdash \texttt{match}\ t\ \texttt{with} \left| \begin{array}{l}\sigma_1\ x_1 \to u_1 \\ \sigma_2\ x_2 \to u_2\end{array}\right. : C}$$

$$\frac{}{\Gamma \vdash () : 1} \qquad \frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma \vdash t : 0}{\Gamma \vdash \texttt{absurd}(t) : A}$$

$\beta\eta$-equivalence

$$(\lambda x.\, t)\; u \;\triangleright_\beta\; t[u/x] \qquad\qquad \pi_i\;(t_1, t_2) \;\triangleright_\beta\; t_i$$

$$\texttt{match}\; \sigma_i\; t\; \texttt{with} \left| \begin{array}{l} \sigma_1\; x_1 \to u_1 \\ \sigma_2\; x_2 \to u_2 \end{array} \right. \;\triangleright_\beta\; u_i[t/x_i]$$

$$\frac{\Gamma \vdash t : A \to B}{t \triangleright_\eta \lambda x.\,(t\; x)} \qquad \frac{\Gamma \vdash t : A_1 \times A_2}{t \triangleright_\eta (\pi_1\; t, \pi_2\; t)} \qquad \frac{\Gamma \vdash t : 1}{t \triangleright_\eta ()}$$

$$\frac{\Gamma \vdash t : A_1 + A_2 \qquad \Gamma \vdash D\,[t] : C}{D\,[t] \triangleright_\eta \begin{array}{l}\texttt{match}\; t\; \texttt{with} \\ \left| \begin{array}{l} \sigma_1\; x_1 \to D\,[\sigma_1\; x_1] \\ \sigma_2\; x_2 \to D\,[\sigma_2\; x_2] \end{array} \right.\end{array}} \qquad \frac{\Gamma \vdash t : 0 \qquad \Gamma \vdash D\,[t] : C}{D\,[t] \triangleright_\eta \texttt{absurd}(t)}$$

(Derived equalities:)

$\beta\eta$-equivalence

$$(\lambda x.\, t)\ u \ \triangleright_\beta\ t[u/x] \qquad\qquad \pi_i\ (t_1, t_2)\ \triangleright_\beta\ t_i$$

$$\texttt{match}\ \sigma_i\ t\ \texttt{with}\ \left|\begin{array}{l}\sigma_1\ x_1 \to u_1\\ \sigma_2\ x_2 \to u_2\end{array}\right. \ \triangleright_\beta\ u_i[t/x_i]$$

$$\frac{\Gamma \vdash t : A \to B}{t \triangleright_\eta \lambda x.\, (t\ x)} \qquad \frac{\Gamma \vdash t : A_1 \times A_2}{t \triangleright_\eta (\pi_1\ t, \pi_2\ t)} \qquad \frac{\Gamma \vdash t : 1}{t \triangleright_\eta ()}$$

$$\frac{\Gamma \vdash t : A_1 + A_2 \qquad \Gamma \vdash D\,[t] : C}{\begin{array}{l}\qquad\texttt{match}\ t\ \texttt{with}\\ D\,[t] \triangleright_\eta \left|\begin{array}{l}\sigma_1\ x_1 \to D\,[\sigma_1\ x_1]\\ \sigma_2\ x_2 \to D\,[\sigma_2\ x_2]\end{array}\right.\end{array}} \qquad \frac{\Gamma \vdash t : 0 \qquad \Gamma \vdash D\,[t] : C}{D\,[t] \triangleright_\eta \texttt{absurd}(t)}$$

(Derived equalities:) $$\frac{\Gamma \vdash t, u : 1}{\Gamma \vdash t \approx_\eta u : 1}$$

$\beta\eta$-equivalence

$$(\lambda x.\, t)\ u \ \rhd_\beta\ t[u/x] \qquad\qquad \pi_i\ (t_1, t_2)\ \rhd_\beta\ t_i$$

$$\texttt{match}\ \sigma_i\ t\ \texttt{with} \left|\begin{array}{l} \sigma_1\ x_1 \to u_1 \\ \sigma_2\ x_2 \to u_2 \end{array}\right. \ \rhd_\beta\ u_i[t/x_i]$$

$$\frac{\Gamma \vdash t : A \to B}{t \rhd_\eta \lambda x.\,(t\ x)} \qquad \frac{\Gamma \vdash t : A_1 \times A_2}{t \rhd_\eta (\pi_1\ t, \pi_2\ t)} \qquad \frac{\Gamma \vdash t : 1}{t \rhd_\eta ()}$$

$$\frac{\Gamma \vdash t : A_1 + A_2 \qquad \Gamma \vdash D\,[t] : C}{D\,[t] \rhd_\eta \begin{array}{l}\texttt{match}\ t\ \texttt{with} \\ \left|\begin{array}{l} \sigma_1\ x_1 \to D\,[\sigma_1\ x_1] \\ \sigma_2\ x_2 \to D\,[\sigma_2\ x_2] \end{array}\right.\end{array}} \qquad \frac{\Gamma \vdash t : 0 \qquad \Gamma \vdash D\,[t] : C}{D\,[t] \rhd_\eta \texttt{absurd}(t)}$$

(Derived equalities:)

$$\frac{\Gamma \vdash t, u : 1}{\Gamma \vdash t \approx_\eta u : 1} \qquad \frac{\Gamma \vdash t : 0 \qquad \Gamma \vdash u_1, u_2 : A}{\Gamma \vdash u_1 \approx_\eta u_2 : A}$$

β-normal forms

Easy in the **negative** fragment $(\to, \times, 1)$. Values and neutrals.

$$v ::= \text{values} \mid \lambda x.\, v \mid (v_1, v_2) \mid n \qquad n ::= \text{neutrals} \mid x \mid \pi_i\ n \mid n\ v$$

Problem: no clear way to add **positives** $(+, 0)$.

β-normal forms

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$$v ::= \text{values} \mid \lambda x.\, v \mid (v_1, v_2) \mid n \mid () \mid \sigma_i\ v \;(?)$$

$$n ::= \text{neutrals} \mid x \mid \pi_i\ n \mid n\ v \mid \texttt{absurd}(n)$$

$$? \mid \texttt{match}\ n\ \texttt{with} \left| \begin{array}{l} \sigma_1\ x_1 \to v_1 \\ \sigma_2\ x_2 \to v_2 \end{array} \right.$$

$$? \mid \texttt{match}\ n\ \texttt{with} \left| \begin{array}{l} \sigma_1\ x_1 \to n_1 \\ \sigma_2\ x_2 \to n_2 \end{array} \right.$$

β-normal forms

Easy in the **negative** fragment $(\to, \times, 1)$. Values and neutrals.

$$\begin{array}{rcl} v & ::= & \text{values} \\ & | & \lambda x.\, v \\ & | & (v_1, v_2) \\ & | & n \end{array} \qquad \begin{array}{rcl} n & ::= & \text{neutrals} \\ & | & x \\ & | & \pi_i\ n \\ & | & n\ v \end{array}$$

Problem: no clear way to add **positives** $(+, 0)$.

$$\begin{array}{rcl} v & ::= & \text{values} \\ & | & \lambda x.\, v \\ & | & (v_1, v_2) \\ & | & n \\ & | & () \\ ? & | & \sigma_i\ v \end{array} \qquad \begin{array}{rcl} n & ::= & \text{neutrals} \\ & | & x \\ & | & \pi_i\ n \\ & | & n\ v \\ & | & \mathtt{absurd}(n) \\ ? & | & \mathtt{match}\ n\ \mathtt{with} \left| \begin{array}{l} \sigma_1\ x_1 \to v_1 \\ \sigma_2\ x_2 \to v_2 \end{array} \right. \\ ? & | & \mathtt{match}\ n\ \mathtt{with} \left| \begin{array}{l} \sigma_1\ x_1 \to n_1 \\ \sigma_2\ x_2 \to n_2 \end{array} \right. \end{array}$$

(Remark on System L)

Section 2

Focusing

A discipline to remove some redundancy in proof representations.

