Stability of boundary measures

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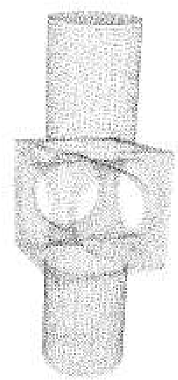
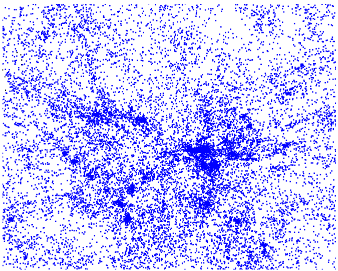
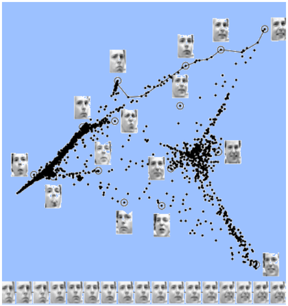
Point cloud geometry

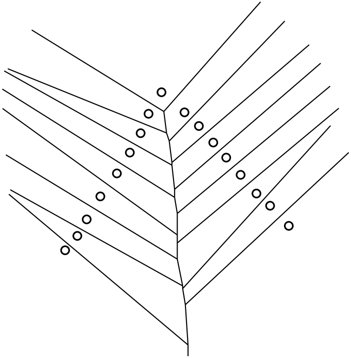






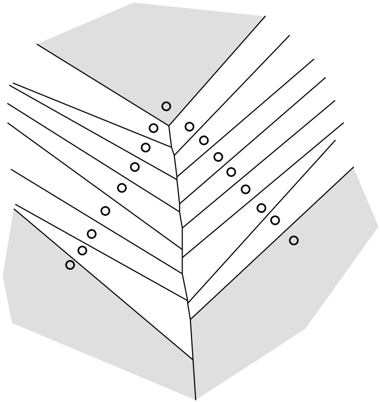
- Given a set of points sampled near an unknown shape, can we infer the geometry of that shape ?

Detecting singularities



- the volume of a cell is very sensitive to perturbation
- but if one consider the union of Voronoï cells whose site is contained in a given ball...

Detecting singularities



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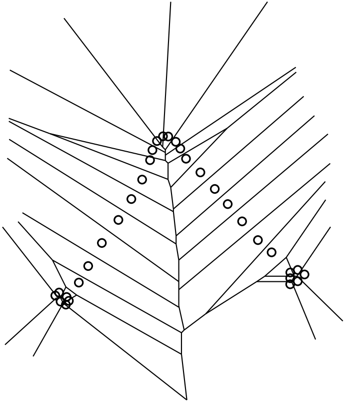

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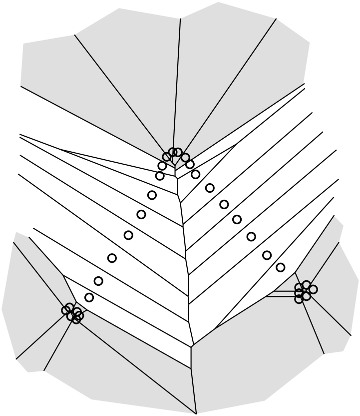

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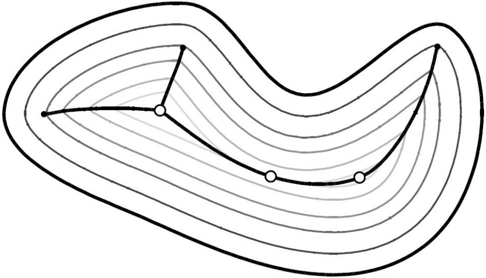
Definition

The projection $p_K : \mathbb{R}^n \to K \subset \mathbb{R}^n$ maps any point $x \in \mathbb{R}^n$ to its closest point in K. It is defined outside of the *medial axis* of K.



Boundary measure

Definition

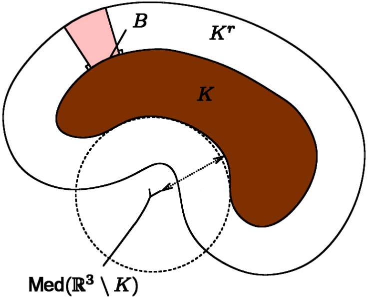
For $E \subset \mathbb{R}^n$, the boundary measure $\mu_{K,E}$ is defined as follows :

$$\forall B \subseteq K,\ \mu_{K,E}(B) = vol^n(\{x \in E \,|\, \mathrm{p}_K(x) \in B\})$$

that is, the n-volume of the part of E that projects on B.

