A survey on algorithms for computing isogenies on low genus curves

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Contents

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I. Motivations

- **Number Theory:**
  - computing algebraic integrals: AGM, etc.
  - classification of curves into isogeny classes (e.g., over a finite field, two curves have the same cardinality).
  - etc.
- **Computational Number Theory:**
  - $g = 1$:
    - First life (1985–1997): crucial role in point counting in Schoof-Elkie-Atkin (SEA), Couveignes, Lercier; still needed for p large; AGM for p small (p-adic methods à la Mestre, Satoh, Kedlaya).
    - Second life (1996–): Kohel, Fouquet/M. (cycles and volcanoes); Couveignes/Henocq, Bröker and Stevenhagen (CM curves using p-adic method).
  - $g \geq 2$: try to extend these previous successes (e.g., modular polynomials).

Motivations (cont'd): cryptologic applications

- $g = 1$ (1999–):
  - speedup for computing $[k]P$ when an "easy" endomorphism is known (Koblitz; Gallant/Lambert/Vanstone + several followers).
  - Special purposes: Smart; Brier & Joye.
  - isogeny graph: $(E_1, E_2) \in \mathcal{E}$ iff E_1 and E_2 are isogenous
    - Galbraith: finding a path between two curves seems difficult;
    - Jao/Miller/Venkatesan: the graph is an expander graph;
    - Galbraith/Hess/Smart: send DL from a hard curve to a weak one;
    - cryptosystems: Teske (hide an easy DLP among harder ones); Rostovtsev/Stolbunov; etc.
    - hash function: Charles/Goren/Lauter use graph of 2-isogenies of supersingular elliptic curves.
- $g \geq 2$:
  - speedups in exponentiations: Kohel/Smith, Takashima, Galbraith/Lin/Scott, etc.
  - $g = 3$: sending DL on $\mathrm{Jac}(H)$ to a weaker one on $\mathrm{Jac}(Q)$ (Smith).

II. Isogenies in theory

Def. An isogeny is a surjective homomorphism of finite kernel between two abelian varieties: $\varphi : \mathcal{A} \to \mathcal{A}'$.

Right away, we will concentrate on jacobians of curves; for simplicity, $g \leq 3$.

Endomorphism: $\mathrm{Jac}' = \mathrm{Jac}$.

The case $g = 1$

Thm. If F is a finite subgroup of $E(\overline{\mathbf{K}})$, then there exists I and $\tilde{E}$ s.t.

$$I : E \to \tilde{E} = E/F, \quad \ker(I) = F.$$

Thm. (dual isogeny) There is a unique $\hat{I} : \tilde{E} \to E$, $\ell = \deg I$ s.t.

$$(*) \qquad \hat{I} \circ I = [\ell]$$

$$\begin{tikzcd} E \arrow[r, "I"] \arrow[dr, "{[\ell]}"'] & \tilde{E} \arrow[d, "\hat{I}"] \\ & E \end{tikzcd}$$

$\Rightarrow$ I is a factor of $[\ell]$, hence I can provide factors of ψ_ℓ
$\Rightarrow$ key to SEA.

Higher genus

$g = 2$: $\mathrm{Jac}(H)/F \sim \mathrm{Jac}(H')$ or $E_1 \times E_2$ (cannot be determined by looking at F only?).

$g = 3$: $\mathrm{Jac}(H)/F \sim \mathrm{Jac}(H')$ or $\mathrm{Jac}(C)$ or $E_1 \times E_2 \times E_3$.

If F has suitable properties, then (*) stands also for some ℓ. Typical example is ℓ prime and $F \sim (\mathbb{Z}/\ell\mathbb{Z})^g$.

First examples and illustrations

1. Separable:

$$[k](x, y) = \left(\frac{\phi_k}{\psi_k^2}, \frac{\omega_k}{\psi_k^3}\right)$$

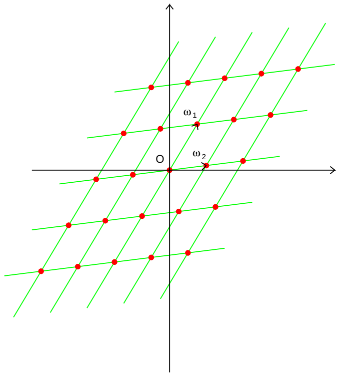
where ψ_k is some division polynomial (i.e., coding the k-torsion). Generalized to division ideals in higher genus.