$\lambda \Longrightarrow$ sequents

$$\frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \pi_i\ t : A_i} \qquad \Rightarrow \qquad \frac{\Gamma \vdash A_1 \times A_2}{\Gamma \vdash A_i} \qquad \Rightarrow \qquad \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \times A_2 \vdash C}$$

$(,)$ is **non**-disjoint union

Sequent calculus

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C} -$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

$$\frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \times A_2 \vdash C} -$$

$$\frac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\Gamma \vdash A_1 \times A_2}$$

$$\frac{\Gamma, A_1 \vdash C \qquad \Gamma, A_2 \vdash C}{\Gamma, A_1 + A_2 \vdash C}$$

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 + A_2} +$$

$$\frac{}{\Gamma, 0 \vdash C} +$$

$$\frac{}{\Gamma \vdash 1} -$$

Invertible vs. non-invertible rules. Positives vs. negatives.

Invertible phase

$$\frac{\dfrac{?}{X + Y \vdash X}}{X + Y \vdash Y + X}$$

If applied too early, non-invertible rules can ruin your proof.

Focusing restriction 1: invertible phases

Invertible rules must be applied as soon and as long as possible
– and their order does not matter.

Invertible phase

$$\dfrac{\dfrac{?}{X + Y \vdash X}}{X + Y \vdash Y + X}$$

If applied too early, non-invertible rules can ruin your proof.

Focusing restriction 1: invertible phases

Invertible rules must be applied as soon and as long as possible
– and their order does not matter.

Imposing this restriction gives a single proof of $(X \to Y) \to (X \to Y)$ instead of two ($\lambda f.\, f$ and $\lambda f.\, \lambda x.\, f\ x$).

After all invertible rules, negative context, positive goal.

Non-invertible phases

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Only step forward: select a formula, apply some non-invertible rules on it.

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Focusing restriction 2: non-invertible phase

When a principal formula is selected for non-invertible rule, they should be applied as long as possible – until its polarity changes.

Non-invertible phases

After all invertible rules, negative context, positive goal.

Only step forward: select a formula, apply some non-invertible rules on it.

Focusing restriction 2: non-invertible phase

When a principal formula is selected for non-invertible rule, they should be applied as long as possible – until its polarity changes.

Completeness: this restriction preserves provability. **Non-trivial !**
Example of removed redundancy:

$$\dfrac{\dfrac{\dfrac{X_2, \quad Y_1 \vdash A}{X_2 \times X_3, \quad Y_1 \vdash A}}{X_2 \times X_3, \quad Y_1 \times Y_2 \vdash A}}{X_1 \times X_2 \times X_3, Y_1 \times Y_2 \vdash A}$$

Demo Time

$$\overline{\vdash (1 \to X \to (Y + Z)) \to X \to (Y \to W) \to (Z + W)}$$

invertible rules

Demo Time

$$\frac{}{(1 \to X \to (Y + Z)) \vdash X \to (Y \to W) \to (Z + W)}$$

invertible rules

Demo Time

$$\frac{}{(1 \to X \to (Y + Z)),\quad X \vdash (Y \to W) \to (Z + W)}$$

invertible rules

Demo Time

$$\frac{}{(1 \to X \to (Y + Z)), \quad X, \quad Y \to W \vdash Z + W}$$

invertible rules

Demo Time

$$\frac{}{(1 \to X \to (Y + Z)), \quad X, \quad Y \to W \ \vdash \ Z + W}$$

choice of focus

Demo Time

$$\frac{}{(1 \to X \to (Y + Z)), \quad X, \quad Y \to W \vdash Z + W}$$

choice of focus

Demo Time

$$\frac{}{(1 \to X \to (Y + Z)), \quad X, \quad Y \to W \vdash Z + W}$$

non-invertible rules

Demo Time

$$\frac{}{(1 \to X \to (Y + Z)), \quad X, \quad Y \to W \ \vdash \ Z + W}$$

non-invertible rules

Demo Time

$$\frac{}{(1 \to X \to (Y + Z)),\quad X,\quad Y \to W \vdash Z + W}$$

invertible rules

Demo Time

$$\frac{Y, Y \to W \vdash Z + W \qquad Z \vdash Z + W}{(1 \to X \to (Y + Z)),\quad X,\quad Y \to W \vdash Z + W}$$

invertible rules

Demo Time

$$\frac{Y, Y \to W \vdash Z + W \qquad Z \vdash Z + W}{(1 \to X \to (Y + Z)),\quad X,\quad Y \to W \vdash Z + W}$$

choice of focus

Demo Time

$$\frac{Y, Y \to W \vdash Z + W \qquad Z \vdash Z + W}{(1 \to X \to (Y + Z)),\quad X,\quad Y \to W \vdash Z + W}$$

conclusion

This was focusing

Focused proofs are structured in alternating phases,
invertible (boring) and non-invertible (focus).

Phases are forced to be as long as possible – to eliminate duplicate proofs.

The idea is independent from the proof system.
Applies to sequent calculus or natural deduction;
intuitionistic, classical, linear, you-name-it logic.

Positive and negative types

A, B	$::= P \mid N \mid X$	all types
N, M	$::=$	negative types
	$\mid A \to B$	function type
	$\mid A \times B$	product
	$\mid 1$	unit
P, Q	$::=$	positive types
	$\mid A + B$	sum
	$\mid 0$	empty
$P_{\mathsf{at}}, Q_{\mathsf{at}}$	$::= P \mid X$	positive or atomic type
$N_{\mathsf{at}}, M_{\mathsf{at}}$	$::= N \mid X$	negative or atomic type

Positive and negative types

Let's pretend that atoms are given a polarity.

$$
\begin{array}{llll}
A, B & ::= & P \mid N \mid X & \text{all types} \\
 & & & \\
N, M & ::= & & \text{negative types} \\
 & & \mid A \to B & \text{function type} \\
 & & \mid A \times B & \text{product} \\
 & & \mid 1 & \text{unit} \\
 & & & \\
P, Q & ::= & & \text{positive types} \\
 & & \mid A + B & \text{sum} \\
 & & \mid 0 & \text{empty} \\
 & & & \\
P_{at}, Q_{at} & ::= & P \mid X & \text{positive or atomic type} \\
N_{at}, M_{at} & ::= & N \mid X & \text{negative or atomic type}
\end{array}
$$

Positive and negative types

A, B	$::= P \mid N$	all types
N, M	$::=$	negative types
	$\mid A \to B$	function type
	$\mid A \times B$	product
	$\mid 1$	unit
	$\mid X^-$	negative atom
P, Q	$::=$	positive types
	$\mid A + B$	sum
	$\mid 0$	empty
	$\mid X^+$	positive atom
$P_{\mathsf{at}}, Q_{\mathsf{at}}$	$::= P \mid X^-$	positive or atomic type
$N_{\mathsf{at}}, M_{\mathsf{at}}$	$::= N \mid X^+$	negative or atomic type

Structural presentation (1)

$$\frac{\Gamma, A \vdash_{\mathsf{inv}} B}{\Gamma \vdash_{\mathsf{inv}} A \to B} \qquad \frac{\Gamma, A_1 \vdash_{\mathsf{inv}} B \qquad \Gamma, A_2 \vdash_{\mathsf{inv}} B}{\Gamma, A_1 + A_2 \vdash_{\mathsf{inv}} B}$$

$$\frac{\Gamma \vdash_{\mathsf{inv}} B_1 \qquad \Gamma \vdash_{\mathsf{inv}} B_2}{\Gamma \vdash_{\mathsf{inv}} B_1 \times B_2} \qquad \frac{}{\Gamma, 0 \vdash_{\mathsf{inv}} B} \qquad \frac{}{\Gamma \vdash_{\mathsf{inv}} 1}$$

$$\frac{\Gamma \text{ negative or atomic} \qquad \Gamma \vdash_{\mathsf{foc}} B \qquad B \text{ positive or atomic}}{\Gamma \vdash_{\mathsf{inv}} B}$$

