

# MPRI – Cours 2.12.2



## Lecture I: Groups for cryptology (part I)

2011/09/13–20–27

The slides are available on <http://www.lix.polytechnique.fr/Labo/Francois.Morain/MPRI/2011>







## Before I forget. . .

- Algorithmic number theory is about algorithms of number theory and they need to be practiced (Maple, Magma, SAGE, pari-gp, etc.).
- A large part of my lectures is taken from a nearly finished book with J.-L. Nicolas.

## Plan

- I. Elementary arithmetic.
- II.  $\mathbb{Z}/N\mathbb{Z}$ ,  $(\mathbb{Z}/N\mathbb{Z})^*$ .
- III. Quadratic reciprocity.
- IV. Finite fields.
- V. Implementation issues.

## Good reading

-  G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Clarendon Press, 5th edition, 1985.
-  D. E. Knuth. *The Art of Computer Programming: Seminumerical Algorithms*. Addison-Wesley, 2nd edition, 1981.
-  H. Cohen. *A course in algorithmic algebraic number theory*, volume 138 of *Graduate Texts in Mathematics*. Springer-Verlag, 4th printing, 2000.
-  P. Ribenboim. *The new book of prime number records*. Springer-Verlag, 1996.
-  R. Crandall and C. Pomerance. *Primes – A Computational Perspective*. Springer Verlag, 2nd edition, 2005.
-  FM. La primalité en temps polynomial [d'après Adleman, Huang; Agrawal, Kayal, Saxena]. Séminaire Bourbaki, Mars 2003.

# I. Elementary arithmetic

## A) Divisibility

**Thm.**[Euclidean Division]  $\forall a, b \neq 0, \exists$  a unique pair  $(q, r)$  s.t.

$$a = bq + r, \quad 0 \leq r < |b|.$$

**Coro.**  $\mathbb{Z}$  is principal: all proper ideal of  $\mathbb{Z}$  are  $a\mathbb{Z}$  for  $a \in \mathbb{N}^*$ .

**Def.**  $b \mid a$  iff  $\exists c, a = bc$ ; otherwise  $b \nmid a$ .

**Prop.** The following are equivalent

1.  $b \mid a$ ;
2. the remainder of the euclidean division of  $a$  by  $b$  is 0;
3.  $a\mathbb{Z} \subset b\mathbb{Z}$ .

**Prop.**

1.  $(d \mid a \text{ and } d \mid b) \Rightarrow d \mid (\lambda a + \mu b)$ ;
2.  $(d \mid b \text{ and } b \mid a) \Rightarrow d \mid a$ ;
3.  $a \mid b \Rightarrow |a| \leq |b|$ .

# Gcd, Lcm

**Prop.**  $a\mathbb{Z} + b\mathbb{Z} = \{au + bv, u \in \mathbb{Z}, v \in \mathbb{Z}\} = d\mathbb{Z}$  with  $d > 0$  (greatest common divisor – gcd).

**Prop.**

$$d = \gcd(a, b) = \max \mathcal{D}(a) \cap \mathcal{D}(b),$$

$$\mathcal{D}(a) \cap \mathcal{D}(b) = \mathcal{D}(d).$$

**Prop.**  $a\mathbb{Z} \cap b\mathbb{Z} = m\mathbb{Z}$  with  $m > 0$  (least common multiple – lcm).

**Prop.**  $a\mathbb{Z} \cap b\mathbb{Z} =$  set of common multiples of  $a$  and  $b$ .

$$\gcd(a, b) = \gcd(b, a), \quad \text{lcm}(a, b) = \text{lcm}(b, a)$$

$$\gcd(a, a) = a, \quad \text{lcm}(a, a) = a$$

$$\gcd(a, 0) = a, \quad \text{lcm}(a, 0) = 0$$

$$\gcd(a, -b) = \gcd(a, |b|), \quad \text{lcm}(a, -b) = \text{lcm}(a, |b|)$$

$$\gcd(ca, cb) = c \gcd(a, b), \quad \text{lcm}(ca, cb) = c \text{lcm}(a, b).$$

**Def.**  $a$  and  $b$  are prime together iff  $\gcd(a, b) = 1$ .

**Rem.** if  $d = \gcd(a, b)$ , then  $a/d$  and  $b/d$  are prime together.

## Euclid's algorithm

**Prop.** If  $a = bq + r$ , with  $0 \leq r < b$ , then  $\gcd(a, b) = \gcd(b, r)$ .

*Proof :*

Define  $(q_i)$  and  $(r_i)$ :

$$a = bq_1 + r_1, \quad 0 \leq r_1 < b_1$$

$$b = r_1q_2 + r_2, \quad 0 \leq r_2 < r_1$$

...

$$r_i = r_{i+1}q_{i+2} + r_{i+2}, \quad 0 \leq r_{i+2} < r_{i+1}$$

$(r_i)$  is strictly decreasing: let  $k$  s.t.  $r_k = 0$ . Then  $\gcd(a, b) = r_{k-1}$ .

