Isogenies in cryptography

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Plan

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II. Elliptic curves and isogenies.

III. Computations and algorithms.

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I. Introduction

A short history:

- 1990ff: Schoof-Elkies-Atkin (SEA), Couveignes-FM, Dewaghe (isogeny cycles);
- Kohel, Galbraith, Fouquet-FM (volcanoes);
- Galbraith-Hess-Smart; Smart; Jao-Miller-Venkatesan; Teske; Rostovtsev-Stolbunov; Charles-Goren-Lauter.
- Quite recently: finding good Edwards curves (FM); CRT methods for computing class (resp. modular) polynomials(Sutherland *et al.*).

Bibliography:

- Silverman; Lang's *Elliptic functions*.
- green book (Blake, Seroussi, Smart). Don't forget to read the original papers, when available...
- Gathen & Gerhard, etc.

Different usages

- Generalize $[m]$ on E:
  - factor f_m: SEA.
  - speed up the computation of $[k]P$ when small degree isogeny exist (Doche-Icart-Kohel).
- Replace E by some sister or cousin having better or stronger properties:
  - find $\tilde{E}$ of the same cardinality, but $Y^2 = X^3 - 3X + b$ (Brier-Joye);
  - preventing the existence of “special points” à la Goubin (Smart);
  - find a convenient Edwards curve (FM).
- Hide a curve in a graph for cryptographic applications (see later).
- New CRT methods for computing modular or class polynomials (Sutherland).

II. Elliptic curves and isogenies

$$E : y^2 = x^3 + Ax + B \text{ over } \mathbf{K}, \mathrm{char}(\mathbf{K}) \notin \{2,3\}.$$

Def. (torsion points) For $n \in \mathbb{N}$, $E[n] = \{P \in E(\overline{\mathbf{K}}), [n]P = O_E\}$.

Division polynomials:

$$[n](x,y) = \left(\frac{\varphi_n(x,y)}{\psi_n(x,y)^2}, \frac{\omega_n(x,y)}{\psi_n(x,y)^3} \right)$$

$$\varphi_n = x\psi_n^2 - \psi_{n+1}\psi_{n-1}$$

$$4y\omega_n = \psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2$$

In $\mathbf{K}[x,y]/(y^2 - (x^3 + Ax + B))$, one has:

$$\psi_{2m+1}(x,y) = f_{2m+1}(x), \quad \psi_{2m} = 2yf_{2m}(x)$$

for some $f_m(x) \in \mathbf{K}[A,B,x]$.

$$f_n(x) = \begin{cases} \psi_n(x,y) & \text{for } n \text{ odd} \\ \psi_n(x,y)/(2y) & \text{for } n \text{ even} \end{cases}$$

$$f_{-1} = -1, \quad f_0 = 0, \quad f_1 = 1, \quad f_2 = 1$$

$$f_3(x,y) = 3x^4 + 6Ax^2 + 12Bx - A^2$$

$$f_4(x,y) = x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3$$

$$f_{2n} = f_n(f_{n+2}f_{n-1}^2 - f_{n-2}f_{n+1}^2)$$

$$f_{2n+1} = \begin{cases} f_{n+2}f_n^3 - f_{n+1}^3 f_{n-1}(16y^4) & \text{if } n \text{ is odd} \\ (16y^4)f_{n+2}f_n^3 - f_{n+1}^3 f_{n-1} & \text{otherwise.} \end{cases}$$

$$\deg(f_n(x)) = \begin{cases} (n^2-1)/2 & \text{if } n \text{ is odd} \\ (n^2-4)/2 & \text{otherwise.} \end{cases}$$

Thm. $P = (x,y) \in E[\ell] \iff [2]P = O_E$ or $f_\ell(x) = 0$.

Isogenies

Def. $\phi : E \to \tilde{E}$, $\phi(O_E) = O_{\tilde{E}}$; induces a morphism of groups.

