On learning statistical mixtures maximizing the complete likelihood

The $k$-MLE methodology using geometric hard clustering

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Finite mixtures: Semi-parametric statistical models

- Mixture $M \sim \text{MM}(W, \Lambda)$ with density $m(x) = \sum_{i=1}^{k} w_i p(x|\lambda_i)$
  - not sum of RVs!
  - $\Lambda = \{\lambda_i\}_i$, $W = \{w_i\}_i$
- Multimodal, universally modeling smooth densities
- Gaussian MMs with support $\mathcal{X} = \mathbb{R}$, Gamma MMs with support $\mathcal{X} = \mathbb{R}^+$ (modeling distances [34])
- Capture sub-populations within an overall population ($k = 2$, crab data [29] in Pearson)
Example of $k = 2$-component mixture [17]

Sub-populations ($k = 2$) within an overall population...

Sub-species in species, etc.  Truncated distributions (what is the support! black swans ?!)

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Sampling from mixtures: Doubly stochastic process

To sample a variate $x$ from a MM:

- Choose a component $l$ according to the weight distribution $w_1, \ldots, w_k$ (multinomial),
- Draw a variate $x$ according to $p(x | \lambda_l)$. 

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Statistical mixtures: Generative data models

Image = 5D xyRGB point set
GMM = *feature descriptor* for information retrieval (IR)
Increase dimension $d$ using color image $s \times s$ patches: $d = 2 + 3s^2$

Low-frequency information encoded into compact statistical model.
Mixtures: $\epsilon$-statistically learnable and $\epsilon$-estimates

Problem statement: Given $n$ IID $d$-dimensional observations $x_1, ..., x_n \sim \text{MM}(\Lambda, W)$, estimate $\text{MM}(\hat{\Lambda}, \hat{W})$:

- **Theoretical Computer Science (TCS) approach:** $\epsilon$-closely parameter recovery ($\pi$: permutation)
  
  $\triangleright$ $|w_i - \hat{w}_{\pi(i)}| \leq \epsilon$
  
  $\triangleright$ $\text{KL}(p(x|\lambda_i) : p(x|\hat{\lambda}_{\pi(i)})) \leq \epsilon$ (or other divergences like TV, etc.)

Consider $\epsilon$-learnable MMs:

$\triangleright$ $\min_i w_i \geq \epsilon$

$\triangleright$ $\text{KL}(p(x|\lambda_i) : p(x|\lambda_i)) \geq \epsilon$, $\forall i \neq j$ (or other divergence)

- **Statistical approach:**
  Define the **best** model/MM as the one maximizing the likelihood function $l(\Lambda, W) = \prod_i m(x_i|\Lambda, W)$. 
Mixture inference: Incomplete versus complete likelihood

- Sub-populations within an overall population: observed data $x_i$ does not include the subpopulation label $l_i$
- $k = 2$: Classification and Bayes error (upper bounded by Chernoff information [24])
- Inference: Assume IID, maximize (log)-likelihood:
  - Complete using indicator variables $z_{i,j}$ (for $l_i$: $z_{i,l_i} = 1$):
    \[
    l_c = \log \prod_{i=1}^{n} \prod_{j=1}^{k} (w_j p(x_i|\theta_j))^{z_{i,j}} = \sum_{i} \sum_{j} z_{i,j} \log(w_j p(x_i|\theta_j))
    \]
  - Incomplete (hidden/latent variables) and log-sum intractability:
    \[
    l_i = \log \prod_{i} m(x|\mathcal{W}, \Lambda) = \sum_{i} \log \left( \sum_{j} w_j p(x_i|\theta_j) \right)
    \]
Mixture learnability and inference algorithms

- Which criterion to maximize? Incomplete or complete likelihood? What kind of evaluation criteria?


Some technicalities:

- Many local maxima of likelihood functions $l_i$ and $l_c$ (EM converges locally and needs a stopping criterion)

- Multimodal density ($\#\text{modes} > k$ [9], ghost modes even for isotropic GMMs)

- Identifiability (permutation of labels, parameter distinctness)

- Irregularity: Fisher information may be zero [6], convergence speed of EM

- etc.
Learning MMs: A geometric hard clustering viewpoint

\[
\max_{W, \Lambda} l_c(W, \Lambda) = \max_{\Lambda} \sum_{i=1}^{n} \max_{j=1}^{k} \log(w_j p(x_i | \theta_j))
\]

\[
\equiv \min_{W, \Lambda} \sum_{i} \min_{j} (- \log p(x_i | \theta_j) - \log w_j)
\]

\[
= \min_{W, \Lambda} \sum_{i=1}^{n} \min_{j=1}^{k} D_j(x_i),
\]

where \(c_j = (w_j, \theta_j)\) (cluster prototype) and \(D_j(x_i) = - \log p(x_i | \theta_j) - \log w_j\) are potential distance-like functions.