## Euclid (cont'd)

**Prop.** The number of divisions is  $\leq 2 \log b / \log 2 + 1$ .

**Thm.** (Knuth)  $a, b$  uniformaly in  $[1, N]$ . Then

1. the number of steps is

$$\left\lceil \frac{\log(\sqrt{5}N)}{\log((1 + \sqrt{5})/2)} \right\rceil - 2 \approx [2.078 \log N + 1.672] - 2;$$

2. the average number of steps is

$$\frac{12 \log 2}{\pi^2} \log N + 1.47 \approx 0.843 \log N + 1.47.$$

There exist  $u, v$  s.t.

$$au + bv = \gcd(a, b).$$

Define  $(u_i), (v_i)$  s.t.  $u_{-1} = 1, v_{-1} = 0, u_0 = 0, v_0 = 1$ , and for  $-1 \leq i \leq k-2$ ,

$$\begin{cases} u_{i+2} = u_i - q_{i+2}u_{i+1} \\ v_{i+2} = v_i - q_{i+2}v_{i+1} \end{cases}$$

$$u_1 = 1, v_1 = -q_1, u_2 = -q_2, v_2 = 1 + q_1q_2$$

**Lem.** For  $-1 \leq i \leq k, r_i = u_i a + v_i b$ .

$\Rightarrow$  take  $u = u_{k-1}$  and  $v = v_{k-1}$ .

**Thm.** (Bachet-Bézout)  $\gcd(a, b) = 1$  iff  $\exists u, v$  s.t.  $au + bv = 1$ .

**Thm.** (Gauss) Let  $\gcd(a, b) = 1, c$  an integer:  $a \mid bc \Rightarrow a \mid c$ .

*Proof:*

**Thm.**  $\gcd(a, b) = 1 \Rightarrow \text{lcm}(a, b) = |ab|$ .

*Proof:*

**Coro.**  $\gcd(a, b)\text{lcm}(a, b) = |ab|$ .

*Proof:*

## B) Prime numbers

**Def. prime number:**  $p$  s.t.  $\text{Card}\mathcal{D}(p) = 2$ ; if not prime, then **composite**.

Smallest primes: 2, 3, 5, 7, 11, 13, ...

**Thm.** the integer  $n$  has at least one prime factor  $p$ .

*Proof:*

**Coro.** if  $n$  is composite,  $n$  has at least one prime factor  $p \leq \sqrt{n}$ .

*Proof:*

**Thm.** (Euclide) There exists an infinite number of primes.

*Proof:*

**Thm.** (Gauss) If  $p \mid a_1 a_2 \cdots a_n$ , there is some  $i$  s.t.  $p \mid a_i$ .

*Proof:* (recurrence)

**Coro.** Let  $p$  be prime and  $1 \leq k \leq p-1$ . Then  $p \mid \binom{p}{k}$ .

*Proof:*

## Factoring into primes

**Thm.** For all integer  $n$ :

$$n = \varepsilon p_1 p_2 \cdots p_k$$

where  $\varepsilon = \pm 1$  and  $p_1 \leq p_2 \leq \cdots \leq p_k$  are prime numbers.

*Proof:* (recurrence on  $n$ )

**Notation.** Sometimes

$$n = \varepsilon p_1^{\alpha_1} \cdots p_k^{\alpha_k}$$

with  $p_1 < p_2 < \cdots < p_k$  and  $\alpha_i > 0$ .

# Valuations

**$p$ -adic valuation:** For  $p$  prime,  $\nu_p(n) =$  largest integer s.t.  $p^{\nu_p(n)} \mid n$ .

$$n = \varepsilon \prod_{p \in \mathbb{P}} p^{\nu_p(n)}.$$

**Prop.**

- $\nu_p(ab) = \nu_p(a) + \nu_p(b);$
- $a \mid b \Leftrightarrow (\forall p \in \mathbb{P}) \nu_p(a) \leq \nu_p(b).$

**Prop.**

$$a = \prod_{p \in \mathbb{P}} p^{\nu_p(a)}, \quad b = \prod_{p \in \mathbb{P}} p^{\nu_p(b)}$$

$\Rightarrow$

$$\gcd(a, b) = \prod_{p \in \mathbb{P}} p^{\min(\nu_p(a), \nu_p(b))},$$

$$\text{lcm}(a, b) = \prod_{p \in \mathbb{P}} p^{\max(\nu_p(a), \nu_p(b))}.$$

# Estimates on prime numbers

**Thm.** (Hadamard, de la Vallée Poussin)

$$\pi(x) = \#\{N \leq x, N \text{ is prime}\} \approx x / \log x.$$

**Thm.** (Rosser & Schoenfeld)

$$\frac{x}{\log x} \left(1 + \frac{1}{2 \log x}\right) < \pi(x) < \frac{x}{\log x} \left(1 + \frac{3}{2 \log x}\right)$$

for  $x \geq 59$ .