First examples

1. Separable:

$$[k](x,y) = \left(\frac{\varphi_k}{\psi_k^2}, \frac{\omega_k}{\psi_k^3} \right),$$

$\tilde{E} = E$ (endomorphism).

2. Complex multiplication: $[i](x,y) = (-x, iy)$ on $E : y^2 = x^3 - x$; $\tilde{E} = E$ (endomorphism).

3. Inseparable: $\pi(x,y) = (x^p, y^p)$, $\mathbf{K} = \mathbb{F}_p$; $\tilde{E} = E^p$.

In the sequel:

- only separable isogenies;
- finite fields.

Properties of isogenies

Thm. If F is a finite subgroup of $E(\overline{\mathbf{K}})$, there exists ϕ and $\tilde{E}$ s.t.

$$\phi : E \to \tilde{E} = E/F, \quad \ker(\phi) = F.$$

Def. (Separable) *degree* of ϕ is $\#F$.

Problem 0: given E, F, compute an equation for $\tilde{E}$ and formulas for ϕ.

Thm. (dual isogeny) There is a unique $\hat{\phi} : \tilde{E} \to E$, $\hat{\phi} \circ \phi = [m]$, $m = \deg \phi$.

$$\begin{tikzcd}
E \arrow[r, "\phi"] \arrow[dr, "{[m]}"'] & \tilde{E} \arrow[d, "\hat{\phi}"] \\
 & E
\end{tikzcd}$$

$\Rightarrow$ we can get a factorization of f_ℓ.

Finding isogenous curves

Key fact: for all integers n there exists a polynomial $\Phi_n(X,Y) \in \mathbb{Z}[X,Y]$ (modular polynomial) s.t. E and E' are n-isogenous iff $\Phi_n(j(E),j(E')) = 0$.

Problem 1: given E, find all roots of $\Phi_n(X,j(E))$ and construct from this all (E',ϕ) that are n-isogenous.

Rem. If ℓ is prime $\deg(\Phi_\ell) = \ell+1$ and can be computed in $\tilde{O}(\ell^3)$ operations (Enge).

Thm. When ℓ is prime, $\Phi_\ell(X,j(E))$ has 0, 1, 2 or $\ell+1$ roots over $\mathbf{K}$.

$\Rightarrow$ we can build a graph of isogenies starting from E.

Atkin and Elkies (1986–1990)

The Frobenius $\pi : (X,Y) \mapsto (X^q, Y^q)$ has minimal equation

$$\pi_\ell^2 \ominus [t]\pi_\ell \oplus [q] = 0, \quad \Delta = t^2 - 4q.$$

If $(\Delta/\ell) = +1$, then over $\mathbb{F}_\ell$,
$\mathrm{Mat}(\pi_\ell) \simeq \left(\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}\right) \Leftrightarrow \exists F, \pi_\ell(F) = F \Leftrightarrow F$ is a cyclic subgroup of order ℓ, defined over $\mathbf{K}$; E is ℓ-isogenous to $\tilde{E} = E/F$.

As a consequence, f_ℓ has a factor of degree $(\ell - 1)/2$.

Computational primitive: $E \mapsto \tilde{E}$ in direction λ.

III. Computations and algorithms

But what does an isogeny look like? Let

$$D(x) = \prod_{Q \in F^*} (x - x_Q) = x^{\ell-1} - \sigma x^{\ell-2} + \sigma_2 x^{\ell-3} - \sigma_3 x^{\ell-4} + \cdots .$$

where $\sigma = \sum_{Q \in F^*} x_Q$.

Rem. When ℓ is odd, $D(x) = g(x)^2$.

