École Polytechnique, January 16th 2008

Towards Persistence-Based Reconstruction in Euclidean Spaces

Frédéric Chazal — Géometrica Group, INRIA Futurs

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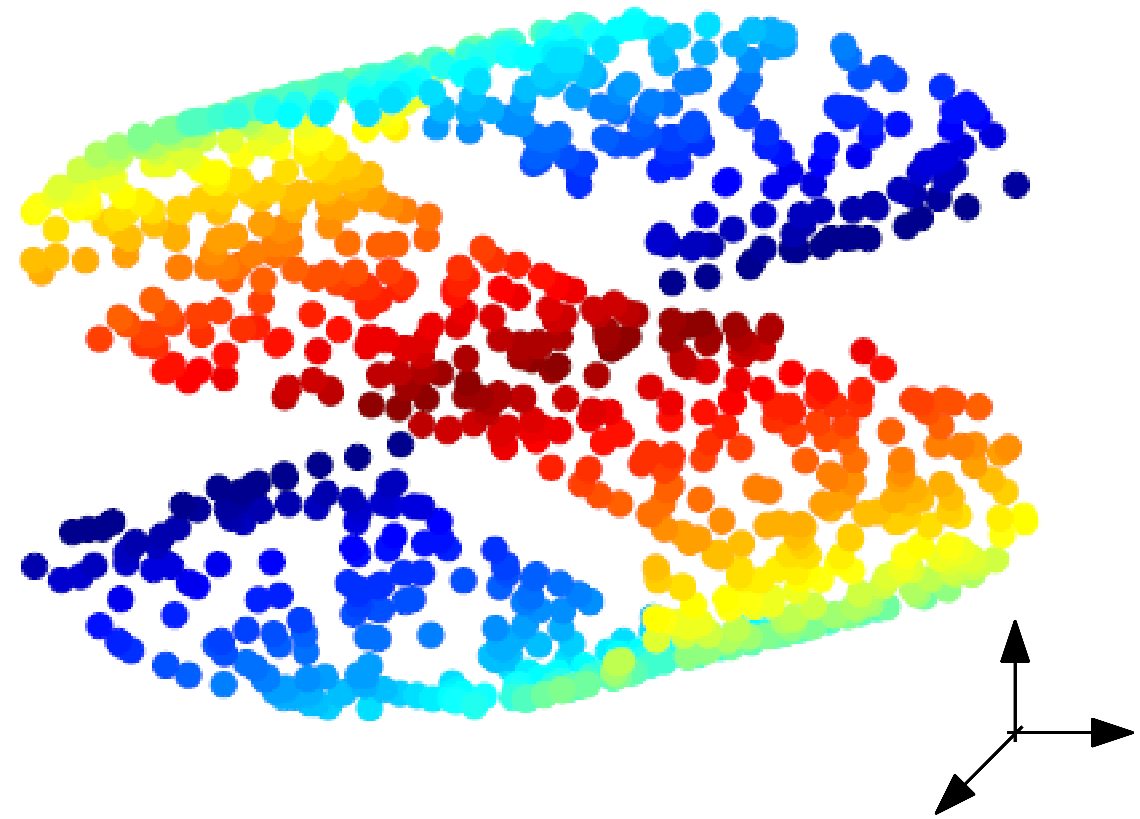
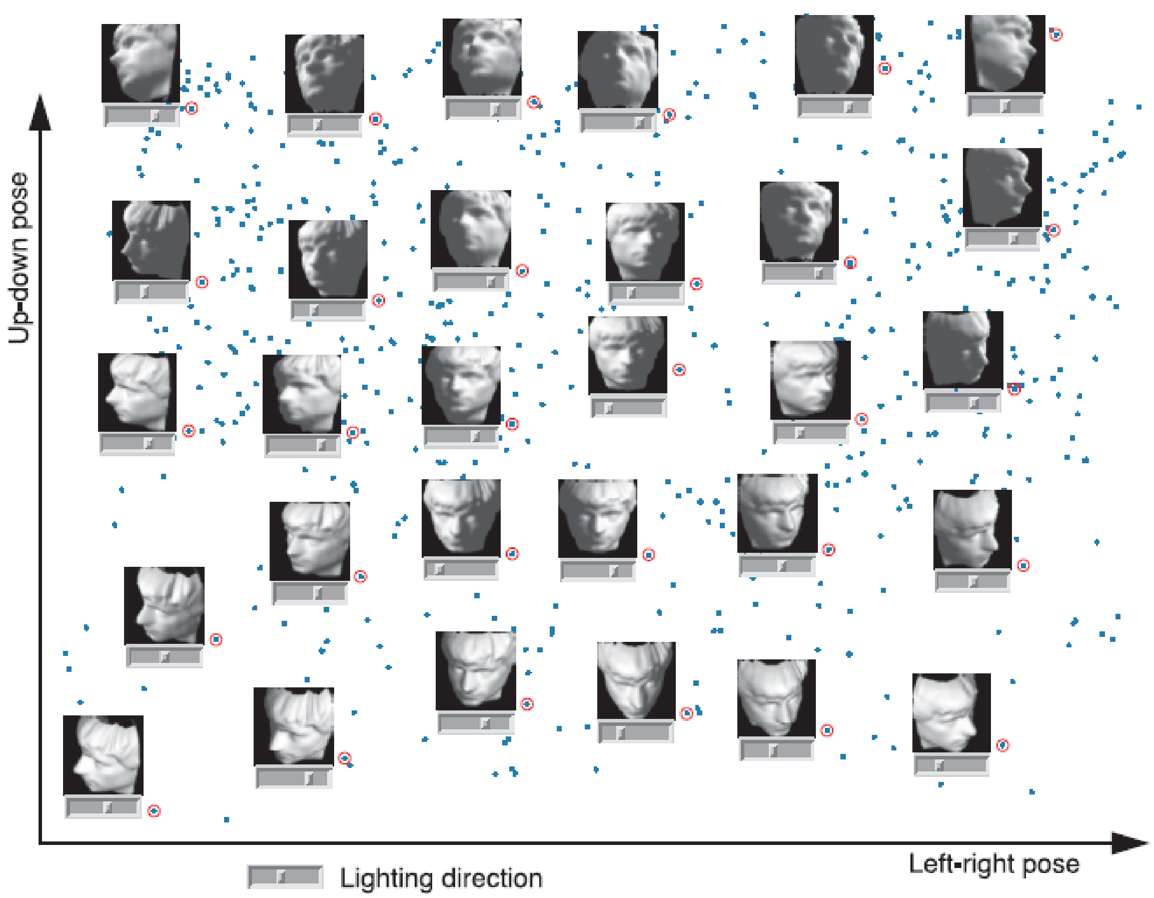
$\longrightarrow$ Special thanks to G. Carlsson, V. de Silva, L. J. Guibas

Goal

Input: a point cloud in a metric space.



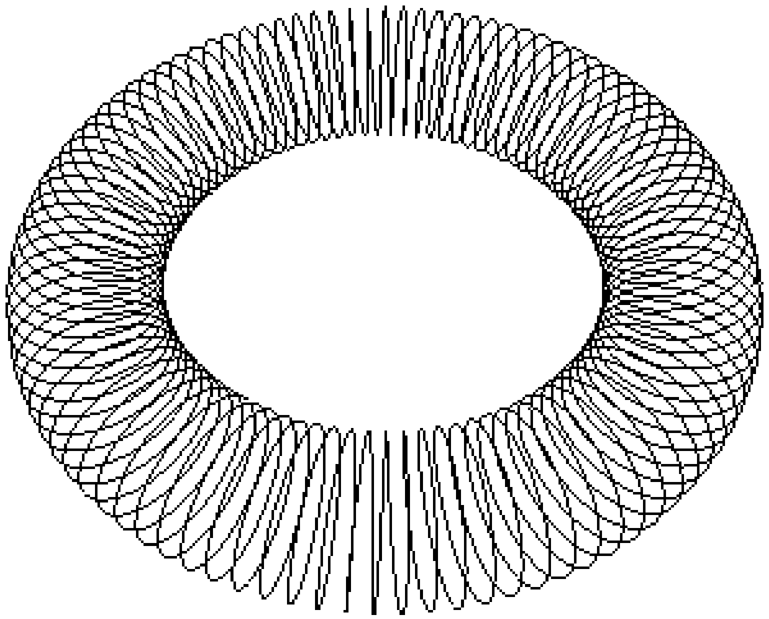
Q Is there structure in the data? What is the topology of the space underlying the data? Can we build some sort of *atlas* of this space?

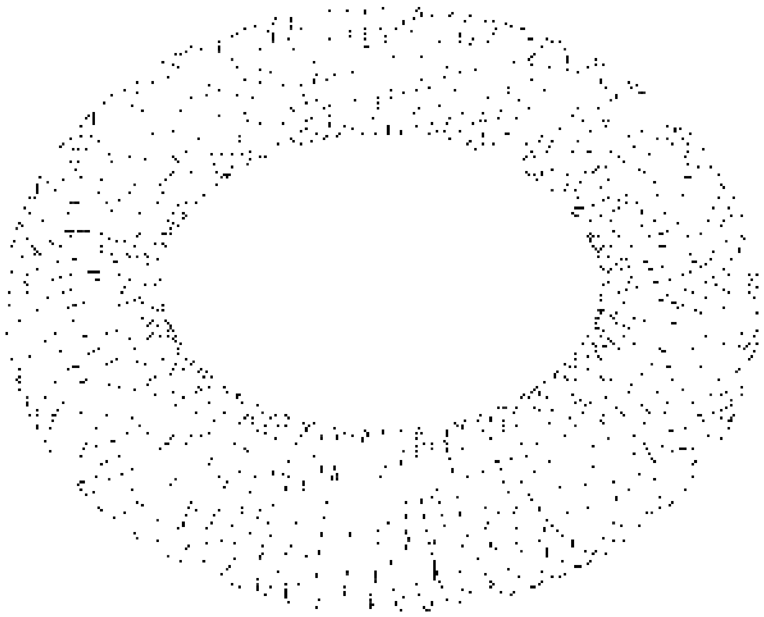


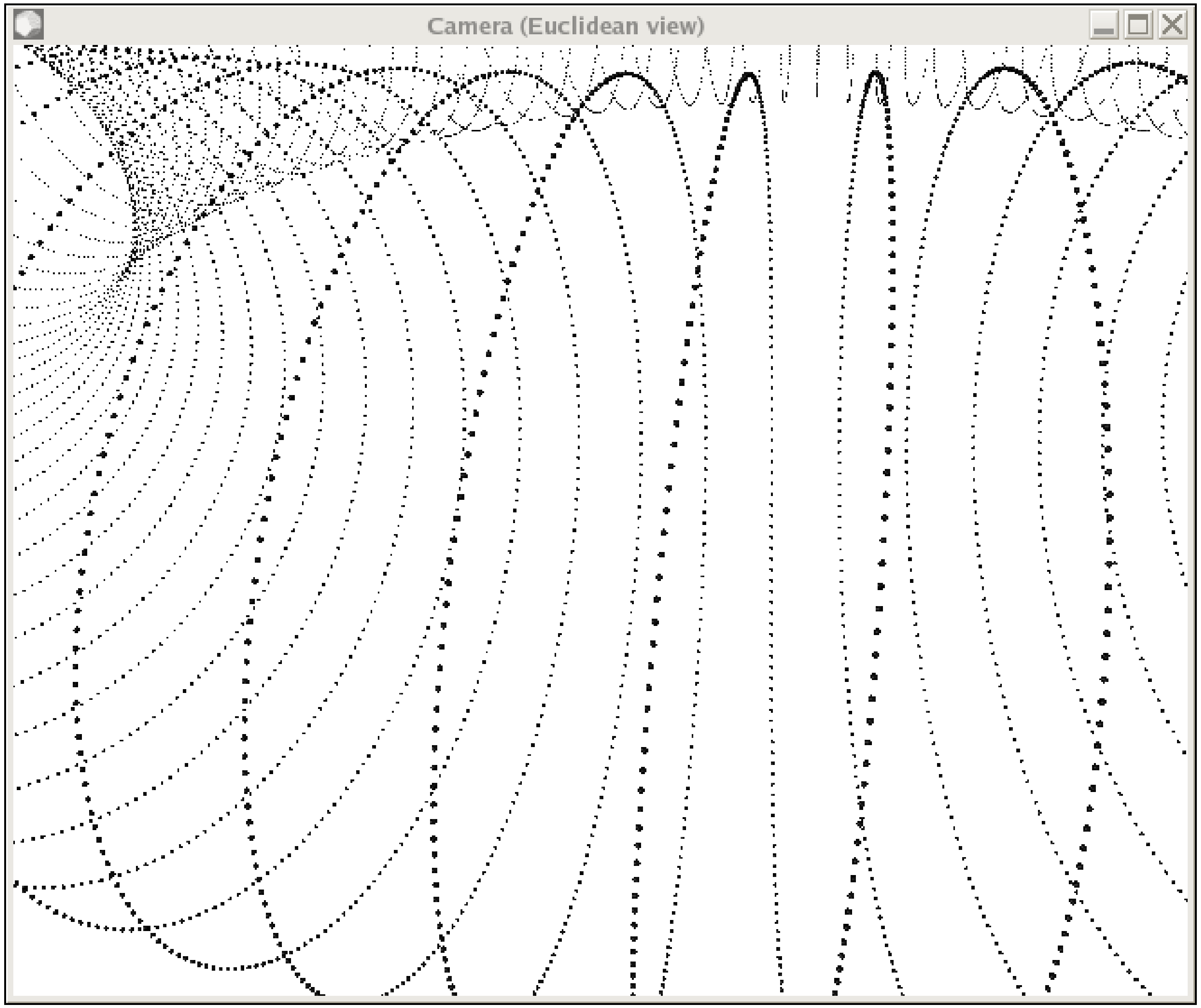
Example: set of 4096-dimensional data points, representing 64x64 pixels images of a same object, seen under various lighting and camera angles. (from Isomap, Science 290).

Theoretical Challenges

What is the reconstruction?



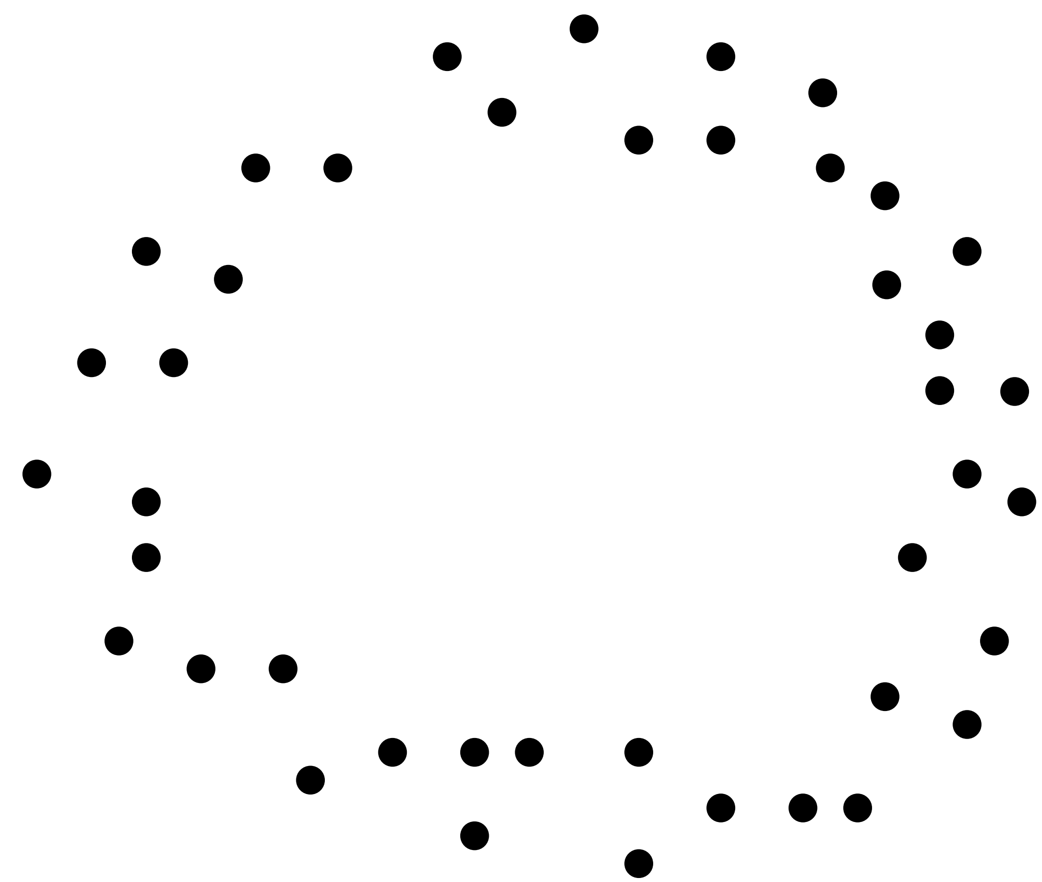




Algorithmic Challenges

Curse of dimensionality:

X smooth k-dimensional manifold, $\varepsilon > 0$. For any mesh M s.t. $\mathsf{d}_H(M, X) \leq \varepsilon$, $|M| \geq c(X)\, \varepsilon^{-\frac{k}{2}}$. [Gruber 1993], [Clarkson 2006]



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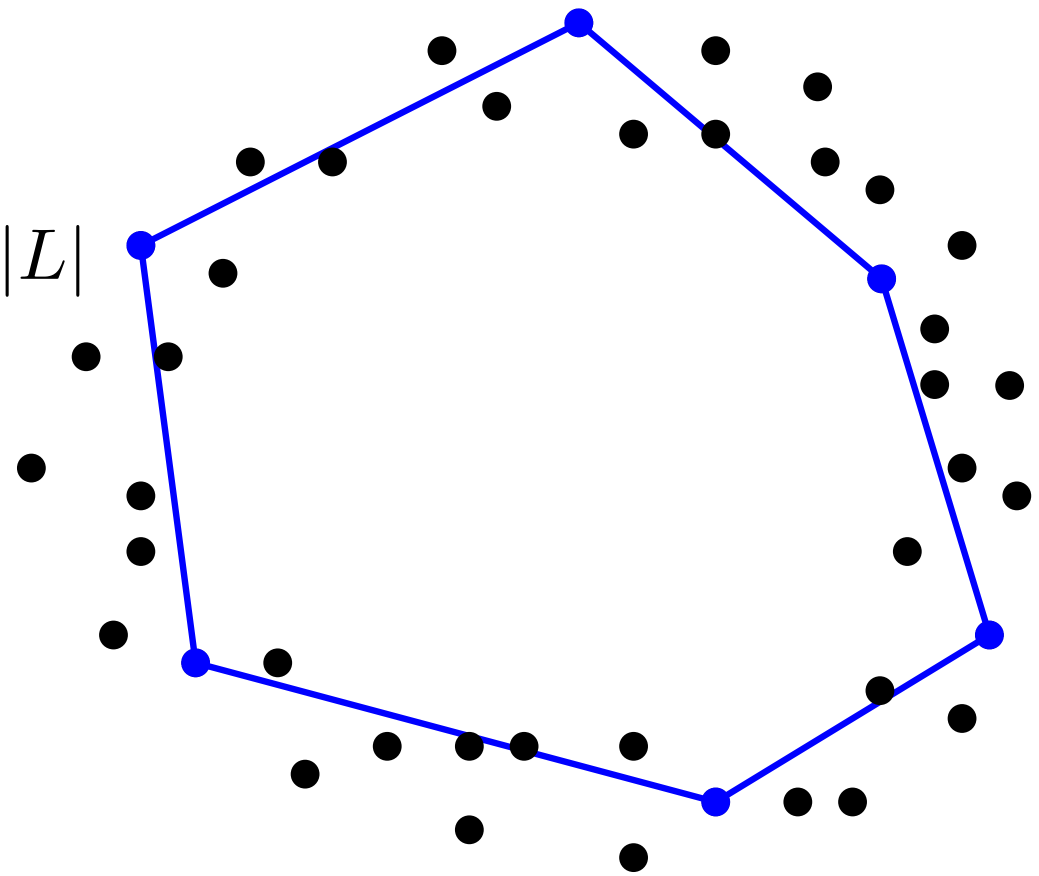
- assume high co-dimension ($k \ll d$)
- use landmarking / multi-scale approach
- Build lightweight data structures, of size $c(k)|L|$
- weaker concepts of reconstruction:

  homology equivalence, persistent homology...



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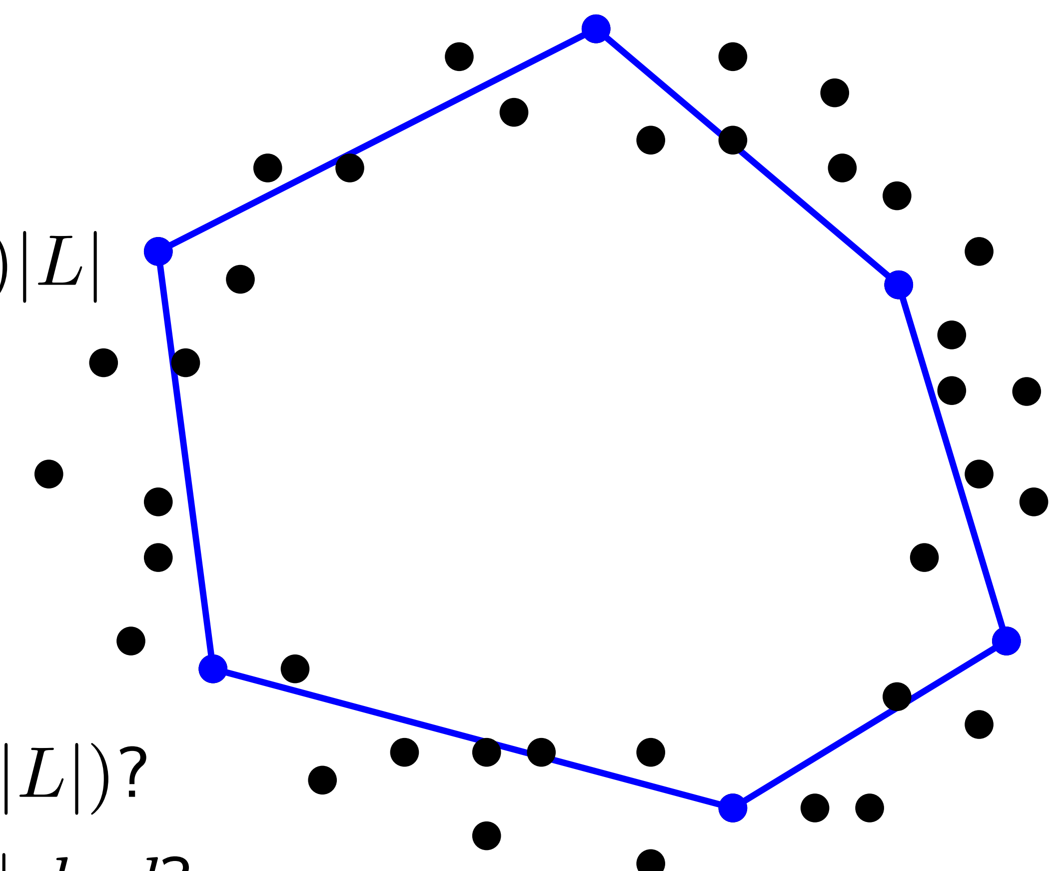

Q can complexity be reduced to $2^{O(k)} Poly(|L|)$?

Q can complexity be made polynomial in $|L|, k, d$?