**Thm.** (See Ellison + Mendès France)

$$\pi(x; k, \ell) = \#\{N \leq x, N = kn + \ell \text{ is prime}\} \sim \frac{x}{\varphi(k) \log x}.$$

**Rem.** A fast way to compute exact (small) values of  $\pi(x)$  is to use Eratosthene's sieve and its improvements (see later). More clever algorithms exist (Lagarias/Miller/Odlyzko, Deléglise/Rivat, Lagarias/Odlyzko).

## II. $\mathbb{Z}/N\mathbb{Z}$ and $(\mathbb{Z}/N\mathbb{Z})^*$

**Def.**  $a \equiv b \pmod N \Leftrightarrow N \mid a - b$ .

$\mathbb{Z}/N\mathbb{Z}$  is a quotient ring; representatives of  $\equiv$  are generally chosen as  $\{0, 1, \dots, N - 1\}$ .

**Prop.**

- $a \equiv b \pmod N$  and  $c \equiv d \pmod N \Rightarrow$

$$a + c \equiv b + d \pmod N, \quad ac \equiv bd \pmod N;$$

- if  $d \mid a, b, N$  then

$$a \equiv b \pmod N \Leftrightarrow a/d \equiv b/d \pmod{N/d}.$$

**Ex.**  $7^2 \pmod{10} \equiv 49 \pmod{10} \equiv 9 \pmod{10}, 7^3 \equiv 7 \times 9 \equiv 63 \equiv 3 \pmod{10}, 7^4 \equiv 7 \times 3 \equiv 1 \pmod{10}.$

## Invertible elements

**Prop.** Invertible elements in  $\mathbb{Z}/N\mathbb{Z}$  form the group

$$(\mathbb{Z}/N\mathbb{Z})^* = \{a \in \mathbb{Z}/N\mathbb{Z}, \gcd(a, N) = 1\}.$$

*Proof:*

**Coro.**  $\mathbb{Z}/N\mathbb{Z}$  is a field iff  $N$  is prime.

**Def. Euler totient function:**  $\varphi(N) = \#(\mathbb{Z}/N\mathbb{Z})^*.$

**Prop.**  $\varphi(p^e) = p^e - p^{e-1}.$

*Proof:*

**Prop.**  $\sum_{d \mid N} \varphi(d) = N.$

*Proof:*

## Order

**Def.**  $\text{ord}_N(a) = \min\{k > 0, a^k \equiv 1 \pmod N\}$ .

**Prop.** Let  $t = \text{ord}_N(a)$ . Then  $a^z \equiv 1 \pmod N \Leftrightarrow t \mid z$ .

**Prop.** Let  $a \in (\mathbb{Z}/N\mathbb{Z})^*$  and  $t = \text{ord}_N(a)$ . Then

$$\text{ord}_N(a^i) = t / \gcd(i, t).$$

*Proof:*

**Prop.** a)  $\text{ord}_N(ab) \mid \text{lcm}(\text{ord}_N(a), \text{ord}_N(b))$ .

b) If the orders are prime together, we have an equality.

*Proof*

**Rem.** Take  $N = 19$ :  $\text{ord}_N(7) = 3$ ,  $\text{ord}_N(7^2) = 3$ , but  $\text{ord}_N(7^3) = 1$ .

## Lagrange, Fermat, Euler

**Thm.**(Lagrange) Let  $G$  be a finite groupe and  $a \in G$ . Then  $\text{ord}_G(a) \mid \text{Card}(G)$ .

**Coro.**[Fermat's little theorem] Let  $p$  be prime and  $a$  prime to  $p$ . Then

$$a^{p-1} \equiv 1 \pmod p.$$

**Coro.**[Euler] Let  $N$  be an integer and  $a$  prime to  $N$ . Then

$$a^{\varphi(N)} \equiv 1 \pmod N.$$

**Prop.**  $\text{ord}_N(a) = t = \prod_{i=1}^r q_i^{e_i}$  iff  $a^t \equiv 1 \pmod N$  and for all  $i$ ,  $a^{t/q_i} \not\equiv 1 \pmod N$ .