- measure supported in K
- contains a lot of geometric information about K

Smooth object



- Let $K \subset \mathbb{R}^n$ be an n-dimensional object with smooth boundary.
- The smallest distance between K and its medial axis is called reach(K).
- Take $E = K^r \equiv \{x \in \mathbb{R}^n; \mathrm{d}(x, K) \leq r\}$, assuming $r <$ reach(K).

Smooth object

Tube formula (Steiner, Weyl, Federer)

If $r < reach(K)$:

$$vol^n(K^r) = vol^n(K) + \sum_{k=1}^{n} const(n,k)[\int_{\partial K} \sigma_{k-1}]\ r^k$$

Smooth object

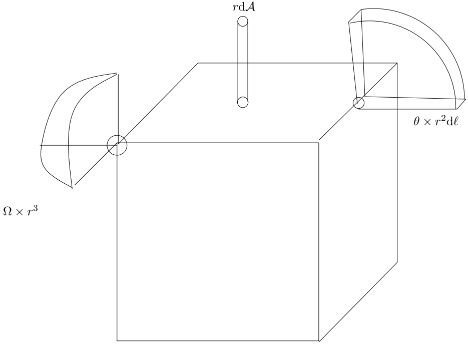
Tube formula (Steiner, Weyl, Federer)

If $r < reach(K)$, for $B \subset K$:

$$\mu_{K,K^r}(B) = \underbrace{vol^n(B)}_{\Phi_K^n(B)} + \sum_{k=1}^{n} const(n,k) \underbrace{[\int_{B \cap \partial K} \sigma_{k-1}]}_{\Phi_K^{n-k}(B)} \; r^k$$

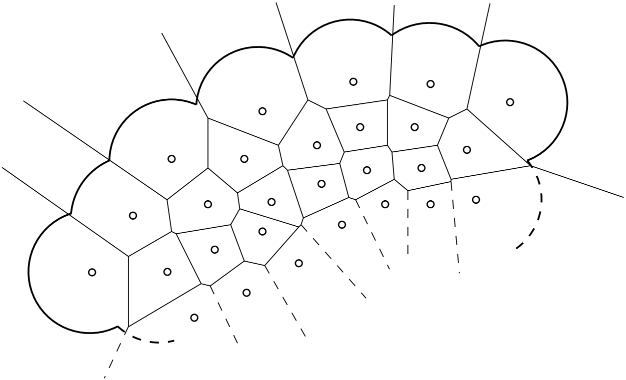
- The Φ_K^i are the (signed) *curvature measures* of K.
- If K is d-dimensional, they vanish identically for $i > d$.
- $\Phi_K^0(K)$ is the Euler characteristic of K.
- They are intrinsic, *i.e.* they do not depend on the embedding.

Convex polyhedron



- the boundary measure of a convex polyhedron K can be decomposed as a sum : $\mu_{K,K^r}(B) = \sum_{k=0}^{n} const(n,k)\ \Phi_{n-k}(B)\ r^k$.
- the curvature measure Φ_K^i is the i-dimensional measure supported on the i-skeleton of K whose density is the local external dihedral angle.

The boundary measure of a point cloud



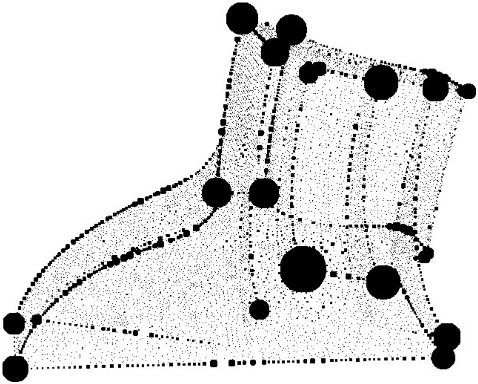
the boundary measure is a sum of weighted Dirac masses :

$$\mu_{C,C^r} = \sum_i vol^n(\mathrm{Vor}(x_i) \cap C^r)\delta_{x_i}$$

Stability and computation of boundary measure

1. is it practically feasible to compute the boundary measure of a point cloud $C \subseteq \mathbb{R}^d$?
2. if C is a good approximation of K (ie dense enough and without too much noise), does the boundary measure μ_{C,C^r} carry approximately the same geometric information as μ_{K,K^r} ?