2. Complex multiplication: $[i](x, y) = (-x, iy)$ on $E : y^2 = x^3 - x$. Ever integer k can be written as $k = k_0 + Ik_1$ where $I^2 \equiv -1 \bmod p$ and $|k_0|, |k_1| \approx \sqrt{p}$
$\Rightarrow$ fast way of evaluating $[k]P$.

3. Inseparable: $\varphi(x, y) = (x^p, y^p)$, $\mathbf{K} = \mathbb{F}_p$.

In the sequel: only separable isogenies.

The classical case: isogenies for curves over $\mathbb{C}$



If $E = \mathbb{C}/L$ and $E' = \mathbb{C}/L'$ and there exists an α s.t. $\alpha L' \subset L$, then E and E' are isogenous.

Modular polynomial: there exists a bivariate polynomial $\Phi_m(X, Y) \in \mathbb{Z}[X, Y]$ such that if L/L' is cyclic of index m then

$$\Phi_m(j(L), j(L')) = \Phi_m(j(E), j(E')) = 0.$$

Examples

Ex. $E : Y^2 = X^3 + bX$, $F = \langle(0, 0)\rangle$; $\tilde{E} : Y^2 = X^3 - 4bX$,

$$I : (x, y) \mapsto \left(\frac{x^3 + bx}{x^2}, y\frac{x^2 - b}{x^2}\right).$$

$$\hat{I}(x) = \frac{x^2 - 4\,b}{x},$$

$$\hat{I} \circ I = 2^2[2] = \frac{x^4 - 2\,x^2 b + b^2}{x\,(x^2 + b)}.$$

Later on: how we can effectively compute such formulas.

A typical isogeny pair: $\tilde{E} = \mathbb{C}/(\omega_1/\ell, \omega_2)$ is ℓ-isogenous to $E = \mathbb{C}/(\omega_1, \omega_2)$. Take as finite subgroup:

$$F = \{O_E\} \cup \left\{(\wp(r\omega_1/\ell), \frac{1}{2}\wp'(r\omega_1/\ell)), 1 \le r \le \ell - 1\right\}.$$

[remember that Weierstrass $\wp$ parametrizes E.]

Complex multiplication

$E = \mathbb{C}/L(1, \tau)$ with quadratic τ in some $\mathbf{K} = \mathbb{Q}(\sqrt{-D})$.

For α an integer in $\mathbf{K}$, Weierstrass $\wp$ gives:

$$\wp(\alpha z) = \frac{N(\wp(z))}{D(\wp(z))}$$

with $\deg(N) = \deg(D) + 1 = \mathrm{Norm}(\alpha)$.

Take $D = 7$ and $E : Y^2 = X^3 - 35X - 98$, $\omega = (-1 + \sqrt{-7})/2$:

$$[\omega](x) = \frac{\left(x^2 + (4 + \omega)\,x + 21\,\omega + 7\right)(-1 + \omega)}{4\,x + 16 + 4\,\omega}.$$

CM generalizes to other genera: theory ok, computations doable in genus 2.

Two strategies for building isogenies

Starting from a kernel:

- given $\mathrm{Jac}(C)$ and F, find the module(s) of $\mathrm{Jac}(C') = \mathrm{Jac}(C)/F$, and then C' [this could be non-trivial];
- compute I.

Using modular polynomials: try to mimic the classical case of

- find the roots $\{j'\}$ of $\Phi_\ell(X, j(E)) = 0$;
- for each j', find E' of invariant j';
- compute I.

En route: examine each of these, starting from the (easy) case of $g = 1$.

III. Computing modular polynomials

A) when $g = 1$

Traditionnal modular polynomial: constructed via lattices and curves over $\mathbb{C}$ (plus modular forms and functions). Remember that

$$j(q) = \frac{1}{q} + 744 + \sum_{n \geq 1} c_n q^n.$$

Then $\Phi_\ell^T(X, Y)$ is such that $\Phi_\ell^T(j(q), j(q^\ell))$ vanishes identically. This polynomial has a lot of properties: symmetrical $\mathbb{Z}[X, Y]$, degree in X and Y is $\ell + 1$ (hence $(\ell+1)^2$ coefficients), etc. and moreover

Thm. [P. Cohen] the height of $\Phi_\ell^T(X, Y)$ is $O((\ell + 1)\log \ell)$.