- Maximizing the complete likelihood amounts to a geometric hard clustering [37, 11] for fixed \(w_j\)'s (distance \(D_j(\cdot)\) depends on cluster prototypes \(c_j\)): \(\min_{\Lambda} \sum_{i} \min_{j} D_j(x_i)\).
- Related to classification EM [5] (CEM), hard/truncated EM
- Solution of \(\arg \max l_c\) to initialize \(l_i\) (optimized by EM)
The $k$-MLE method: $k$-means type clustering algorithms

**$k$-MLE:**

1. Initialize weight $W$ (in open probability simplex $\Delta_k$)
2. Solve $\min_{\Lambda} \sum_i \min_j D_j(x_i)$ (center-based clustering, $W$ fixed)
3. Solve $\min_W \sum_i \min_j D_j(x_i)$ ($\Lambda$ fixed)
4. Test for convergence and go to step 2) otherwise.

$\Rightarrow$ group coordinate ascent (ML)/descent (distance) optimization.
k-MLE: Center-based clustering, $W$ fixed

Solve $\min \sum_i \min_j D_j(x_i)$

$k$-means type convergence proof for assignment/relocation:

- **Data assignment:**
  $\forall i, l_i = \arg \max_j w_j p(x|\lambda_j) = \arg \min_j D_j(x_i), \ C_j = \{x_i|l_i = j\}$

- **Center relocation:** $\forall j, \lambda_j = \text{MLE}(C_j)$

**Farthest Maximum Likelihood (FML) Voronoi diagram:**

$\text{Vor}_{\text{FML}}(c_i) = \{x \in \mathcal{X} : w_i p(x|\lambda_i) \geq w_j p(x|\lambda_j), \ \forall i \neq j\}$

$\text{Vor}(c_i) = \{x \in \mathcal{X} : D_i(x) \leq D_j(x), \ \forall i \neq j\}$

FML Voronoi $\equiv$ **additively weighted Voronoi** with:

$D_i(x) = -\log p(x|\lambda_i) - \log w_i$
**k-MLE: Example for mixtures of exponential families**

Exponential family:

Component density

\[
p(x|\theta) = \exp(t(x)^\top \theta - F(\theta) + k(x))
\]

is log-concave with:

- \(t(x)\): sufficient statistic in \(\mathbb{R}^D\), \(D\): family order.
- \(k(x)\): auxiliary carrier term (wrt Lebesgue/counting measure)
- \(F(\theta)\): log-normalized, cumulant function, log-partition.

\(D_j(x)\) is convex: Clustering \(k\)-means wrt convex “distances”.

**Farthest ML Voronoi** \(\equiv\) *additively-weighted Bregman Voronoi* \([4]\):

\[
- \log p(x; \theta) - \log w = F(\theta) - t(x)^\top \theta - k(x) - \log w
\]

\[
= B_{F^*}(t(x) : \eta) + F^*(t(x)) + k(x) - \log w
\]

\(F^*(\eta) = \max_{\theta}(\theta^\top \eta - F(\theta))\): Legendre-Fenchel convex conjugate
Exponential families: Rayleigh distributions [36, 25]

Application: IntraVascular UltraSound (IVUS) imaging:

Rayleigh distribution:
\[ p(x; \lambda) = \frac{x}{\lambda^2} e^{-\frac{x^2}{2\lambda^2}} \]

\( x \in \mathbb{R}^+ = X \)
\( d = 1 \) (univariate)
\( D = 1 \) (order 1)
\( \theta = -\frac{1}{2\lambda^2} \)
\( \Theta = (-\infty, 0) \)
\( F(\theta) = -\log(-2\theta) \)
\( t(x) = x^2 \)
\( k(x) = \log x \)
(Weibull for \( k = 2 \))