Existing Techniques

Delaunay

- restricted Delaunay
- ε-sampling theory
- H-manifolds
- Witness complex

Persistent Homology

- filtrations (Čech, α-shape)
- persistence algorithm
- stability of diagrams

Distance Functions

- offsets of compact sets
- critical point theory
- λ-medial axis

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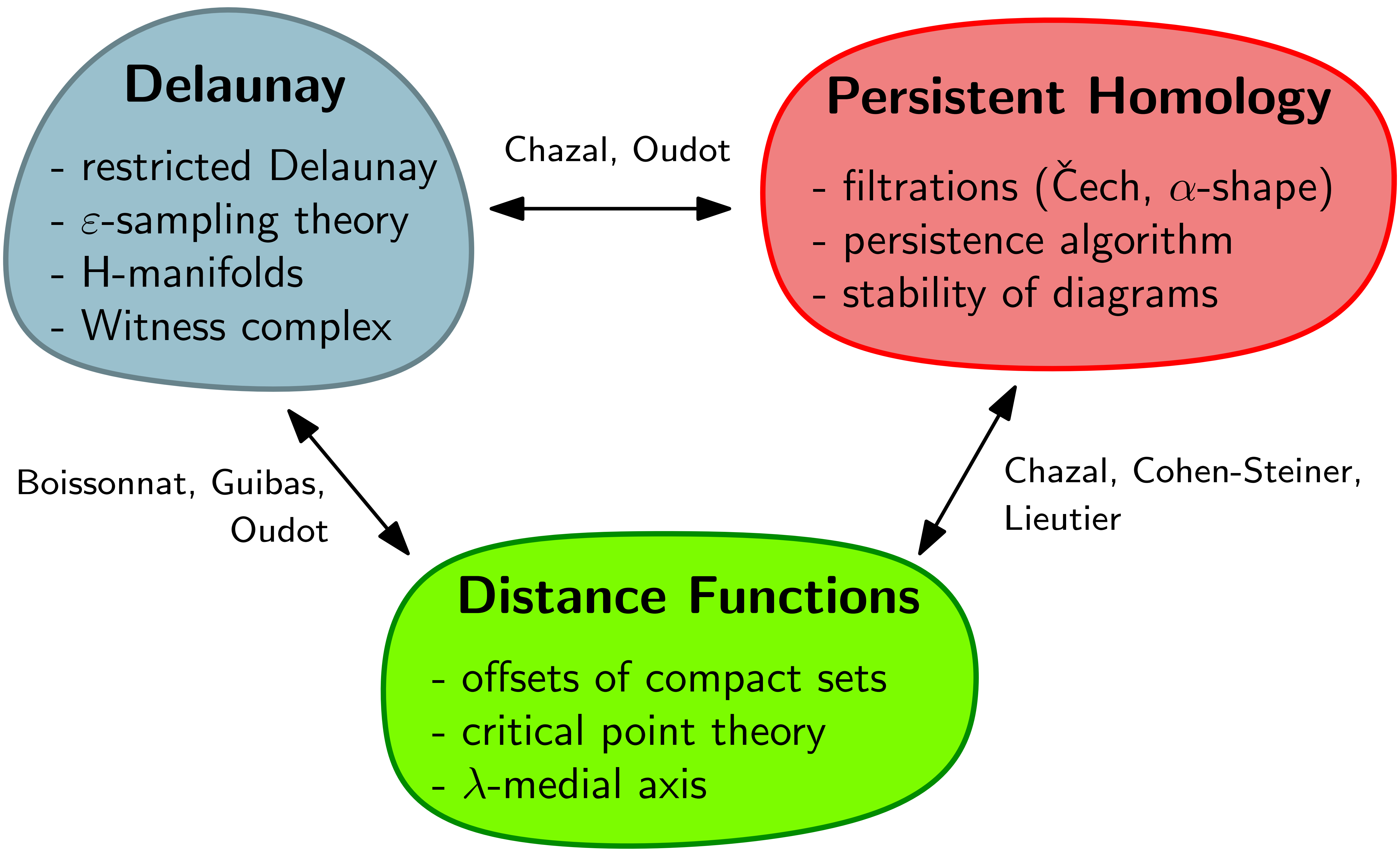

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Chazal, Oudot

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Persistence-based reconstruction

complexity $2^{2^{O(k)}} Poly(|L|)$

Boissonnat, Guibas, Oudot

Chazal, Cohen-Steiner, Lieutier

Distance Functions

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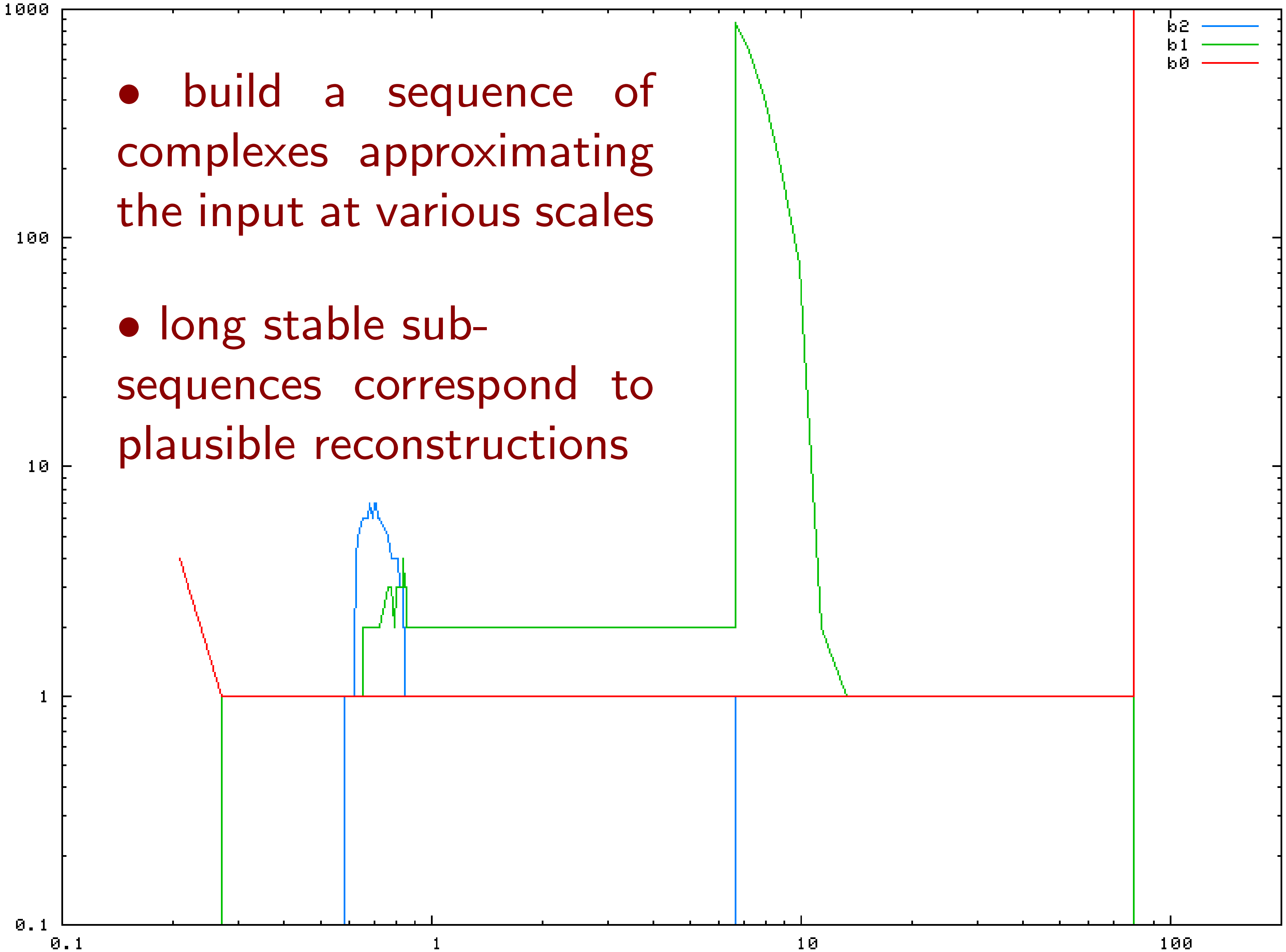
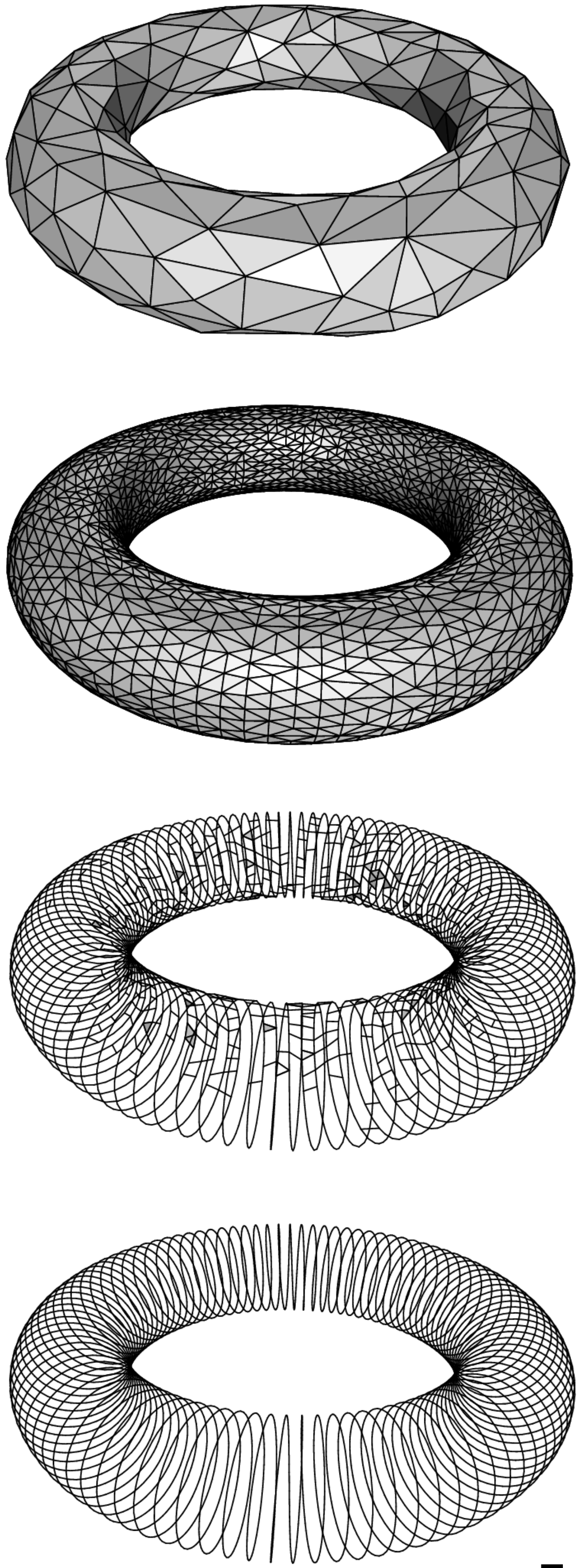
Multiscale Reconstruction

[Guibas, O. 07]

- build a sequence of complexes approximating the input at various scales
- long stable sub-sequences correspond to plausible reconstructions



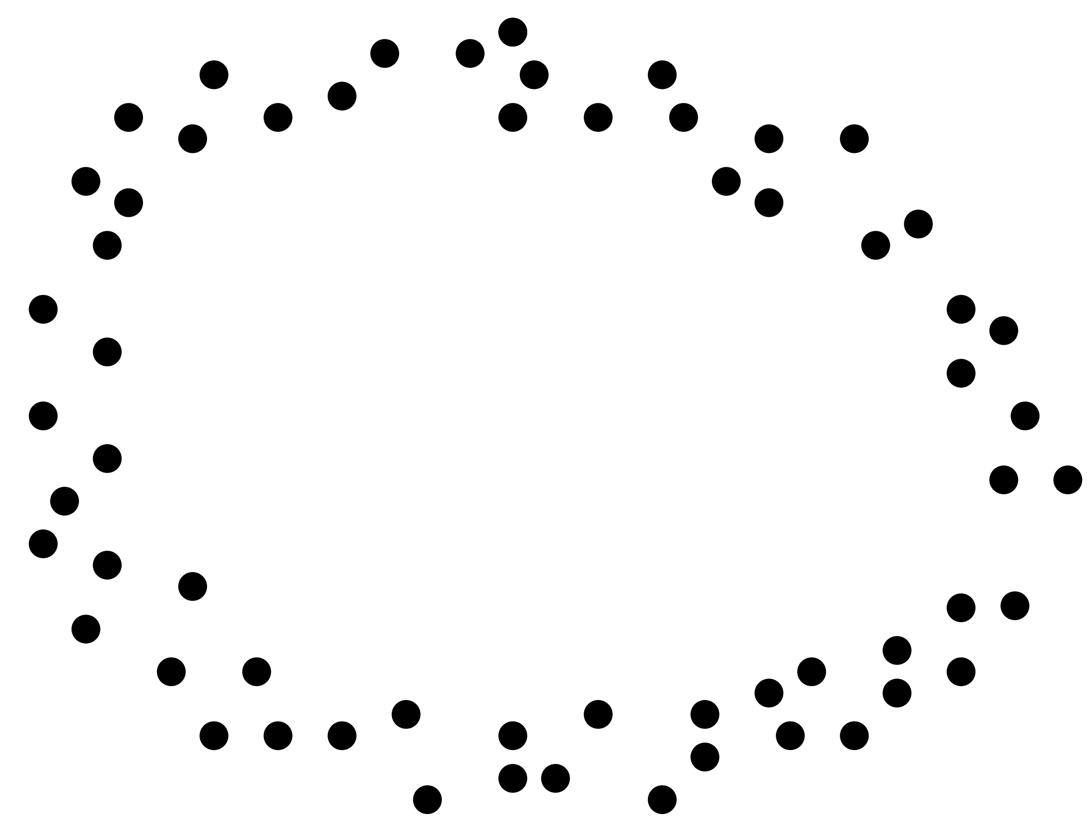


Algorithm (low dimensions)

[Guibas, O. 07]

Input: a finite point set W in $\mathbb{R}^2$ or $\mathbb{R}^3$.

$\longrightarrow$ build $L \subseteq W$ iteratively, and maintain its witness complex $\mathcal{C}_W(L)$.



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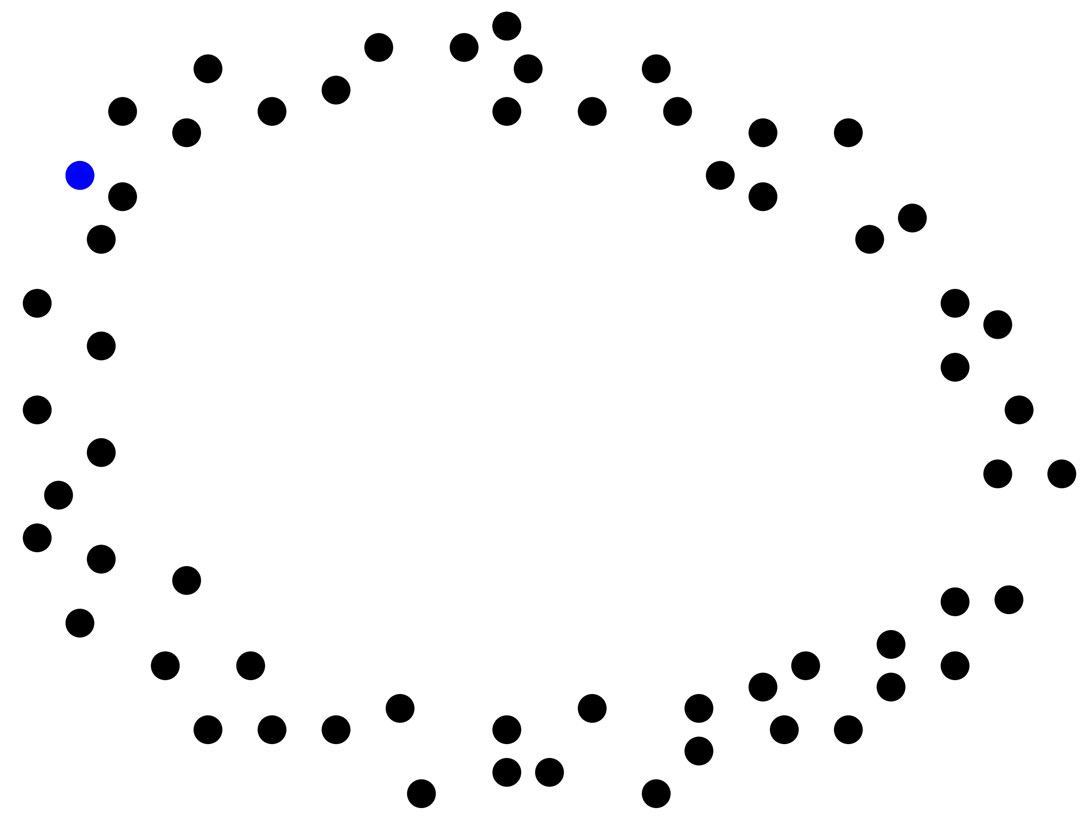

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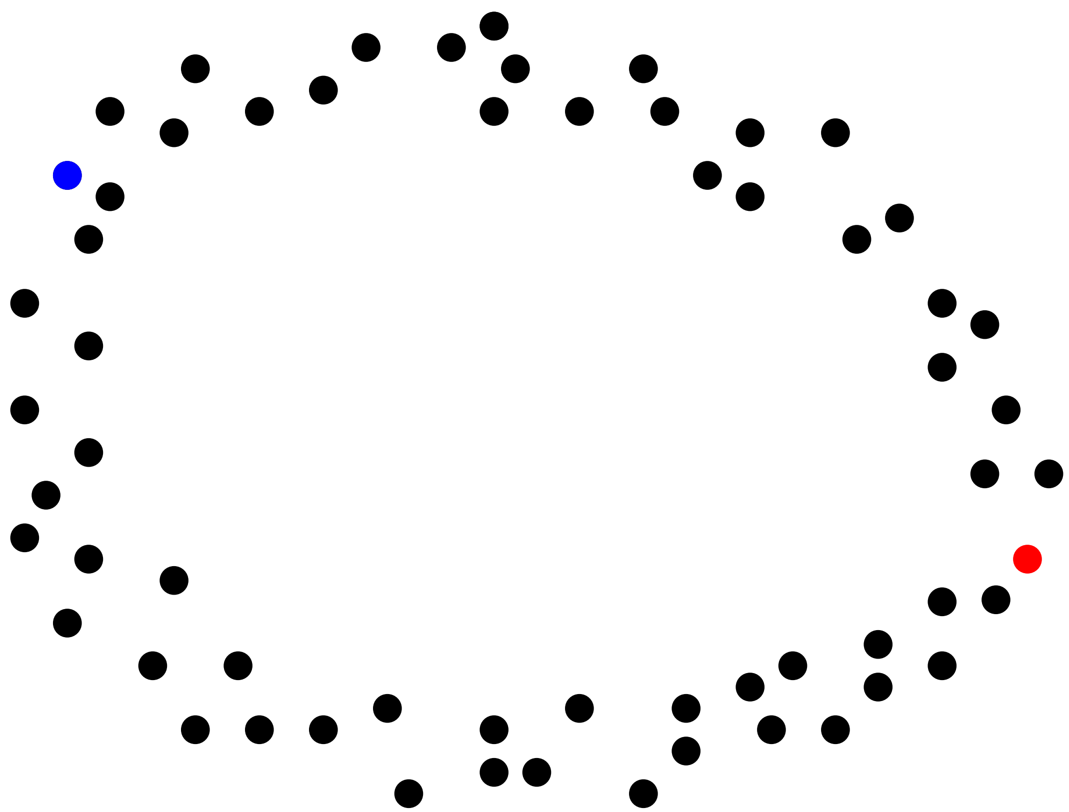

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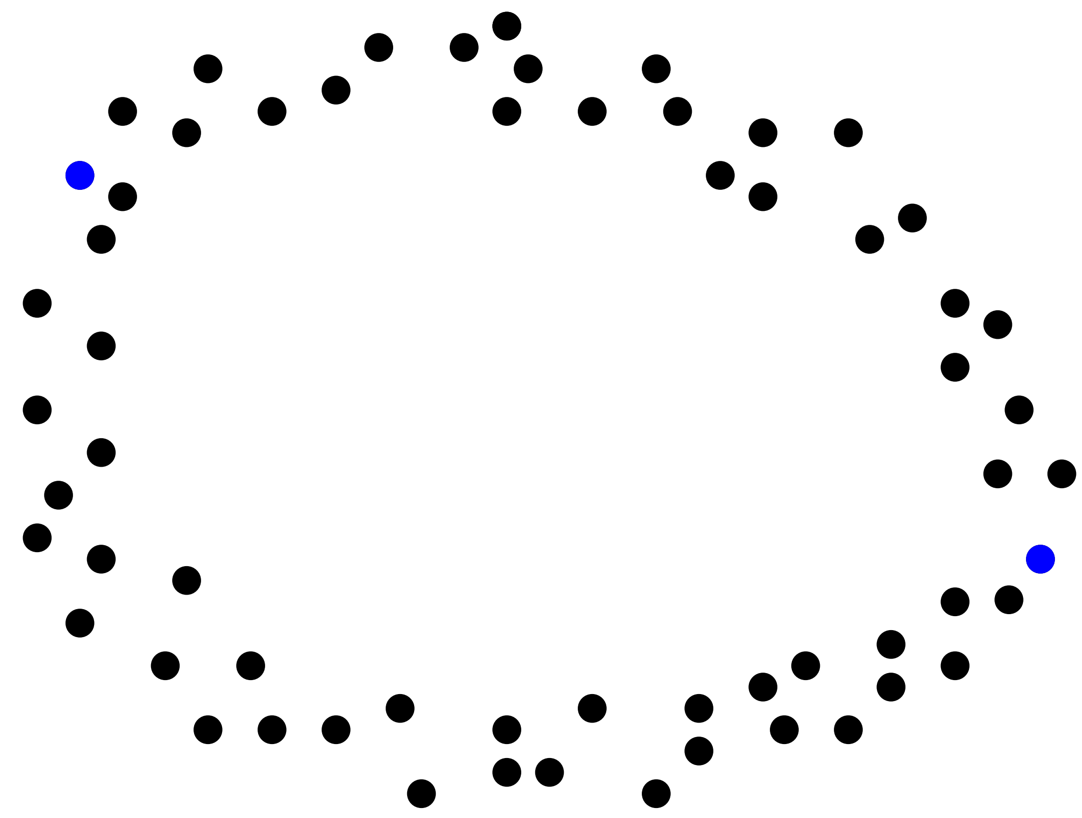

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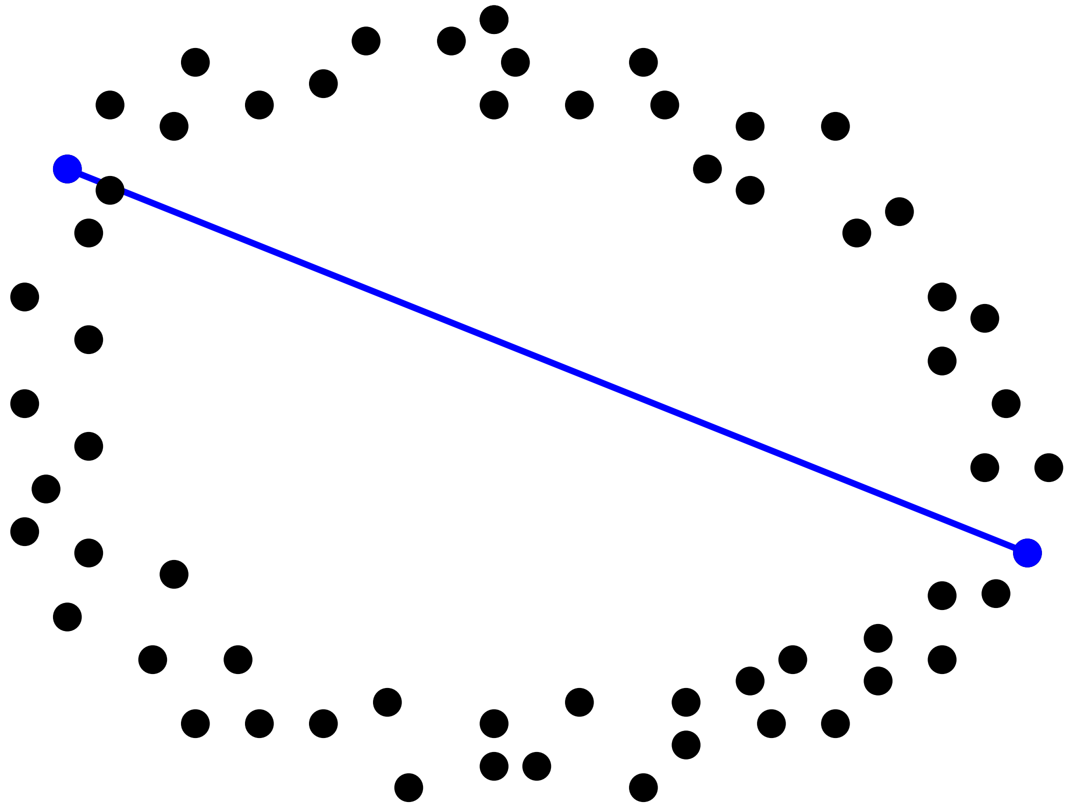