Fundamental proposition. $\tilde{E} : Y^2 = X^3 + \tilde{A}X + \tilde{B}$ where $\tilde{A} = A - 5t$, $\tilde{B} = B - 7w$ with

$$t = A(\ell-1) + 3(\sigma^2 - 2\sigma_2), w = 3A\sigma + 2B(\ell-1) + 5(\sigma^3 - 3\sigma\sigma_2 + 3\sigma_3);$$

$$\phi(x,y) = \left(\frac{N(x)}{D(x)}, y\left(\frac{N(x)}{D(x)}\right)'\right),$$

$$\frac{N(x)}{D(x)} = \ell x - \sigma - (3x^2 + A)\frac{D'(x)}{D(x)} - 2(x^3 + Ax + B)\left(\frac{D'(x)}{D(x)}\right)'$$

Numerical examples

Ex 1. $E : y^2 = x^3 + bx$, $F = \langle (0,0) \rangle$;

$$\tilde{E} : y^2 = x^3 - 4bx,$$

$$\phi : (x,y) \mapsto \left(\frac{x^3 + bx}{x^2}, y\frac{x^2 - b}{x^2} \right).$$

A curiosity: $E : y^2 = x^3 + x + 3$ defined over $\mathbb{F}_{1009}$; E is 6-isogenous to

$$\tilde{E} : y^2 = x^3 + 830x + 82$$

and $\sigma = 739$ (formulas for prime ℓ valid here too!!!) for which

$$\frac{N(x)}{D(x)} = \frac{x^6 + 270x^5 + 325x^4 + 566x^3 + 382x^2 + 555x + 203}{x^5 + 270x^4 + 289x^3 + 659x^2 + 533x + 399}.$$

The denominator factors as

$$(x - 66)(x - 23)^2 (x - 818)^2.$$

$x = 66$ is the abscissa of a point of 2-torsion;
23 is the abscissa of a point of 3-torsion;
818 is the abscissa of a primitive point of 6-torsion.

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Isogenies and complexity (1/2)

Goal: compute an isogenous curve of degree ℓ over $\mathbb{F}_{p^n}$.

Basic algorithm:
Given p, n, $E/\mathbb{F}_{p^n}$,
for j_0 a root of $\Phi_\ell(X, j(E))$, compute $\tilde{E}$ and $\phi : E \to \tilde{E}$.

- Case $p \gg \ell$: (Elkies, Atkin), finding $\tilde{E}$ costs $O(\ell)$ operations; finding ϕ costs $O(\ell^2) + O(\mathsf{M}(\ell))$ (Bostan-FM-Salvy-Schost).
- Case $p \ll \ell$:
  - $p = 2$: super fast algorithm by Lercier, complexity not proven $O(\ell^2)$ or $O(\ell^3)$?.
  - $p > 2$: Couveignes's Artin Schreier approach (remember L. De Feo's talk, joint work with É. Schost), $\tilde{O}(\ell^2)$. To be confronted with Lercier-Sirvent's p-adic approach.

Isogenies and complexity (2/2)

Two easy problems:

- Problem 0: given E, F, compute an equation for $\tilde{E}$ and formulas for ϕ (Vélu's formulas).
- Problem 1: given E, find all roots of $\Phi_n(X, j(E))$ and construct from this all (E', ϕ) that are n-isogenous (Elkies, Atkin, etc., but $\tilde{O}(n^3)$).

Two difficult problems:

- Problem 2: given E_1 and E_2, are they isogenous? (modular polynomials can help if bound on degree or very efficient SEA to use Tate's theorem).
- Problem 3: given that E_1 and E_2 are isogenous, find an isogeny between them (probably best to solve Problem 0 and use an isomorphism from $\tilde{E}$ to E_2; bound required).

IV. Isogeny graphs and cryptographic applications

Def. $G = (\mathscr{V}, \mathscr{E})$ where $(E_1, E_2) \in \mathscr{E}$ if and only if E_1 and E_2 are isogenous.

Thm. (Tate) isogenous curves (over $\mathbb{F}_q$) have the same cardinality.

$\Rightarrow$ G is complete. It is more interesting to study particular paths between curves.

For instance: graph of ℓ-isogenies for ℓ fixed.