$\Rightarrow$ total size is $\tilde{O}(\ell^3)$.

Example:

$$\Phi_2^T(X, Y) = X^3 + X^2\left(-Y^2 + 1488\,Y - 162000\right) + X\left(1488\,Y^2 + 40773375\,Y + 8748000000\right)$$
$$+Y^3 - 162000\,Y^2 + 8748000000\,Y - 157464000000000.$$

Choosing another modular equation

Why? Always good to have the smallest polynomial so as not to fill the disks too rapidly...

Key point: any function on $\Gamma_0(\ell)$ (or $\Gamma_0(\ell)/\langle w_\ell \rangle$) will do. In particula if

$$f(q) = q^{-v} + \cdots$$

then there will exist a polynomial $\Phi_\ell[f](X, Y)$ s.t.

$$\Phi_\ell[f](j(q), f(q)) \equiv 0.$$

This polynomial will have $(v+1)(\ell+1)$ coefficients, and height $O(v \log \ell)$, still in $\tilde{O}(\ell^3)$.

Choosing f

Atkin:

- canonical choice $f(q)$ using some power of $\eta(q)/\eta(q^\ell)$ where $\eta(q) = q^{1/24}\prod_{n \geq 1}(1 - q^n)$. E.g.
$$\Phi_2^c(J, F) = F^3 + 48F^2 + 768F - JF + 4096.$$
- a difficult method (the laundry method) for finding (conjecturally) the f with smallest v (that can rewritten as θ-functions with characters).

Müller: for (small) integer r, use

$$\frac{T_r(\eta\eta_\ell)}{\eta\eta_\ell}$$

where T_r is the Hecke operator

$$(T_r|f)(\tau) = f(r\tau) + \frac{1}{r}\sum_{k=0}^{r-1} f\left(\frac{\tau + k}{r}\right).$$

Alternatively: one may use some linear algebra on functions obtained via Hecke operators.

Computing $\Phi_\ell[f]$ given f

- **Atkin** (analysis by Elkies): use q-expansion of j and f with $O(v$ terms, compute power sums of roots of $\Phi_\ell[f]$, write them as polynomials in J and go back to coefficients of $\Phi_\ell[f](X, J)$ via Newton's formulas; use CRT on small primes. $\tilde{O}(\ell^3 \mathsf{M}(p))$; used for $\ell \leq 1000$ fifteen years ago.
- **Charles+Lauter (2005):** compute Φ_ℓ^T modulo p using supersingular invariants mod p, Mestre *méthode des graphes*, torsion points defined over $\mathbb{F}_{p^{O(\ell)}}$ and interpolation. $\tilde{O}(\ell^4 \mathsf{M}(p))$
- **Enge (2004); Dupont (2004):** use complex floating point evaluation and interpolation. $\tilde{O}(\ell^3)$

Write

$$\Phi_\ell^T(X,J) = X^{\ell+1} + \sum_{u=0}^{\ell} c_u(J)X^u$$

where $c_u(J) \in \mathbb{Z}[J]$, $\deg(c_u(J)) \leq \ell+1$. All computations are done using precision $H = O(\ell \log \ell)$.

1. **for** $\ell+1$ values of z_i **do**:

 1.1 Compute floating point approximations to the $\ell+1$ roots $f_r(z_i)$ of $\Phi_\ell[f](X, j(z_i))$ to precision H;

 1.2 Build $\prod_{r=1}^{\ell+1}(X - f_r(z_i)) = X^{\ell+1} + \sum_{u=0}^{\ell} c_u(j(z_i))X^u$; $O(\mathsf{M}(\ell)\log\ell)$ ops.

2. Perform $\ell+1$ interpolations for the c_u's: $O((\ell+1)\mathsf{M}(\ell)\log\ell)$ ops.

All 1.2 + 2 has cost $O(\ell \mathsf{M}(\ell)(\log\ell)\mathsf{M}(H)) = \tilde{O}(\ell^3)$.