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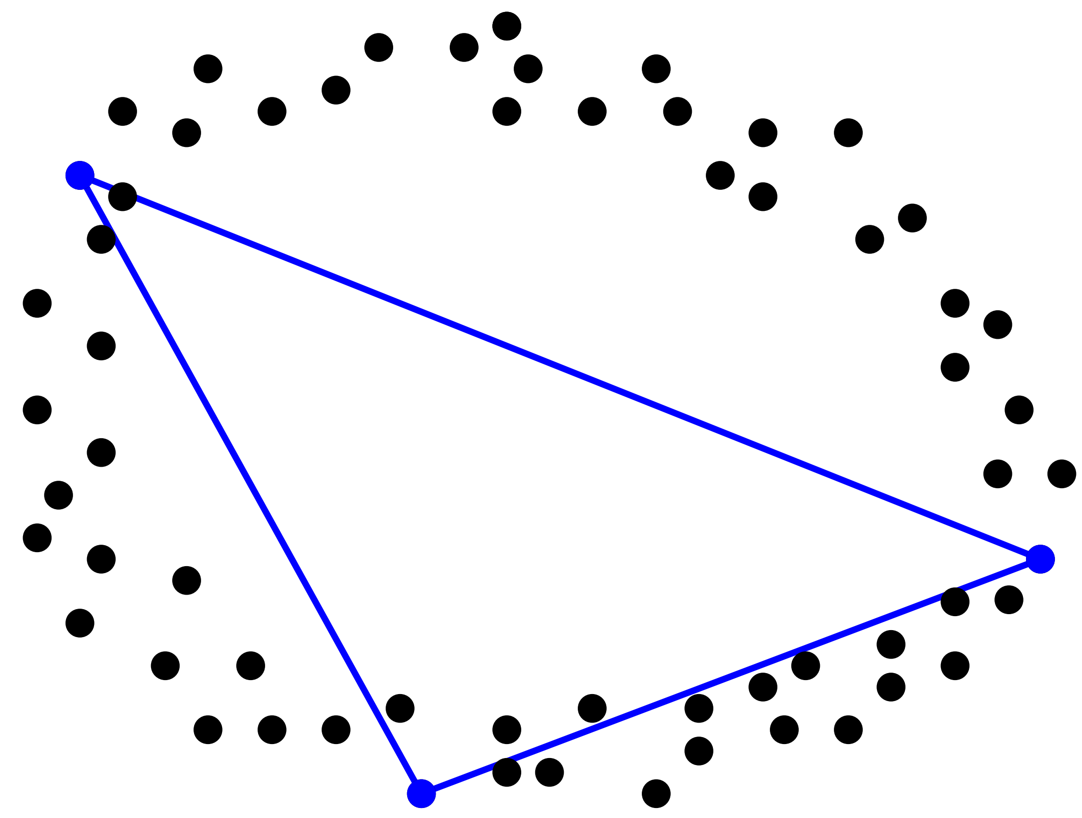

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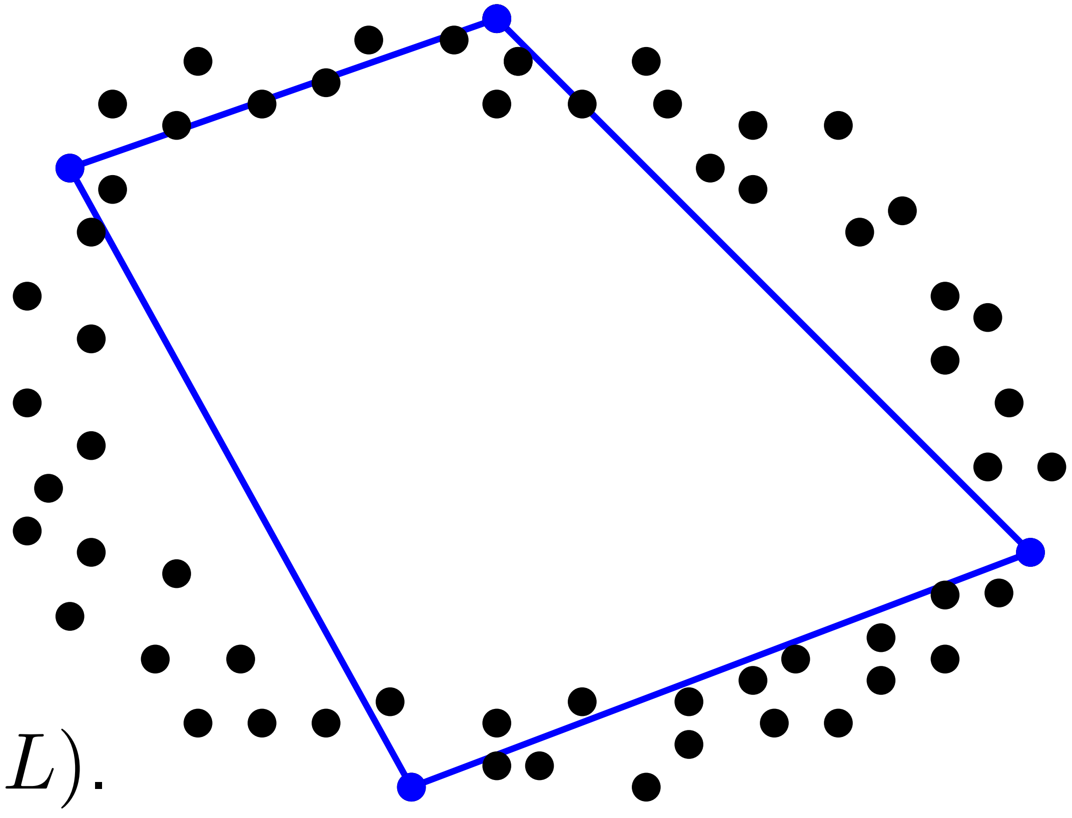

Key Structural Property

If X is a closed k-manifold smoothly embedded in $\mathbb{R}^d$, then, under reasonable sampling conditions, $\mathcal{C}_W(L) = \mathcal{D}_X(L) \approx X$

- Case $k = 1$:
  - $\mathcal{C}_W(L) = \mathcal{D}_X(L) \approx X$
- Case $k = 2$:
  - $\mathcal{C}_W(L) \subseteq \mathcal{D}_X(L) \approx X$
  - $\mathcal{C}_W(L) \not\supseteq \mathcal{D}_X(L)$
- Case $k \geq 3$:
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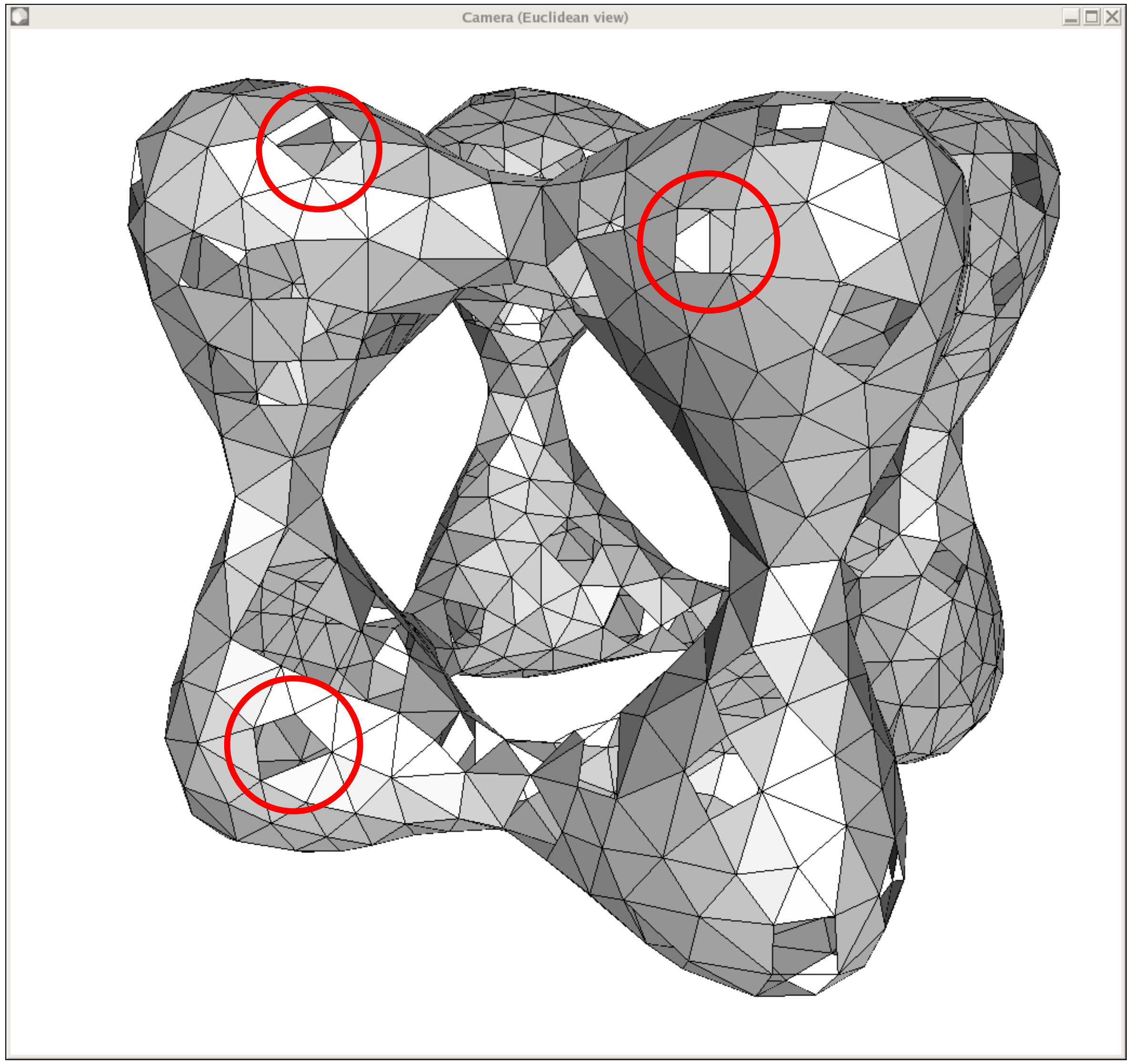

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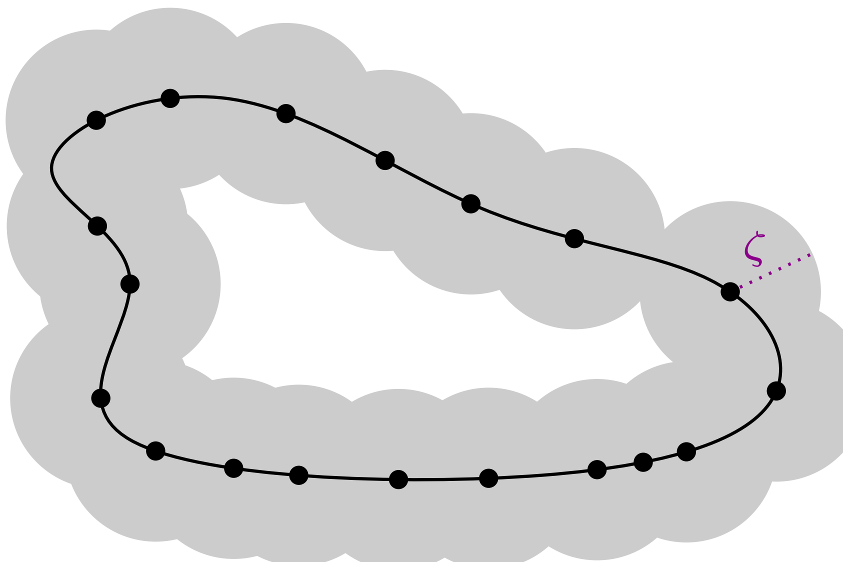

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  - $\mathcal{C}_W(L) \not\supseteq \mathcal{D}_X(L)$ ⟶ dilate W so that $W^\zeta \supseteq X$
- Case $k \geq 3$:
  - $\mathcal{C}_W(L) \not\subseteq \mathcal{D}_X(L)$
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  ⟶ assign weights to the landmarks to remove all slivers from the vicinity of $\mathcal{D}_X(L)$ [Cheng *et al.* 00]

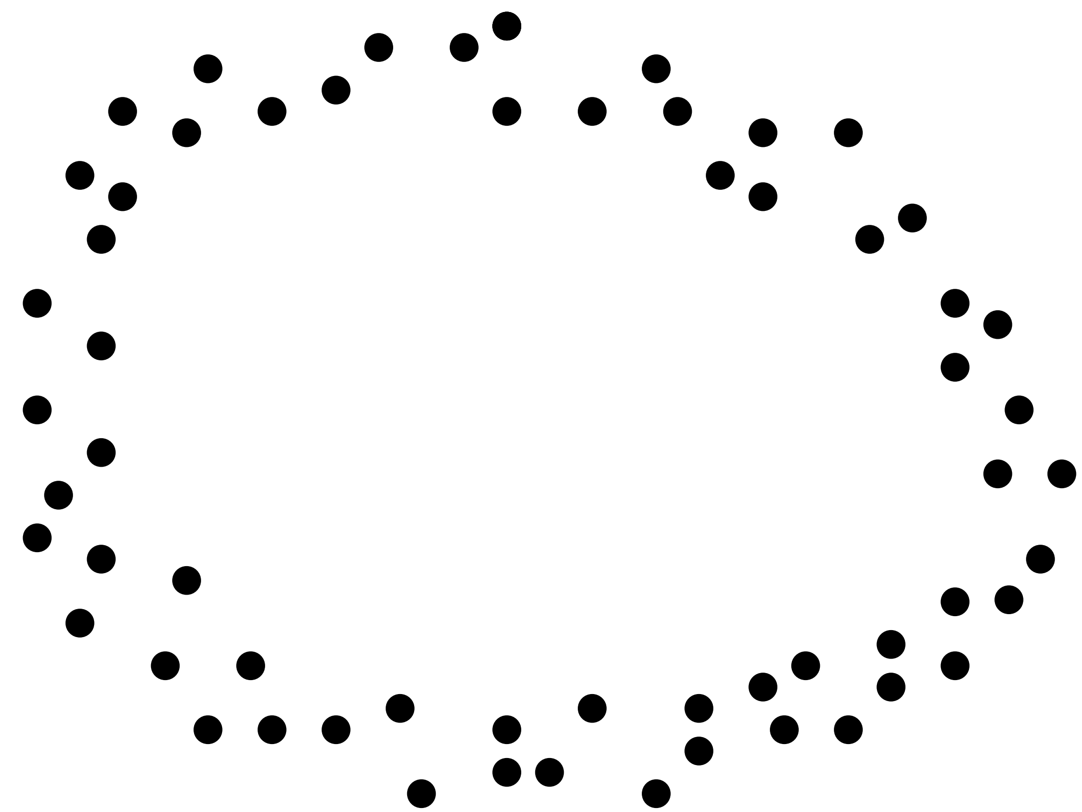
$\longrightarrow$ Source of problems: slivers

Algorithm (arbitrary dimensions)

[Boissonnat, Guibas, O. 07]

Input: a finite point set $W \subset \mathbb{R}^d$.

$\rightarrow$ maintain $\mathcal{C}^{\omega}_{W^{\zeta}}(L)$ for some carefully-chosen ζ, ω.



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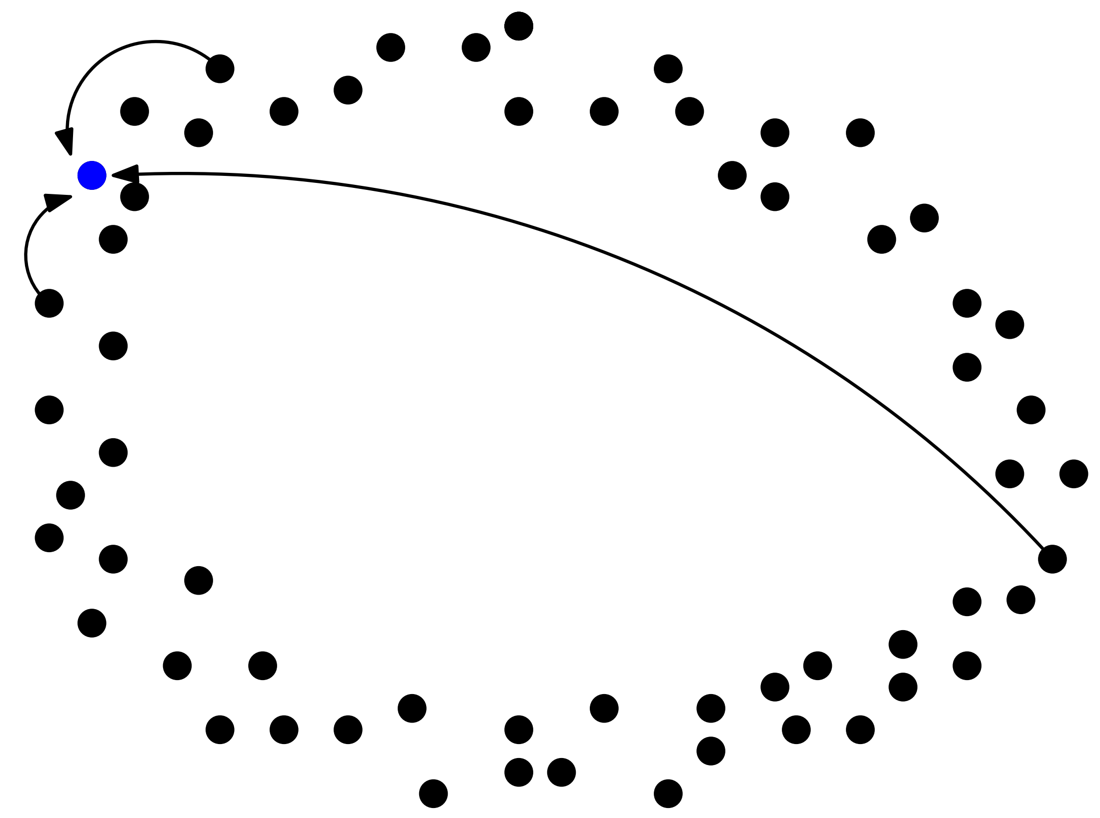

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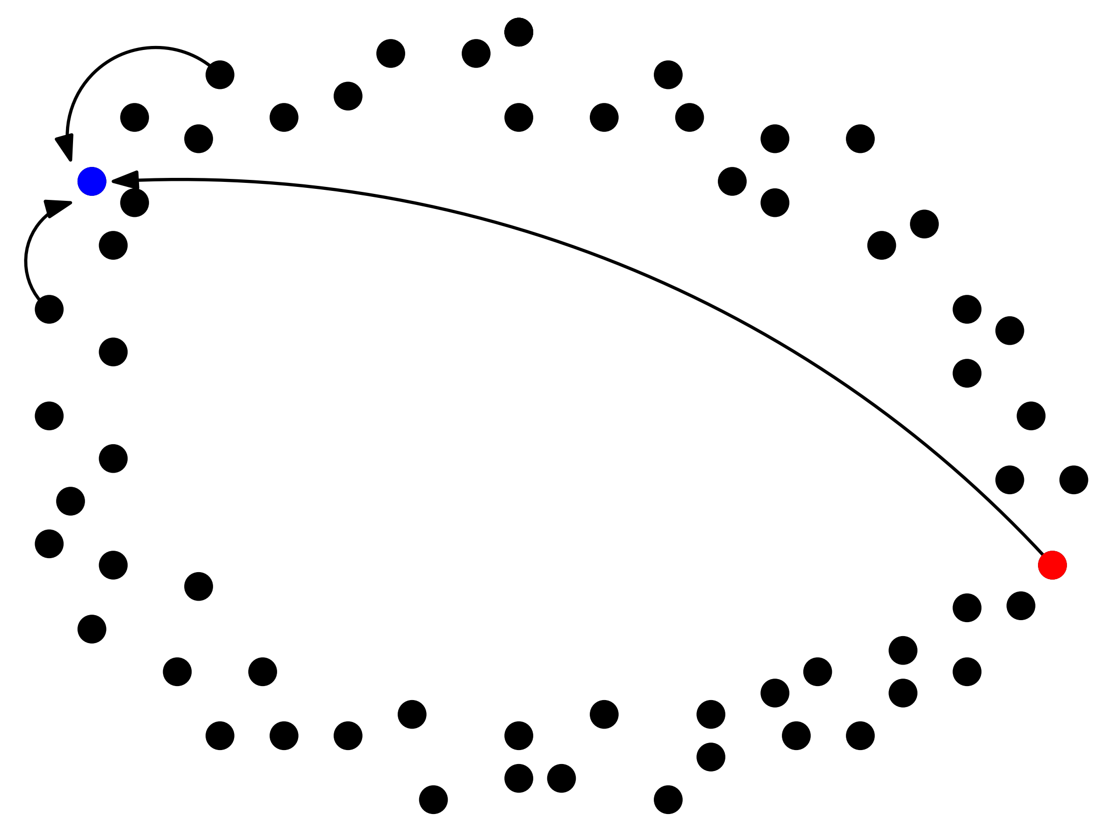

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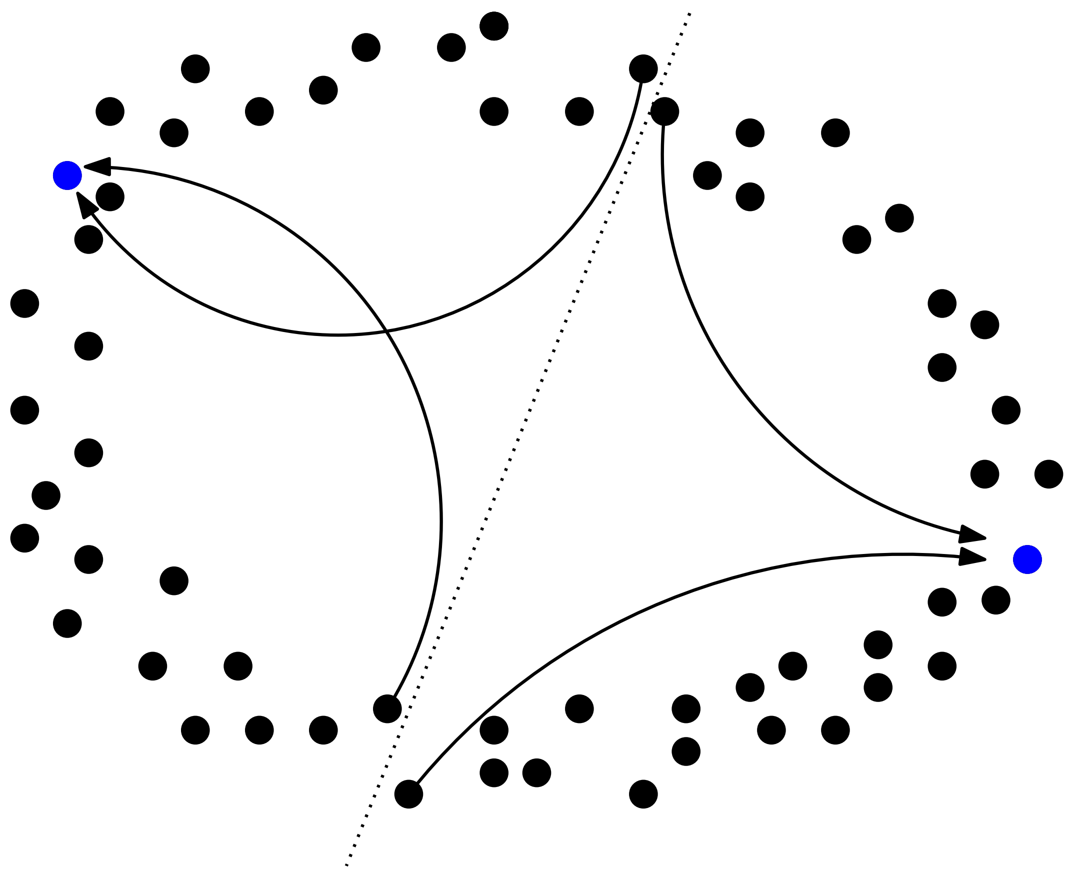