*Proof:*

## Structure of $(\mathbb{Z}/p\mathbb{Z})^*$

**Thm.** When  $p$  is prime,  $(\mathbb{Z}/p\mathbb{Z})^*$  is cyclic: there exists  $g$  s.t.  $\{1, 2, \dots, p-1\} = \{g^i \pmod p, 1 \leq i \leq p-1\}$ .

**Lem.** For all  $d \geq 1$ , the number of elements of order  $d$  is  $\leq \varphi(d)$ .

*Proof:*

*Proof of the thm*

**Rem.** a) When  $d \mid p-1$ , there are exactly  $\varphi(d)$  elements of order  $d$ .

b) In particular, if  $g$  has order  $p-1$ , the other generators are  $g^i$  for  $\gcd(i, p-1) = 1$ .

**Rem.** finding a generator is not easy.

## Structure of $(\mathbb{Z}/p^e\mathbb{Z})^*$

**Prop.** For all odd  $p$  and  $e \geq 1$ ,  $(\mathbb{Z}/p^e\mathbb{Z})^*$  is cyclic. Let  $g$  be a generator of  $(\mathbb{Z}/p\mathbb{Z})^*$ . If  $g^{p-1} \not\equiv 1 \pmod{p^2}$ , then  $g$  is a generator of  $(\mathbb{Z}/p^e\mathbb{Z})^*$ . Otherwise, take  $g+p$ .

*Proof:*

**Lem.** Let  $k \not\equiv 0 \pmod p$  ( $\neq 2$ ) and  $s \geq 0$ . Then

$$(1+kp)^{p^s} = 1 + k_s p^{s+1},$$

with  $p \nmid k_s$ .

*Proof*

*Proof*

**Rem.** The only  $p$ 's for which  $2^{p-1} \equiv 1 \pmod{p^2}$  for  $p \leq 1.25 \times 10^{15}$  are 1093 and  $p = 3511$ .

## The case $p = 2$

**Prop.**  $(\mathbb{Z}/2\mathbb{Z})^* = \{1\}$ ,  $(\mathbb{Z}/4\mathbb{Z})^* = \{1, 3\} = \langle -1 \rangle$ . If  $e \geq 3$ ,  $(\mathbb{Z}/2^e\mathbb{Z})^* \simeq \langle 5 \rangle \times \langle -1 \rangle$ , with  $\text{ord}(5) = 2^{e-2}$ . Equivalently, all  $\text{dein}(\mathbb{Z}/2^e\mathbb{Z})^*$  s'can be written uniquely as

$$x \equiv (-1)^{u(x)} 5^{v(x)} \pmod{2^e}$$

with  $u(x) \in \{0, 1\}$  and  $1 \leq v(x) \leq 2^{e-2}$ . Moreover,  $u(x) \equiv (x - 1)/2 \pmod{2}$ .

*Proof:*

**Lem.**  $5^{2^t} = 1 + h_t 2^{t+2}$ , with  $h_t$  odd.

*Proof:*

## Structure of $(\mathbb{Z}/N\mathbb{Z})^*$

**Thm. (Chinese remaindering theorem)**

$$\mathbb{Z}/N\mathbb{Z} \sim \prod_{p^e \parallel N} \mathbb{Z}/p^e\mathbb{Z}$$

*Proof:*

**Coro.** Let  $m_1, \dots, m_k$  pairwise primes and let  $x_1, \dots, x_k$  be integers. There exists  $x$  s.t.

$$x \equiv x_i \pmod{m_i}$$

for all  $i$ . Moreover,  $x$  is unique modulo  $M = \prod_{i=1}^k m_i$ .

## Fundamental theorem

**Thm.**  $(\mathbb{Z}/N\mathbb{Z})^*$  is cyclic iff  $N = 2, 4, p^\alpha, 2p^\alpha$ ,  $p$  an odd prime and  $\alpha \geq 1$ .

**Def.**  $\{k, \forall a \in G, a^k = 1\}$ .

**Prop.**  $\text{Exp}(G) = \text{lcm}\{\text{ord}_G(a), a \in G\}$ .

**Prop.**  $\text{Exp}(G_1 \times G_2 \times \dots \times G_n) = \text{lcm}(\text{Exp}(G_1), \text{Exp}(G_2), \dots, \text{Exp}(G_n))$ .

## Carmichael function

**Def. (Carmichael function)**  $\lambda(N) = \text{Exp}((\mathbb{Z}/N\mathbb{Z})^*)$ .

**Prop.**

a) For all  $a \in (\mathbb{Z}/N\mathbb{Z})^*$ :  $a^{\lambda(N)} \equiv 1 \pmod{N}$ .

b)  $a^t \equiv 1 \pmod{N}$  for all  $a$  prime to  $N$  iff  $\lambda(N) \mid t$ .