Structural presentation (2)

$$\frac{\Gamma \vdash_{\mathsf{foc.r}} [B]}{\Gamma \vdash_{\mathsf{foc}} B} \qquad \frac{\Gamma, [A] \vdash_{\mathsf{foc.l}} B}{\Gamma, A \vdash_{\mathsf{foc}} B}$$

Structural presentation (2)

$$\frac{\Gamma \vdash_{\mathsf{foc.r}} [B]}{\Gamma \vdash_{\mathsf{foc}} B} \qquad \frac{\Gamma, [A] \vdash_{\mathsf{foc.l}} B}{\Gamma, A \vdash_{\mathsf{foc}} B}$$

$$\frac{\Gamma \vdash_{\mathsf{foc.r}} [B_i]}{\Gamma \vdash_{\mathsf{foc.r}} [B_1 + B_2]} \qquad \frac{}{\Gamma, X^+ \vdash_{\mathsf{foc.r}} [X^+]} \qquad \frac{B \text{ negative} \qquad \Gamma \vdash_{\mathsf{inv}} B}{\Gamma \vdash_{\mathsf{foc.r}} [B]}$$

Structural presentation (2)

$$\frac{\Gamma \vdash_{\text{foc.r}} [B]}{\Gamma \vdash_{\text{foc}} B} \qquad \frac{\Gamma, [A] \vdash_{\text{foc.l}} B}{\Gamma, A \vdash_{\text{foc}} B}$$

$$\frac{\Gamma \vdash_{\text{foc.r}} [B_i]}{\Gamma \vdash_{\text{foc.r}} [B_1 + B_2]} \qquad \frac{}{\Gamma, X^+ \vdash_{\text{foc.r}} [X^+]} \qquad \frac{B \text{ negative} \qquad \Gamma \vdash_{\text{inv}} B}{\Gamma \vdash_{\text{foc.r}} [B]}$$

$$\frac{\Gamma, [A_i] \vdash_{\text{foc.l}} B}{\Gamma, [A_1 \times A_2] \vdash_{\text{foc.l}} B} \qquad \frac{\Gamma \vdash_{\text{foc.r}} [B] \qquad \Gamma, [A] \vdash_{\text{foc.l}} C}{\Gamma, [B \to A] \vdash_{\text{foc.l}} C}$$

$$\frac{}{\Gamma, [X^-] \vdash_{\text{foc.l}} X^-} \qquad \frac{A \text{ positive} \qquad \Gamma, A \vdash_{\text{inv}} B}{\Gamma, [A] \vdash_{\text{foc.l}} B}$$

Section 3

Focused λ-calculus

Two angles:

1. focusing in natural deduction
2. normal forms for a mixed-polarity world

Normality without loss of generality

Any ($t : A \to B$) may be η-expanded into a λ-abstraction: $\lambda x. (t\ x)$

We can ask all (open) values of type ($A \to B$) to be of the form $\lambda x. t$

We **cannot** ask all (open) values of type $A + B$ to be of the form $\sigma_i\ t$

$$x : X + Y \vdash ? : Y + X$$

An invertible rule is an **inversion** principle.

Focused normal forms: invertible rules

Inversion judgment ($\Gamma \vdash_{\mathsf{inv}} v : A$).

$$\frac{\Gamma, x : A \vdash_{\mathsf{inv}} v : B}{\Gamma \vdash_{\mathsf{inv}} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \vdash_{\mathsf{inv}} v_1 : A_1 \qquad \Gamma \vdash_{\mathsf{inv}} v_2 : A_2}{\Gamma \vdash_{\mathsf{inv}} (v_1, v_2) : A_1 \times A_2}$$

Focused normal forms: invertible rules

Inversion judgment ($\Gamma \vdash_{\text{inv}} v : A$).

$$\frac{\Gamma, x : A \vdash_{\text{inv}} v : B}{\Gamma \vdash_{\text{inv}} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \vdash_{\text{inv}} v_1 : A_1 \qquad \Gamma \vdash_{\text{inv}} v_2 : A_2}{\Gamma \vdash_{\text{inv}} (v_1, v_2) : A_1 \times A_2}$$

$$\frac{\Gamma, x : A_1 \vdash_{\text{inv}} v_1 : C \qquad \Gamma, x : A_2 \vdash_{\text{inv}} v_2 : C}{\Gamma, x : A_1 + A_2 \vdash_{\text{inv}} \texttt{match}\ x\ \texttt{with}\ \mid \sigma_1\, x \to v_1 \mid \sigma_2\, x \to v_2 : C}$$

Focused normal forms: invertible rules

Inversion judgment ($\Gamma \vdash_{\mathsf{inv}} v : A$).

$$\frac{\Gamma, x : A \vdash_{\mathsf{inv}} v : B}{\Gamma \vdash_{\mathsf{inv}} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \vdash_{\mathsf{inv}} v_1 : A_1 \qquad \Gamma \vdash_{\mathsf{inv}} v_2 : A_2}{\Gamma \vdash_{\mathsf{inv}} (v_1, v_2) : A_1 \times A_2}$$

$$\frac{\Gamma, x : A_1 \vdash_{\mathsf{inv}} v_1 : C \qquad \Gamma, x : A_2 \vdash_{\mathsf{inv}} v_2 : C}{\Gamma, x : A_1 + A_2 \vdash_{\mathsf{inv}} \mathtt{match}\ x\ \mathtt{with}\ \mid \sigma_1\, x \to v_1 \mid \sigma_2\, x \to v_2 : C}$$

$$D[x] \rhd_\eta \mathtt{match}\ x\ \mathtt{with} \left|\begin{array}{l} \sigma_1\, x_1 \to D[\sigma_1\, x_1] \\ \sigma_2\, x_2 \to D[\sigma_2\, x_2] \end{array}\right.$$

Focused normal forms: invertible rules

Inversion judgment ($\Gamma \vdash_{\text{inv}} v : A$).

$$\frac{\Gamma, x : A \vdash_{\text{inv}} v : B}{\Gamma \vdash_{\text{inv}} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \vdash_{\text{inv}} v_1 : A_1 \qquad \Gamma \vdash_{\text{inv}} v_2 : A_2}{\Gamma \vdash_{\text{inv}} (v_1, v_2) : A_1 \times A_2}$$

$$\frac{\Gamma, x : A_1 \vdash_{\text{inv}} v_1 : C \qquad \Gamma, x : A_2 \vdash_{\text{inv}} v_2 : C}{\Gamma, x : A_1 + A_2 \vdash_{\text{inv}} \texttt{match}\ x\ \texttt{with}\ \mid \sigma_1\, x \to v_1 \mid \sigma_2\, x \to v_2 : C}$$

$$\frac{}{\Gamma \vdash_{\text{inv}} () : 1} \qquad \frac{}{\Gamma, x : 0 \vdash_{\text{inv}} \texttt{absurd}(x) : A}$$

Focused normal forms: invertible rules

Inversion judgment ($\Gamma \vdash_{\text{inv}} v : A$).

$$\frac{\Gamma, x : A \vdash_{\text{inv}} v : B}{\Gamma \vdash_{\text{inv}} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \vdash_{\text{inv}} v_1 : A_1 \qquad \Gamma \vdash_{\text{inv}} v_2 : A_2}{\Gamma \vdash_{\text{inv}} (v_1, v_2) : A_1 \times A_2}$$

$$\frac{\Gamma, x : A_1 \vdash_{\text{inv}} v_1 : C \qquad \Gamma, x : A_2 \vdash_{\text{inv}} v_2 : C}{\Gamma, x : A_1 + A_2 \vdash_{\text{inv}} \texttt{match}\ x\ \texttt{with}\ \mid \sigma_1\, x \to v_1 \mid \sigma_2\, x \to v_2 : C}$$

$$\frac{}{\Gamma \vdash_{\text{inv}} () : 1} \qquad \frac{}{\Gamma, x : 0 \vdash_{\text{inv}} \texttt{absurd}(x) : A}$$

$$\frac{\Gamma \text{ negative or atomic} \qquad \Gamma \vdash_{\text{foc}} v : A \qquad A \text{ positive or atomic}}{\Gamma \vdash_{\text{inv}} v : A}$$