Coronary plaques: fibrotic tissues, calcified tissues, lipidic tissues

Rayleigh Mixture Models (RMMs):

for segmentation and classification tasks
Exponential families: Multivariate Gaussians [14, 25]

Gaussian Mixture Models (GMMs).
(Color image interpreted as a 5D xyRGB point set)

Gaussian distribution $p(x; \mu, \Sigma)$:

$$
\frac{1}{(2\pi)^{d/2} \sqrt{|\Sigma|}} e^{-\frac{1}{2} D_{\Sigma^{-1}}(x-\mu, x-\mu)}
$$

Squared Mahalanobis distance:

$$
D_Q(x, y) = (x - y)^T Q (x - y)
$$

$x \in \mathbb{R}^d = X$

$d$ (multivariate)

$$
D = \frac{d(d+3)}{2} \quad \text{(order)}
$$

$$
\theta = (\Sigma^{-1} \mu, \frac{1}{2} \Sigma^{-1}) = (\theta_v, \theta_M)
$$

$$
\Theta = \mathbb{R} \times S^d_{++}
$$

$$
F(\theta) = \frac{1}{4} \theta_v^T \theta_M^{-1} \theta_v - \frac{1}{2} \log |\theta_M| + \frac{d}{2} \log \pi
$$

$$
t(x) = (x, -xx^T)
$$

$$
k(x) = 0
$$
The \textit{k-MLE} method for exponential families

\textit{k-MLEE}:  
\begin{enumerate}
  \item Initialize weight $W$ (in open probability simplex $\Delta_k$)
  \item Solve $\min_{\Lambda} \sum_i \min_j (B_{F^*}(t(x) : \eta_j) - \log w_j)$
  \item Solve $\min_{W} \sum_i \min_j D_j(x_i)$
  \item Test for convergence and go to step 2) otherwise.
\end{enumerate}

Assignment condition in Step 2: additively-weighted Bregman Voronoi diagram.
$k$-MLE: Solving for weights given component parameters

Solve

$$\min_W \sum_i \min_j D_j(x_i)$$

Amounts to \( \arg \min_W -n_j \log w_j = \arg \min_W -\frac{n_j}{n} \log w_j \) where \( n_j = \#\{x_i \in \text{Vor}(c_j)\} = |C_j| \).

$$\min_{W \in \Delta_k} H^\times(N : W)$$

where \( N = (\frac{n_1}{n}, ..., \frac{n_k}{n}) \) is cluster point proportion vector \( \in \Delta_k \).

**Cross-entropy** \( H^\times \) is minimized when \( H^\times(N : W) = H(N) \) that is \( W = N \).

**Kullback-Leibler divergence:**

\( \text{KL}(N : W) = H^\times(N : W) - H(N) = 0 \) when \( W = N \).
MLE for exponential families

Given a ML farthest Voronoi partition, computes MLEs $\theta_j$'s:

$$\hat{\theta}_j = \arg\max_{\theta \in \Theta} \prod_{x_i \in \text{Vor}(c_j)} p_F(x_i; \theta)$$

is unique (***) maximum since $\nabla^2 F(\theta) \succ 0$:

**Moment equation**: $\nabla F(\hat{\theta}_j) = \eta(\hat{\theta}_j) = \frac{1}{n_j} \sum_{x_i \in \text{Vor}(c_j)} t(x_i) = \bar{t} = \hat{\eta}$

MLE is consistent, efficient with asymptotic normal distribution:

$$\hat{\theta}_j \sim N \left( \theta_j, \frac{1}{n_j} I^{-1}(\theta_j) \right)$$

Fisher information matrix

$$I(\theta_j) = \text{var}[t(X)] = \nabla^2 F(\theta_j) = (\nabla^2 F^*)^{-1}(\eta_j)$$

MLE may be biased (eg, normal distributions).
Existence of MLEs for exponential families (***)

For minimal and full EFs, MLE guaranteed to exist [3, 21] provided that matrix:

\[
T = \begin{bmatrix}
1 & t_1(x_1) & \ldots & t_D(x_1) \\
: & : & : & : \\
1 & t_1(x_n) & \ldots & t_D(x_n)
\end{bmatrix}
\]

of dimension \(n \times (D + 1)\) has \(\text{rank } D + 1\) [3].

For example, problems for MLEs of MVNs with \(n < d\) observations (undefined with likelihood \(\infty\)).