**Prop.**

- (i)  $\lambda(p^e) = \varphi(p^e)$  if  $p$  is odd and  $e \geq 1$
- (ii)  $\lambda(2) = 1, \lambda(4) = 2, \lambda(2^e) = 2^{e-2}$  if  $e \geq 3$
- (iii)  $\lambda(\prod_i p_i^{\alpha_i}) = \text{lcm}(\lambda(p_i^{\alpha_i}))$

**Prop.** If  $N$  is squarefree, then for all  $a \in \mathbb{Z}$ ,  $a^{\lambda(N)+1} \equiv a \pmod{N}$ .

*Proof:*

**Coro.** RSA is valid: for all  $x$ ,  $x^{ed} \equiv x \pmod{N}$ .

*Proof:*

**Prop.** If  $\gcd(N, M) = 1$ , then  $\varphi(NM) = \varphi(N)\varphi(M)$ .  
( $\varphi$  is *multiplicative*).

**Lem.**  $x$  is invertible modulo  $NM$  iff  $x$  is invertible modulo  $N$  and modulo  $M$ .

**Coro.** If

$$N = p_1^{\alpha_1} \cdots p_k^{\alpha_k},$$

$$\varphi(N) = \prod_{i=1}^k \varphi(p_i^{\alpha_i}) = N \prod_{i=1}^k (1 - 1/p_i).$$

## III. Quadratic reciprocity

**Legendre symbol:** for prime odd  $p$  and  $a \in \mathbb{Z}$

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a \\ 1 & \text{if } \exists x \text{ s.t. } a \equiv x^2 \pmod{p} \\ -1 & \text{otherwise.} \end{cases}$$

**Easy properties:**

(i)  $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}$ ;

(ii)  $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$ ;

(iii)  $\left(\frac{a}{p}\right) = \left(\frac{a \pmod{p}}{p}\right)$ ;

(iv)  $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$ ;

**Not so easy properties:** (Gauss)

(v)  $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$ ;

(vi) (Quadratic reciprocity law)  $p$  and  $q$  odd primes:

$$\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \times \frac{q-1}{2}} \left(\frac{p}{q}\right).$$

**Jacobi symbol:**  $n \in \mathbb{Z}$ ,  $m = \prod_{i=1}^k p_i \in \mathbb{Z}$  odd,

$$\left(\frac{n}{m}\right) = \prod_{i=1}^k \left(\frac{n}{p_i}\right).$$

**Properties:** same as for the Legendre symbol.

**Ex.** Show that  $\left(\frac{n}{m}\right) = 0$  iff  $\gcd(n, m) > 1$ .

## IV. Finite fields

**Thm. (characteristic)** Let  $\mathbb{F}$  be a finite field. There exists a smallest  $p > 1$  s.t.  $p \cdot 1_{\mathbb{F}} = 0$ ;  $p$  is prime. The set  $\{k \cdot 1_{\mathbb{F}}, 0 \leq k < p\}$  is the smallest subfield of  $\mathbb{F}$ ; it is isomorphic to  $\mathbb{F}_p$  (prime subfield of  $\mathbb{F}$ ).

**Thm.**

$$\begin{array}{ccc} \mathbb{F} \times \mathbb{F} & \rightarrow & \mathbb{F} \\ (x, y) & \mapsto & x + y \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{F}_p \times \mathbb{F} & \rightarrow & \mathbb{F} \\ (a, x) & \mapsto & ax \end{array}$$

turn  $\mathbb{F}$  into a  $\mathbb{F}_p$ -vector space. If  $n$  is the dimension of this space,  $\mathbb{F}$  has  $p^n$  elements.

**Thm.**  $\mathbb{F}^\times$  is cyclic.

## Frobenius

**Thm.**  $\sigma_{\mathbb{F}}$  :

$$\begin{array}{ccc} \mathbb{F} & \rightarrow & \mathbb{F} \\ x & \mapsto & x^p. \end{array}$$

It is a field automorphism, i.e.

$$\sigma_{\mathbb{F}}(1) = 1, \quad \sigma_{\mathbb{F}}(x + y) = \sigma_{\mathbb{F}}(x) + \sigma_{\mathbb{F}}(y), \quad \sigma_{\mathbb{F}}(xy) = \sigma_{\mathbb{F}}(x)\sigma_{\mathbb{F}}(y).$$

Fixed points are the elements of  $\mathbb{F}_p$ .

*Proof:*

## Properties of $\mathbf{K}[X]$

Properties close to that of  $\mathbb{Z}$ , we do not give proofs.

