Computing isogenies in small or medium characteristic

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Motivation

The quest for efficient algorithms to perform computations in finite fields $\mathbb{F}_{p^n}$ (i.e. asymptotically fast as a function of $\log q$, $q = p^n$), is (too) often restricted to the two cases :

- n fixed and p tends to the infinity ($\mathbb{F}_p$ is typical) or
- p fixed and n tends to the infinity ($\mathbb{F}_{2^n}$ is typical).

Hence, until now, not that much people designed algorithms with reasonable behaviors when n and p both tend to ∞.

But, recently, new algorithms for computing discrete logarithms in $\mathbb{F}_{p^n}$ [JL06, JLSV06] yield surprisingly low complexities in this setting...

How about counting points on elliptic curves ?

Introduction

Let p be a prime, $\mathbb{F}_q$ a finite field with $q = p^n$ elements, E be an elliptic curve defined over $\mathbb{F}_q$. Let P be a point in $E(\overline{\mathbb{F}}_q)$ and denote with ϕ_q the Frobenius endomorphism, then $\phi_q(P) = P$ if and only if P is $\mathbb{F}_q$-rational.

How to efficiently compute $\#E(\mathbb{F}_q)$?

Thanks to Schoof, an algorithm with polynomial time complexity is known whatever the way p and n tend to ∞ [Schoof85, Schoof95].

Weil's dream

The general strategy is based on ideas introduced by Weil, Serre, Grothendieck, Dwork, . . . in order to prove the Weil conjectures.

The dream of Weil was to construct a good cohomology theory such that the number of fixed points of ϕ_q is given by a Lefschetz fixed point formula known in the complex setting as

$$\#\{P \in M \mid f(P) = P\} = \sum_i (-1)^i \operatorname{Tr}(f_* | H^i_{DR}(M)) .$$

(M be a compact complex analytic manifold, $f : M \to M$ an analytic map, f only has isolated non-degenerate fixed points, the $H^i_{DR}(M)$ are called the de Rham cohomology groups of M and are finite dimensional vector spaces over $\mathbb{C}$ on which f induces a linear map f^*)

Grothendieck's breakthrough

Very briefly. . . very restricted the elliptic curve case. . .

Let $\ell \neq p$ and let $\mathbb{Q}_\ell$ be the field of ℓ-adic numbers. Grothendieck introduced the ℓ-adic cohomology groups $H^i(E, \mathbb{Q}_\ell)$ s.t.

$$\#E(\mathbb{F}_q) = \sum_{i=0}^{2} (-1)^i \operatorname{Tr}(\phi_q^*; H^i(E, \mathbb{Q}_\ell)) .$$

The ℓ-adic cohomology groups $H^i(E, \mathbb{Q}_\ell)$ are finite dimensional vector spaces over $\mathbb{Q}_\ell$, which are non-trivial only for $i = 1$,

$$H^1(E, \mathbb{Q}_\ell) \cong T_\ell(E) .$$

Schoof's algorithm

Let $\ell \neq p$ be a prime, let $\chi_E(\phi_q)$ is the characteristic polynomial of ϕ_q on the Tate module $T_\ell(E)$. The main idea is to approximate $T_\ell(E)$ by the ℓ-torsion points $E[\ell]$.

The ℓ-torsion is a 2 dimensional $\mathbb{Z}/\ell\mathbb{Z}$ vector space, and the restriction of ϕ_q to $E[\ell]$ is linear. Let $P_\ell(T)$ denote the characteristic polynomial of this restriction, then $P_\ell(T) \equiv \chi_E(\phi_q)(T) \pmod{\ell}$.

Only one coefficient a_1 of $\chi_E(\phi_q)$ is needed and we have $|a_1| \leq q^{1/2}$ (Riemann hypothesis). Using the Chinese remainder theorem, we can therefore uniquely recover $\chi_E(\phi_q)$ from $P_\ell(T)$ for primes ℓ such that

$$\prod_{\text{primes } \ell, \gcd(\ell,q)=1} \ell > q^{1/2}.$$

This yields a $\widetilde{O}((\log q)^5)$ time complexity (with nothing in n or p hidden in the O) and $O((\log q)^3)$ in space.