Condition: \(\bar{t} = \frac{1}{n_j} \sum_{x_i \in \text{Vor}(c_j)} t(x_i) \in \text{int}(C)\), where \(C\) is closed convex support.
MLE of EFs: Observed point in IG/Bregman 1-mean

$$\hat{\theta} = \arg \max_\theta \prod_{i=1}^n p_F(x_i; \theta) = \arg \max_\theta \sum_{i=1}^n \log p_F(x_i; \theta)$$

$$\arg \max_\theta \sum_{i=1}^n -B_{F^*}(t(x_i) : \eta) + F^*(t(x_i)) + k(x_i)$$

$$\equiv \arg \min_\theta \sum_{i=1}^n B_{F^*}(t(x_i) : \eta)$$

Right-sided Bregman centroid = center of mass:

$$\hat{\eta} = \frac{1}{n} \sum_{i=1}^n t(x_i)$$

$$\bar{l} = \frac{1}{n} \sum_{i=1}^n (-B_{F^*}(t(x_i) : \hat{\eta}) + F^*(t(x_i)) + k(x_i))$$

$$= \langle \hat{\eta}, \hat{\theta} \rangle - F(\hat{\theta}) + \bar{k} = F^*(\hat{\eta}) + \bar{k}$$

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The $k$-MLE method: Heuristics based on $k$-means

$k$-means is NP-hard (non-convex optimization) when $d > 1$ and $k > 1$ and solved exactly using dynamic programming [26] in $O(n^2k)$ and $O(n)$ memory when $d = 1$.

Heuristics:

- Kanungo et al. [18] swap: yields a $(9 + \epsilon)$-approximation
- Global seeds: random seed (Forgy [12]), $k$-means++ [2], global $k$-means initialization [38],
- Local refinements: Lloyd batched update [19], MacQueen iterative update [20], Hartigan single-point swap [16], etc.
- etc.
Generalized \textit{k-MLE}

Weibull or generalized Gaussians are \textit{parametric families of exponential families} [35]: $F(\gamma)$. Fixing some parameters yields \textit{nested families} of (sub)-exponential families [34]: obtain one free parameter with convex conjugate $F^*$ approximated by line search (Gamma distributions/generalized Gaussians).

![Diagram of Original k-MLE](a)  Original k-MLE  

![Diagram of Extended k-MLE](b)  Extended k-MLE
Generalized $k$-MLE

$k$-GMLE:

1. Initialize weight $W \in \Delta_k$ and family type $(F_1, \ldots, F_k)$ for each cluster
2. Solve $\min_{\Lambda} \sum_i \min_j D_j(x_i)$ (center-based clustering for $W$ fixed) with potential functions:
   $$D_j(x_i) = -\log p_{F_j}(x_i | \theta_j) - \log w_j$$
3. Solve family types maximizing the MLE in each cluster $C_j$ by choosing the parametric family of distributions $F_j = F(\gamma_j)$ that yields the best likelihood:
   $$\min_{F_1=F(\gamma_1), \ldots, F_k=F(\gamma_k) \in F(\gamma)} \sum_i \min_j D_{w_j, \theta_j, F_j}(x_i).$$
4. Update $W$ as the cluster point proportion
5. Test for convergence and go to step 2) otherwise.

$$D_{w_j, \theta_j, F_j}(x) = - \log p_{F_j}(x; \theta_j) - \log w_j$$
Generalized $k$-MLE: Convergence

- Lloyd’s batched generalized $k$-MLE maximizes **monotonically** the complete likelihood.
- Hartigan single-point relocation generalized $k$-MLE maximizes **monotonically** the complete likelihood [32], improves over Lloyd local maxima, and avoids the problem of the existence of MLE inside clusters by ensuring $n_j \geq D$ in general position ($T \text{ rank } D + 1$).
- **Model selection**: Learn $k$ automatically using DP $k$-means [32] (Dirichlet Process)
### $k$-MLE [23] versus EM for Exponential Families [1]

<table>
<thead>
<tr>
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<tbody>
<tr>
<td><strong>Memory</strong></td>
<td>lighter $O(n)$</td>
<td>heavier $O(nk)$</td>
</tr>
<tr>
<td><strong>Assignment</strong></td>
<td>NNs with VP-trees [27], BB-trees [30]</td>
<td>all $k$-NNs</td>
</tr>
<tr>
<td><strong>Conv.</strong></td>
<td>always finitely</td>
<td>$\infty$, need stopping criterion</td>
</tr>
</tbody>
</table>

Many (probabilistically) guaranteed initialization for $k$-MLE [18, 2, 28]
Rayleigh, Poisson or (nested) univariate normal with constant \( \sigma \) are order 1 EFs \((D = 1)\).