**Thm.** for all  $A, B$  in  $\mathbf{K}[X]$ ,  $B \neq 0$ . there exists a unique pair  $(Q, R)$  in  $\mathbf{K}[X]$  s.t.

$$A = BQ + R, \quad \text{with } R = 0 \text{ or } \deg(R) < \deg(B).$$

**Thm. (Bézout)** There exists  $U$  and  $V$  in  $\mathbf{K}[X]$  s.t.

$$AU + BV = \gcd(A, B).$$

**Def.**  $A(X)$  is irreducible if its degree is  $\geq 1$ , and all its divisors are constant, or  $aA(X)$  with  $a \in \mathbf{K}^*$ .

**Thm.** We may factor polynomials

$$P = a \prod_{i=1}^r P_i^{\alpha_i},$$

where  $a \in \mathbf{K}$ ,  $P_i$  monic irreducible and  $\alpha_i \in \mathbb{N}$ .

## Quotient ring

**Def.**  $A \equiv B \pmod{f}$  iff  $A - B$  is a multiple of  $f$ .

**Def.**  $\mathbf{K}[X]/f\mathbf{K}[X]$  or  $\mathbf{K}[X]/(f)$

Let  $\bar{P}$  be the class of  $P$ .  $\mathbf{K}[X]/(f)$  together with  $\bar{A} + \bar{B} = \overline{A+B}$ ,  $\bar{A}\bar{B} = \overline{AB}$ , is a ring.

Canonical representant: for all  $P$ , there is a unique  $R$  of degree  $n - 1$  s.t.  $P \equiv R \pmod{f}$ .

With

$$\lambda \cdot \bar{A} = \overline{\lambda A},$$

$\mathbf{K}[X]/(f)$  is  $\mathbf{K}$ -vector space of dimension  $n$ . The set  $\{\bar{1}, \bar{X}, \dots, \overline{X^{n-1}}\}$  is a basis for this vector space.

**Thm.**  $A$  is invertible modulo  $f$  iff  $\gcd(A, f) = 1$ .

**Coro.**  $\mathbf{K}[X]/(f)$  is a field iff  $f$  is irreducible. Moreover, if  $\mathbf{K} = \mathbb{F}_p$ ,  $\mathbb{F}_p[X]/(f)$  is a field of cardinality  $p^{\deg(f)}$ .



## More properties

**Thm. [CRT]** Let  $A_1, A_2, \dots, A_r$  in  $\mathbf{K}[X]$  and  $Q_1, Q_2, \dots, Q_r$  polynomials that are pairwise prime. There exists  $A \in \mathbf{K}[X]$  s.t.

$$\begin{cases} P \equiv A_1 \pmod{Q_1} \\ P \equiv A_2 \pmod{Q_2} \\ \vdots \\ P \equiv A_r \pmod{Q_r} \end{cases}$$

is equivalent to

$$P \equiv A \pmod{Q_1 Q_2 \dots Q_r}.$$

$$\mathbf{K}[X]/(Q_1 Q_2 \dots Q_r) \simeq \mathbf{K}[X]/(Q_1) \times \mathbf{K}[X]/(Q_2) \times \dots \times \mathbf{K}[X]/(Q_r).$$

## Building finite fields

**Thm. (the canonical way)** Let  $f(X)$  be an irreducible polynomial of degree  $n$  over  $\mathbb{F}_p$ . Then  $\mathbb{F}_p[X]/(f(X))$  is a finite field of degree  $n$  and cardinality  $p^n$ , noted  $\mathbb{F}_{p^n}$ .

**Ex.** Build  $\mathbb{F}_{41^2}$ , using a quadratic non-residue modulo 41.

$$\begin{aligned} \left(\frac{7}{41}\right) &= (-1)^{(41-1)/2 \times (7-1)/2} \left(\frac{41}{7}\right) \\ &= \left(\frac{41}{7}\right) = \left(\frac{41 \bmod 7}{7}\right) \\ &= \left(\frac{6}{7}\right) = \left(\frac{2}{7}\right) \left(\frac{3}{7}\right) = \left(\frac{3}{7}\right) = (-1) \left(\frac{7}{3}\right) \\ &= -\left(\frac{1}{3}\right) = -1 \end{aligned}$$

Hence use  $\mathbf{K}_1 = \mathbb{F}_{41^2} \simeq \mathbb{F}_{41}[X]/(X^2 - 7)$ . This is a vector space of dimension 2 over  $\mathbb{F}_{41}$ . Let  $\theta = \bar{X}$ . All elements can be written  $a + b\theta$  where  $a, b$  are in  $\mathbb{F}_{41}$ .