It turns out that endomorphisms are important:
$\mathrm{End}(E) = \{I : E \to E\}$.

First task: classify curves according to their endomorphism ring.

A) Fixing ℓ: volcanoes

Thm. If E is ordinary, write $\#E = q+1-t$ and $t^2-4q = d = f^2 D$. Then $\mathrm{End}(E)$ is an order $\mathscr{O}$ in $\mathbf{K} = \mathbb{Q}(\sqrt{D})$ where $D = \mathrm{disc}(\mathbf{K}) < 0$.

General picture: $\mathbb{Z}[\pi] = \mathbb{Z}[(d+\sqrt{d})/2] \subset \mathrm{End}(E) \subset \mathscr{O}_K$.

Class polynomial: $H_d(X) = \prod_{\mathrm{End}(E)=\mathscr{O}}(X - j(E)) \in \mathbb{Z}[X]$.

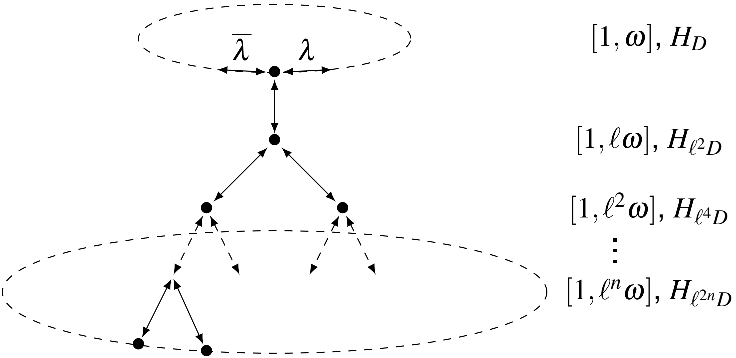
Important result: (Deuring, Waterhouse, Schoof) number of isomorphism classes of curves having the same cardinal is

$$H(d) = \sum_{\mathbb{Z}[\pi]\subset\mathscr{O}\subset\mathscr{O}_K} h(\mathscr{O}).$$

$\Rightarrow \#\mathscr{V}$ is reasonably large ($h(\Delta) = O(|\Delta|^{1/2+\varepsilon})$).

Volcano

Most interesting case is $\left(\frac{D}{\ell}\right) = +1$ and $\ell^{2n} \mid\mid \mathrm{disc}(\pi) = t^2 - 4q$:



Navigating in the structure is relatively easy, using modular polynomials: solve $\Phi_\ell(X, j(E_i))$ to get E_{i+1} in direction λ; see Kohel, Fouquet-FM.

B) Varying ℓ

Problem: given $E_1, E_2 \in \mathscr{V}$, find a path from E_1 to E_2.

Thm. (Galbraith, over $\mathbb{F}_p$) there exists a probabilistic algorithm that builds an isogeny $I : E_1 \to E_2$ requiring $O(p^{3/2} \log p)$ expected time and expected space $O(p \log p)$ at worse.

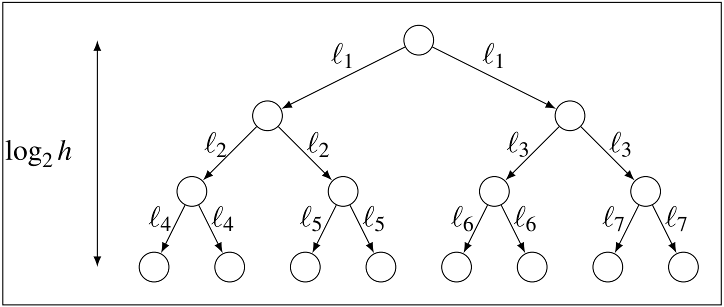
Algorithm:
INPUT: E_1 and E_2 which are isogenous.
OUTPUT: an isogeny path from E_1 to E_2.
1. Find E_i' isogenous to E_i s.t. $\mathrm{End}(E_i') = \mathscr{O}_K$.
2. Find two paths from E_1' and E_2' that meet in some point.
3. Assemble the isogeny.