The boundary measure of a point cloud



Here, the volumes of the Voronoï cells are evaluated using a Monte-Carlo method. Cost scales linearly with ambient dimension. Approximation error does not depend on the ambient dimension.

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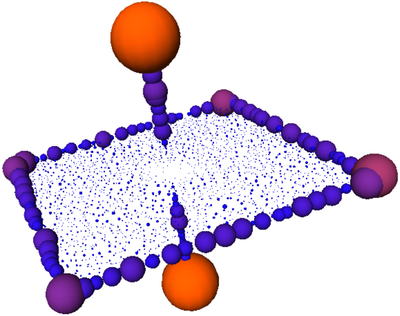

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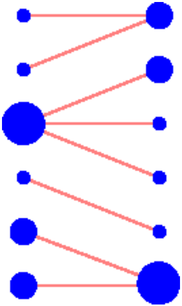
Stability

- Assume point cloud C samples compact K well, *e.g.* $d_H(K, C) \leq \varepsilon$. This means that

$$C \subset K^\varepsilon \text{ and } K \subset C^\varepsilon$$

- Are $\mu_{K,E}$ and $\mu_{C,E}$ close ? In which sense ?



- Assume measures μ and ν are discrete :
  $\mu = \sum_i c_i \delta_{x_i}$, $\nu = \sum_j d_j \delta_{y_j}$
  we suppose that $\text{mass}(\mu) = \text{mass}(\nu)$
- a transport plan between is a set of nonnegative coefficients p_{ij} specifying the amount of mass which is transported from x_i to y_j, with

$$\sum_i p_{ij} = d_j \text{ and } \sum_j p_{ij} = c_i$$

- the cost of a transport plan is
  $C(p) = \sum_{ij} \|x_i - y_j\| \, p_{ij}$
- $W(\mu, \nu) = \inf_p C(p)$

Kantorovich-Rubinstein theorem

Definition

Let μ and ν be two measures on $\mathbb{R}^d$ having the same total mass $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$.

$$W(\mu, \nu) = \inf_{X,Y} \mathbb{E}[d(X, Y)]$$

where the infimum is taken on all pairs of $\mathbb{R}^d$-valued random variables X and Y whose law are μ and ν respectively.

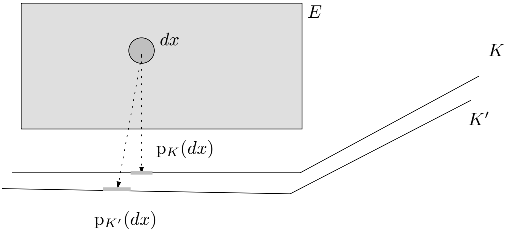
Theorem

For two measures μ and ν with common finite mass and bounded support,

$$W(\mu, \nu) = \sup_f \left| \int f \, d\mu - \int f \, d\nu \right|$$

where the sup is taken over all 1-Lipschitz functions $\mathbb{R}^n \to \mathbb{R}$.

Wasserstein distance between boundary measures



We consider the following transport plan : the element of mass $\mathrm{p}_K(x)\mathrm{d}x$ coming from an element of mass dx at $x \in E$ will be transported to $p_{K'}(x)\mathrm{d}x$.
the total cost of this transport is :

$$\int_E \|p_K(x) - p_{K'}(x)\| \,\mathrm{d}x = \left\|p_K - p_K'\right\|_{\mathrm{L}^1(E)}$$

$$\Rightarrow \quad W(\mu_{K,E}, \mu_{K',E}) \leq \left\|p_K - p_K'\right\|_{\mathrm{L}^1(E)}$$

A L^1 stability theorem for projections

Theorem

If E is an open set of $\mathbb{R}^n$ with rectifiable boundary, and K and K' are two *close enough* compact subsets :

$$\|p_K - p_{K'}\|_{L^1(E)} := \int_E \|p_K - p_{K'}\|$$
$$\leq C(n)[vol^n(E) + \mathrm{diam}(K) vol^{n-1}(\partial E)] \sqrt{R_K \mathrm{d}_H(K, K')}$$

where $R_K = \sup_{x \in E} \mathrm{d}(x, K)$.