Examples

Data for $T_r(\eta\eta_\ell)/\eta\eta_\ell$ (courtesy Enge)

ℓ	r	H	deg(J)	eval(s)	interp(s)	tot (d)	Mb gz
3011	5	7560	200				368
3079	97	9018	254	7790	640	23	547
3527	13	9894	268	799	1440	3	746
3517	97	10746	290	12400	1110	42	850
4003	13	11408	308	1130	2320	4	1127
5009	5	13349	334	880	3110	3	1819
6029	5	16418	402	1550	6370	7	3251
7001	5	19473	466	2440	11700	13	5182
8009	5	22515	534	3500	20000	22	7905
9029	5	25507	602	5030	33100	35	11460
10079	5	28825	672	7690	56300	61	16152

An algebraic alternative: Charlap/Coley/Robbins

Over some $\mathbf{K}$, write

$$\psi_\ell(X) = \prod_{1\leq r,s\leq \ell-1} (X - \wp((r\omega_1 + s\omega_2)/\ell)).$$

The factor we build is:

$$D(x) = \prod_{1\leq r\leq \ell-1} (X - \wp(r\omega_1/\ell))$$

and all its coefficients are in $\mathbf{K}[\sigma]$ where $\sigma = \sum_r \wp(r\omega_1/\ell)$.

$$\begin{array}{cc} \mathbf{K}[x]/(\psi_\ell(x)) & \\ | & \ell-1 \\ \mathbf{K}[x]/(M_\sigma(x)) & \\ | & \ell+1 \\ \mathbf{K}[x] & \end{array}$$

If σ is rational over $\mathbf{K}$, then $D(x)$ will have rational coefficients.

CCR (cont'd)

Another modular equation: $M_\sigma(x) = \Phi_\ell(x, j(E))$.
It has the same properties as the traditional one (e.g., factorization patterns) and can be used as is in SEA.
To find $\tilde{A}$ and $\tilde{B}$, we need two more polynomials + some tedious matching of roots.

The first values are:

$$U_3(X) = X^4 + 2AX^2 + 4BX - A^2/3,$$

$$V_3(X) = X^4 + 84AX^3 + 246A^2X^2 + (-63756A^3 - 432000B^2)X$$
$$+576081A^4 + 3888000B^2A,$$

$$W_3(X) = X^4 + 732BX^3 + (171534B^2 + 25088A^3)X^2$$
$$+(11009548B^3 + 1630720BA^3)X - 297493504/27A^6$$
$$-437245479B^4 - 139150592B^2A^3,$$

$$U_5(X) = X^6 + 20AX^4 + 160BX^3 - 80A^2X^2 - 128ABX - 80B^2.$$

B) Modular polynomials when $g = 2$

- **Gaudry + Schost:** the algebraic alternative is generic (Ξ_ℓ)
  - total degree is $d = (\ell^4 - 1)/(\ell - 1)$;
  - number of monomials is $O(\ell^{12})$;
  - can do $\ell = 3$: 50k but a lot of computing time (weblink still active);
  - use its factorization patterns à la Atkin to speedup cardinality computations.
- **The classical modular approach:**
  - Poincaré $\rightarrow$ Siegel (dim $2g$);
  - replace j by $(j_1, j_2, j_3) \Rightarrow$ triplet of modular polynomials, coefficients are rational fractions in j_i's;
  - Dupont (experimental conjectures proven more recently by Bröker+Lauter): stuck at $\ell = 2$ with 26.8 Mbgz (just the beginning of $\ell = 3$); uses evaluation/interpolation again.

C) Modular polynomials when $g = 3$

Gaudry + Schost $\Rightarrow d = (\ell^{2g} - 1)/(\ell - 1)$.

And then: ?????

IV. Computing the isogeny

A) the case $g = 1$: Vélu's formulas

Vélu suggests to use

$$x_{I(P)} = x_P + \sum_{Q \in F^*} (x_{P+Q} - x_Q)$$

and derives equations for $\tilde{E}$ and I in terms of symmetric functions in the x_Q, the abscissas of points in F. (Plus more properties, like the isogeny is strict.)

How does an isogeny look like?

