Determining Ellipsoidal Outer and Inner Enclosures of Nonlinear Mappings of Ellipsoidal Domains

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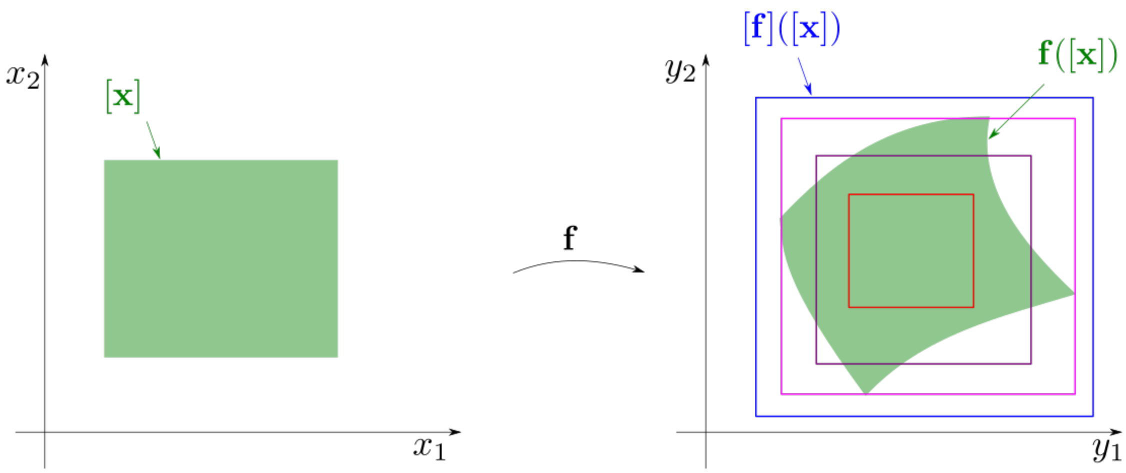
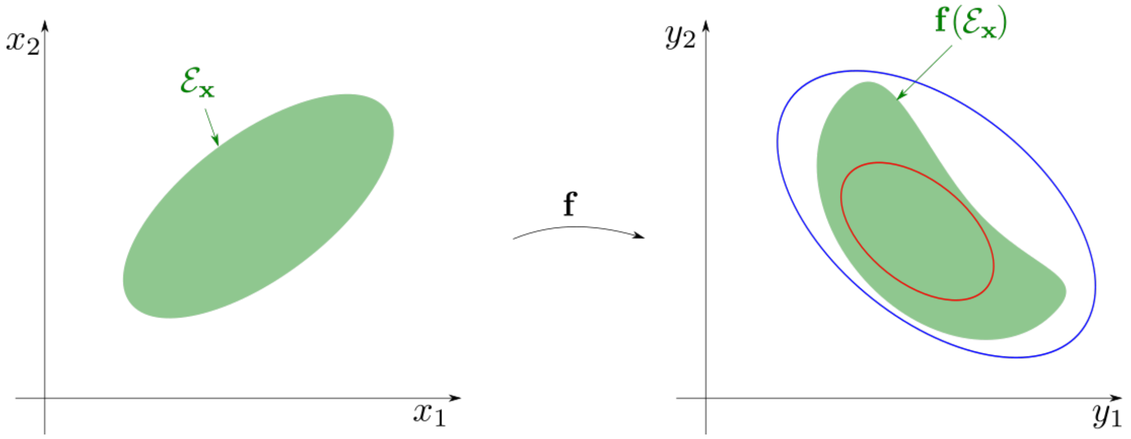
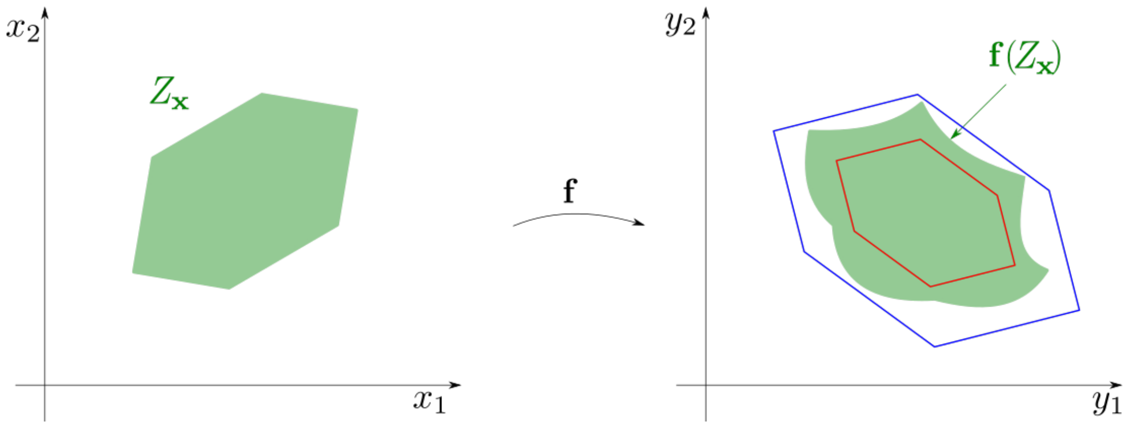
Wrappers