Clustering problem: Dual 1D Bregman clustering [1] on 1D scalars \( y_i = t(x_i) \).

FML Voronoi diagrams have **connected cells**: Optimal clustering yields *interval clustering*.

1D \( k \)-means (with additive weights) can be solved exactly using **dynamic programming** in \( O(n^2k) \) time [26]. Then update the weights \( W \) (cluster point proportion) and reiterate...
Dynamic programming for $D = 1$-order mixtures [26]

Consider $W$ fixed. $k$-MLE cost: $\sum_{j=1}^{k} l(C_j)$ where $C_j$ are clusters.

MLE$_{k-1}(x_{1,j-1})$  MLE$_1(x_{j,n})$

Dynamic programming optimality equation:

$$\text{MLE}_k(x_1, \ldots, x_n) = \max_{j=2}^{n} (\text{MLE}_{k-1}(x_{1,j-1}) + \text{MLE}_1(x_{j,n}))$$

$\mathcal{X}_{l,r} : \{x_l, x_{l+1}, \ldots, x_{r-1}, x_r\}$.

- Build dynamic programming table from $l = 1$ to $l = k$ columns, $m = 1$ to $m = n$ rows.
- Retrieve $C_j$ from DP table by backtracking on the $\arg \max_j$.
- For $D = 1$ EFs, $O(n^2k)$ time [26].
Experiments with: 1D Gaussian Mixture Models (GMMs)

gmm_1 score = −3.075 (Euclidean $k$-means, $\sigma$ fixed)
gmm_2 score = −3.038 (Bregman $k$-means, $\sigma$ fitted, better)
Summary: *k*-MLE methodology for learning mixtures

Learn MM from **sequences of geometric hard clustering** [11].

- Hard *k*-MLE (≡ dual Bregman hard clustering for EFs) versus soft EM (≡ soft Bregman clustering [1] for EFs):
  - *k*-MLE maximizes the **complete likelihood** $l_c$.
  - EM maximizes locally the **incomplete likelihood** $l_i$.

- The component parameters $\eta$ geometric clustering (Step 2.) can be implemented using any Bregman *k*-means heuristic on conjugate $F^*$

- Consider **generalized *k*-MLE** when $F^*$ not available in closed form: nested exponential families (eg., Gamma)

- Initialization can be performed using *k*-means initialization: *k*-MLE++, etc.

- Exact solution with dynamic programming for order 1 EFs (with prescribed weight proportion $W$).

- Avoid **unbounded likelihood** (eg., $\infty$ for location-scale member with $\sigma \to 0$: Dirac) using Hartigan’s heuristic [32]
Discussion: Learning statistical models FAST!

- (EF) Mixture Models allow one to approximate universally smooth densities
- A single (multimodal) EF can approximate any smooth density too [7] but $F$ not in closed-form
- Which criterion to maximize is best/realistic: incomplete or complete, or parameter distortions? **Leverage** many recent results on $k$-means clustering to learning mixture models.
- Alternative approach: Simplifying mixtures from kernel density estimators (KDEs) is one **fine-to-coarse** solution [33]
- **Open problem**: How to constrain the MMs to have a prescribed number of modes/antimodes?
Thank you.

Experiments and performance evaluations on generalized $k$-MLE:

- $k$-GMLE for generalized Gaussians [35]
- $k$-GMLE for Gamma distributions [34]
- $k$-GMLE for singly-parametric distributions [26]

(compared with Expectation-Maximization [8])

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Bibliography I

Arindam Banerjee, Srujana Merugu, Inderjit S. Dhillon, and Joydeep Ghosh.
Clustering with Bregman divergences.

Anup Bhattacharya, Ragesh Jaiswal, and Nir Ailon.
A tight lower bound instance for $k$-means++ in constant dimension.

Krzysztof Bogdan and Małgorzata Bogdan.
On existence of maximum likelihood estimators in exponential families.

Jean-Daniel Boissonnat, Frank Nielsen, and Richard Nock.
Bregman Voronoi diagrams.