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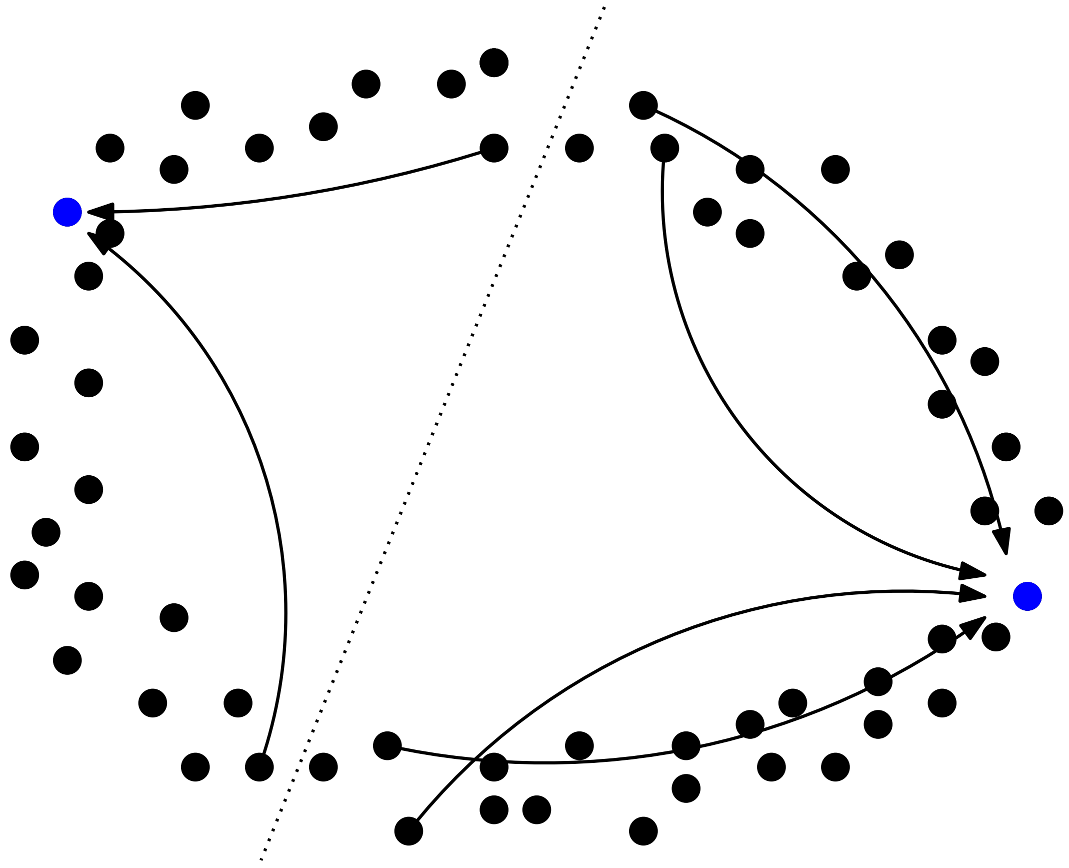

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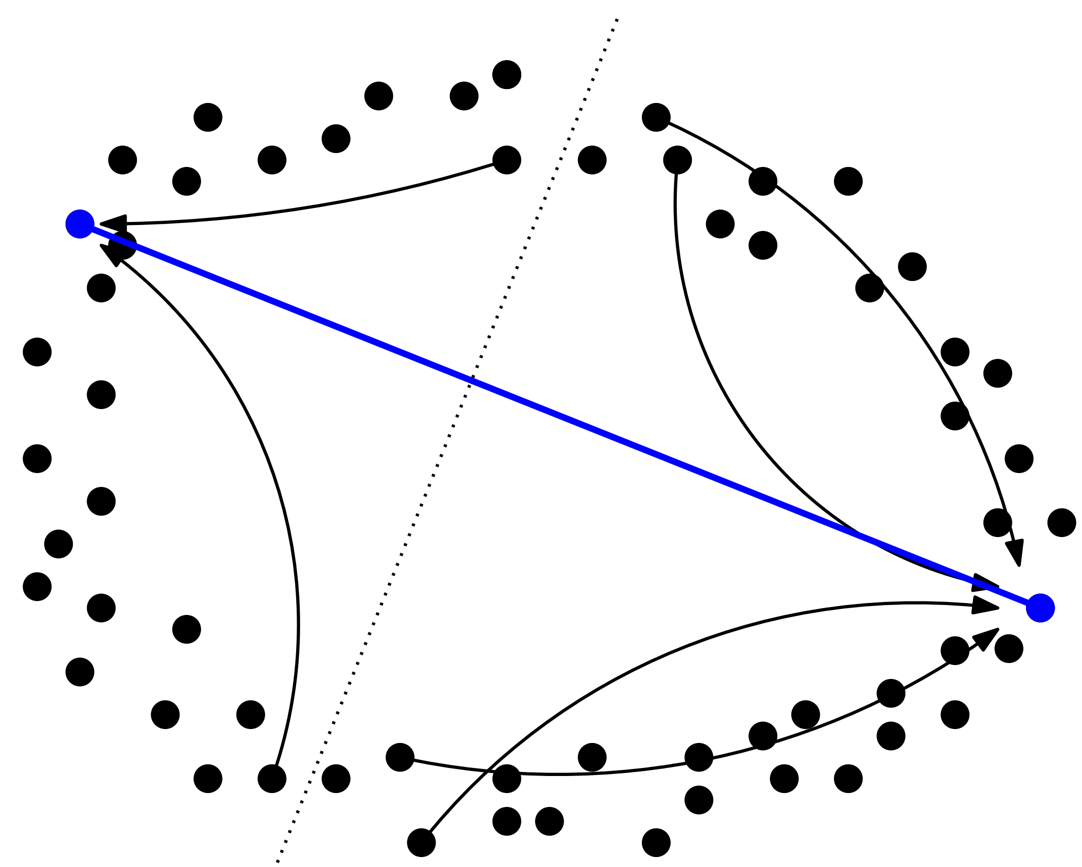

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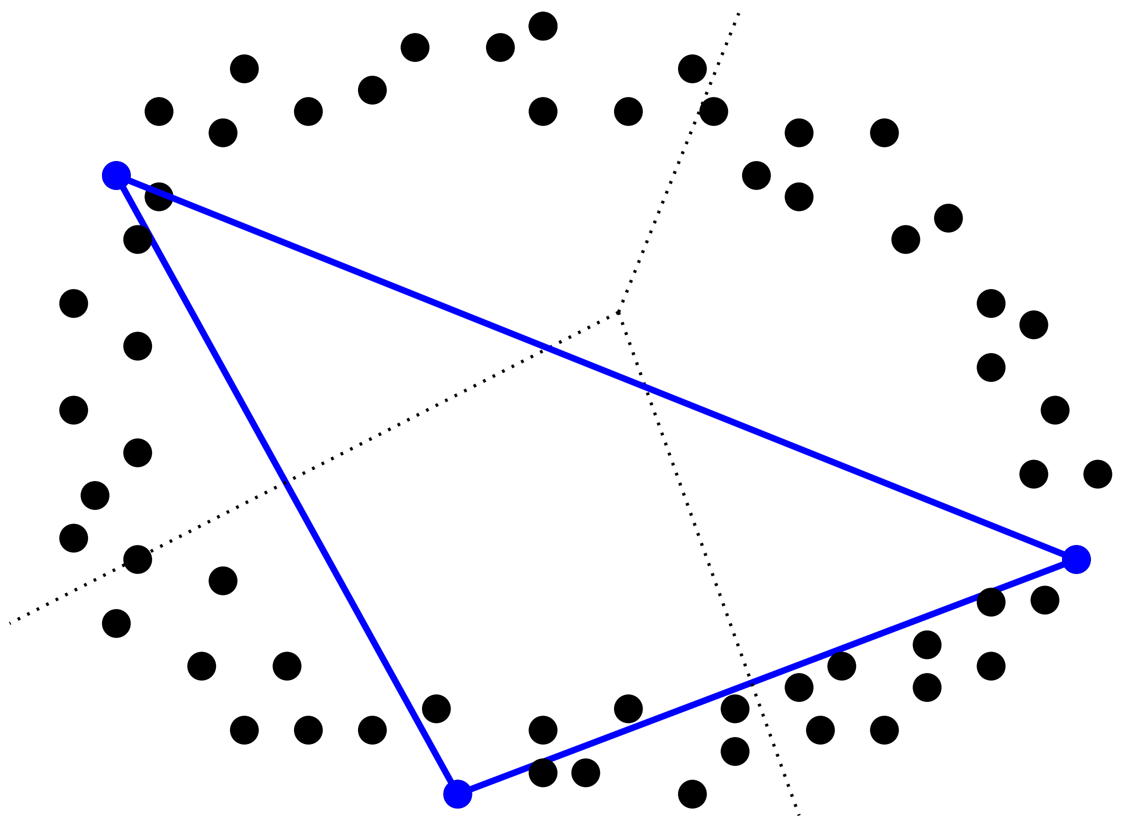

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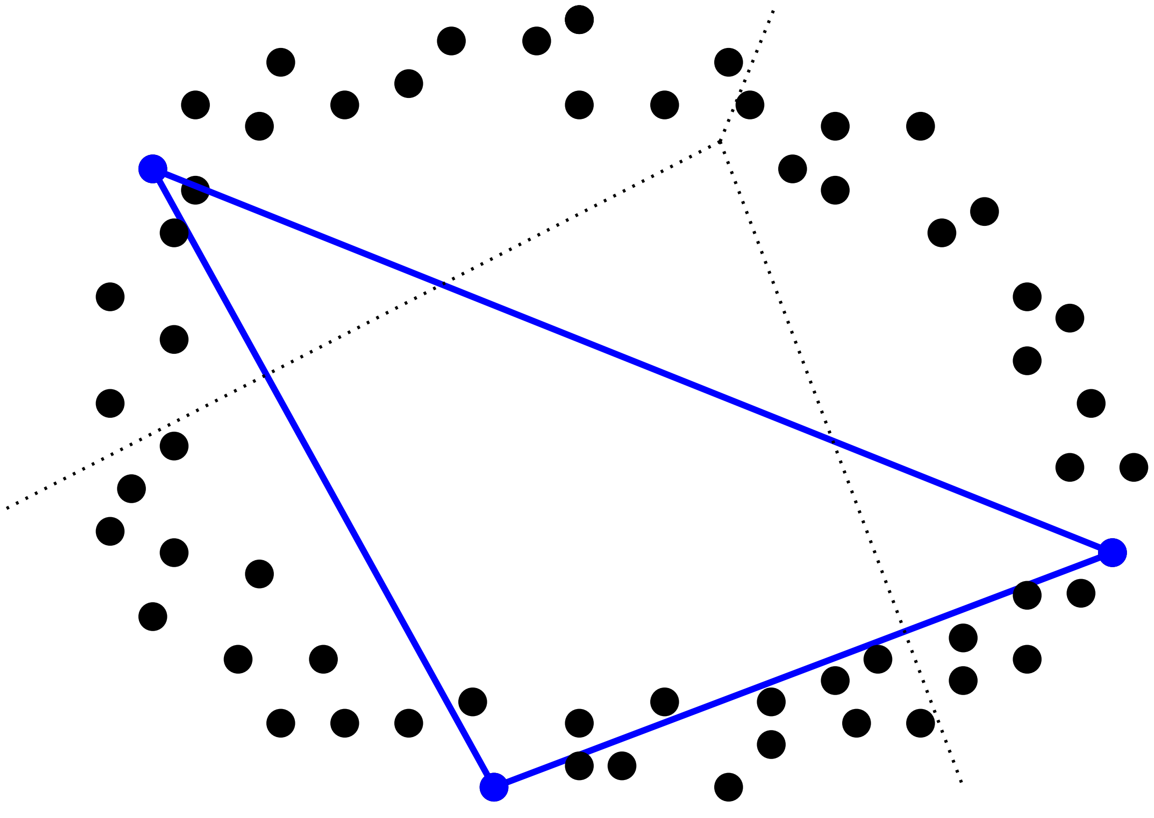

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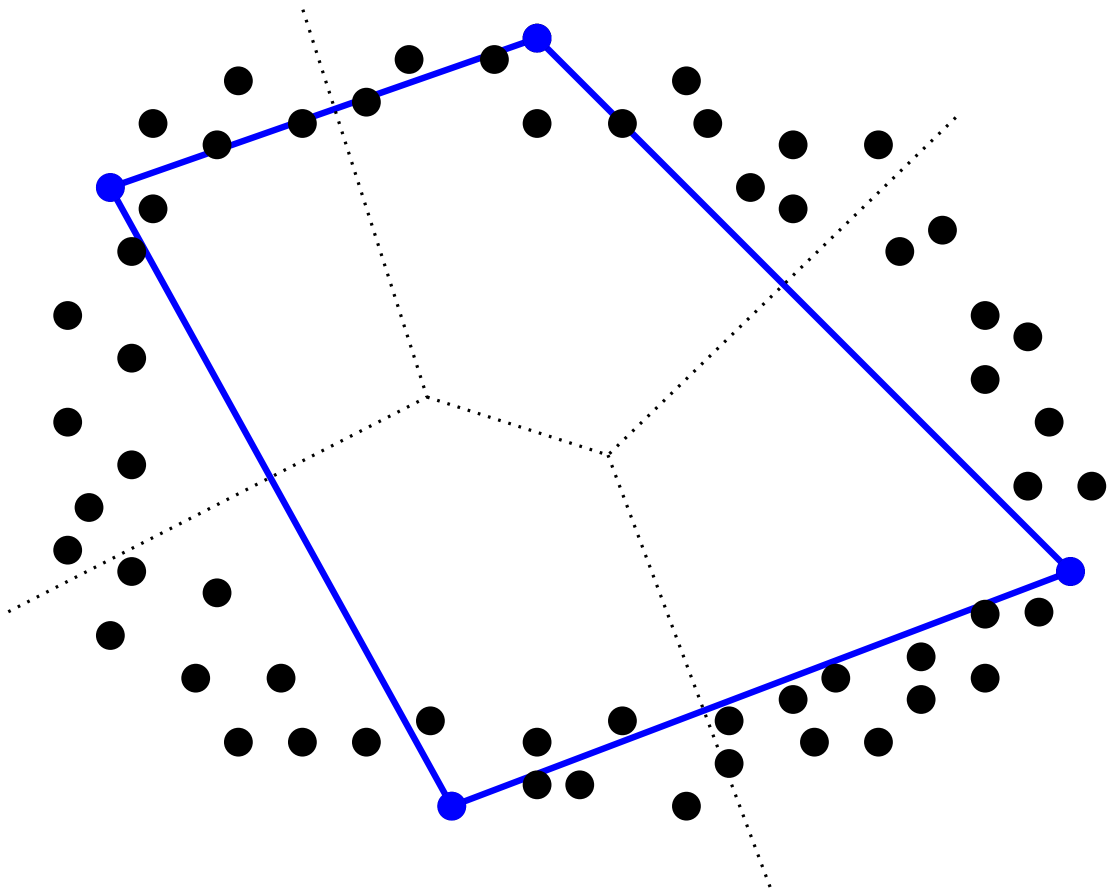

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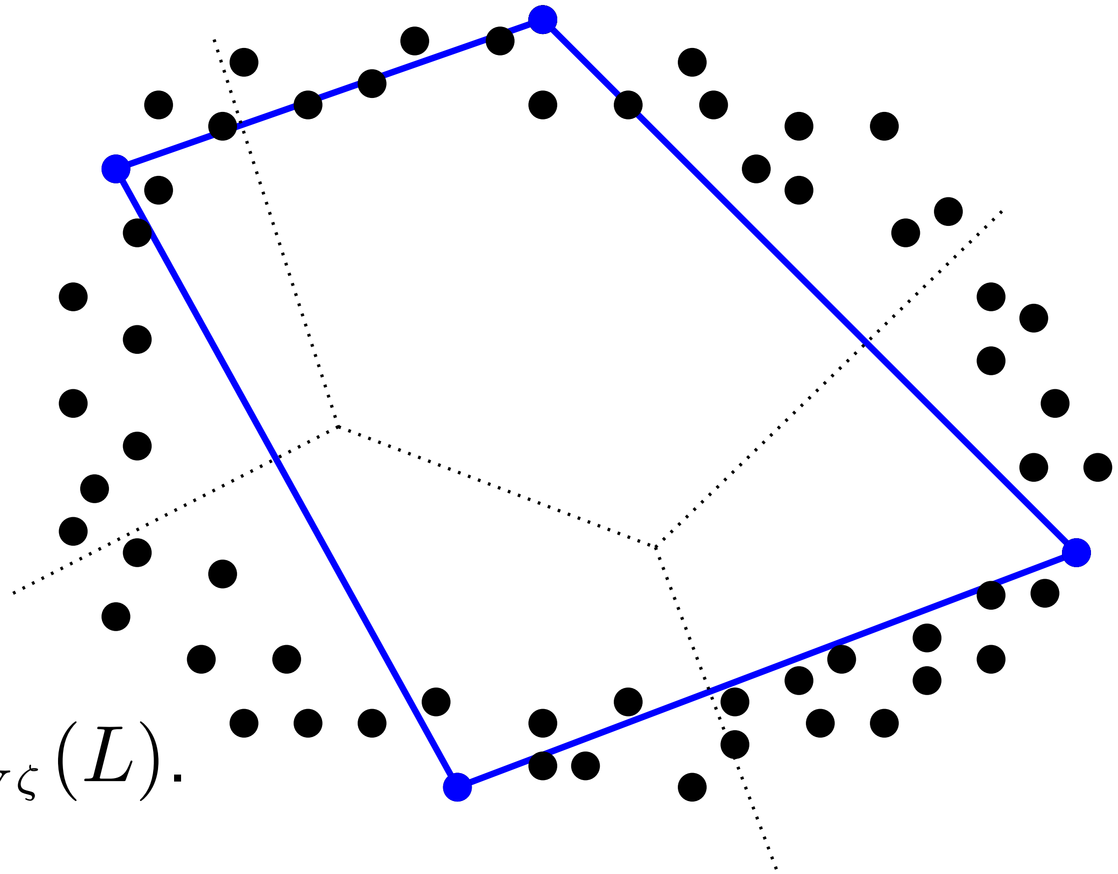

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time $\sim 2^d |W|^2$

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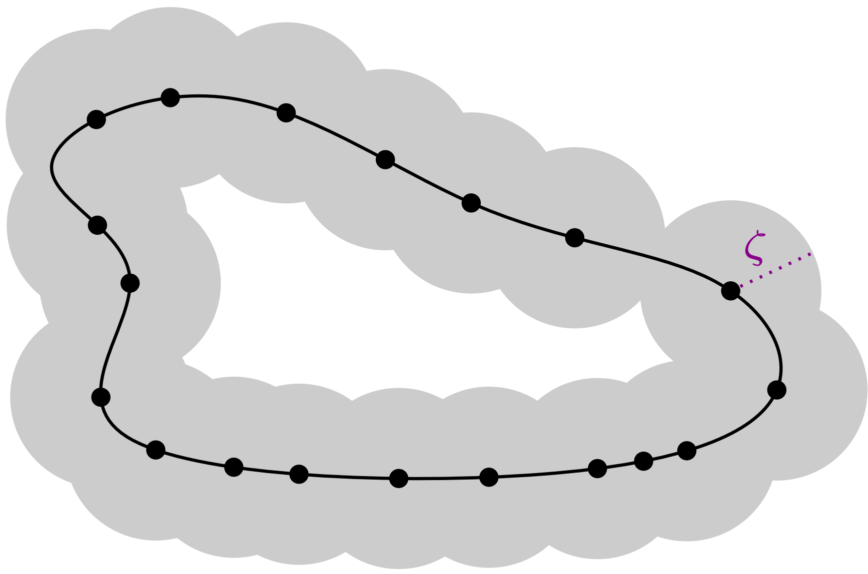

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Persistence-Based Algorithm

[Chazal, O. 08]

Input: a finite point set $W \subset \mathbb{R}^d$.

$\longrightarrow$ maintain a nested pair of *easily-computable* complexes, $\mathcal{C}^1(L) \subseteq \mathcal{C}^2(L)$.

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update $\mathcal{C}^1(L)$ and $\mathcal{C}^2(L)$;

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- easy to implement
- bounded complexity
- theoretical guarantees

Easy-to-compute Complexes

Let $L \subset \mathbb{R}^d$ be finite and let $\alpha \geq 0$.

1. Vietoris-Rips complex:

- Given $v_0, \cdots, v_k \in L$ and $\alpha \in \mathbb{R}$, $[v_0, \cdots, v_k]$ is a simplex of $\mathcal{R}^\alpha(L)$ iff we have $\|v_i - v_j\| < \alpha$ for all $0 \leq i < j \leq k$.

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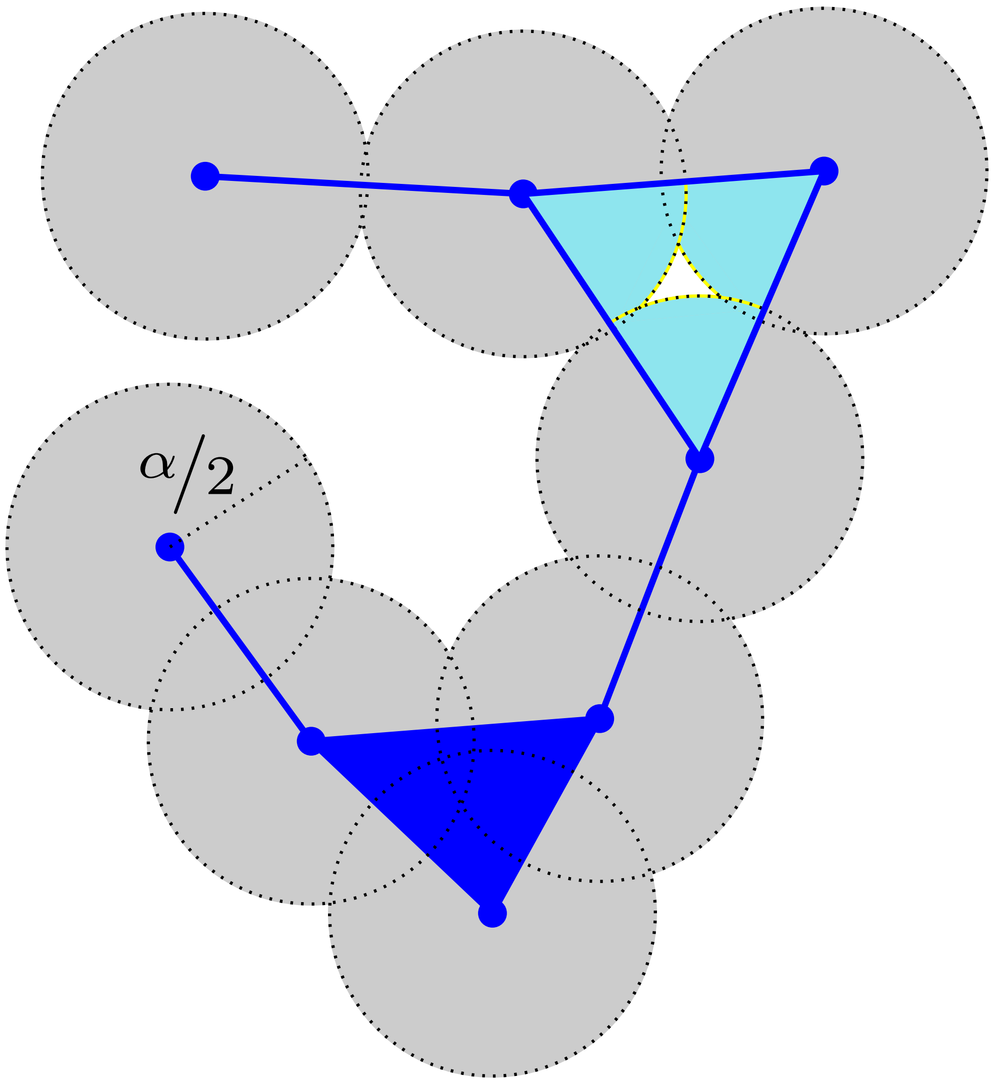

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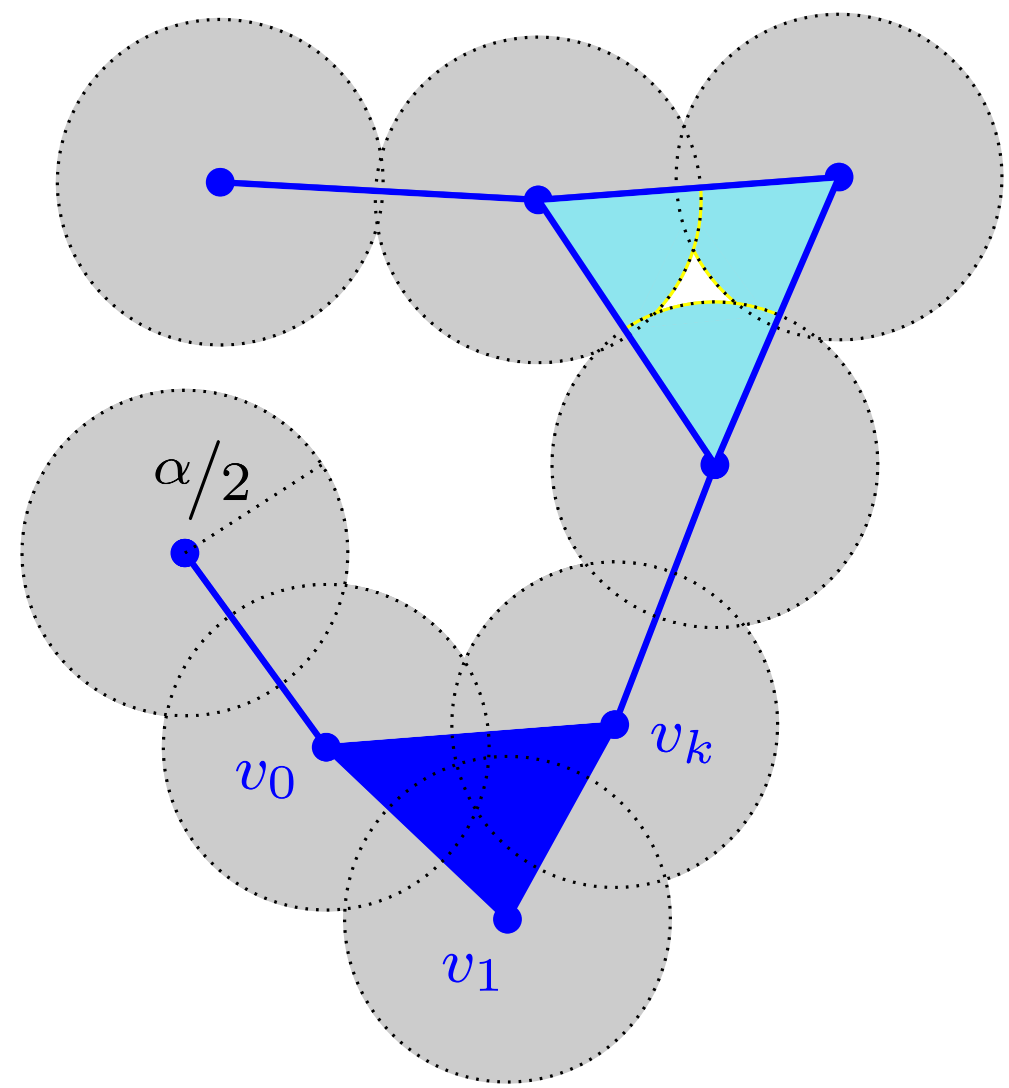
Prop. $\forall L \subset \mathbb{R}^d$, $\forall \alpha > 0$,
$\mathcal{C}^{\frac{\alpha}{2}}(L) \subseteq \mathcal{R}^\alpha(L) \subseteq \mathcal{C}^\alpha(L)$.