Focused normal forms: invertible rules

Inversion judgment ($\Gamma \vdash_{\text{inv}} v : A$).

$$\frac{\Gamma, x : A \vdash_{\text{inv}} v : B}{\Gamma \vdash_{\text{inv}} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \vdash_{\text{inv}} v_1 : A_1 \qquad \Gamma \vdash_{\text{inv}} v_2 : A_2}{\Gamma \vdash_{\text{inv}} (v_1, v_2) : A_1 \times A_2}$$

$$\frac{\Gamma, x : A_1 \vdash_{\text{inv}} v_1 : C \qquad \Gamma, x : A_2 \vdash_{\text{inv}} v_2 : C}{\Gamma, x : A_1 + A_2 \vdash_{\text{inv}} \texttt{match}\ x\ \texttt{with}\ \mid \sigma_1\, x \to v_1 \mid \sigma_2\, x \to v_2 : C}$$

$$\frac{}{\Gamma \vdash_{\text{inv}} () : 1} \qquad \frac{}{\Gamma, x : 0 \vdash_{\text{inv}} \texttt{absurd}(x) : A}$$

$$\frac{\Gamma \text{ negative or atomic} \qquad \Gamma \vdash_{\text{foc}} v : A \qquad A \text{ positive or atomic}}{\Gamma \vdash_{\text{inv}} v : A}$$

(discuss ordering)

Focused normal forms: non-invertible rules (right)

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash_{\mathsf{inv}} v : N}{\Gamma \vdash_{\mathsf{foc.r}} [N]}}{\Gamma \vdash_{\mathsf{foc.r}} [N + 1]}}{\Gamma \vdash_{\mathsf{foc.r}} [0 + (N + 1)]}}{\Gamma \vdash_{\mathsf{foc}} 0 + (N + 1)}$$

Focused normal forms: non-invertible rules (right)

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash_{\mathsf{inv}} v : N}{\Gamma \vdash_{\mathsf{foc.r}} [N]}}{\Gamma \vdash_{\mathsf{foc.r}} [N+1]}}{\Gamma \vdash_{\mathsf{foc.r}} [0+(N+1)]}}{\Gamma \vdash_{\mathsf{foc}} 0+(N+1)}$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash_{\mathsf{inv}} v : N}{\Gamma \vdash_{\mathsf{ne}} v \Uparrow N}}{\Gamma \vdash_{\mathsf{ne}} \sigma_1\ v \Uparrow N+1}}{\Gamma \vdash_{\mathsf{ne}} \sigma_2\ (\sigma_1\ v) \Uparrow 0+(N+1)}$$

Non-invertible right-introductions: positive constructors.

Focused normal forms: non-invertible rules (right)

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash_{\text{inv}} v : N}{\Gamma \vdash_{\text{foc.r}} [N]}}{\Gamma \vdash_{\text{foc.r}} [N + 1]}}{\Gamma \vdash_{\text{foc.r}} [0 + (N + 1)]}}{\Gamma \vdash_{\text{foc}} 0 + (N + 1)}$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash_{\text{inv}} v : N}{\Gamma \vdash_{\text{ne}} v \Uparrow N}}{\Gamma \vdash_{\text{ne}} \sigma_1\ v \Uparrow N + 1}}{\Gamma \vdash_{\text{ne}} \sigma_2\ (\sigma_1\ v) \Uparrow 0 + (N + 1)}$$

Non-invertible right-introductions: positive constructors.

$$\dfrac{\Gamma \vdash_{\text{ne}} p \Uparrow A}{\Gamma \vdash_{\text{ne}} \sigma_i\ p \Uparrow A_1 + A_2} \qquad \dfrac{(x : X^+) \in \Gamma}{\Gamma \vdash_{\text{ne}} x \Uparrow X^+} \qquad \dfrac{\Gamma \vdash_{\text{inv}} v : N \qquad N \text{ negative}}{\Gamma \vdash_{\text{ne}} v \Uparrow N}$$

Focused normal forms: non-invertible rules (left)

$$\dfrac{\Gamma \vdash_{\mathsf{foc.r}} [P] \qquad \dfrac{\dfrac{\Gamma, Q_2 \vdash_{\mathsf{inv}} A}{\Gamma, [Q_2] \vdash_{\mathsf{foc.l}} A}}{\Gamma, [Q_1 \times Q_2] \vdash_{\mathsf{foc.l}} A}}{\dfrac{\Gamma, [P \to (Q_1 \times Q_2)] \vdash_{\mathsf{foc.l}} A}{\Gamma \ni x : P \to (Q_1 \times Q_2) \vdash_{\mathsf{foc}} A}}$$

Focused normal forms: non-invertible rules (left)

$$\dfrac{\Gamma \vdash_{\mathsf{foc.r}} [P] \qquad \dfrac{\dfrac{\Gamma, Q_2 \vdash_{\mathsf{inv}} A}{\Gamma, [Q_2] \vdash_{\mathsf{foc.l}} A}}{\Gamma, [Q_1 \times Q_2] \vdash_{\mathsf{foc.l}} A}}{\dfrac{\Gamma, [P \to (Q_1 \times Q_2)] \vdash_{\mathsf{foc.l}} A}{\Gamma \ni x : P \to (Q_1 \times Q_2) \vdash_{\mathsf{foc}} A}}$$

$$\dfrac{\dfrac{\dfrac{(x : P \to (Q_1 \times Q_2)) \in \Gamma}{\Gamma \vdash_{\mathsf{ne}} x \Downarrow P \to (Q_1 \times Q_2)} \qquad \Gamma \vdash_{\mathsf{ne}} p \Uparrow P}{\Gamma \vdash_{\mathsf{ne}} x\ p \Downarrow Q_1 \times Q_2}}{\Gamma \vdash_{\mathsf{ne}} \pi_2\ (x\ p) \Downarrow Q_2}$$

Non-invertible left-introductions: negative destructors/eliminators.

Focused normal forms: non-invertible rules (left)

$$\dfrac{\Gamma \vdash_{\text{foc.r}} [P] \qquad \dfrac{\dfrac{\Gamma, Q_2 \vdash_{\text{inv}} A}{\Gamma, [Q_2] \vdash_{\text{foc.l}} A}}{\Gamma, [Q_1 \times Q_2] \vdash_{\text{foc.l}} A}}{\dfrac{\Gamma, [P \to (Q_1 \times Q_2)] \vdash_{\text{foc.l}} A}{\Gamma \ni x : P \to (Q_1 \times Q_2) \vdash_{\text{foc}} A}}$$

$$\dfrac{\dfrac{\dfrac{(x : P \to (Q_1 \times Q_2)) \in \Gamma}{\Gamma \vdash_{\text{ne}} x \Downarrow P \to (Q_1 \times Q_2)} \qquad \Gamma \vdash_{\text{ne}} p \Uparrow P}{\Gamma \vdash_{\text{ne}} x\ p \Downarrow Q_1 \times Q_2}}{\Gamma \vdash_{\text{ne}} \pi_2\ (x\ p) \Downarrow Q_2}$$

Non-invertible left-introductions: negative destructors/eliminators.

$$\dfrac{(x : N) \in \Gamma}{\Gamma \vdash_{\text{ne}} x \Downarrow N} \qquad \dfrac{\Gamma \vdash_{\text{ne}} n \Downarrow A_1 \times A_2}{\Gamma \vdash_{\text{ne}} \pi_i\ n \Downarrow A_i} \qquad \dfrac{\Gamma \vdash_{\text{ne}} n \Downarrow A \to B \qquad \Gamma \vdash_{\text{ne}} p \Uparrow A}{\Gamma \vdash_{\text{ne}} n\ p \Downarrow B}$$