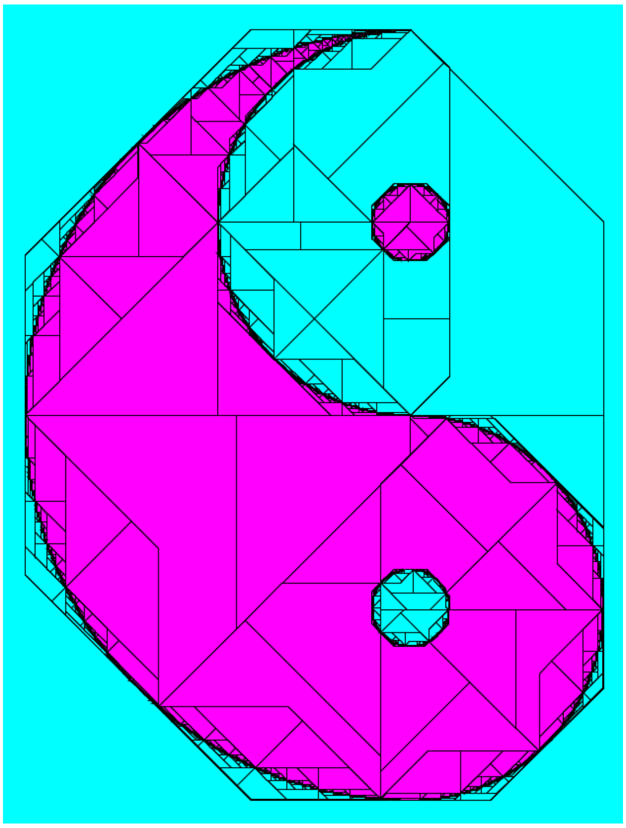
Consider a function $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$.
Set-membership technics propose methods to approximate $\mathbf{f}(\mathbb{X}), \mathbb{X} \subset \mathbb{R}^n$


x_2
$[\mathbf{x}]$
x_1
$\mathbf{f}$
$[\mathbf{f}]([\mathbf{x}])$
$\mathbf{f}([\mathbf{x}])$
y_2
y_1


x_2
$\mathcal{E}_\mathbf{x}$
x_1
$\mathbf{f}$
y_2
$\mathbf{f}(\mathcal{E}_\mathbf{x})$
y_1


x_2
$Z_{\mathbf{x}}$
x_1
$\mathbf{f}$
y_2
$\mathbf{f}(Z_{\mathbf{x}})$
y_1

	Boxes	Ellipsoids	octogons	polygons	LMI
Closed by $\cap$	Yes	No	Yes	No	Yes
Hull	Yes	No	Yes	No	No
Bissectable	Yes	No	Yes	Yes	Yes
Parametric	Yes	**Yes**	Yes	No	No
linear perfect	No	**Yes**	No	Yes	Yes
Stabilizing	No	**Yes**	No	Yes	Yes



Ellipsoidal enclosure: linear case

Consider the ellipsoid

$$\mathscr{E}_{\mathbf{x}} : (\mathbf{x}-\bar{\mathbf{x}})^{\mathsf{T}}\mathbf{Q}^{-1}(\mathbf{x}-\bar{\mathbf{x}}) \leq 1$$