- If the $B(v_i, \frac{\alpha}{2}), B(v_j, \frac{\alpha}{2})$ pairwise intersect, then the v_i, v_j are at most α away from one another.

- In addition, if v_0 is at distance α of $v_1, \cdots, v_k$, then $v_0 \in \bigcap_{i=0}^k B(v_i, \alpha)$.



Easy-to-compute Complexes

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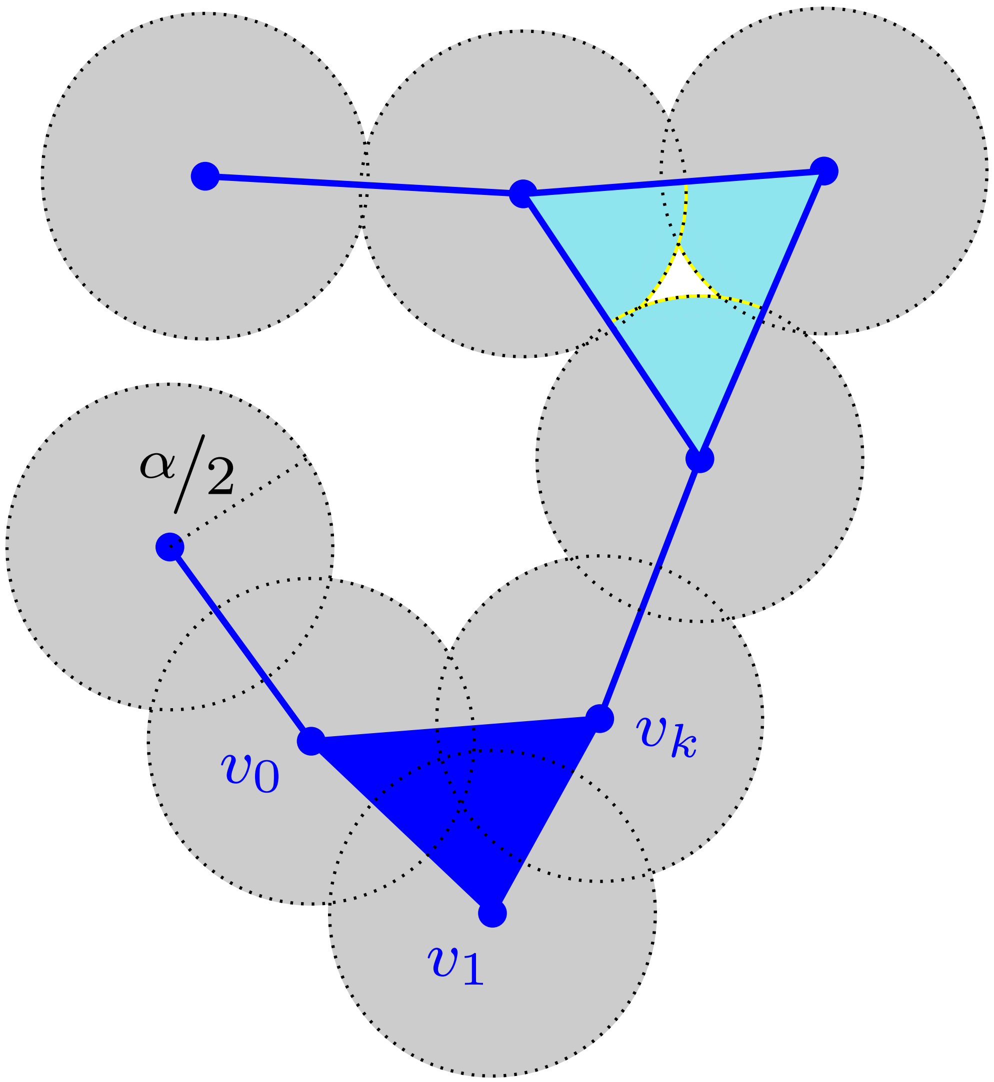
1. Vietoris-Rips complex:

- Given $v_0, \cdots, v_k \in L$ and $\alpha \in \mathbb{R}$, $[v_0, \cdots, v_k]$ is a simplex of $\mathcal{R}^\alpha(L)$ iff we have $\|v_i - v_j\| < \alpha$ for all $0 \leq i < j \leq k$.
- $\mathcal{C}^{\frac{\alpha}{2}}(L)$ is the nerve of $L^{\frac{\alpha}{2}}$.

Prop. $\forall L \subset \mathbb{R}^d$, $\forall \alpha > 0$,
$\mathcal{C}^{\frac{\alpha}{2}}(L) \subseteq \mathcal{R}^\alpha(L) \subseteq \mathcal{C}^\alpha(L)$.

- holds in arbitrary metric spaces, where the bounds are tight.

- Tight bounds in $\mathbb{R}^d$ [de Silva, Ghrist 07]:
$\mathcal{C}^{\frac{\alpha}{2}}(L) \subseteq \mathcal{R}^\alpha(L) \subseteq \mathcal{C}^{\frac{\alpha}{\sqrt{2}}}(L)$.



Easy-to-compute Complexes

Let $L \subset \mathbb{R}^d$ be finite and let $\alpha \geq 0$.

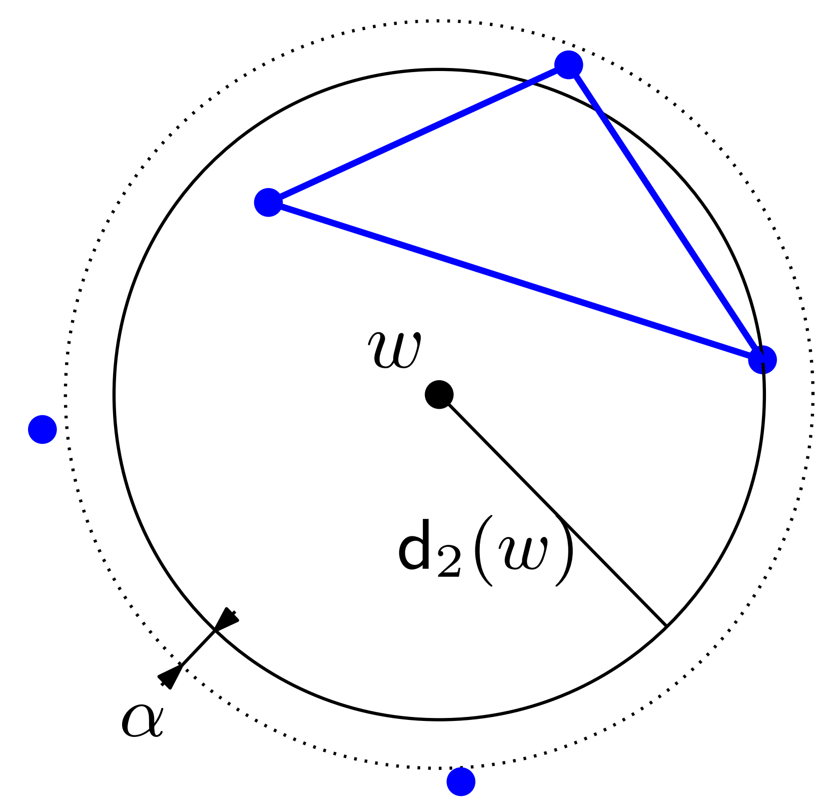
2. Relaxed witness complex [Carlsson, de Silva 04]:

Let $W \subseteq \mathbb{R}^d$.

- Given $v_0, \cdots, v_k \in L$ and $\alpha \in \mathbb{R}$, $w \in W$ is an α-witness of $[v_0, \cdots, v_k]$ if the v_i belong to the ball $B(w, \mathsf{d}_{k+1}(w) + \alpha)$, where $\mathsf{d}_{k+1}(w)$ is the Euclidean distance between w and its $(k+1)$th nearest landmark.

- Given $\alpha \in \mathbb{R}$, $\mathcal{C}_W^\alpha(L)$ is the maximum abstract simplicial complex with vertices in L, whose simplices are α-witnessed by points of W.

Note: $\mathcal{C}_W^0(L) = \mathcal{C}_W(L)$.



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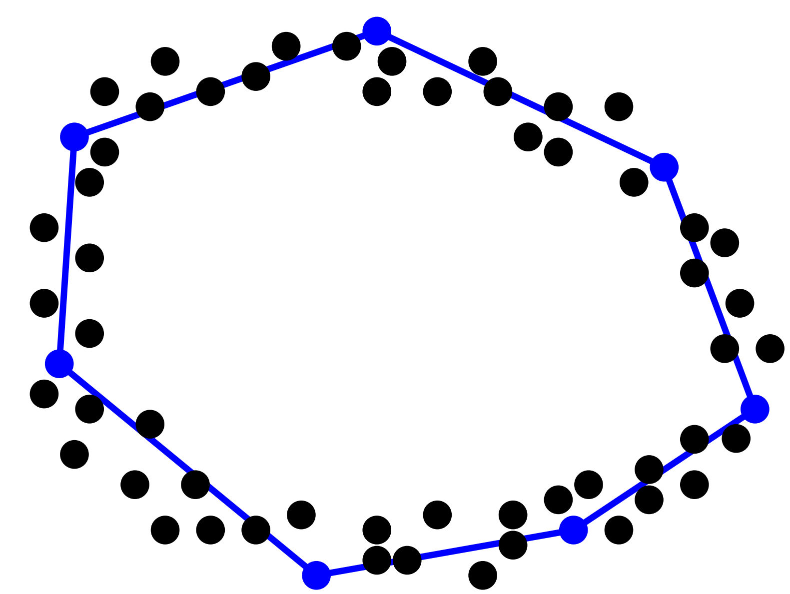


Thm.: if X is a connected compact subset of $\mathbb{R}^d$, s.t. $\mathrm{d_H}(X, W) \leq \mathrm{d_H}(W, L) < \frac{1}{8}\,\mathrm{diam}(X)$, then: $\forall \alpha \geq 2\mathrm{d_H}(W, L)$, $\mathcal{C}^{\frac{\alpha}{4}}(L) \subseteq \mathcal{C}_W^\alpha(L) \subseteq \mathcal{C}^{8\alpha}(L)$.

- holds in arbitrary metric spaces, where the bounds are tight.

Easy-to-compute Complexes

Let $L \subset \mathbb{R}^d$ be finite and let $\alpha \geq 0$. Let $W \subseteq \mathbb{R}^d$.

$\longrightarrow$ Intertwined filtrations:

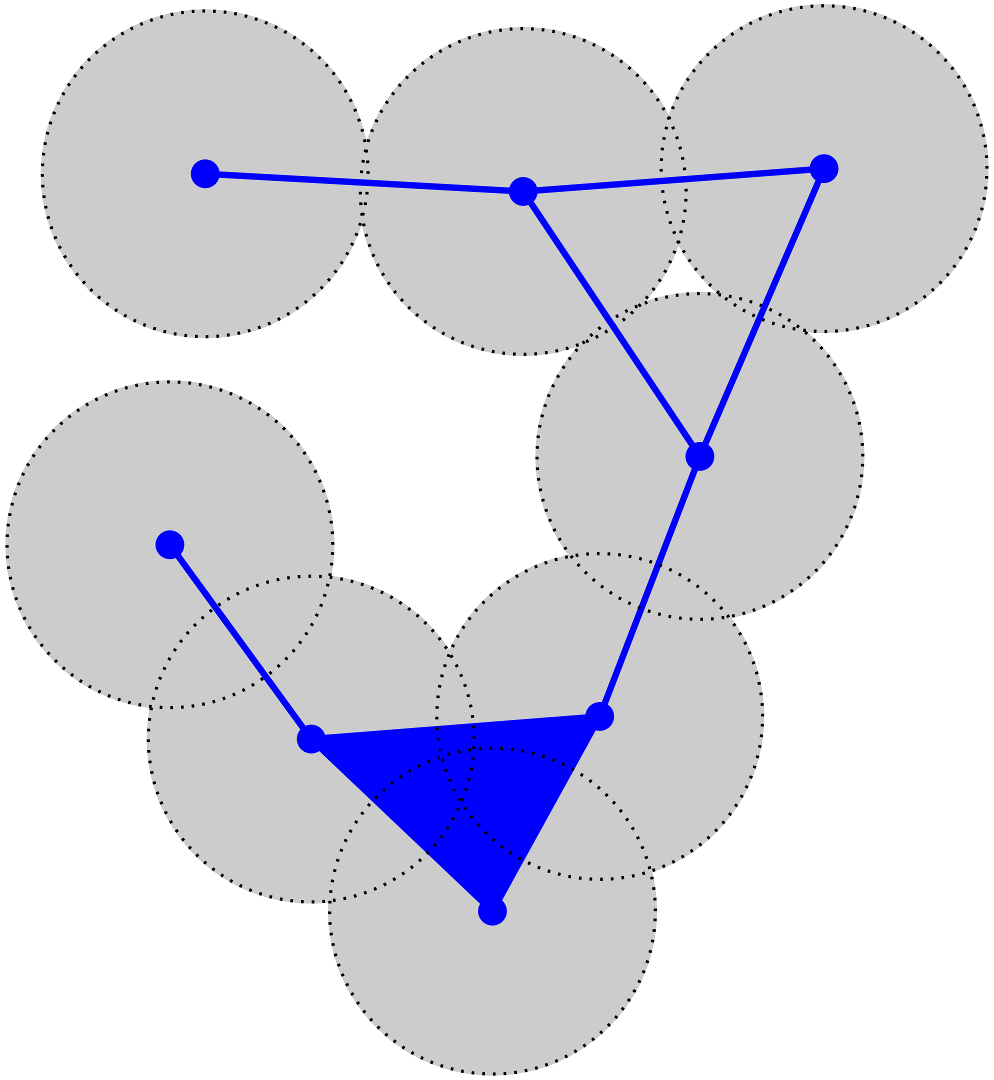
$$\mathcal{C}^{\frac{\alpha}{2}}(L) \hookrightarrow \mathcal{R}^{\alpha}(L) \hookrightarrow \mathcal{C}^{\alpha}(L) \hookrightarrow \mathcal{R}^{2\alpha}(L) \hookrightarrow \mathcal{C}^{2\alpha}(L) \hookrightarrow \cdots$$

$$\mathcal{C}^{\frac{\alpha}{4}}(L) \hookrightarrow \mathcal{C}_W^{\alpha}(L) \hookrightarrow \mathcal{C}^{8\alpha}(L) \hookrightarrow \mathcal{C}_W^{32\alpha}(L) \hookrightarrow \mathcal{C}^{256\alpha}(L) \hookrightarrow \cdots$$

$\longrightarrow$ Our goal: study the homomorphisms induced by $\mathcal{C}^{\alpha}(L) \hookrightarrow \mathcal{C}^{\alpha'}(L)$.

Topology of Unions of Balls

Recall that $\mathcal{C}^\alpha(L)$ is the nerve of the union of balls L^α.



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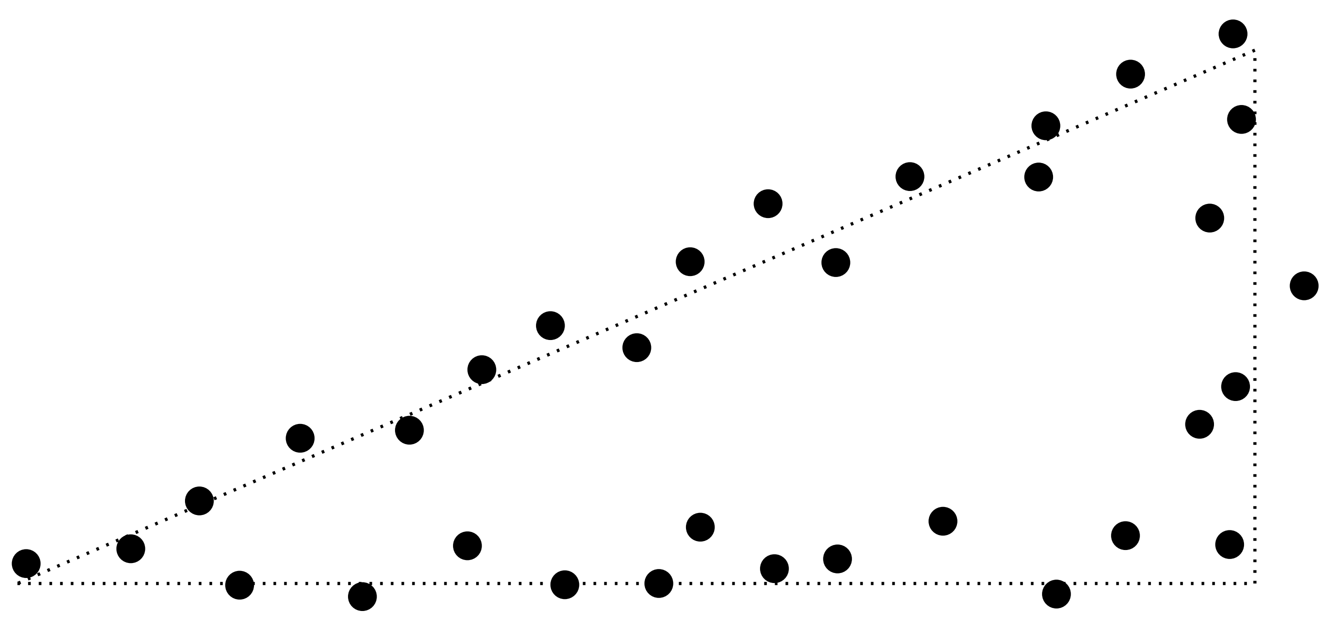
Thm [Chazal, Lieutier 05], [Cohen-Steiner, Edelsbrunner, Harer 05]
If $X \subset \mathbb{R}^d$ is a compact set with positive *weak feature size*, and if $\mathrm{d_H}(X, L) = \varepsilon < \frac{1}{4}\ \mathrm{wfs}(X)$, then, for all $\alpha, \alpha' \in [\varepsilon,\ \mathrm{wfs}(X) - \varepsilon]$ such that $\alpha' \geq \alpha + 2\varepsilon$, and for all $\lambda \in (0, \mathrm{wfs}(X))$, we have: $\forall k \in \mathbb{N}$, $H_k(X^\lambda) \cong \mathrm{im}\ i_*$, where $i_* : H_k(L^\alpha) \to H_k(L^{\alpha'})$ is the homomorphism induced by $L^\alpha \hookrightarrow L^{\alpha'}$.



(from [Chazal, Cohen-Steiner 07])

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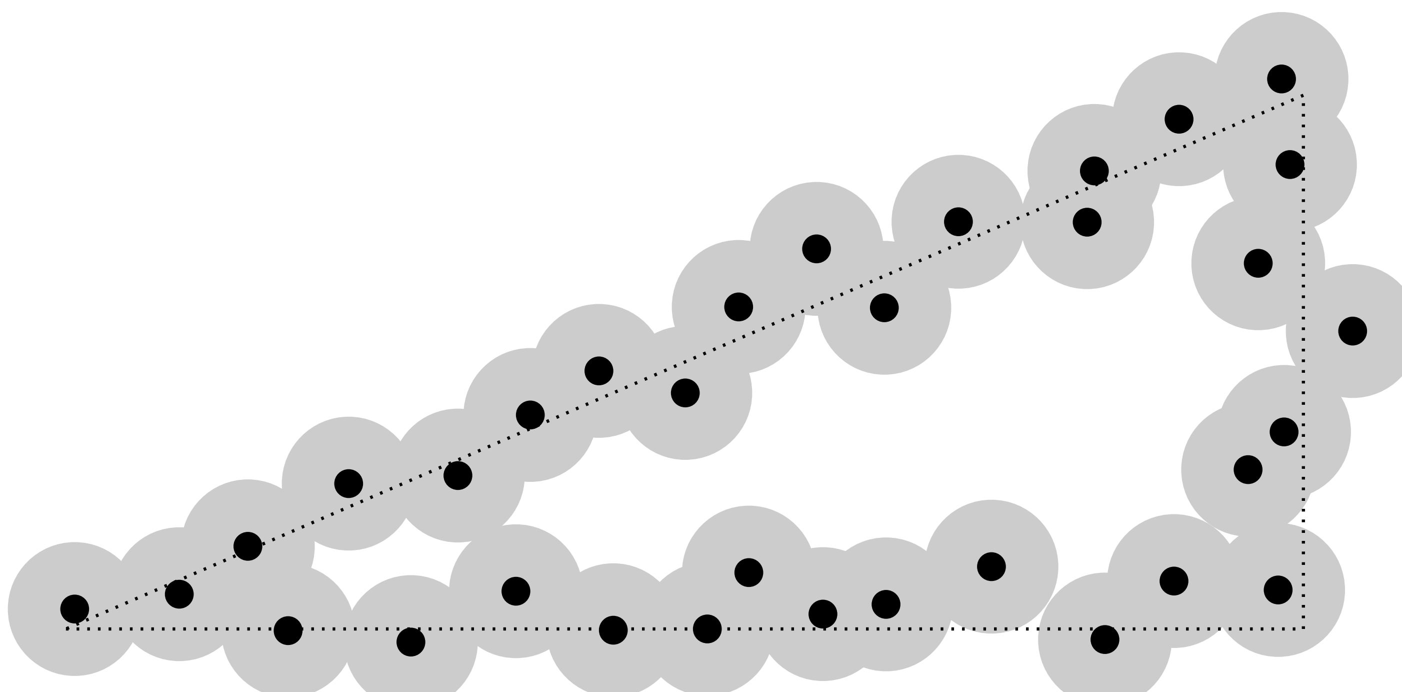