1. *close enough* means that $\mathrm{d}_H(K, K')$ does not exceed $\min(R_K, \mathrm{diam}(K), \mathrm{diam}(K)^2 / R_K)$
2. $C(n) = O(\sqrt{n})$

Ingredients of proof

Lemma

Function $v_K : x \mapsto \|x\|^2 - d_K^2(x)$ is convex and $\nabla v_K = 2p_K$ almost everywhere.

Lemma

If $f, g : E \to \mathbb{R}$ are convex and $k = \mathrm{diam}(\nabla f(E) \cup \nabla g(E))$ then

$$\|\nabla f - \nabla g\|_{\mathrm{L}^1(E)} \leq C(n)\left[vol^n(E) + k\ vol^{n-1}(\partial E)\right]\ \|f - g\|_\infty^{1/2}$$

$$+ C(n)\ vol^{n-1}(\partial E)\ \|f - g\|_\infty$$

+ integral geometry arguments.

Stability of boundary measures

Theorem

If K is a fixed compact set, and E an open set with smooth boundary, then

$$\mathrm{W}(\mu_{K,E}, \mu_{K',E}) \leq C(n, E, K)\, \mathrm{d}_H(K, K')^{1/2}$$

as soon as K' is close enough to K.

A similar result holds for μ_{K,K^r} and μ_{K',K'^r}.

Estimating curvature measures

1. for any K with positive reach, there exists measures $\Phi_{K,i}$ such that for $r < \mathrm{reach}(K)$, $\mu_{K,r}(B) = \sum_{i=1}^{n} \Phi_K^{n-i}(B) r^i$
2. can be computed knowing only the boundary measures for $n+1$ values $r_0 < \ldots < r_n$: denote the result by $\Phi_{K,i}^{(r)}$.

Corollary

If $\mathrm{reach}(K) > r_n$ and K' is *close* to K, there is a constant $C(K, n, (r))$ such that

$$\mathrm{d_{bL}}\left(\Phi_{K,i}, \Phi_{K',i}^{(r)}\right) \leq C(K, n, (r))\, \mathrm{d}_H(K, K')^{1/2}$$

$\mathcal{C}^0$ (Hausdorff) closeness implies closeness of differential properties at a given scale.

Computing boundary measures

algorithm

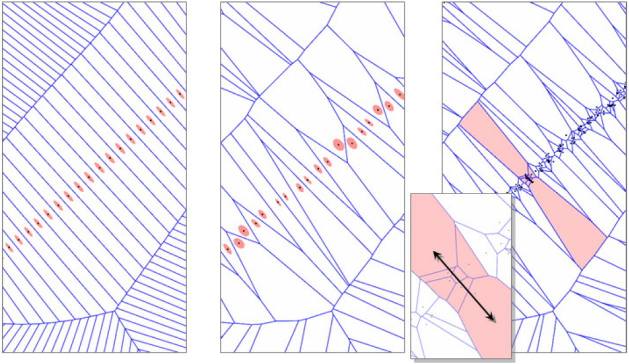
Input : a point cloud C, a scalar r, a number N
Output : an approximation of μ_{C,C^r} in the form $\frac{1}{N}\sum n(p_i)\delta_{p_i}$

while $k \leq N$
I. Choose a random point X with probability distribution $\frac{1}{\mathcal{H}^d(C^r)}\,\mathcal{H}^d\big|_{C^r}$
II. Finds its closest point p_i in the cloud C, add 1 to $n(p_i)$
end while
III. Multiply each $n(p_i)$ by $\mathcal{H}^d(C^r)$.

Hoeffding's inequality :

$$\mathbb{P}\left[d_{bL}\left(\mu_N, \mu\right) \geq \varepsilon\right] \leq 2\exp\left(\ln(16/\varepsilon)\mathcal{N}(K, \varepsilon/16) - N\varepsilon^2/2\right)$$

Directions



Replace **volumes** of Voronoï cells by their **covariance matrices**. This gives a tensor-valued measure.

- small/large eigenvalues $\simeq$ tangent/normal space
- principal curvatures/directions ?

Discussion

1. The boundary measure and its tensor version encode "much' of the geometry of a compact set.
2. these measures depend continuously on the compact set for the Hausdorff distance.
3. what can we do when the underlying shape has zero reach ?
4. what happens if we replace nearest neighbors by approximate nearest neighbors ?
5. how can we deal with outliers ?