Elkies' ideas

For primes ℓ s.t.

- there exists a rational isogeny of degree ℓ defined on E (half the primes),
- the kernel of which can be efficiently computed,

we can compute $P_\ell(T)$ on an (only) 1 dimensional $\mathbb{Z}/\ell\mathbb{Z}$ sub-vector space of $E[\ell]$...

...and we can hope a $\widetilde{O}((\log q)^4)$ time complexity.

Computing isogenies

There exists two classes of algorithms following the characteristic p.

- If $p > \ell$, first algorithms by Elkies, Charlap-Coley-Robbins, the best algorithms [BoMoSaSc06] have $\widetilde{O}((\ell \log q))$ time complexity.
- If $p < \ell$, the previous method yields obstructions. But, when p is fixed, we may consider three other algorithms, mainly :
  - [Couveignes94]. Time $\widetilde{O}(\ell^3 \log q)$, space $O(\ell^2 \log q)$.
  - [Couveignes96]. Time $\widetilde{O}(n\ell \log q)$, space $O(\ell^2 \log q)$.
  - [Lercier96]. For $p = 2$, heuristic time $O(n\ell^3)$, space $O(n\ell^2)$, more efficient for practical use.

Computing isogenies for $\ell \simeq p$

In the worst point counting situation, that is $p \simeq n(\simeq \log q \to \infty)$, one has to compute isogenies of degree $\ell \simeq p$.
[Lercier96] or [BoMoSaSc06] are not an option. It remains Couveignes algorithms but a careful look at their complexities reveal that we have bad powers of p "hidden" in the O constant.

For instance, in [Couveignes96],

- the complexity is at least the cost of computing an isomorphism between two Artin-Schreier extensions defined over $\mathbb{F}_q$,
- if we precompute the inverse of a $pn \times pn$ $\mathbb{F}_p$-matrix,
- then, each isogeny kernel computation involves a matrix-vector multiplication by such a matrix, that is $O((pn)^2)$ bit operations.

SEA algorithm runs thus in $\widetilde{O}(\log^5 q)$ bit operations ...again...

Canonical lift

At the end of 1999, Satoh introduced the p-adic approach to compute the number of points on an ordinary elliptic curve over a finite field [Satoh00].

Let $\overline{\mathcal{A}}$ be an abelian variety defined over $\mathbb{F}_q$ with $q = p^n$. Let $\mathbb{Q}_q$ be an unramified extension of $\mathbb{Q}_p$ of degree n with valuation ring $\mathbb{Z}_q$ and residue field $\mathbb{Z}_q/(p\mathbb{Z}_q) \simeq \mathbb{F}_q$. Consider a lift $\mathcal{A}$ of $\overline{\mathcal{A}}$ defined over $\mathbb{Z}_q$, then in general there exists none $\mathcal{F} \in \mathrm{End}(\mathcal{A})$ that reduces to the Frobenius $\phi_q \in \mathrm{End}(\overline{\mathcal{A}})$.

A canonical lift of an abelian variety $\overline{\mathcal{A}}$ over $\mathbb{F}_q$ is an abelian variety $\mathcal{A}$ over $\mathbb{Z}_q$ such that $\mathcal{A}$ reduces to $\overline{\mathcal{A}}$ modulo p and the ring homomorphism $\mathrm{End}(\mathcal{A}) \longrightarrow \mathrm{End}(\overline{\mathcal{A}})$ induced by reduction modulo p is an isomorphism.

Lubin–Serre-Tate Canonical lift

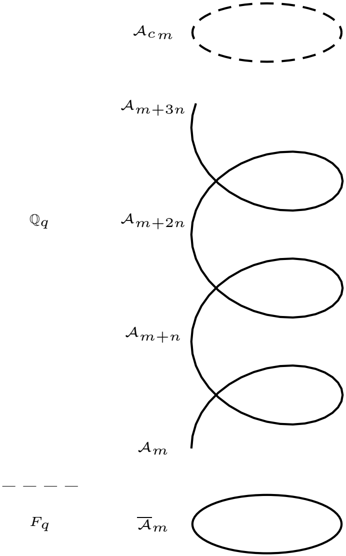
Theorem (Lubin–Serre-Tate) : Let $\overline{A}$ be an ordinary abelian variety over $\mathbb{F}_q$ (i.e. $\overline{\mathcal{A}}[p] \simeq (\mathbb{Z}/p\mathbb{Z})^{\dim(\overline{\mathcal{A}})}$). Then there exists a canonical lift $\mathcal{A}_c$ of $\overline{A}$ over $\mathbb{Z}_q$ and $\mathcal{A}_c$ is unique up to isomorphism.