Gilles Celeux and Gérard Govaert.
A classification EM algorithm for clustering and two stochastic versions.

Jiahua Chen.
Optimal rate of convergence for finite mixture models.

Loren Cobb, Peter Koppstein, and Neng Hsin Chen.
Estimation and moment recursion relations for multimodal distributions of the exponential family.
Bibliography II

Arthur Pentland Dempster, Nan M. Laird, and Donald B. Rubin.  
Maximum likelihood from incomplete data via the EM algorithm.  

Herbert Edelsbrunner, Brittany Terese Fasy, and Günter Rote.  
Add isotropic Gaussian kernels at own risk: more and more resilient modes in higher dimensions.  

Dan Feldman, Matthew Faulkner, and Andreas Krause.  
Scalable training of mixture models via coresets.  

Dan Feldman, Morteza Monemizadeh, and Christian Sohler.  
A PTAS for $k$-means clustering based on weak coresets.  

Edward W. Forgy.  
Cluster analysis of multivariate data: efficiency vs interpretability of classifications.  

Francis Galton.  
*Hereditary genius.*  
Macmillan and Company, 1869.

Vincent Garcia and Frank Nielsen.  
Simplification and hierarchical representations of mixtures of exponential families.  
Moritz Hardt and Eric Price.  
Sharp bounds for learning a mixture of two gaussians.  

John A. Hartigan.  
*Clustering Algorithms*.  

Adam Tauman Kalai, Ankur Moitra, and Gregory Valiant.  
Disentangling gaussians.  

Tapas Kanungo, David M. Mount, Nathan S. Netanyahu, Christine D. Piatko, Ruth Silverman, and Angela Y. Wu.  
A local search approximation algorithm for $k$-means clustering.  

Stuart P. Lloyd.  
Least squares quantization in PCM.  

James B. MacQueen.  
Some methods of classification and analysis of multivariate observations.  

Weiwen Miao and Marjorie Hahn.  
Existence of maximum likelihood estimates for multi-dimensional exponential families.  
Ankur Moitra and Gregory Valiant.
Settling the polynomial learnability of mixtures of Gaussians.

Frank Nielsen.
$k$-MLE: A fast algorithm for learning statistical mixture models.

Frank Nielsen.
Generalized Bhattacharyya and Chernoff upper bounds on Bayes error using quasi-arithmetic means.

Frank Nielsen and Vincent Garcia.

Frank Nielsen and Richard Nock.
Optimal interval clustering: Application to bregman clustering and statistical mixture learning.

Frank Nielsen, Paolo Piro, and Michel Barlaud.
Bregman vantage point trees for efficient nearest neighbor queries.

Rafail Ostrovsky, Yuval Rabani, Leonard J. Schulman, and Chaitanya Swamy.
The effectiveness of Lloyd-type methods for the $k$-means problem.
Karl Pearson.
Contributions to the mathematical theory of evolution.

Paolo Piro, Frank Nielsen, and Michel Barlaud.
Tailored Bregman ball trees for effective nearest neighbors.
In *European Workshop on Computational Geometry (EuroCG)*, LORIA, Nancy, France, March 2009. IEEE.

Adolphe Quetelet.
*Lettres sur la théorie des probabilités, appliquée aux sciences morales et politiques*.
Hayez, 1846.

Christophe Saint-Jean and Frank Nielsen.
Hartigan’s method for k-MLE: Mixture modeling with Wishart distributions and its application to motion retrieval.

Olivier Schwander and Frank Nielsen.
Model centroids for the simplification of kernel density estimators.

Olivier Schwander and Frank Nielsen.
Fast learning of Gamma mixture models with k-mle.
In *Similarity-Based Pattern Recognition (SIMBAD)*, pages 235–249, 2013.

Olivier Schwander, Aurélien J. Schutz, Frank Nielsen, and Yannick Berthoumieu.
k-MLE for mixtures of generalized Gaussians.
Jose Seabra, Francesco Ciompi, Oriol Pujol, Josepa Mauri, Petia Radeva, and Joao Sanchez.
Rayleigh mixture model for plaque characterization in intravascular ultrasound.

Marc Teboulle.
A unified continuous optimization framework for center-based clustering methods.

Juanying Xie, Shuai Jiang, Weixin Xie, and Xinbo Gao.
An efficient global $k$-means clustering algorithm.