Idea: build paths using ℓ-isogenies of prime degree $\ell \leq L = O((\log D)^2$ (under GRH).

Conjecture: this will terminate after $O(\log h_K)$ iterations.

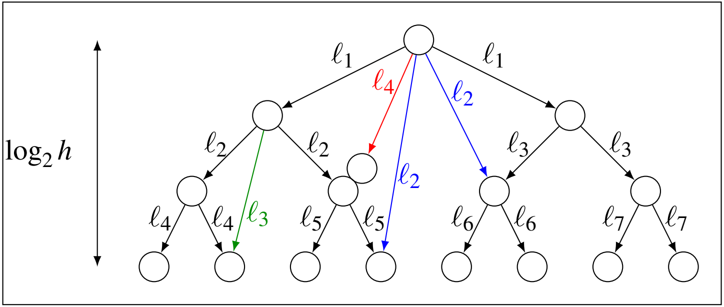
Building a binary tree

Start from any curve and build a tree, at each node selecting some ℓ at random (this is needed since for fixed ℓ, we find a cycle).



Classical property of binary trees: if height is $\log_2 h$, then the total number of nodes is h, half of which are leaves.

Building a “bushy” tree



At each iteration ℓ, for each vertex j, compute the roots of $\Phi_\ell(X,j)$. Expect the tree to have size $O(\sqrt{h})$ after $O(\log h)$ iterations.

Using two trees and a birthday-paradox approach, there exists a common vertex in both trees after $O(\log h)$ iterations.

Build the respective paths and that's it.

Jao, Miller, Venkatesan (ASIACRYPT 2005)

$\mathscr{G} = (\mathscr{V}, \mathscr{E})$ where $(E_1, E_2) \in \mathscr{E}$ if and only if $\exists I : E_1 \to E_2, \deg(I) = \ell \in O((\log q)^{2+\delta})$ for some $\delta > 0$.

Prop. $\mathscr{G}$ is an expander graph, hence there is a rapid mixing property for random walks.

Prop. Let G be a regular graph of degree k on h vertices. Suppose that the eigenvalue λ of any nonconstant eigenvector satisfies the bound $|\lambda| \leq c$ for some $c < k$. Let S be any subset of the vertices of G, and x be any vertex in G. Then a random walk of any length at least $\frac{\log(2h/|S|^{1/2})}{\log(k/c)}$ starting from x will land in S with probability at least $\frac{|S|}{2h} = \frac{|S|}{2|G|}$.

Coro. ECDLP is not stronger among an isogeny class.

Some cryptographic applications

Where is the difficult problem? Given two isogenous curves E_1 and E_2, build an explicit isogeny $I : E_1 \to E_2$.

Only known way: Galbraith's in $O(\sqrt{h})$.

Using the graphs:

- Key exchange: (Rostovtsev, Stolbunov) using two routes and $R_A(R_B(E)) = R_B(R_A(E))$.
- ECDLP: the GHS attack is not invariant under isogeny, hence we could dream of finding an isogenous curve E_2 for which the GHS is more (resp. less) successful. Confirmed by JaMiVe05. $\Rightarrow$ key for trapdoors, see E. Teske's (*J. Cryptology*).
- Hash function: (D. Charles, E. Goren, K. Lauter): $H(m_0 m_1 \ldots m_{k-1})$: start from a given (supersingular) E; use m_i to decide to go left or right at each step; hash value is the last curve.

Conclusions

- Isogenies prove their interest outside classical number theory, and even outside the original SEA context.
- Not all algorithmic problems solved: see the current cleaning of Couveignes's algorithm, the use of p-adic methods, etc.
- New applications appear: CRT again; more crypto things?
- Higher genus: almost everything has to be done (see FM's slides for ANTS8).

$\Rightarrow$ not the end of the story!