The construction of a p-adic approximation of $\mathcal{A}_c$ given $\overline{\mathcal{A}}$ is as follows:

- Let $\mathcal{A}_0$ be a lift of $\overline{A}$ to $\mathbb{Z}_q$ and $\mathcal{A}_0[p]^{\mathrm{loc}} = \mathcal{A}_0[p] \cap \mathrm{Ker}(\pi_1)$ be the p-torsion points on $\mathcal{A}_0$ that reduce to the neutral element of $\overline{\mathcal{A}}$.
- Then, $\mathcal{A}_1 = \mathcal{A}_0/\mathcal{A}_0[p]^{\mathrm{loc}}$ is again an abelian variety s.t. its reduction is ordinary and there exists an isogeny $I_0 : \mathcal{A}_0 \longrightarrow \mathcal{A}_1$ which reduces to the small Frobenius morphism $\sigma : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}^{\sigma}$.
- Repeating this construction, we get a sequence of abelian varieties and isogenies $\mathcal{A}_0 \xrightarrow{I_0} \mathcal{A}_1 \xrightarrow{I_1} \dots$.

Clearly $\mathcal{A}_{kn}$ reduces to $\overline{\mathcal{A}}$ modulo p; furthermore, $\{\mathcal{A}_{kn}\}_{k\in\mathbb{N}}$ converges to the canonical lift $\mathcal{A}_c$ and the convergence is linear.

Riemann iterations



p fixed, $O(n^3)$ time complexity

Algorithm `AGM` [Mestre01]

Algorithm to compute the trace of an ordinary elliptic curve $E/\mathbb{F}_{2^n} : y^2 + xy = x^3 + \alpha$.

INPUT: $\alpha \in \mathbb{F}_{2^n}$.
OUTPUT: The trace c of E.

Lift phase

1. $a := 1 + 8\alpha \in \mathbb{Z}_q; b := 1 \in \mathbb{Z}_q;$
2. **for** $(i := 1; i < n/2 + O(1); i := i + 1)$ {
3. $\quad a, b := \frac{a+b}{2}, \sqrt{ab}$;
4. }

Norm phase

5. A := a; B := b;
6. **for** $(i := 1; i < n; i := i + 1)$ {
7. $\quad a, b := \frac{a+b}{2}, \sqrt{ab}$;
8. }
9. **return** $\frac{A}{a}$ mod 2^n as a signed integer in $[-2\sqrt{2^n}, 2\sqrt{2^n}]$.

$O(n^2)$ time complexity, but...

Thanks to numerous people in this field, when p id fixed, $O(n^2)$ time complexity can be achieved.

For non fixed p... one bottleneck is that we can not avoid the calculation of the p-torsion part of the curve and this involves the computation in the p-adics of the p-th division polynomial.

The best algorithms run finally in $\widetilde{O}(p^2 n^2)$ bit operations, but requires a $O(p^2 n^2)$ memory.

One may think to Kedlaya's algorithm in this setting, but again the complexity, both in time and space, is reported to be $\widetilde{O}(pn^3)$.

$\widetilde{O}(\log^4 q)$ bit operations, but a (too large) $O(\log^4 q)$ in storage too.

The idea

In our worst point counting case, $p \simeq n (\simeq \log q \to \infty)$, we would like to get rid of the obstructions which arise with isogeny algorithms for $\ell > p$.

In the same spirit as for counting points in the Satoh-Mestre like fashion, we propose to lift the isogeneous elliptic curves in an unramified extension denoted $\mathbb{Q}_q$ of the p-adic (corresponding to the extension $\mathbb{F}_q$ of $\mathbb{F}_p$).

So that the inversions by p which may occur in the isogeny algorithms are no longer a problem (at least for computations with with enough precision).