Consider a function $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $\mathbf{y} = \mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{c}$ where $\mathbf{A}$ is invertible. Then

$$\mathbf{f}(\mathscr{E}_{\mathbf{y}}) : (\mathbf{y}-\bar{\mathbf{y}})^{\mathsf{T}}\mathbf{R}^{-1}(\mathbf{y}-\bar{\mathbf{y}}) \leq 1$$

with

$$\begin{aligned} \mathbf{R} &= \mathbf{A}\mathbf{Q}\mathbf{A}^{\mathsf{T}} \\ \bar{\mathbf{y}} &= \mathbf{f}(\bar{\mathbf{x}}) = \mathbf{A}\bar{\mathbf{x}} + \mathbf{c} \end{aligned}$$

Ellipsoidal enclosure: non-linear case

Proposition [3]. Consider $\mathbf{x}$ inside

$$\mathscr{E}_{\mathbf{x}} : (\mathbf{x}-\bar{\mathbf{x}})^{\mathsf{T}}\mathbf{Q}^{-1}(\mathbf{x}-\bar{\mathbf{x}}) \leq 1$$

Consider a function $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $\mathbf{y} = \mathbf{f}(\mathbf{x})$ where $\mathbf{A} = \frac{d\mathbf{f}}{d\mathbf{x}}(\bar{\mathbf{x}})$ is invertible. Then

$$(\mathbf{y}-\bar{\mathbf{y}})^{\mathsf{T}}\left((1+\rho)\mathbf{R}\right)^{-1}(\mathbf{y}-\bar{\mathbf{y}}) \leq 1$$

with

$$\begin{array}{ccc} \mathbf{R} & = & \mathbf{A}\mathbf{Q}\mathbf{A}^{\mathsf{T}} \\ \bar{\mathbf{y}} & = & \mathbf{f}(\bar{\mathbf{x}}) \\ \rho & = & \max_{\mathbf{x}\in\mathscr{E}_{\mathbf{x}}} 2\mathbf{b}(\mathbf{x})^{\mathsf{T}}\mathbf{R}^{-1}(\mathbf{f}(\mathbf{x})-\bar{\mathbf{y}}) \\ \mathbf{b}(\mathbf{x}) & = & \mathbf{f}(\mathbf{x}) - (\bar{\mathbf{y}} + \mathbf{A}\cdot(\mathbf{x}-\bar{\mathbf{x}})) \end{array}$$

Smallest box which encloses an ellipsoid

To enclose $\max_{\mathbf{x}\in\mathscr{E}_{\mathbf{x}}}$, we need first to find a box which encloses the ellipsoid $\mathscr{E}_{\mathbf{x}}$.

Consider the ellipsoid:

$$(\mathbf{x}-\bar{\mathbf{x}})^{\mathsf{T}}\,\mathbf{Q}^{-1}\,(\mathbf{x}-\bar{\mathbf{x}}) \leq 1$$

The smallest box which encloses $\mathscr{E}_{\mathbf{x}}$ is

$$\bar{\mathbf{x}} + \prod_{i\in\{1,\ldots,n\}} \sqrt{\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}\mathbf{e}_i}\cdot[-1,1]$$

Algorithm.

$$
\begin{array}{lcl}
\text{In:} & & \mathbf{f}, \mathscr{E}_{\mathbf{x}} : \{\bar{\mathbf{x}}, \mathbf{Q}_{\mathbf{x}}\} \\
\mathbf{A} & = & \frac{d\mathbf{f}}{d\mathbf{x}}(\bar{\mathbf{x}}) \\
\bar{\mathbf{y}} & = & \mathbf{f}(\bar{\mathbf{x}}) \\
\mathbf{R} & = & \mathbf{A}\mathbf{Q}_{\mathbf{x}}\mathbf{A}^{\mathsf{T}} \\
[\mathbf{x}] & = & \bar{\mathbf{x}} + \prod_{i \in \{1,\dots,n\}} \sqrt{\mathbf{e}_i^{\mathsf{T}}\mathbf{Q}_{\mathbf{x}}\mathbf{e}_i} \cdot [-1, 1] \\
[\mathbf{b}] & = & \left(\left[\frac{d\mathbf{f}}{d\mathbf{x}} \right]([\mathbf{x}]) - \mathbf{A} \right) \cdot ([\mathbf{x}] - \bar{\mathbf{x}}) \\
[\rho] & = & 2[\mathbf{b}]^{\mathsf{T}}\mathbf{R}^{-1}([\mathbf{f}]([\mathbf{x}]) - \bar{\mathbf{y}}) \\
\mathbf{Q}_{\mathbf{y}} & = & (1 + \rho^{+}) \cdot \mathbf{R} \\
\text{Out:} & & \mathscr{E}_{\mathbf{y}} : \{\bar{\mathbf{y}}, \mathbf{Q}_{\mathbf{y}}\}
\end{array}
$$