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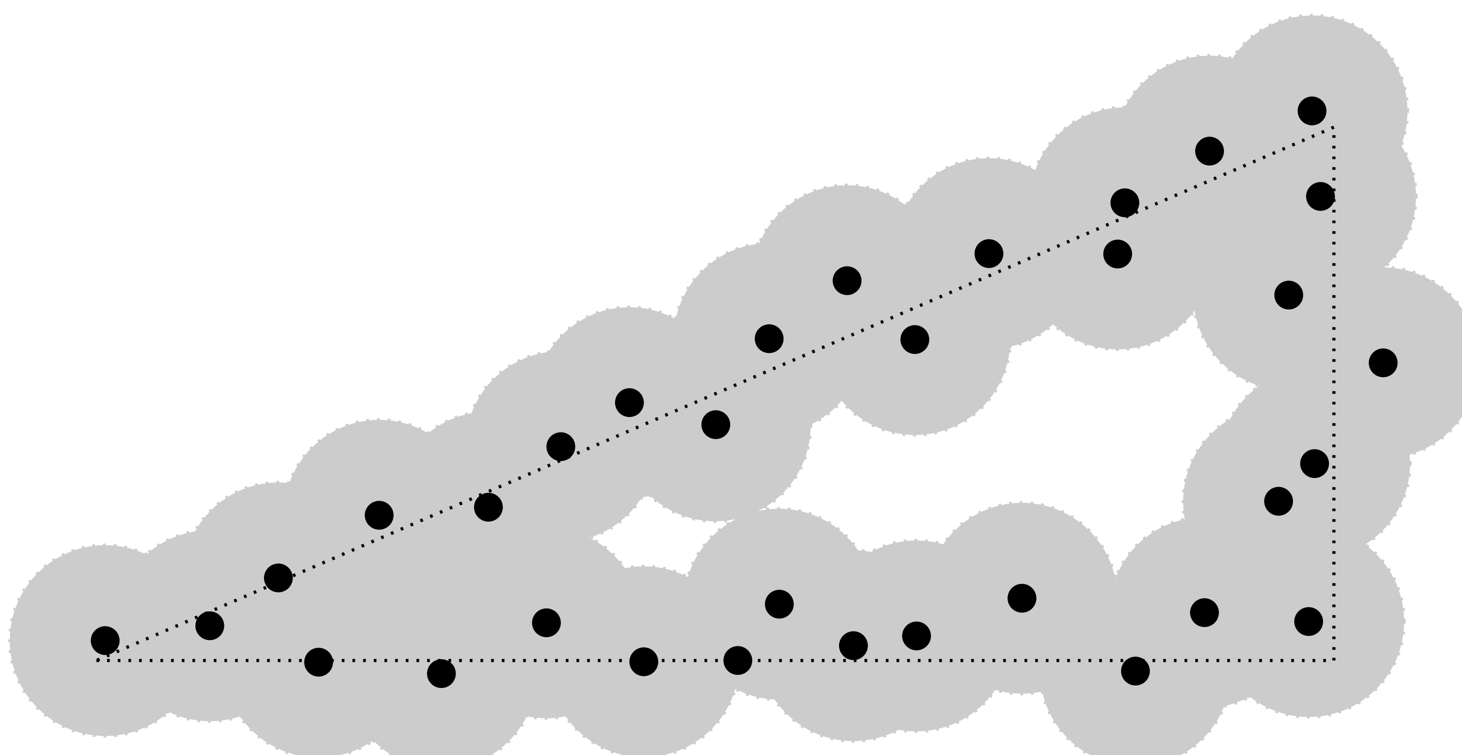

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- α-complex:

  Thm. [Edelsbrunner 93] $\forall \alpha > 0$, L^α deformation retracts onto $\alpha(L)$.

$$
\begin{array}{ccc}
L^\alpha & \hookrightarrow & L^{\alpha'} \\
\uparrow & & \uparrow \\
\alpha(L) & \hookrightarrow & \alpha'(L)
\end{array}
$$

- vertical arrows are homotopy equivalences
- canonical inclusions commute

Topology of Unions of Balls

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- Čech complex:

  Thm (Nerve) $\forall \alpha > 0$, $\mathcal{C}^\alpha(L)$ is homotopy equivalent to L^α.

$$\begin{array}{ccc} L^\alpha & \hookrightarrow & L^{\alpha'} \\ \uparrow & & \uparrow \\ \mathcal{C}^\alpha(L) & \hookrightarrow & \mathcal{C}^{\alpha'}(L) \end{array}$$

- vertical arrows are homotopy equivalences
- diagram might not commute

About the Nerve Theorem

Thm Let $L \subset \mathbb{R}^d$ be finite, and let $0 < \alpha \leq \alpha'$. Then, there exist homotopy equivalences $\mathcal{C}^\alpha(L) \to L^\alpha$ and $\mathcal{C}^{\alpha'}(L) \to L^{\alpha'}$ that make the previous diagram commute at homology level.

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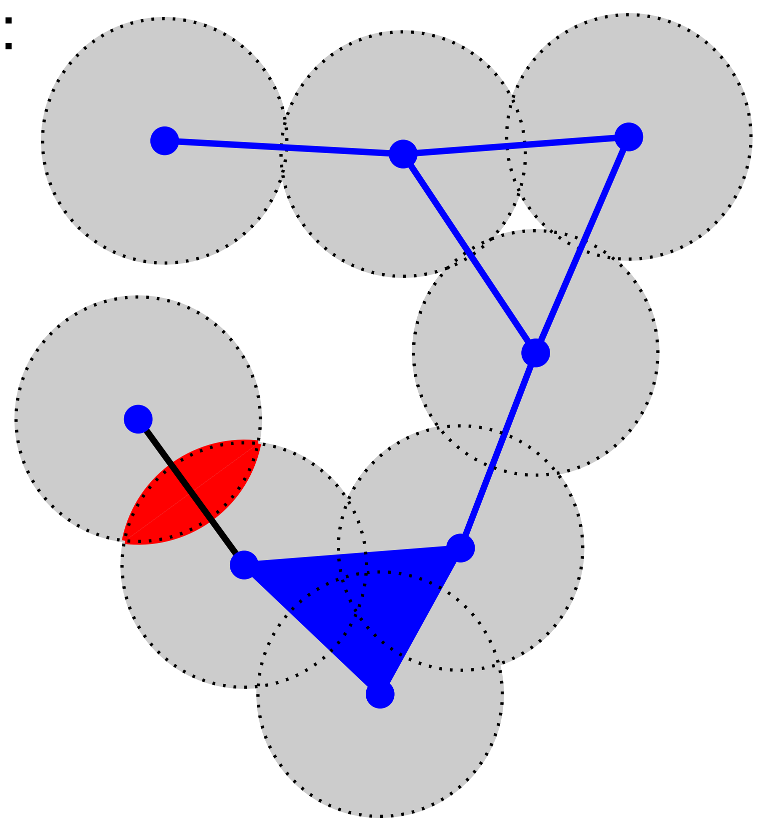
Proof: Review of the proof of the Nerve theorem [Hatcher 01, Sec. 4G].

- Fact: balls of L^α intersect along convex ($\Rightarrow$ contractible) subspaces, if at all.

- Let $n = \#L - 1$, and let $\Delta L^\alpha \subseteq X \times \Delta^n$ be defined by:

$$\Delta L^\alpha := \bigcup_{\emptyset \neq S \subseteq L} B_S(\alpha) \times [S]$$



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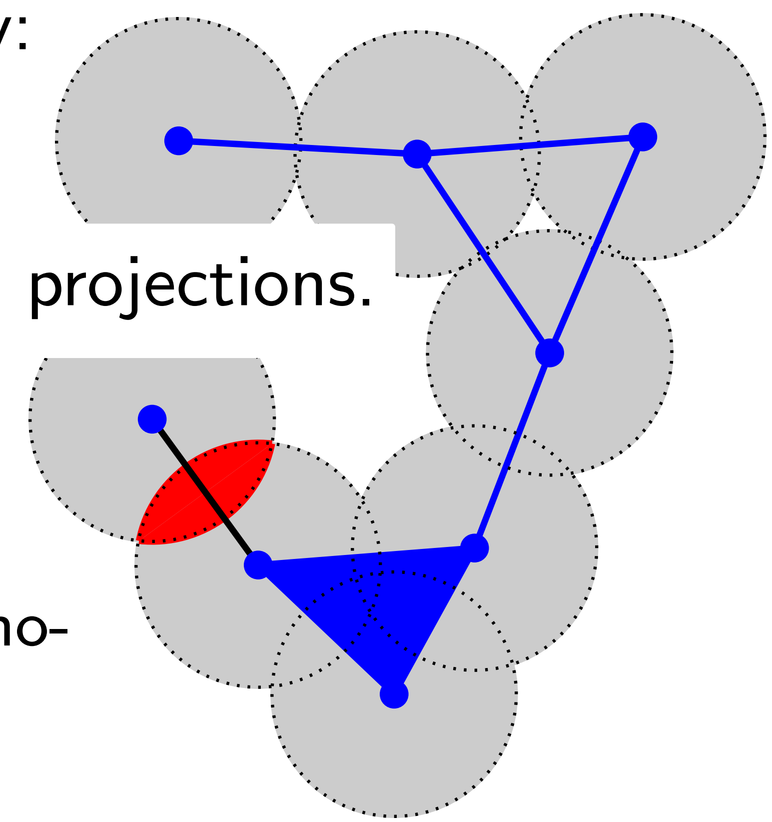
- Let $p_\alpha : \Delta L^\alpha \to L^\alpha$ and $q_\alpha : \Delta L^\alpha \to \mathcal{C}^\alpha(L)$ be natural projections.



$$\begin{array}{ccc} L^\alpha & \hookrightarrow & L^{\alpha'} \\ p_\alpha \uparrow & & \uparrow p_{\alpha'} \\ \Delta L^\alpha & \hookrightarrow & \Delta L^{\alpha'} \\ q_\alpha \downarrow & & \downarrow q_{\alpha'} \\ \mathcal{C}^\alpha(L) & \hookrightarrow & \mathcal{C}^{\alpha'}(L) \end{array}$$

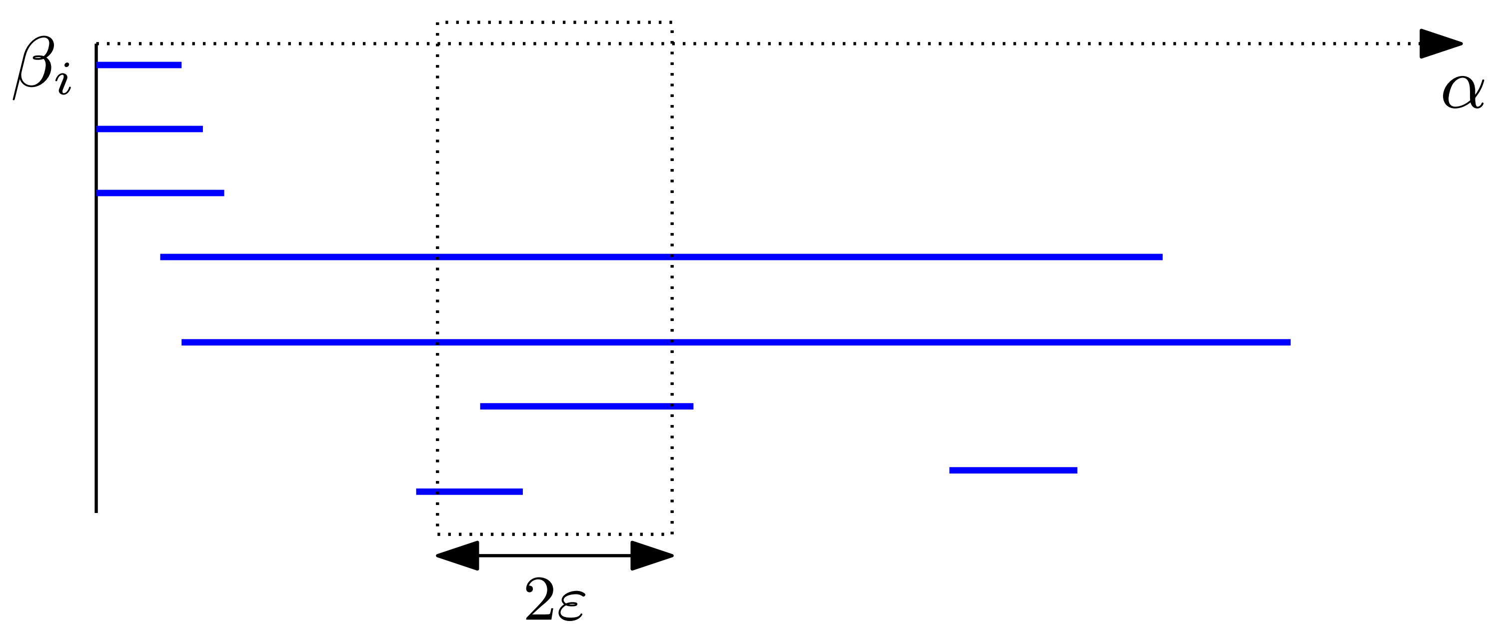
- the diagram commutes
- vertical arrows are homotopy equivalences

$\square$

About the Nerve Theorem

Thm Let $L \subset \mathbb{R}^d$ be finite, and let $0 < \alpha \leq \alpha'$. Then, there exist homotopy equivalences $\mathcal{C}^\alpha(L) \to L^\alpha$ and $\mathcal{C}^{\alpha'}(L) \to L^{\alpha'}$ that make the previous diagram commute at homology level.

Corollary If $X \subset \mathbb{R}^d$ is a compact set with positive *weak feature size*, and if $\mathrm{d_H}(X, L) = \varepsilon < \frac{1}{4}\,\mathrm{wfs}(X)$, then, for all $\alpha, \alpha' \in [\varepsilon,\ \mathrm{wfs}(X) - \varepsilon]$ such that $\alpha' \geq \alpha + 2\varepsilon$, and for all $\lambda \in (0, \mathrm{wfs}(X))$, we have: $\forall k \in \mathbb{N}$, $H_k(X^\lambda) \cong \mathrm{im}\; i_*$, where $i_* : H_k(\mathcal{C}^\alpha(L)) \to H_k(\mathcal{C}^{\alpha'}(L))$ is the homomorphism induced by $\mathcal{C}^\alpha(L) \hookrightarrow \mathcal{C}^{\alpha'}(L)$.



Effect on Intertwined Filtrations

- Rips filtration: Let $\alpha \geq 2\varepsilon$.

$$\mathcal{C}^{\frac{\alpha}{2}}(L) \hookrightarrow \mathcal{R}^{\alpha}(L) \hookrightarrow \mathcal{C}^{\alpha}(L) \hookrightarrow \mathcal{C}^{2\alpha}(L) \hookrightarrow \mathcal{R}^{4\alpha}(L) \hookrightarrow \mathcal{C}^{4\alpha}(L)$$

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$$\begin{array}{rcl} \dim H_k(X^\lambda) & = & \operatorname{rank} H_k(\mathcal{C}^{\frac{\alpha}{2}}(L)) \rightarrow H_k(\mathcal{C}^{4\alpha}(L)) \\ & & \leq \\ & & \operatorname{rank} H_k(\mathcal{R}^{\alpha}(L)) \rightarrow H_k(\mathcal{R}^{4\alpha}(L)) \\ & & \leq \\ & & \operatorname{rank} H_k(\mathcal{C}^{\alpha}(L)) \rightarrow H_k(\mathcal{C}^{2\alpha}(L)) \quad = \quad \dim H_k(X^\lambda) \end{array}$$

$\Rightarrow \operatorname{im} H_k(\mathcal{R}^{\alpha}(L)) \rightarrow H_k(\mathcal{R}^{4\alpha}(L)) \cong H_k(X^\lambda)$, since our ring of coefficients is a field.

Effect on Intertwined Filtrations

- Witness complex filtration: Let $\alpha \geq 4\varepsilon$.

$$\mathcal{C}^{\frac{\alpha}{4}}(L) \hookrightarrow \mathcal{C}_W^{\alpha}(L) \hookrightarrow \mathcal{C}^{8\alpha}(L) \hookrightarrow \mathcal{C}^{9\alpha}(L) \hookrightarrow \mathcal{C}_W^{36\alpha}(L) \hookrightarrow \mathcal{C}^{288\alpha}(L)$$

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Effect on Intertwined Filtrations

- Intertwined filtration: Let $\alpha \geq \frac{1}{a}\varepsilon$.

$$\mathcal{C}^{a\alpha}(L) \hookrightarrow \mathcal{F}^{\alpha}(L) \hookrightarrow \mathcal{C}^{b\alpha}(L) \hookrightarrow \mathcal{C}^{(b+1)\alpha}(L) \hookrightarrow \mathcal{F}^{c\alpha}(L) \hookrightarrow \mathcal{C}^{d\alpha}(L)$$

$$\begin{aligned}
\dim H_k(X^\lambda) \quad = \quad & \operatorname{rank} H_k(\mathcal{C}^{a\alpha}(L)) \rightarrow H_k(\mathcal{C}^{d\alpha}(L)) \\
& \leq \\
& \operatorname{rank} H_k(\mathcal{F}^{\alpha}(L)) \rightarrow H_k(\mathcal{F}^{c\alpha}(L)) \\
& \leq \\
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\end{aligned}$$

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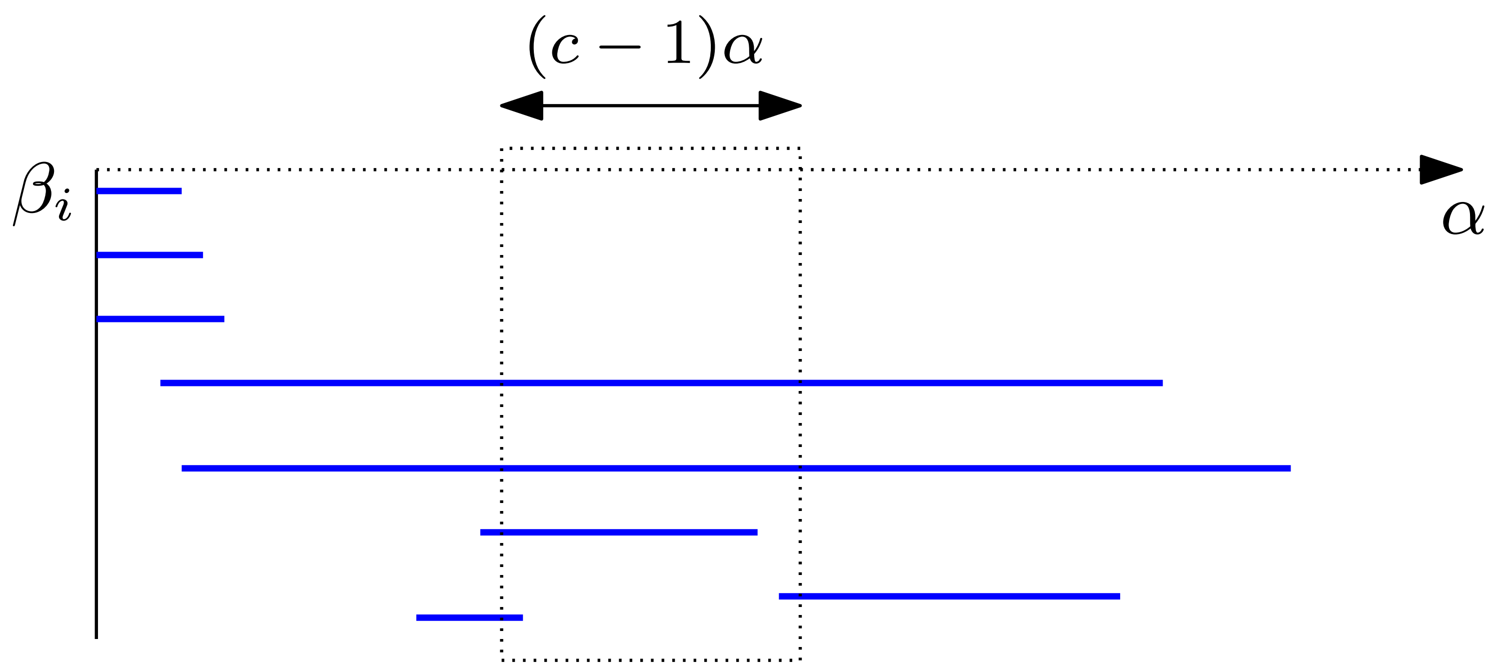

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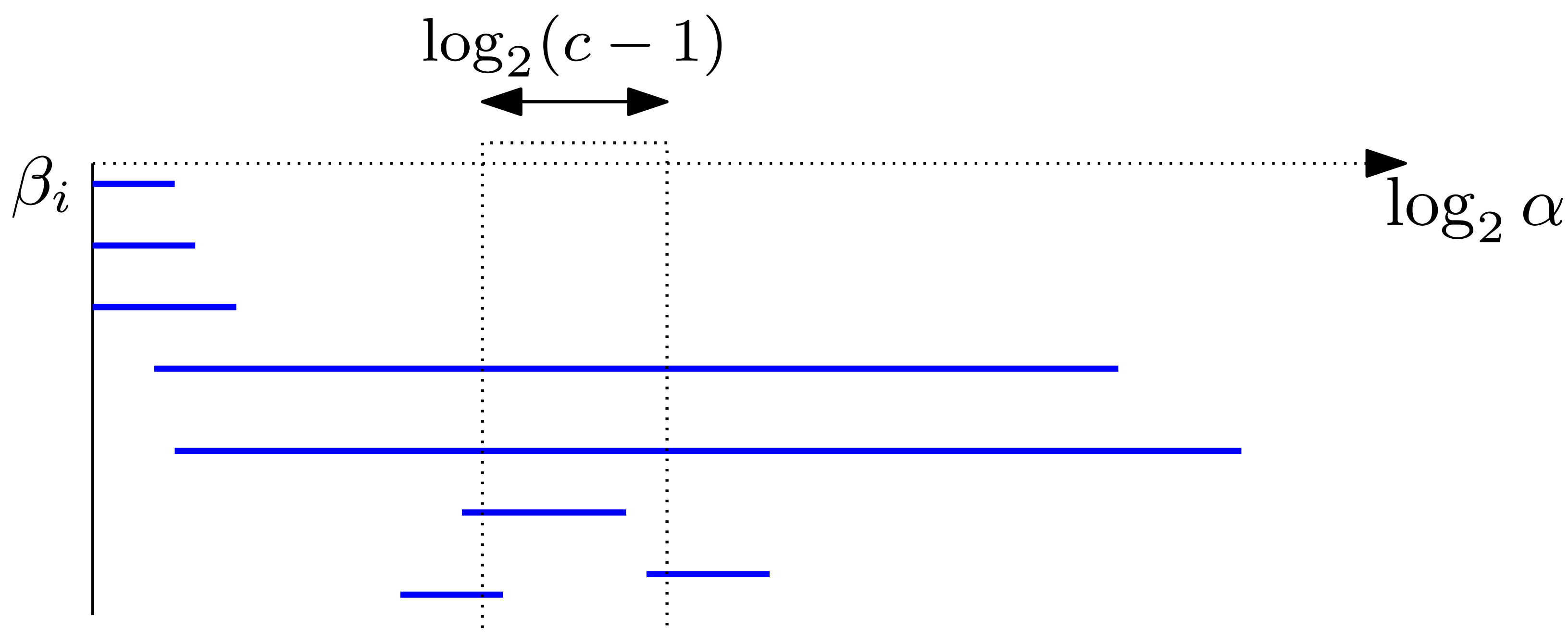

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Back to the Algorithm

Input: a finite point set $W \subset \mathbb{R}^d$.

$\longrightarrow$ maintain the nested pair $\mathcal{R}^{4\varepsilon}(L) \subseteq \mathcal{R}^{16\varepsilon}(L)$.

Init: $L := \emptyset$, $\varepsilon := \infty$;

WHILE $L \subsetneq W$

insert $p = \mathrm{argmax}_{w \in W} \mathsf{d}(w, L)$ in L;

update $\varepsilon := \max_{w \in W} \mathsf{d}(w, L)$;

update $\mathcal{R}^{4\varepsilon}(L)$ and $\mathcal{R}^{16\varepsilon}(L)$;

compute persistence $(\mathcal{R}^{4\varepsilon}(L) \hookrightarrow \mathcal{R}^{16\varepsilon}(L))$;

END_WHILE

Output: sequence of persistent Betti numbers of $\mathcal{R}^{4\varepsilon}(L) \hookrightarrow \mathcal{R}^{16\varepsilon}(L)$.

Back to the Algorithm

Input: a finite point set $W \subset \mathbb{R}^d$.

Thm If W is a δ-sample of some compact set $X \subset \mathbb{R}^d$, such that $\delta < \frac{1}{18}\mathrm{wfs}(X)$, then, at all iteration such that $\delta < \varepsilon < \frac{1}{18}\mathrm{wfs}(X)$, one has: $\forall \lambda \in (0, \mathrm{wfs}(X))$, $\forall k \in \mathbb{N}$, $\beta_k(X^\lambda) = \beta_k^p(\mathcal{R}^{4\varepsilon}(L) \hookrightarrow \mathcal{R}^{16\varepsilon}(L))$.