Algorithm `IsogenyLifted` [JL06a]

Algorithm to compute separable kernels of isogenies of degree ℓ

INPUT: An non-supersingular elliptic curve given over $\mathbb{F}_q$ by $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_x + a_6$ and an Elkies prime ℓ.
OUTPUT: Two polynomials in $\mathbb{F}_q[X]$ of degree $\lfloor \ell/2 \rfloor$ the roots of which are x-coordinates of points of $E[\ell]$.

1. Let $w = O(\ell/p)$ be a p-adic precision.

2. Lift E in $\mathbb{Q}_q$ in a arbitrary way.

3. Compute an isomorphic Weierstraß model $\mathcal{E} : y^2 = x^3 + A_4x + A_6$ isomorphic by λ to $E/\mathbb{Q}_q$.

4. Use Atkin-Elkies' algorithm to get at precision w in $\mathbb{Q}_q$ two isogeneous curves $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}'$.

5. Use Atkin-Elkies' algorithm to get at precision w in $\mathbb{Q}_q$ the sums p_1 and p_1'.

6. Take the isogeny algorithm of your choice to get from $(\tilde{\mathcal{E}}, p_1)$ and $(\tilde{\mathcal{E}}', p_1')$ two polynomials $\mathcal{H}_\ell(X)$ and $\mathcal{H}_\ell'(X)$.

7. **return** $\{\lambda^{-1}(\mathcal{H}_\ell(X)) \bmod p,\ \lambda^{-1}(\mathcal{H}'_\ell(X)) \bmod p\}$.

Complexity analysis

We studied the Charlap-Coley-Robbins algorithm in this case.

Mainly :

- the precision needed depends on the number of non invertible elements that the algorithm will encounter, that is

$$\lfloor 15 + 3\ell/2 \rfloor \text{ if } p = 2, 5 + \ell \text{ if } p = 3 \text{ and } \lfloor 1 + 2\ell/p \rfloor \text{ otherwise.}$$

- the complexity in time of the algorithm is still $\widetilde{O}(\ell^2)$ multiplications... but in $\mathbb{Q}_q$, at precision $\widetilde{O}(\ell/p)$. We therefore have a total complexity in time equal to $\widetilde{O}((1 + \ell/p)\ell^2 \log q)$. The complexity in space is $O((1 + \ell/p)\ell \log q)$.

Example, a degree 13 isogeny in $\mathbb{F}_{23}$

Let $E/\mathbb{F}_{23}\ : y^2 = x^3 + 6\,x + 17.$ We take as Weierstraß model isomorphic to E in $\mathbb{Q}_{23}$ at precision 2 the curve

$$\mathcal{E} : y^2 = x^3 + (6 + O(23^2))\,x + (17 + O(23^2)).$$

Atkin-Elkies' algrithms then enable us to find that $\mathcal{E}$ is 13-isogeneous to a curve approximated by $\tilde{\mathcal{E}} : y^2 = x^3 + (99 + O(23^2))\,x$.
Charlap-Coley-Robbins algorithm applied to these inputs yields

$$\begin{aligned}\mathcal{H}_{23}(X) = X^6 + (19 + O(23^2))\,X^5 - (50 + O(23^2))\,X^4 + (208 + O(23^2))\,X^3 \\ - (119 + O(23^2))\,X^2 - (252 + O(23^2))\,X - 231 + O(23^2).\end{aligned}$$

Reducing the result modulo 23, we finally find that

$$h_{23}(X) = X^6 + 19\,X^5 + 19\,X^4 + X^3 + 19\,X^2 + X + 22.$$

Conclusion

In our main concern, $p \simeq n \simeq \log q$, plugging this isogeny algorithm in the SEA framework yields, for the first time, a nice $\widetilde{O}(\log^4 q)$ complexity in time, and a at most $\widetilde{O}(\log^3 q)$ complexity in space.

With a more efficient isogeny algorithm, as the one of [BoMoSaSc06], we *might expect* to reduce the isogeny complexity phase as low as $\widetilde{O}((1 + \ell/p)\ell \log q)$, but we may not have such a low complexity for the overall computation, due to the computation of the isogenous curves in $\mathbb{Q}_q$, via modular polynomials.

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