Focused normal forms: choosing a focus

On the right, rien à signaler.

$$\frac{\Gamma \vdash_{\mathsf{foc.r}} [P]}{\Gamma \vdash_{\mathsf{foc}} P} \qquad\qquad \frac{\Gamma \vdash_{\mathsf{ne}} p \Uparrow P}{\Gamma \vdash_{\mathsf{foc}} p : P}$$

Focused normal forms: choosing a focus

On the right, rien à signaler.

$$\frac{\Gamma \vdash_{\mathsf{foc.r}} [P]}{\Gamma \vdash_{\mathsf{foc}} P} \qquad \frac{\Gamma \vdash_{\mathsf{ne}} p \Uparrow P}{\Gamma \vdash_{\mathsf{foc}} p : P}$$

Left is upside-down: starting a left focus corresponds to sequent release rules.

$$\frac{N \in \Gamma \qquad \Gamma, [N] \vdash_{\mathsf{foc.l}} A}{\Gamma \vdash_{\mathsf{foc}} A} \qquad \frac{}{\Gamma, [X^-] \vdash_{\mathsf{foc.l}} X^-} \qquad \frac{\Gamma, P \vdash_{\mathsf{inv}} A}{\Gamma, [P] \vdash_{\mathsf{foc.l}} A}$$

Focused normal forms: choosing a focus

On the right, rien à signaler.

$$\frac{\Gamma \vdash_{\text{foc.r}} [P]}{\Gamma \vdash_{\text{foc}} P} \qquad \frac{\Gamma \vdash_{\text{ne}} p \Uparrow P}{\Gamma \vdash_{\text{foc}} p : P}$$

Left is upside-down: starting a left focus corresponds to sequent release rules.

$$\frac{N \in \Gamma \qquad \Gamma, [N] \vdash_{\text{foc.l}} A}{\Gamma \vdash_{\text{foc}} A} \qquad \frac{}{\Gamma, [X^-] \vdash_{\text{foc.l}} X^-} \qquad \frac{\Gamma, P \vdash_{\text{inv}} A}{\Gamma, [P] \vdash_{\text{foc.l}} A}$$

$$\frac{\Gamma \vdash_{\text{ne}} n \Downarrow X^-}{\Gamma \vdash_{\text{foc}} n : X^-} \qquad \frac{\Gamma \vdash_{\text{ne}} n \Downarrow P \qquad \Gamma, x : P \vdash_{\text{inv}} v : A}{\Gamma \vdash_{\text{foc}} \texttt{let}\ x = n\ \texttt{in}\ v : A}$$

Focused normal forms: summary

(Grammar with type annotations)

$$
\begin{aligned}
v ::=\ & \text{values} \\
\mid\ & \lambda x.\, v \\
\mid\ & (v_1, v_2) \\
\mid\ & \texttt{match}\ x\ \texttt{with} \left|\begin{array}{l} \sigma_1\ x \to v_1 \\ \sigma_2\ x \to v_2 \end{array}\right. \\
\mid\ & \texttt{absurd}(x) \\
\mid\ & (f : P_{\text{at}})
\end{aligned}
$$

$$
\begin{aligned}
f ::=\ & \text{focused forms} \\
\mid\ & \texttt{let}\ (x : P) = n\ \texttt{in}\ v \\
\mid\ & (n : X^-) \\
\mid\ & (p : P)
\end{aligned}
$$

$$
\begin{aligned}
n ::=\ & \text{negative neutrals} \\
\mid\ & (x : N) \\
\mid\ & \pi_i\ n \\
\mid\ & n\ p
\end{aligned}
$$

$$
\begin{aligned}
p ::=\ & \text{positive neutrals} \\
\mid\ & (x : X^+) \\
\mid\ & \sigma_i\ p \\
\mid\ & (v : N)
\end{aligned}
$$

Focused normal forms: summary

(Grammar with type annotations)

$$
\begin{aligned}
v ::=\ & \text{values} \\
\mid\ & \lambda x.\, v \\
\mid\ & (v_1, v_2) \\
\mid\ & \texttt{match}\ x\ \texttt{with} \left| \begin{array}{l} \sigma_1\ x \to v_1 \\ \sigma_2\ x \to v_2 \end{array} \right. \\
\mid\ & \texttt{absurd}(x) \\
\mid\ & (f : P_{\text{at}})
\end{aligned}
$$

$$
\begin{aligned}
f ::=\ & \text{focused forms} \\
\mid\ & \texttt{let}\ (x : P) = n\ \texttt{in}\ v \\
\mid\ & (n : X^-) \\
\mid\ & (p : P)
\end{aligned}
$$

$$
\begin{aligned}
n ::=\ & \text{negative neutrals} \\
\mid\ & (x : N) \\
\mid\ & \pi_i\ n \\
\mid\ & n\ p
\end{aligned}
$$

$$
\begin{aligned}
p ::=\ & \text{positive neutrals} \\
\mid\ & (x : X^+) \\
\mid\ & \sigma_i\ p \\
\mid\ & (v : N)
\end{aligned}
$$

(Cut-free)

Focused normal forms: summary

(Grammar with type annotations)

$$
\begin{aligned}
v ::=\ & \text{values} \\
\mid\ & \lambda x.\, v \\
\mid\ & (v_1, v_2) \\
\mid\ & \texttt{match}\ x\ \texttt{with} \left|\ \begin{aligned} &\sigma_1\ x \to v_1 \\ &\sigma_2\ x \to v_2 \end{aligned}\right. \\
\mid\ & \texttt{absurd}(x) \\
\mid\ & (f : P_{\text{at}})
\end{aligned}
$$

$$
\begin{aligned}
f ::=\ & \text{focused forms} \\
\mid\ & \texttt{let}\ (x : P) = n\ \texttt{in}\ v \\
\mid\ & (n : X^-) \\
\mid\ & (p : P)
\end{aligned}
$$

$$
\begin{aligned}
n ::=\ & \text{negative neutrals} \\
\mid\ & (x : N) \\
\mid\ & \pi_i\ n \\
\mid\ & n\ p
\end{aligned}
$$

$$
\begin{aligned}
p ::=\ & \text{positive neutrals} \\
\mid\ & (x : X^+) \\
\mid\ & \sigma_i\ p \\
\mid\ & (v : N)
\end{aligned}
$$

(Cut-free) (System L)

Remark

Focused forms as normal forms (stronger than β-normal).
Inversion is η-expansion.

Remark: After inversion, β-normal forms may be non-neutral.

$$\pi_i \left(\begin{array}{l} \texttt{match}\ f\ ()\ \texttt{with} \\ \mid \sigma_1\ x_1 \to x_1 \\ \mid \sigma_2\ x_2 \to x_2 \end{array} \right) \qquad \begin{array}{l} \texttt{match}\ f\ ()\ \texttt{with} \\ \mid \sigma_1\ x \to \pi_1\ x \\ \mid \sigma_2\ x \to \sigma_1\ () \end{array} \qquad g\ (\texttt{absurd}(f\ ()))$$

`let`-extrusion ($\texttt{let}\ x = (n : P)\ \texttt{in}\ v$) saves the day.

Section 4

(Maximal) Multi-focusing

Focused forms do not solve positive η-equivalence

$$x : 1 \to (X + X) \vdash_{\mathsf{inv}} ? : 0 + (X \times X)$$

Focused forms do not solve positive η-equivalence

$$x : 1 \to (X + X) \vdash_{\mathsf{inv}} ? : 0 + (X \times X)$$

$$\texttt{let } y = x\,() \texttt{ in match } y \texttt{ with } \left| \begin{array}{l} \sigma_1\ z \to \sigma_2\ (z, z) \\ \sigma_2\ z \to \sigma_2\ (z, z) \end{array} \right.$$

$$\sigma_2 \left(\begin{array}{l} \texttt{let } y_1 = x\,() \texttt{ in match } y_1 \texttt{ with } \left| \begin{array}{l} \sigma_1\ z \to z \\ \sigma_2\ z \to z \end{array} \right. \\ , \\ \texttt{let } y_2 = x\,() \texttt{ in match } y_2 \texttt{ with } \left| \begin{array}{l} \sigma_1\ z \to z \\ \sigma_2\ z \to z \end{array} \right. \end{array} \right)$$

Neutrals of positive type may be split at difference places, a different number of times.