WHILE $L \subsetneq W$

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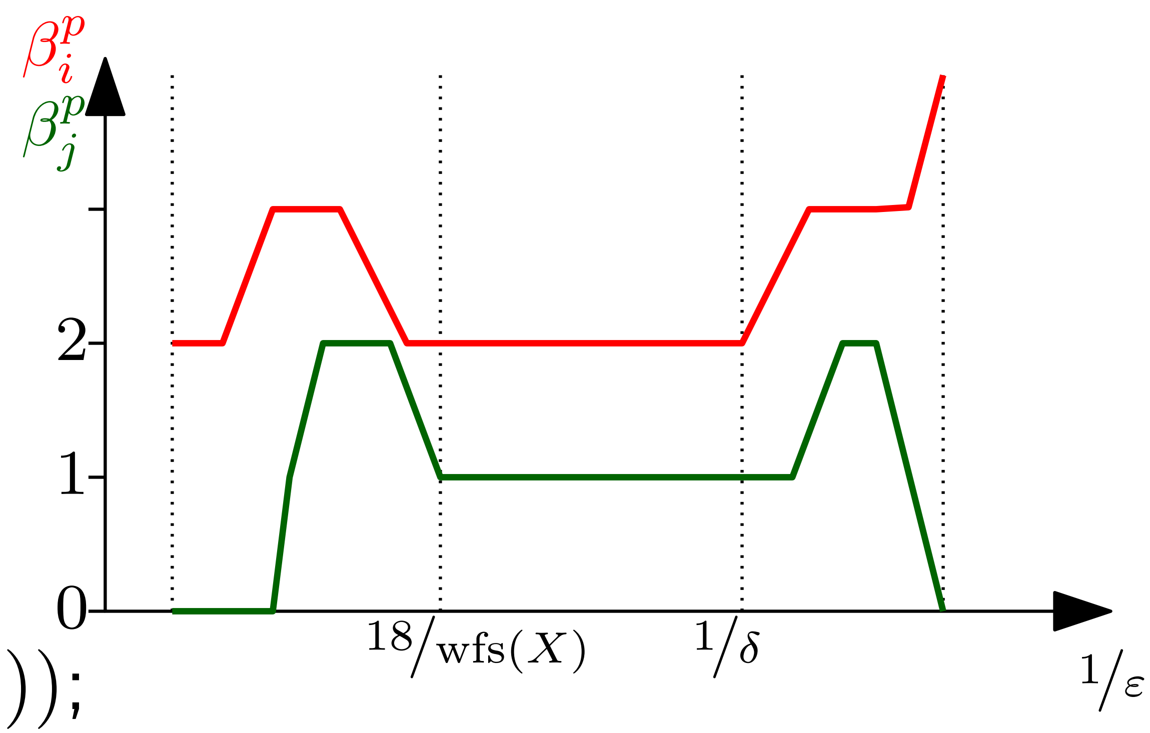

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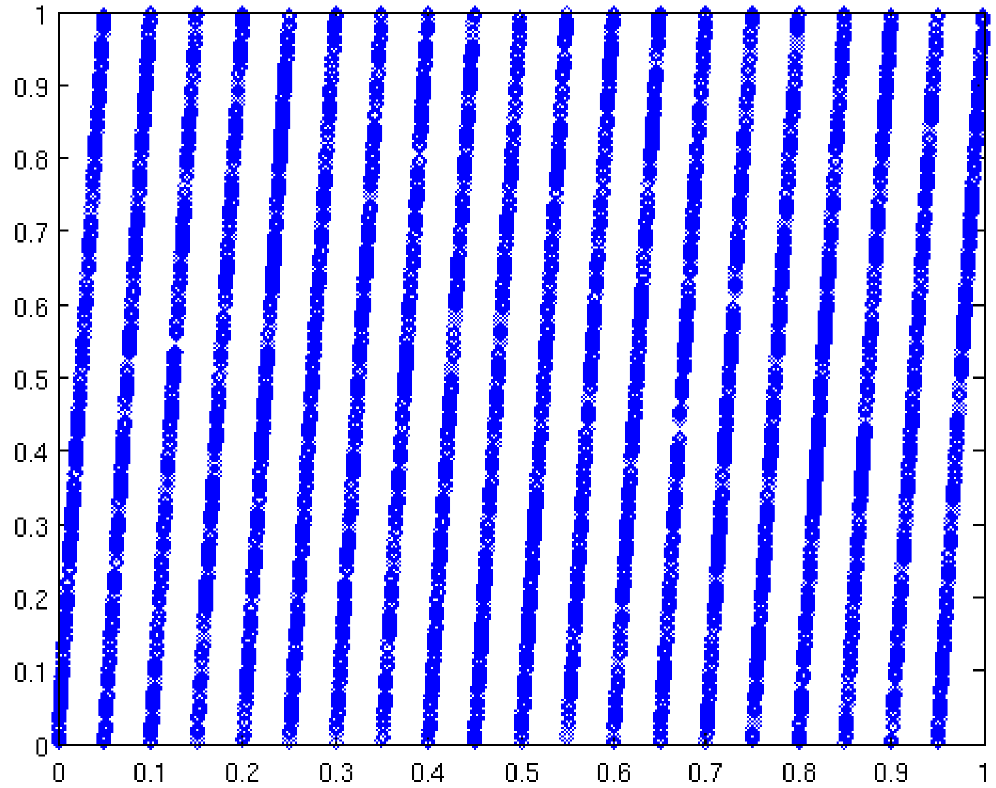
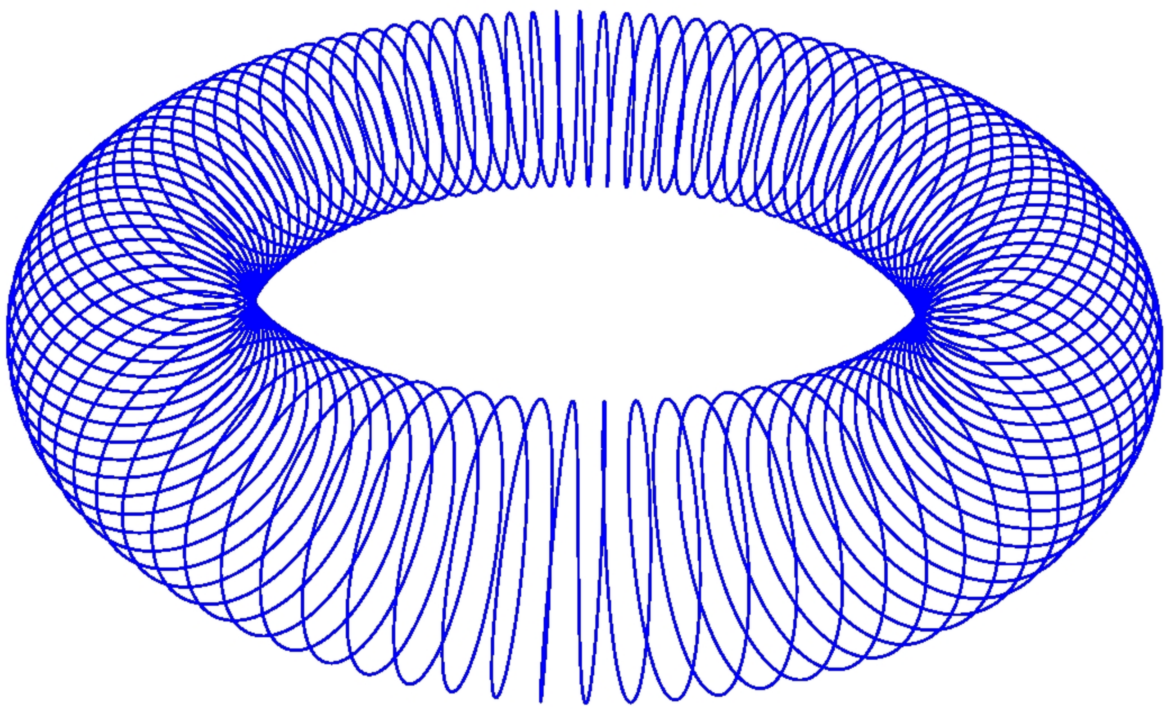
A Toy Example

$[0, 1] \times [0, 1]$

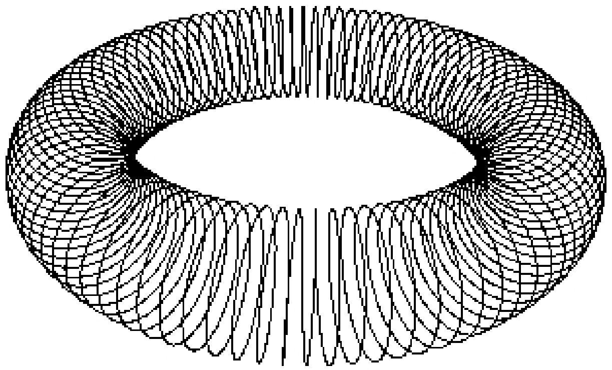
$\mathbb{R}^4$





$$(u, v) \mapsto \frac{1}{2}(\cos 2\pi u, \sin 2\pi u, \cos 2\pi v, \sin 2\pi v)$$

$10,000$ points sampled uniformly at random from a curve drawn on Clifford's torus.



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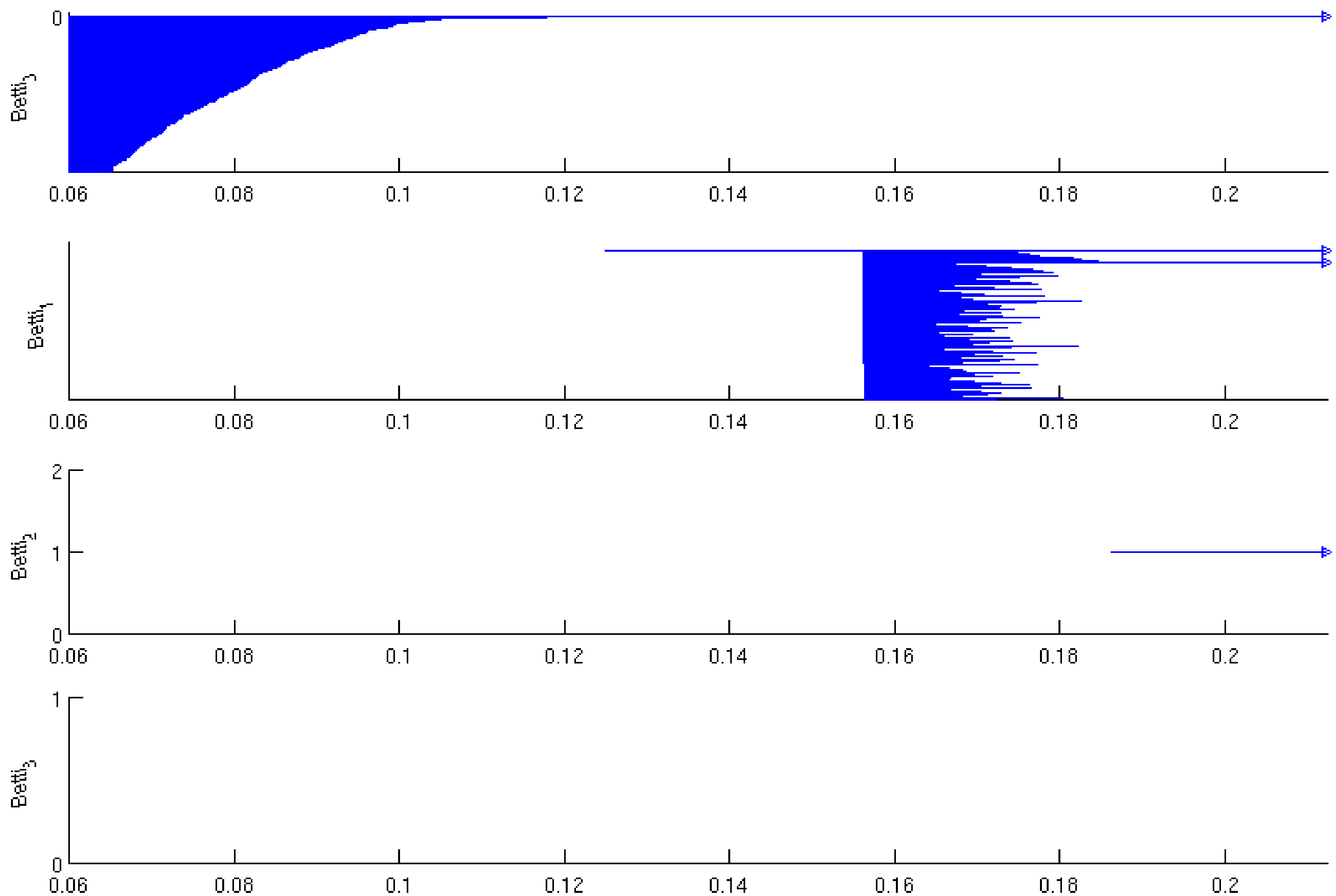
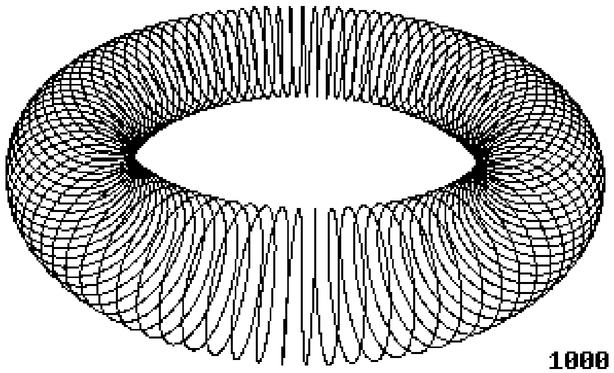
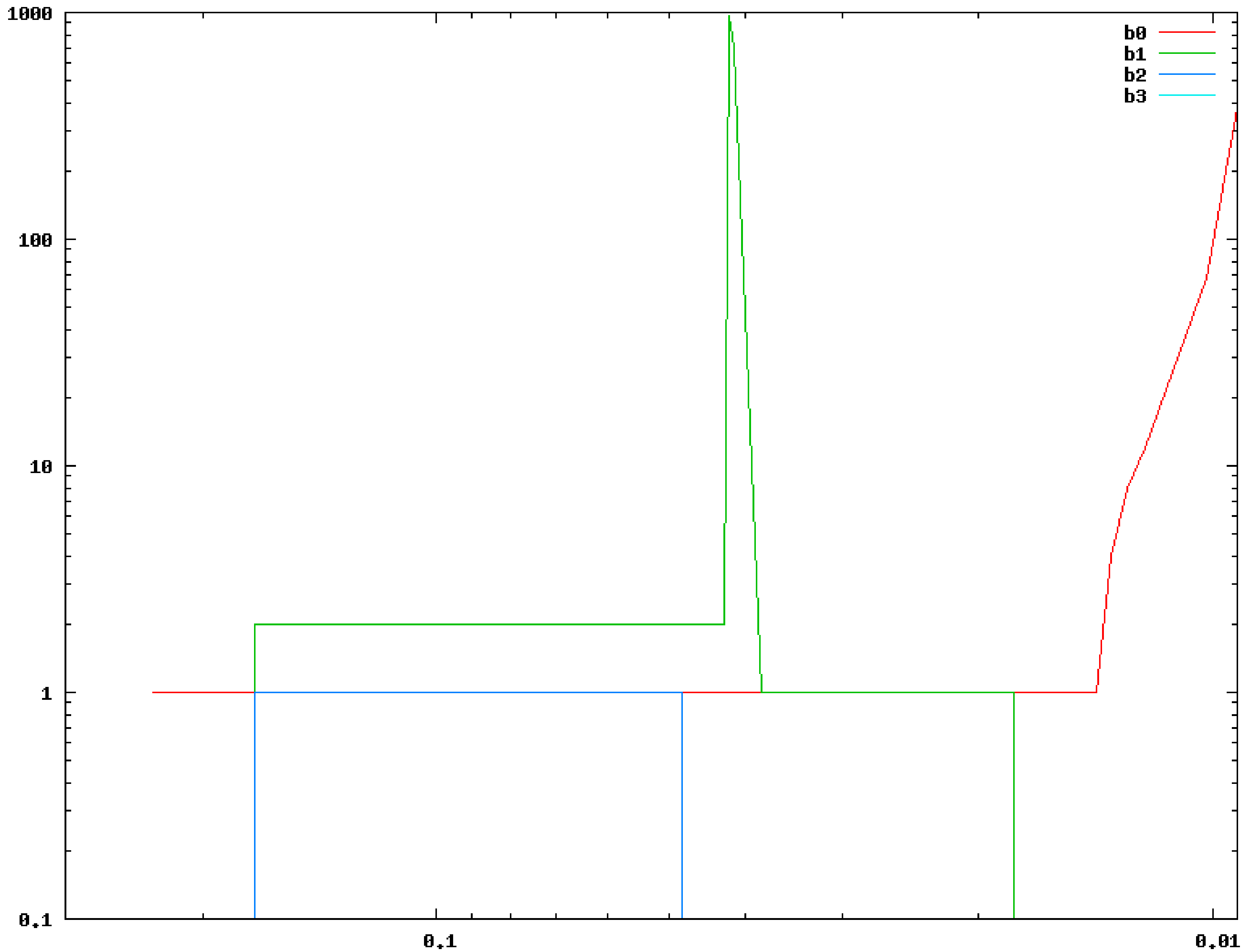


900 carefully-chosen landmarks, $\varepsilon = 0.0483$, Rips filtration up to 6ε (linear scale).

(result provided by Plex)

A Toy Example





Output of the algorithm, applied *blindly* to the input point cloud.

Complexity

WHILE $L \subsetneq W$

insert $p = \operatorname{argmax}_{w \in W} \mathsf{d}(w, L)$ in L;

update $\varepsilon := \max_{w \in W} \mathsf{d}(w, L)$;

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update $\mathcal{R}^{4\varepsilon}(L)$ and $\mathcal{R}^{16\varepsilon}(L)$;

compute persistence $(\mathcal{R}^{4\varepsilon}(L) \hookrightarrow \mathcal{R}^{16\varepsilon}(L))$;

END_WHILE

Hypothesis: $W \subset \mathbb{R}^d$.

• At the end of each iteration, the points of L are at least ε away from one another. $\Rightarrow$ they are centers of pairwise-disjoint balls of radius $\frac{\varepsilon}{2}$.

Complexity

WHILE $L \subsetneq W$

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$\varepsilon \ll \mathrm{rch}(X)$.

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• Neighbors in the Rips complex are at most 16ε away from each other, and close to the tangent spaces of X. $\Rightarrow$ by a packing argument, each vertex v has at most 35^m neighbors.

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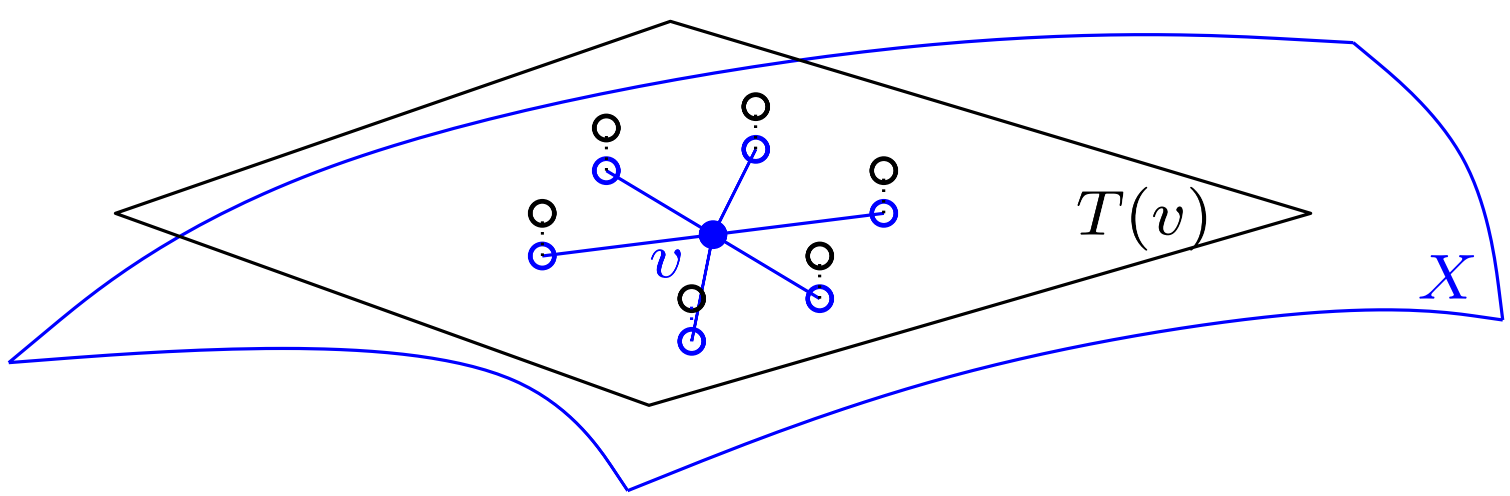

Complexity

```
WHILE L ⊊ W
    insert p = argmax_{w∈W} d(w, L) in L;
    update ε := max_{w∈W} d(w, L);
    update R^{4ε}(L) and R^{16ε}(L);
    compute persistence (R^{4ε}(L) ↪ R^{16ε}(L));
END_WHILE
```

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- *transition* phase: L coarse, $|\mathcal{R}^{16\varepsilon}(L)|$ scales up with d
- *stable* phase: L dense, $|\mathcal{R}^{16\varepsilon}(L)|$ scales up with m

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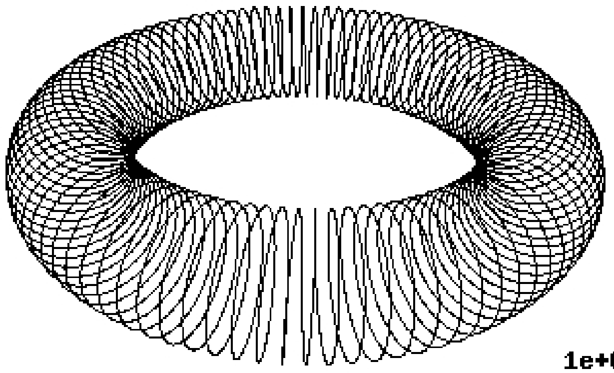
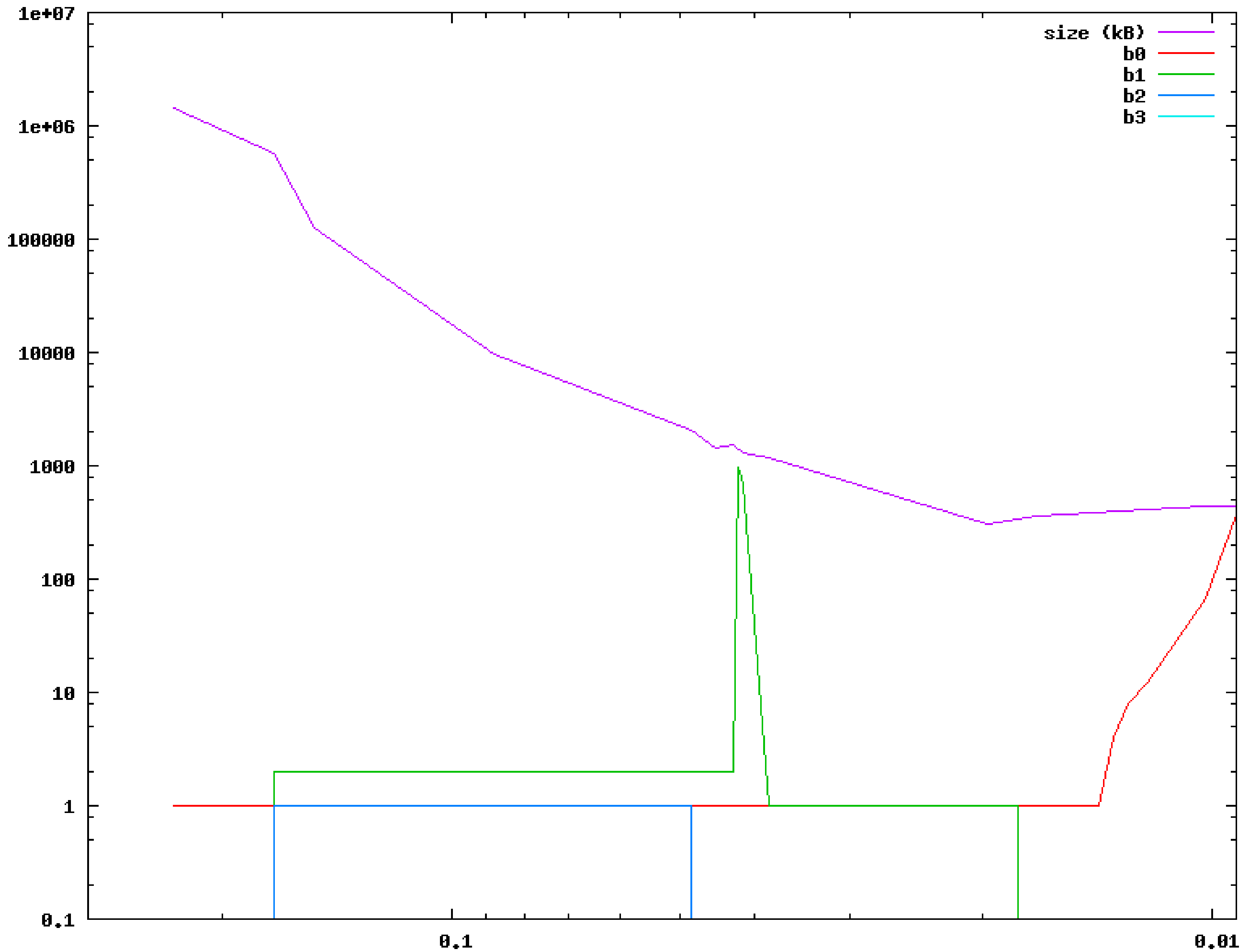
$\longrightarrow$ with a backtracking strategy, the complexity scales up with m.