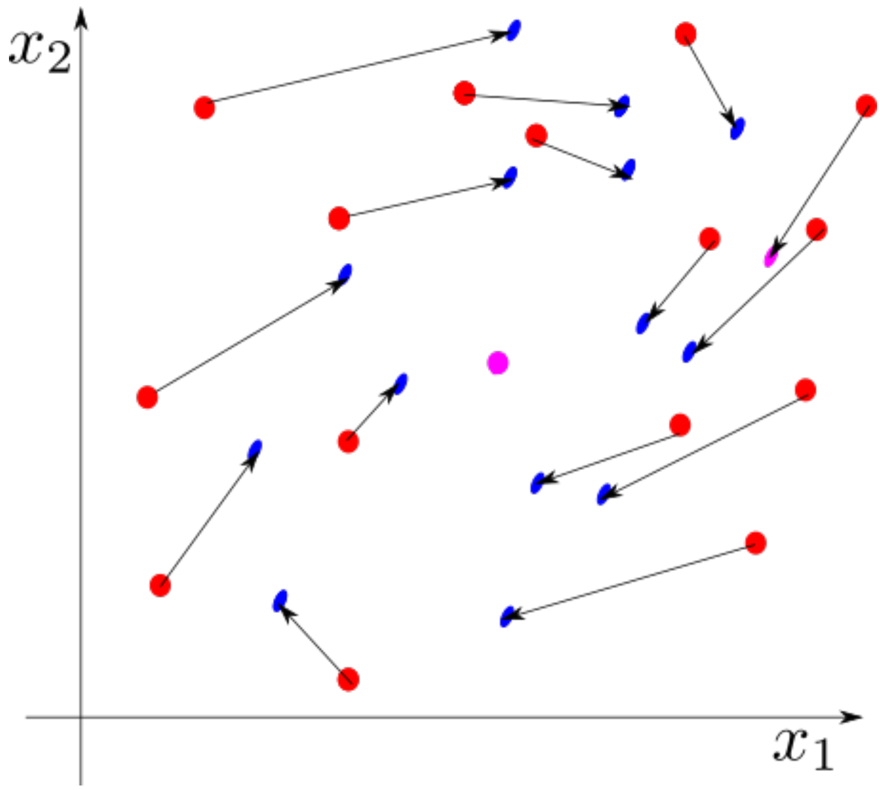
En inner ellipse can also be obtained [4]
The online Python program can be found here:
https://replit.com/@aulin/ellipse