Canonical neutral introduction

Idea: introduce each neutral $(n : P)$ as **early** as possible.

This suggests an "even more normal" form, **maximally** focused.

Canonical neutral introduction

Idea: introduce each neutral $(n : P)$ as **early** as possible.

This suggests an "even more normal" form, **maximally** focused.

Multi-focusing:

$$\frac{\Gamma \vdash_{\mathsf{ne}} \bar{n} \Downarrow \bar{P} \qquad \Gamma, \bar{x} : \bar{P} \vdash_{\mathsf{ne}} p_{\mathsf{at}} \Uparrow_{\mathsf{at}} P_{\mathsf{at}}}{\Gamma \vdash_{\mathsf{foc}} \texttt{let } \bar{x} = \bar{n} \texttt{ in } p_{\mathsf{at}} : P_{\mathsf{at}}}$$

$$(p_{\mathsf{at}} : P_{\mathsf{at}}) ::= (p : P) \mid (v : P_{\mathsf{at}})$$

Maximal multi-focusing: no n can be moved **down** in the term.

Maximal multi-focusing solves positive η-equivalence

$$x : 1 \to (X + X) \vdash_{\mathsf{inv}} ? : 0 + (X, X)$$

$$\texttt{let}\ y = x\ ()\ \texttt{in match}\ z\ \texttt{with} \left|\ \begin{array}{l} \sigma_1\ z \to \sigma_2\ (z, z) \\ \sigma_2\ z \to \sigma_2\ (z, z) \end{array}\right.$$

$$\sigma_2 \left(\begin{array}{l} \texttt{let}\ y_1 = x\ ()\ \texttt{in match}\ y_1\ \texttt{with} \left|\ \begin{array}{l} \sigma_1\ z \to z \\ \sigma_2\ z \to z \end{array}\right. \\ , \\ \texttt{let}\ y_2 = x\ ()\ \texttt{in match}\ y_2\ \texttt{with} \left|\ \begin{array}{l} \sigma_1\ z \to z \\ \sigma_2\ z \to z \end{array}\right. \end{array} \right)$$

Neutrals of positive type have a canonical place.

Section 5

First application: equivalence of (focused) λ-terms

Equivalence : intro

Algorithm for equivalence of focused values.
Idea: perform maximal multi-focusing on the fly.

Mutual judgments:

- $\Gamma \vdash_{\mathsf{inv}} v \sim_{\mathsf{alg}} v' : A$
- $\Gamma \vdash_{\mathsf{sat}} f \sim_{\mathsf{alg}} f' : A$
- $\Gamma \vdash_{\mathsf{ne}} n \sim_{\mathsf{alg}} n' \Downarrow A$
- $\Gamma \vdash_{\mathsf{ne}} p \sim_{\mathsf{alg}} p' \Uparrow A$

Depends on a consistency checking judgment $\Gamma \vdash 0$
Definable as $\exists t, (\Gamma \vdash t : 0)$.
Decidable in propositional logic (key!)

Equivalence: inversion

(Terms modulo commuting conversions, or enforced ordering)

$$\frac{\Gamma, x : A \vdash_{\mathsf{inv}} v \sim_{\mathsf{alg}} v' : B}{\Gamma \vdash_{\mathsf{inv}} \lambda x.\, v \sim_{\mathsf{alg}} \lambda x.\, v' : A \to B}$$

Equivalence: inversion

(Terms modulo commuting conversions, or enforced ordering)

$$\frac{\Gamma, x : A \vdash_{\text{inv}} v \sim_{\text{alg}} v' : B}{\Gamma \vdash_{\text{inv}} \lambda x.\, v \sim_{\text{alg}} \lambda x.\, v' : A \to B}
\qquad
\frac{\Gamma \vdash_{\text{inv}} v_1 \sim_{\text{alg}} v_1' : A_1 \quad \Gamma \vdash_{\text{inv}} v_2 \sim_{\text{alg}} v_2' : A_2}{\Gamma \vdash_{\text{inv}} (v_1, v_2) \sim_{\text{alg}} (v_1', v_2') : A_1 \times A_2}$$

$$\frac{\Gamma, x : A_1 \vdash_{\text{inv}} v_1 \sim_{\text{alg}} v_1' : C \quad \Gamma, x : A_2 \vdash_{\text{inv}} v_2 \sim_{\text{alg}} v_2' : C}{\Gamma, x : A_1 + A_2 \vdash_{\text{inv}} \begin{array}{l}\texttt{match } x \texttt{ with} \\ \mid \sigma_1\, x \to v_1 \\ \mid \sigma_2\, x \to v_2\end{array} \sim_{\text{alg}} \begin{array}{l}\texttt{match } x \texttt{ with} \\ \mid \sigma_1\, x \to v_1' \\ \mid \sigma_2\, x \to v_2'\end{array} : C}$$

$$\frac{}{\Gamma \vdash_{\text{inv}} () \sim_{\text{alg}} () : 1}
\qquad
\frac{}{\Gamma, x : 0 \vdash_{\text{inv}} \texttt{absurd}(x) \sim_{\text{alg}} \texttt{absurd}(x) : A}$$

$$\frac{\Gamma \text{ negative or atomic} \qquad \Gamma \vdash_{\text{sat}} v \sim_{\text{alg}} v' : A \qquad A \text{ positive or atomic}}{\Gamma \vdash_{\text{inv}} v \sim_{\text{alg}} v' : A}$$

Equivalence: neutrals

Just follow the constructors. (Failure cases)

$$\frac{(x : N) \in \Gamma}{\Gamma \vdash_{\mathsf{ne}} x \sim_{\mathsf{alg}} x \Downarrow N} \qquad \frac{\Gamma \vdash_{\mathsf{ne}} n \sim_{\mathsf{alg}} n' \Downarrow A_1 \times A_2}{\Gamma \vdash_{\mathsf{ne}} \pi_i\ n \sim_{\mathsf{alg}} \pi_i\ n' \Downarrow A_i}$$

Equivalence: neutrals

Just follow the constructors. (Failure cases)

$$\frac{(x : N) \in \Gamma}{\Gamma \vdash_{\text{ne}} x \sim_{\text{alg}} x \Downarrow N} \qquad \frac{\Gamma \vdash_{\text{ne}} n \sim_{\text{alg}} n' \Downarrow A_1 \times A_2}{\Gamma \vdash_{\text{ne}} \pi_i\ n \sim_{\text{alg}} \pi_i\ n' \Downarrow A_i}$$

$$\frac{\Gamma \vdash_{\text{ne}} n \sim_{\text{alg}} n' \Downarrow A \to B \qquad \Gamma \vdash_{\text{ne}} p \sim_{\text{alg}} p' \Uparrow A}{\Gamma \vdash_{\text{ne}} n\ p \sim_{\text{alg}} n'\ p' \Downarrow B}$$

$$\frac{\Gamma \vdash_{\text{ne}} p \sim_{\text{alg}} p' \Uparrow A}{\Gamma \vdash_{\text{ne}} \sigma_i\ p \sim_{\text{alg}} \sigma_i\ p' \Uparrow A_1 + A_2} \qquad \frac{(x : X^+) \in \Gamma}{\Gamma \vdash_{\text{ne}} x \sim_{\text{alg}} x \Uparrow X^+}$$

$$\frac{\Gamma \vdash_{\text{inv}} v \sim_{\text{alg}} v' : N \qquad N \text{ negative}}{\Gamma \vdash_{\text{ne}} v \sim_{\text{alg}} v' \Uparrow N}$$