Complexity
(example)



Space complexity blows up when $|L| < 300$, and becomes intractable when $|L| = 100$.

Witness Complex vs. Čech, Rips Filtrations

Conjecture: [Carlsson, de Silva 04]
The witness complex filtration should have *cleaner* persistence barcodes than Čech or Rips filtrations, at least on smooth submanifolds of $\mathbb{R}^d$.

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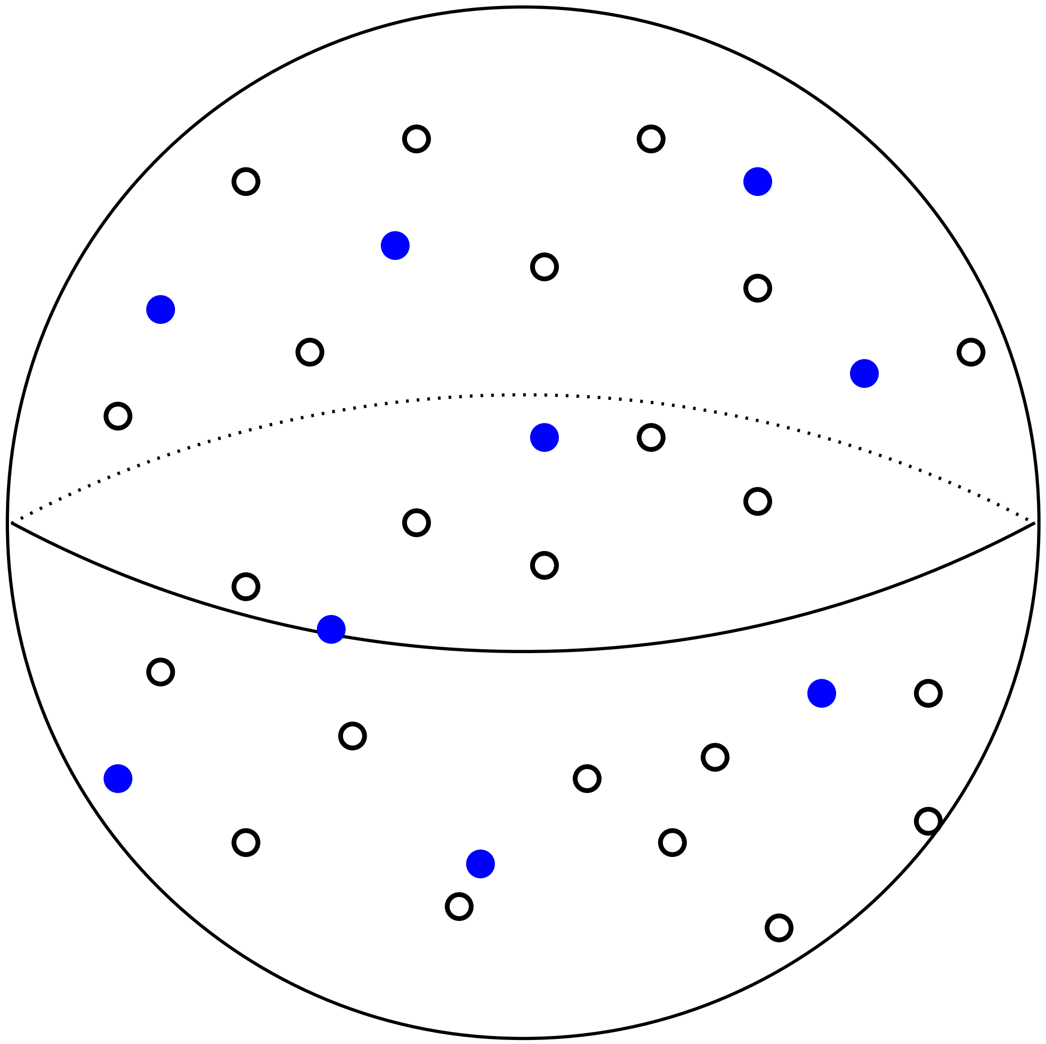
Toy example:

1000 points sampled uniformly at random on the unit 2-sphere

15 well-separated landmarks

rest of points used as witnesses (for witness complex only)



From V. de Silva, *Topological Estimation using Witness Complexes*, SPBG'04 talk.

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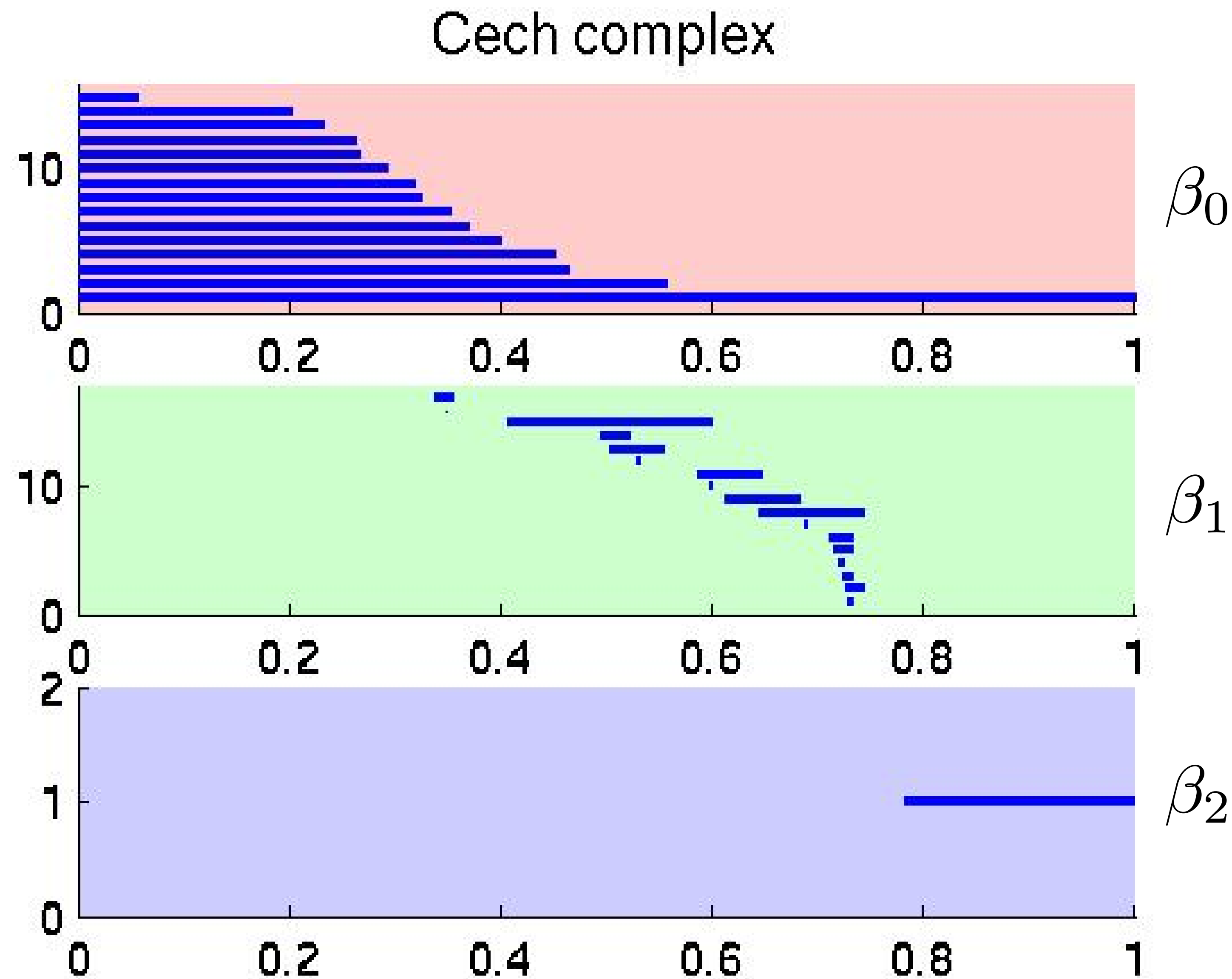

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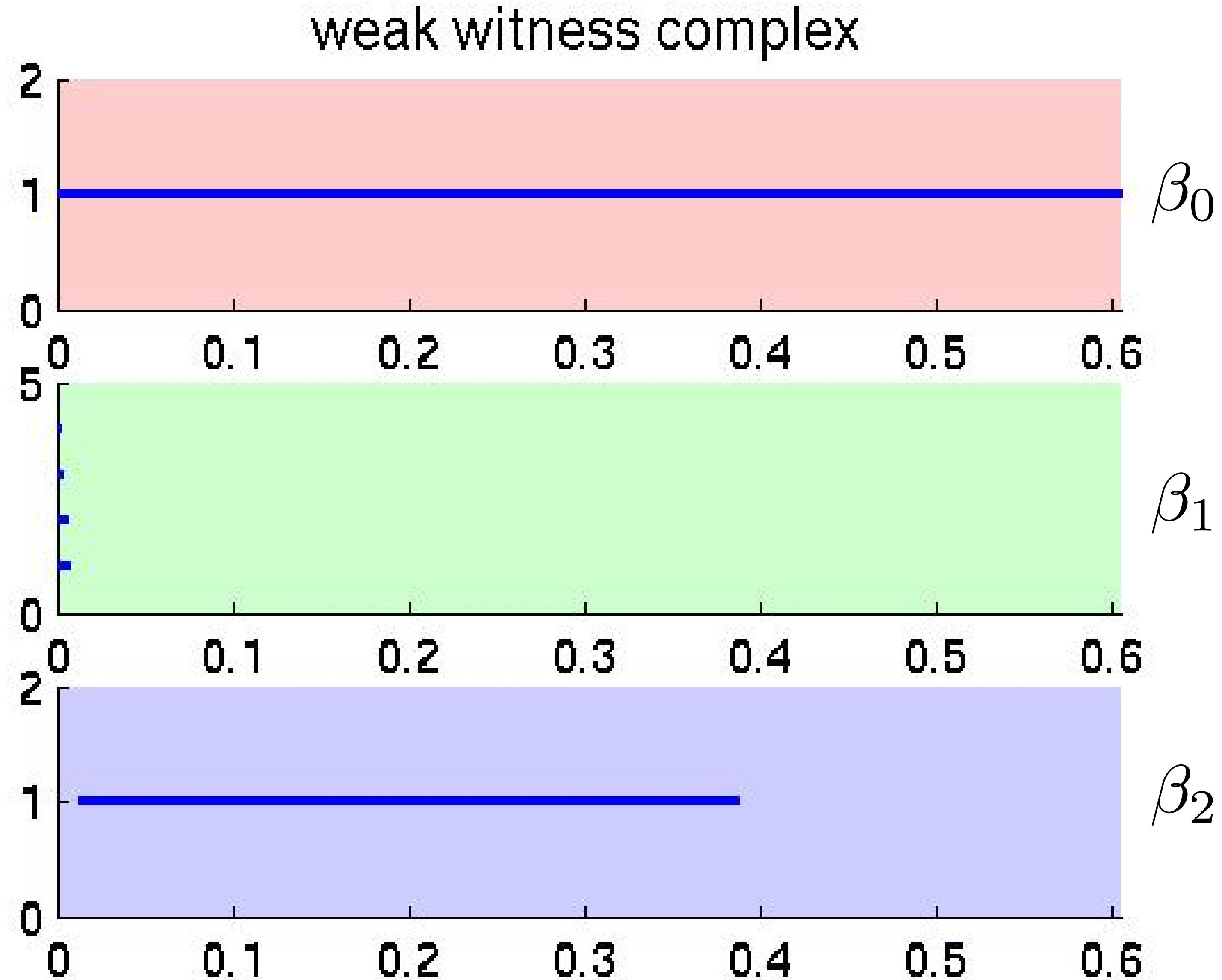

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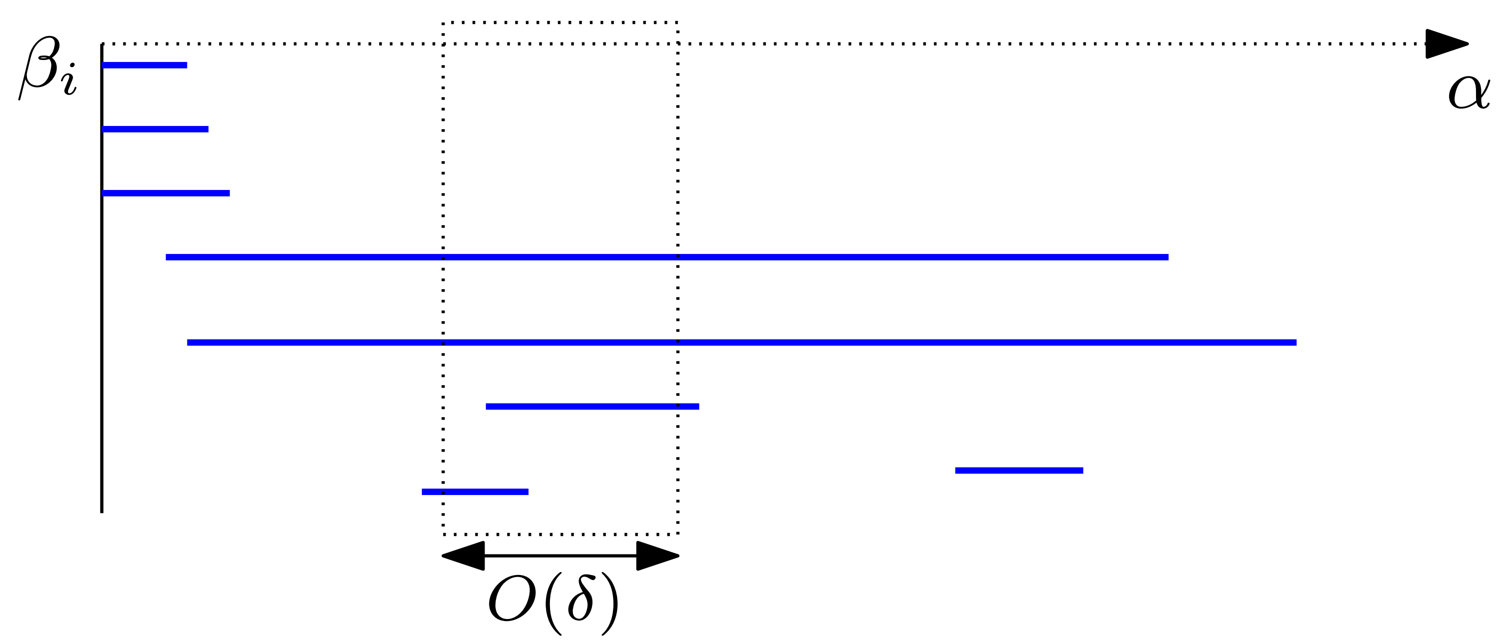
Q If W is a δ-sample of some smooth submanifold X, and L is a uniform ε-sample of W, does the topological noise in the barcode of the filtration $\{\mathcal{C}_W^\alpha(L)\}_{\alpha\geq 0}$ depend solely on δ?



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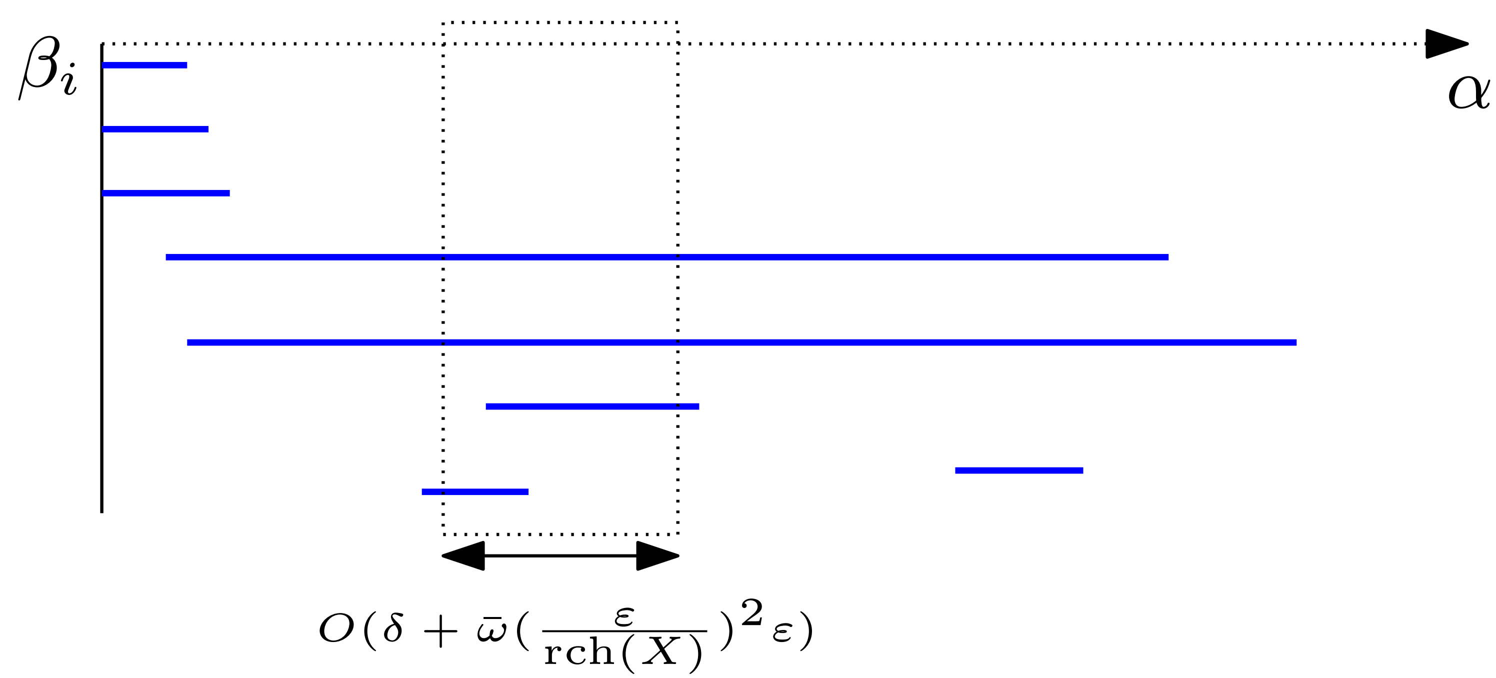
The witness complex filtration should have *cleaner* persistence barcodes than Čech or Rips filtrations, at least on smooth submanifolds of $\mathbb{R}^d$.

Thm There exist a constant $\varrho > 0$ and a non-decreasing continuous map $\bar{\omega} : [0, \varrho) \to [0, \frac{1}{2})$, s.t. for all $0 < \delta \leq \varepsilon < \varrho \operatorname{rch}(X)$, and for all $\alpha \in \left[\frac{8}{3}(\delta + \bar{\omega}(\frac{\varepsilon}{\operatorname{rch}(X)})^2\varepsilon),\ \frac{1}{2}\operatorname{rch}(X) - O(\varepsilon+\delta)\right)$, $\mathcal{C}_W^\alpha(L)$ contains a subcomplex $\mathcal{D}$ homeomorphic to X and such that $\mathcal{D} \hookrightarrow \mathcal{C}_W^\alpha(L)$ induces monomorphisms at homology level.



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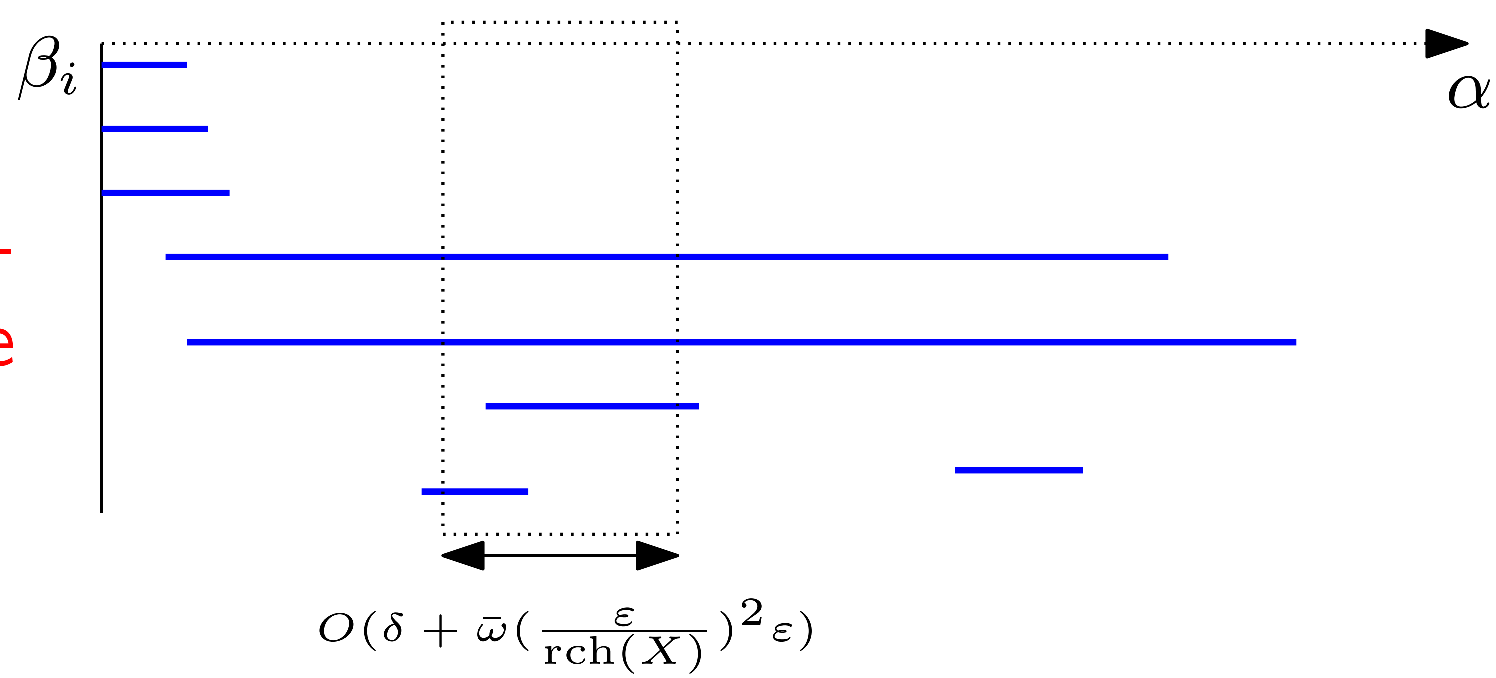
$\oplus$ the bound on the amplitude of the topological noise cannot depend solely on δ



Concluding Remarks

- **A weaker concept of reconstruction:**
  - stands in-between classical reconstruction and topological estimation,
  - complexity scales up with intrinsic dimension of the data,
  - comes with theoretical guarantees on a large class of compact sets.

- **New stability results for a class of filtrations**:
  - Čech filtration versus unions of Euclidean balls,
  - filtrations intertwined with Čech filtration (Rips, witness complex),
  - superiority of the witness complex on smooth submanifolds.

- **A few (of many) open questions**:
  - can a single complex be extracted from $\mathcal{R}^{4\varepsilon}(L) \hookrightarrow \mathcal{R}^{16\varepsilon}(L)$?
  - can the computation of the entire Rips complex be avoided?
  - what is the exact power of the witness complex filtration?