Extending Vélu, Dewaghe (for $E : Y^2 = X^3 + AX + B$):

$$D(x) = \prod_{Q \in F^*} (x - x_Q) = x^{\ell - 1} - \sigma x^{\ell - 2} + \cdots .$$

Fundamental proposition. The isogeny I can be written as

$$I(x, y) = \left(\frac{N(x)}{D(x)}, y \left(\frac{N(x)}{D(x)} \right)' \right),$$

$$\frac{N(x)}{D(x)} = \ell x - \sigma - (3x^2 + A) \frac{D'(x)}{D(x)} - 2(x^3 + Ax + B) \left(\frac{D'(x)}{D(x)} \right)'$$

$$= \ell x - \sigma - 2\sqrt{x^3 + Ax + B} \left(\sqrt{x^3 + Ax + B}\, \frac{D'(x)}{D(x)} \right)'.$$

Elkies92/98

1. Compute the h_i's of

$$\frac{N(x)}{D(x)} = x + \sum_{i\geq 1} \frac{h_i}{x^i}$$

in $O(\ell^2)$ operations using

$$(3x^2+A)\left(\frac{N(x)}{D(x)}\right)' + 2(x^3+Ax+B)\left(\frac{N(x)}{D(x)}\right)'' = 3\left(\frac{N(x)}{D(x)}\right)^2 + \tilde{A}.$$

2. deduce power sums p_i of $D(x)$ in $O(\ell)$ operations using also $\tilde{A}$ and $\tilde{B}$;

3. use fast Newton in $O(\mathsf{M}(\ell))$ to get $D(x)$.

$\Rightarrow$ very fast for small ℓ's.

Bostan/M./Salvy/Schost

Prop. $O(\mathsf{M}(\ell))$ method to get the h_i's given $\tilde{A}$, $\tilde{B}$, σ.

Some ideas: there exists a series $S(x)$ s.t.

$$\frac{N(x)}{D(x)} = \frac{1}{S\left(\frac{1}{\sqrt{x}}\right)^2}.$$

$$S(x) = x + \frac{\tilde{A}-A}{10}x^5 + \frac{\tilde{B}-B}{14}x^7 + O(x^9) \in x + x^3\mathbf{K}[[x^2]]$$

is such that

$$(Bx^6+Ax^4+1)\,S'(x)^2 = 1 + \tilde{A}\,S(x)^4 + \tilde{B}\,S(x)^6.$$

Use fast algorithm for solving this differential equation.

Rem. See *Math. Comp.* paper that includes survey of known methods for isogeny computations.

The case of finite fields of small characteristic

- **Couveignes:** formal groups; Artin-Schreier towers; time $\tilde{O}(\ell^2)$ but bad dependancy on p (see on-going work of L. De Feo).
- **Lercier/Joux** (2006): medium p using p-adic lifting.
- **Lercier/Sirvent** (2008): small p using p-adic lifting + BMSS $\Rightarrow$ complexity of $O(\mathsf{M}(\ell))$ in all cases.

B) The case $g = 2$

Probably not complete list:

- Gaudry+Schost: $\mathrm{Jac}(C) \to E_1 \times E_2$ for a $(2,2)$-isogeny of kerne $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
- $\ell = 2$ (AGM): Richelot, Humbert.
- $\ell \geq 3$: Dolgachev/Lehavi; general result for $F = (\mathbb{Z}/\ell\mathbb{Z})^2$; completely explicit for $\ell = 3$; more work needed for $\ell > 3$. Som hope?

C) And for $g = 3$?

Again, lack of general formulas:

- $\ell = 2$ (AGM): Donagi/Livné (+ negative results for $g > 3$); explicit methods by Lehavi + Ritzenthaler.
- Smith (Eurocrypt 2008):
  - $\varphi : \mathrm{Jac}(H) \to \mathrm{Jac}(C)$ where H is hyperelliptic and C smooth plane quartic;
  - intricate construction but relatively simple formulas in the end: uses Recilla's trigonal construction + theorem of Donagi and Livné;
  - works for 18.57% of smooth plane quartics;
  - nice crypto application (DL in $\mathrm{Jac}(C)$ easier than in $\mathrm{Jac}(H)$).

V. Conclusions

- $g = 1$: morally solved.
- $g > 1$:
  - scattered results;
  - curves are not so frequent and/or easy in higher genus;
  - objects are exponentially big (moduli space of hec has dim $g(g+1)/2$): even with sophisticated computer algebra techniques, this sounds difficult.