Equivalence: saturation

The hard stuff.

$$\frac{n \in f_i \qquad \Gamma \vdash_{\mathsf{ne}} n \Downarrow P \qquad \bar{n'} \stackrel{\text{def}}{=} \{n' \in f_j \mid \Gamma \vdash_{\mathsf{ne}} n \sim_{\mathsf{alg}} n' \Downarrow P\} \qquad \Gamma, x : P \vdash_{\mathsf{inv}} f_1[x/\bar{n'}] \sim_{\mathsf{alg}} f_2[x/\bar{n'}] : A}{\Gamma \vdash_{\mathsf{sat}} f_1 \sim_{\mathsf{alg}} f_2 : A}$$

Equivalence: saturation

The hard stuff.

$$\frac{n \in f_i \qquad \Gamma \vdash_{\text{ne}} n \Downarrow P \qquad \bar{n'} \stackrel{\text{def}}{=} \{n' \in f_j \mid \Gamma \vdash_{\text{ne}} n \sim_{\text{alg}} n' \Downarrow P\} \qquad \Gamma, x : P \vdash_{\text{inv}} f_1[x/\bar{n'}] \sim_{\text{alg}} f_2[x/\bar{n'}] : A}{\Gamma \vdash_{\text{sat}} f_1 \sim_{\text{alg}} f_2 : A}$$

$$\frac{\Gamma \vdash 0}{\Gamma \vdash_{\text{sat}} f_1 \sim_{\text{alg}} f_2 : A}$$

Equivalence: saturation

The hard stuff.

$$\frac{n \in f_i \qquad \Gamma \vdash_{\text{ne}} n \Downarrow P \qquad \bar{n'} \stackrel{\text{def}}{=} \{n' \in f_j \mid \Gamma \vdash_{\text{ne}} n \sim_{\text{alg}} n' \Downarrow P\} \qquad \Gamma, x : P \vdash_{\text{inv}} f_1[x/\bar{n'}] \sim_{\text{alg}} f_2[x/\bar{n'}] : A}{\Gamma \vdash_{\text{sat}} f_1 \sim_{\text{alg}} f_2 : A}$$

$$\frac{\Gamma \vdash 0}{\Gamma \vdash_{\text{sat}} f_1 \sim_{\text{alg}} f_2 : A}$$

$$\frac{\neg(\exists n \in f_i,\ \Gamma \vdash_{\text{ne}} n \Downarrow P) \qquad \neg(\Gamma \vdash 0) \qquad \Gamma \vdash_{\text{ne}} f_1 \sim_{\text{alg}} f_2 \Downarrow X^-}{\Gamma \vdash_{\text{sat}} f_1 \sim_{\text{alg}} f_2 : X^-}$$

$$\frac{\neg(\exists n \in f_i,\ \Gamma \vdash_{\text{ne}} n \Downarrow P) \qquad \neg(\Gamma \vdash 0) \qquad \Gamma \vdash_{\text{ne}} f_1 \sim_{\text{alg}} f_2 \Uparrow P}{\Gamma \vdash_{\text{sat}} f_1 \sim_{\text{alg}} f_2 : P}$$

(negative side-conditions: no backtracking)

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$\mathtt{match}\ x\ \mathtt{with}\ \sigma_i\ y \to^i$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$\mathtt{match}\ x\ \mathtt{with}\ \sigma_i\ y \to^i$

$\quad \mathtt{let}\ x_1 = f\ (\sigma_i\ y)\ \mathtt{in}$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l} \mathtt{match}\ x\ \mathtt{with}\ \sigma_i\ y \to^{i} \\ \quad \mathtt{let}\ x_1 = f\ (\sigma_i\ y)\ \mathtt{in} \\ \quad\ \ \mathtt{match}\ x_1\ \mathtt{with} \\ \quad\ \ \left|\begin{array}{l} \sigma_1\ y_1 \to \sigma_1\ y_1 \\ \sigma_2\ y_1 \to \sigma_2\ y_1 \end{array}\right. \end{array} \qquad \sim_{\mathsf{alg}}$$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ y \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ y)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad \mid \sigma_1\ y_1 \to \sigma_1\ y_1 \\ \quad\quad \mid \sigma_2\ y_1 \to \sigma_2\ y_1 \end{array} \quad \sim_{\mathsf{alg}} \quad \begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ y \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ y)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ y_1 \to^j \end{array}$$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l}\texttt{match}\ x\ \texttt{with}\ \sigma_i\ y \to^i \\ \quad \texttt{let}\ x_1 =\ f\ (\sigma_i\ y)\ \ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad \mid\ \sigma_1\ y_1 \to \sigma_1\ y_1 \\ \quad\quad \mid\ \sigma_2\ y_1 \to \sigma_2\ y_1 \end{array} \quad \sim_{\mathsf{alg}} \quad \begin{array}{l}\texttt{match}\ x\ \texttt{with}\ \sigma_i\ y \to^i \\ \quad \texttt{let}\ x_1 =\ f\ (\sigma_i\ y)\ \ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ y_1 \to^j \\ \quad\quad\quad \texttt{let}\ x_2 =\ f\ (\sigma_j\ y_1)\ \ \texttt{in} \end{array}$$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

$$\begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ y \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ y)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with} \\ \quad\quad \mid \sigma_1\ y_1 \to \sigma_1\ y_1 \\ \quad\quad \mid \sigma_2\ y_1 \to \sigma_2\ y_1 \end{array} \quad \sim_{\mathsf{alg}} \quad \begin{array}{l} \texttt{match}\ x\ \texttt{with}\ \sigma_i\ y \to^i \\ \quad \texttt{let}\ x_1 = f\ (\sigma_i\ y)\ \texttt{in} \\ \quad\quad \texttt{match}\ x_1\ \texttt{with}\ \sigma_j\ y_1 \to^j \\ \quad\quad\quad \texttt{let}\ x_2 = f\ (\sigma_j\ y_1)\ \texttt{in} \\ \quad\quad\quad\quad \texttt{match}\ x_2\ \texttt{with}\ \sigma_k\ y_2 \to^k \end{array}$$

Equivalence: demo

$$\emptyset \vdash \lambda f.\, \lambda x.\, f\ x \approx_{\beta\eta} \lambda f.\, \lambda x.\, f\ (f\ (f\ x)) : (\mathbb{B} \to \mathbb{B}) \to (\mathbb{B} \to \mathbb{B})$$

$f : \mathbb{B} \to \mathbb{B}, x : \mathbb{B} \vdash_{\mathsf{inv}}$

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\end{array}
\sim_{\mathsf{alg}}
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\quad\quad\quad \texttt{let}\ x_2 = f\ (\sigma_j\ y_1)\ \texttt{in} \\
\quad\quad\quad\quad \texttt{match}\ x_2\ \texttt{with}\ \sigma_k\ y_2 \to^k \\
\quad\quad\quad\quad\quad \texttt{let}\ x_3 = f\ (\sigma_k\ y_2)\ \texttt{in}
\end{array}$$

Equivalence: demo

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$$

Equivalence: demo

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\end{array}$$

$$i \overset{?}{\neq} j \overset{?}{\neq} k \in \{1, 2\}$$

Equivalence: demo

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$$j = i \quad \Longrightarrow \quad f(x) = x \quad \Longrightarrow \quad f(f(\underline{f(x)})) = f(\underline{f(x)}) = f(x)$$

Equivalence: demo

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$$j = i \implies f(x) = x \implies f(f(\underline{f(x)})) = f(\underline{f(x)}) = f(x)$$

$$k = i \implies f(f(x)) = x \implies f(\underline{f(f(x))}) = f(x)$$

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$$j = i \implies f(x) = x \implies f(f(\underline{f(x)})) = f(\underline{f(x)}) = f(x)$$

$$k = i \implies f(f(x)) = x \implies f(\underline{f(f(x))}) = f(x)$$

$$k = j \implies f(f(x)) = f(x) \implies f(\underline{f(f(x))}) = \underline{f(f(x))} = f(x)$$

Equivalence: summary

Addition of 0 is conceptually very simple once the right point of view is in place. It does not complicate the proofs.