$\theta^2 - 7 = \bar{X}^2 - 7 = 0$ . We get

$$\theta^2 = 7, \theta^3 = 7\theta, \theta^4 = 8, \dots, \theta^{80} = 1,$$

so that  $\theta$  does not generate  $\mathbf{K}^*$ . The number  $\theta + 10$  is found to be a generator.

## One application (1/2)

**Pb.** Given  $\left(\frac{a}{p}\right) = 1$ , compute  $\sqrt{a} \bmod p$ .

**Case**  $p \equiv 3 \pmod{4}$ :  $r = a^{(p+1)/4} \bmod p$ .

**Case**  $p \equiv 1 \pmod{4}$ : find  $b$  s.t.  $\Delta = b^2 - 4a$  is not a square.

$$\alpha = (-b + \sqrt{\Delta})/2 \Rightarrow \alpha^p = (-b - \sqrt{\Delta})/2 \Rightarrow \alpha\alpha^p = a$$

since  $\sqrt{\Delta^p} = \left(\frac{\Delta}{p}\right)\sqrt{\Delta}$ .

Let  $\beta = \alpha^{(p+1)/2} \bmod (p, X^2 + bX + a)$ . Then

$$\beta^2 = \alpha^{p+1} = a.$$

Moreover

$$\beta^p = \beta(\beta^2)^{(p-1)/2} = \beta a^{(p-1)/2} = \beta$$

and  $\beta$  is in  $(\mathbb{Z}/p\mathbb{Z})^*$ .

## One application (2/2)

Let  $a = 2 \bmod 41$ , which a square;  $b = 1$  is s.t.  $\Delta = 1 - 4 \times 2 = -7$  which is not a square; hence  $\mathbb{F}_{41^2} \simeq \mathbb{F}_{41}[X]/(X^2 + X + 2)$ .

$$\alpha = X, \quad \alpha^p = 40X + 40, \quad \alpha\alpha^p = 2.$$

$$\beta = X^{(p+1)/2} = 17, \quad 17^2 \equiv 2 \pmod{41}.$$

## Further properties

**Thm.** Let  $\mathbb{F}$  be a finite field of cardinality  $q$ .

$$X^q - X = \prod_{a \in \mathbb{F}} (X - a).$$

**Thm.**

$$X^{p^n} - X = \prod_{d|n} \prod_{\substack{\deg g=d \\ g \text{ monic irreducible}}} g$$

**Thm.** The number  $I_{n,p}$  of irreducible polynomials of degree  $n$  over  $\mathbb{F}_p$  is always  $> 0$  and

$$I_{n,p} \approx \frac{p^n}{n}.$$

**Thm.** For all  $p$  and  $n \geq 1$ , there exists a finite field of cardinality  $p^n$ . Two finite fields of the same cardinality are isomorphic.

## V. Implementation issues

### A) implementing $\mathbb{Z}$

**How to code an integer:** a positive integer  $a$  is stored as an array containing its digits in base  $B = 2^\beta$ ,  $\beta = 32, 64$ :

$$a = a_{n-1}a_{n-2} \dots a_0 = \sum_{i=0}^{n-1} a_i B^i.$$

The digits are recovered using successive euclidean divisions of  $a$  by  $B$ .

**Operations:**

- $\pm$ : idem base 10.
- $\times$ : base 10, Karatsuba, FFT.
- $/$ : idem base 10 (?); Newton.

## Multiplication (1/2)

**School boy method:**

$$A(X) = a_0 + a_1X, B(X) = b_0 + b_1X$$

$$P(X) = p_0 + p_1X + p_2X^2$$

$$p_0 = a_0b_0, \quad p_1 = a_0b_1 + a_1b_0, \quad p_2 = a_1b_1$$

**Cost:** 4 multiplications.

$\Rightarrow O(lm)$  operations.

**Karatsuba:**

$$p_0 = a_0b_0, \quad p_2 = a_1b_1$$

$$p_1 = (a_0 + a_1)(b_0 + b_1) - p_0 - p_2.$$

**Cost:** 3 multiplications, 2 additions, 2 subtractions.

## Multiplication (2/2)

$$n = 2^t$$

$$A(X) = a_0 + a_1X + \dots + a_{n-1}X^{n-1} = A_0(X) + X^{n/2}A_1(X),$$

$$B(X) = b_0 + b_1X + \dots + b_{n-1}X^{n-1} = B_0(X) + X^{n/2}B_1(X),$$

$$\deg(A_i) = \deg(B_i) = n/2 - 1$$

$$P_0 = A_0B_0, \quad P_1 = (A_0 + A_1)(B_0 + B_1), \quad P_\infty = A_1B_1$$

$$P(X) = P_0 + X^{n/2}(P_1 - P_0 - P_\infty) + X^n P_\infty.$$

**Analysis:**

$$M(n) = 3M(n/2)$$

$$M(n) = O(n^{\log_2 3}) = O(n^{1.585})$$

**Rem.** take care to memory management; branch in naive method for small operands.

**Even better:** FFT in  $O(n^{1+\epsilon})$ .

## a) Classical Gcd:

---

```
function gcd(a, b) (* return gcd of a and b *)
1. while b <> 0 do
    r:=a mod b;
    a:=b;
    b:=r;
endwhile;
2. return a.
3. end.
```