Correctness: immediate.

Completeness: maximal multi-focusing (WIP, will be in my thesis).

Termination: easy on the focused structure.

Obstacles to older approaches

Trying to decide equivalence by “small-step” rewriting of pieces of programs (or in fact any method relying only on comparing the two terms) does not scale to 0.

Neil Ghani, who first solved the sum case (without 0) in his 1995 PhD thesis, had the right intuition:

> *The η_0-reducts of a term are determined more by the consistency of the context in which the term was typed, rather than the term itself. [...] Context inconsistency, term typability and other important issues in the study of the η_0-rewrite rule are decidable [in our setting].*

What was lacking was the framework to make this **easy**.

Section 6

Second application: which types have a unique inhabitant?

Question

Given a typing (Γ, A), is there a **unique** t such that $\Gamma \vdash t : A$ holds?

Unique modulo $\beta\eta$-equivalence.

Motivation: type-directed code inference.

Idea: enumerate maximal multi-focused proofs.

Problem

Maximality of multi-focusing is a **non-local** criterion.

Easily applicable to equivalence: search for subterms to extrude.

Doing **goal-directed** search of maximal proofs is more difficult.

$$\Gamma \vdash_{\mathsf{foc}} ? : P_{\mathsf{at}}$$

Saturation

We don't know which neutrals will be used in subterms – to split on them.

Let us split on **all of them**.

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Let us split on **all of them**.

$$\frac{\overset{\text{DRAFT-DO-NOT-DISTRIBUTE}}{(\bar{n}, \bar{P}) \overset{\text{def}}{=} \{(n, P) \mid (\Gamma_{at} \vdash_{ne} n \Downarrow P)\}} \qquad \Gamma_{at}, \bar{x} : \bar{P} \vdash_{inv} t : Q_{at}}{\Gamma_{at} \vdash_{foc} \texttt{let } \bar{x} = \bar{n} \texttt{ in } t : Q_{at}}$$

Two issues:

- saturate twice on the same neutral $\Longrightarrow$ wrong answer
- infinitary rule; termination?

Issue 1: Structural youth

$$\frac{\Gamma \quad , x : A \vdash_{\text{inv}} v : B}{\Gamma \quad \vdash_{\text{inv}} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \quad \vdash_{\text{inv}} v_1 : A_1 \qquad \Gamma \quad \vdash_{\text{inv}} v_2 : A_2}{\Gamma \quad \vdash_{\text{inv}} (v_1, v_2) : A_1 \times A_2}$$

$$\frac{\Gamma \quad , x : A_1 \vdash_{\text{inv}} v_1 : C \qquad \Gamma \quad , x : A_2 \vdash_{\text{inv}} v_2 : C}{\Gamma \quad , x : A_1 + A_2 \vdash_{\text{inv}} \texttt{match}\ x\ \texttt{with}\ \mid \sigma_1\, x \to v_1 \mid \sigma_2\, x \to v_2 : C}$$

$$\frac{}{\Gamma \quad \vdash_{\text{inv}} () : 1} \qquad \frac{}{\Gamma \quad , x : 0 \vdash_{\text{inv}} \texttt{absurd}(x) : A}$$

$$\frac{\Gamma \quad \text{negative or atomic} \qquad \Gamma \quad \vdash_{\text{foc}} v : A \qquad A \text{ positive or atomic}}{\Gamma \quad \vdash_{\text{inv}} v : A}$$

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$$(\bar{n}, \bar{P}) \stackrel{\text{def}}{=} \{(n, P) \mid (\Gamma \quad \vdash_{\text{ne}} n \Downarrow P) \quad \}$$

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Issue 1: Structural youth

$$\frac{\Gamma \quad ; \Gamma', x : A \vdash_{\text{inv}} v : B}{\Gamma \quad ; \Gamma' \vdash_{\text{inv}} \lambda x.\, v : A \to B} \qquad \frac{\Gamma \quad \vdash_{\text{inv}} v_1 : A_1 \qquad \Gamma \quad \vdash_{\text{inv}} v_2 : A_2}{\Gamma \quad \vdash_{\text{inv}} (v_1, v_2) : A_1 \times A_2}$$

$$\frac{\Gamma \quad , x : A_1 \vdash_{\text{inv}} v_1 : C \qquad \Gamma \quad , x : A_2 \vdash_{\text{inv}} v_2 : C}{\Gamma \quad , x : A_1 + A_2 \vdash_{\text{inv}} \texttt{match } x \texttt{ with } \mid \sigma_1\, x \to v_1 \mid \sigma_2\, x \to v_2 : C}$$

$$\frac{}{\Gamma \quad \vdash_{\text{inv}} () : 1} \qquad \frac{}{\Gamma \quad , x : 0 \vdash_{\text{inv}} \texttt{absurd}(x) : A}$$

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Issue 1: Structural youth

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$$\frac{\Gamma' \quad \text{negative or atomic} \qquad \Gamma_{at}; \Gamma' \vdash_{foc} v : A \qquad A \text{ positive or atomic}}{\Gamma_{at}; \Gamma' \vdash_{inv} v : A}$$

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$$\frac{(\bar{n}, \bar{P}) \stackrel{\text{def}}{=} \{(n, P) \mid (\Gamma_{at}, \Gamma' \vdash_{ne} n \Downarrow P) \ \wedge\ n\ \texttt{uses}\ \Gamma'\} \qquad \Gamma_{at}; \Gamma', \bar{x} : \bar{P} \vdash_{inv} t : Q_{at}}{\Gamma_{at}; \Gamma' \vdash_{foc} \texttt{let}\ \bar{x} = \bar{n}\ \texttt{in}\ t : Q_{at}}$$

Issue 2: two-or-more

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$$\frac{(\bar{n}, \bar{P}) \stackrel{\text{def}}{=} \{(n, P) \mid (\Gamma_{\text{at}}, \Gamma' \vdash_{\text{ne}} n \Downarrow P) \;\wedge\; n \,\texttt{uses}\, \Gamma'\} \qquad \Gamma_{\text{at}}; \Gamma', \bar{x} : \bar{P} \vdash_{\text{inv}} t : Q_{\text{at}}}{\Gamma_{\text{at}}; \Gamma' \vdash_{\text{foc}} \texttt{let}\ \bar{x} = \bar{n}\ \texttt{in}\ t : Q_{\text{at}}}$$

For unicity, no need to keep more than **two** variables at each type.

Contexts are multisets over multiplicities $\{0, 1, \bar{2}\}$.

Finite search space $\Longrightarrow$ search terminates.

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$$\frac{\begin{array}{c}\text{DRAFT-DO-NOT-DISTRIBUTE} \\ (\bar{n}, \bar{P}) \stackrel{\text{def}}{=} \{(n, P) \mid (\Gamma_{\mathsf{at}}, \Gamma' \vdash_{\mathsf{ne}} n \Downarrow P) \ \wedge \ n \texttt{ uses } \Gamma'\} \\ \Gamma_{\mathsf{at}}; \Gamma', \bar{x} : \bar{P} \vdash_{\mathsf{inv}} t : Q_{\mathsf{at}}\end{array}}{\Gamma_{\mathsf{at}}; \Gamma' \vdash_{\mathsf{foc}} \texttt{let } \bar{x} = \bar{n} \texttt{ in } t : Q_{\mathsf{at}}}$$

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`https://gitlab.com/gasche/unique-inhabitant`

Conclusion

Focusing: an idea of proof theory that applies to programming language theory.

Maximal multi-focusing: deeper understanding of program equivalence and identity.

"Focusing is like bidirectional type checking, but better"