---

b) binary algorithm:  $\gcd(a, b) = \gcd(a, a - b)$  and when  $b$  is odd:

$$\gcd(2^k a, b) = \gcd(a, b)$$

## Example:

```
a=01111011110
b=01001101000
a=00111101111
b=00001001101
t=00011010001
a=00011010001
b=00001001101
t=00000100001
a=00000100001
b=00001001101
t=00000001011
a=00000100001
b=00000001011
t=00000001011
a=00000001011
b=00000001011
t=00000000000
```

c) Other methods: Brent; Lehmer; Jebelean, etc.

## B) $\mathbb{Z}/N\mathbb{Z}$

On top of  $\mathbb{Z}$ , reduce all operands; Montgomery arithmetic is division free.

- $\mathbb{Z}/p\mathbb{Z}$ : choice between a global  $p$  and a  $p$  per element.
- $\mathbb{Z}/N\mathbb{Z}$ : use exceptions to catch divisors of composite  $N$  (lazy computations).

## Finite fields $\mathbb{F}_{p^n}$

**General principle:** choose an irreducible polynomial  $f$  of degree  $n$  over  $\mathbb{F}_p$ .

Apart from some families and particular cases (e. g., binomials), this is not known to be doable in deterministic polynomial time.

Identify elements of  $\mathbb{F}_{p^n}$  to elements of  $\mathbb{F}_p^n$  via

$$\overline{a_0 + a_1X + \cdots + a_{n-1}X^{n-1}} \mapsto (a_0, a_1, \dots, a_{n-1}).$$

Addition, subtraction are easy; multiplication of polynomials over  $\mathbb{F}_p$ ; inverse via the extended euclidean algorithm.

**Rem.** When  $\mathbb{F}_q$  is small, fix a generator  $g$  and represent elements via  $g^i$ .

**Rem.** Composite extensions are not that easy to deal with.

## Computing $a^e$ : binary methods

**Pb:** compute  $a^e$  (in any ring).

$$e = 2e' + \varepsilon, \varepsilon \in \{0, 1\}$$

$$a^e = (a^2)^{e'} a^\varepsilon$$

or

$$a^e = (a^{e'})^2 a^\varepsilon.$$

$$a^{11} = (a^2)^5 a = (((a^2)^2)^2 \times a^2)^2 a$$

$$a^{11} = (a^5)^2 a = ((a^2)^2 a)^2 a.$$

**Cost:** 5 operations (multiplications) to evaluate  $a^{11}$ , instead of 10.

## The left-right binary method

$$e = e_t 2^t + e_{t-1} 2^{t-1} + \dots + e_0, e_t = 1$$

Horner:  $e = ((\dots (e_t 2 + e_{t-1}) 2 + e_{t-2}) \dots) 2 + e_0$ .

$$a^e = a^{(((\dots (e_t 2 + e_{t-1}) 2 + e_{t-2}) \dots) 2 + e_0)} = (((\dots (a^{e_t})^2 a^{e_{t-1}})^2 a^{e_{t-2}}) \dots)^2 a^{e_0}.$$

---

```
function ModExp3(a, N, te, nbits)
(* compute de  $a^e \bmod N$  with bits of e *)
(* in te[0..nbits] *)
1. [initialization] b:=a;
2. for i:=nbits-1 .. 0 do
    b:=(b*b) mod N;
    if te[i] = 1 then b:=(b*a) mod N; endif;
endfor;
3. return b;
4. end.
```

---

## Analysis

$\nu(e)$  number of nonzero bits of  $e$ ,  $\lambda(e) = \lfloor \log_2(e) \rfloor$

$$C(e) = \lambda(e) T_{\square} + (\nu(e) - 1) T_{\times a}.$$

On average:

$$\bar{C}(e) = \lambda(e) (T_{\square} + T_{\times a} / 2).$$

If  $T_{\square} = T_{\times a}$ , the mean cost of the algorithm is  $3/2 \log_2(e) T_{\times a}$ .

If  $a$  is small, the cost of  $T_{\times a}$  is negligible.

## Real life

Always a good thing to try to implement its own bignum library.  
However, very difficult to be better than GMP; available in any  
mathematical system (MAPLE, MAGMA, etc.; pari-gp, Sage, etc.).