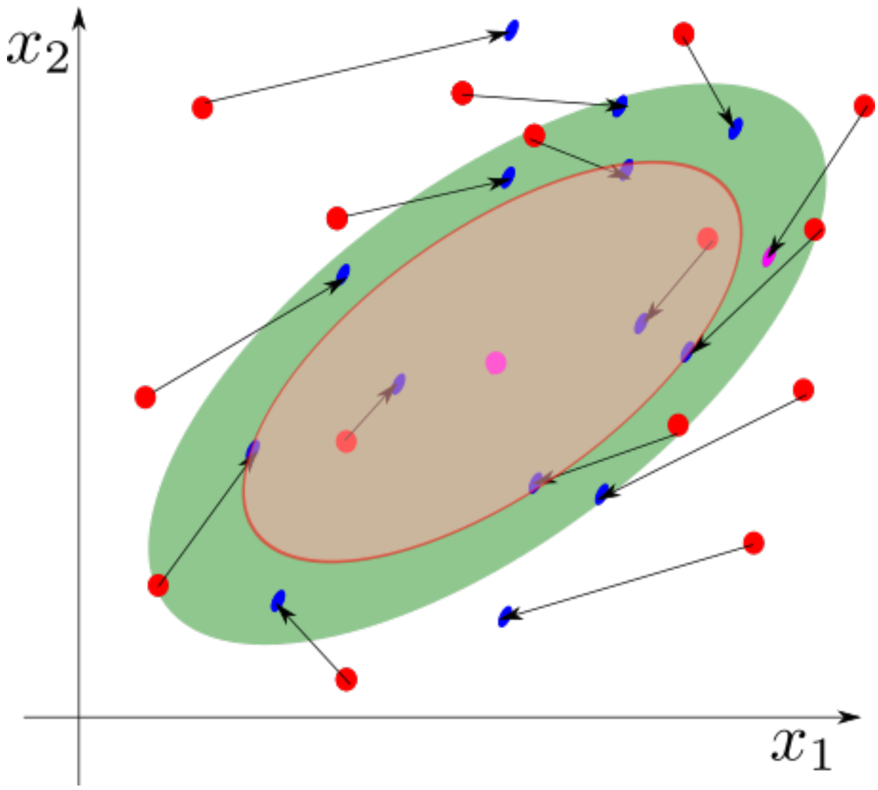
Positive invariant sets

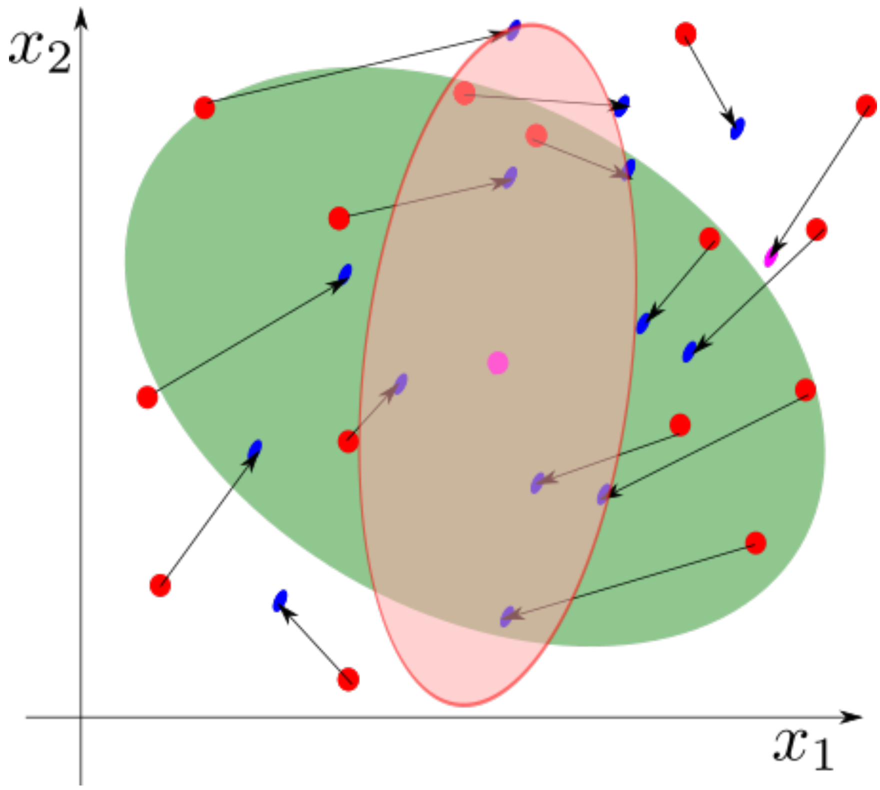
Consider the discrete time system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$$

with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$.


x_2
x_1


x_2
x_1


x_2
x_1

We have to find

$$\mathscr{E}_{\mathbf{x}} : \mathbf{x}^{\mathsf{T}} \cdot \mathbf{P} \cdot \mathbf{x} \leq \varepsilon$$

Such that

$$\mathbf{f}(\mathscr{E}_{\mathbf{x}}) \subset \mathscr{E}_{\mathbf{x}}$$

If the system is stable and linear

$$\mathbf{x}_{k+1} = \mathbf{A} \cdot \mathbf{x}_k$$

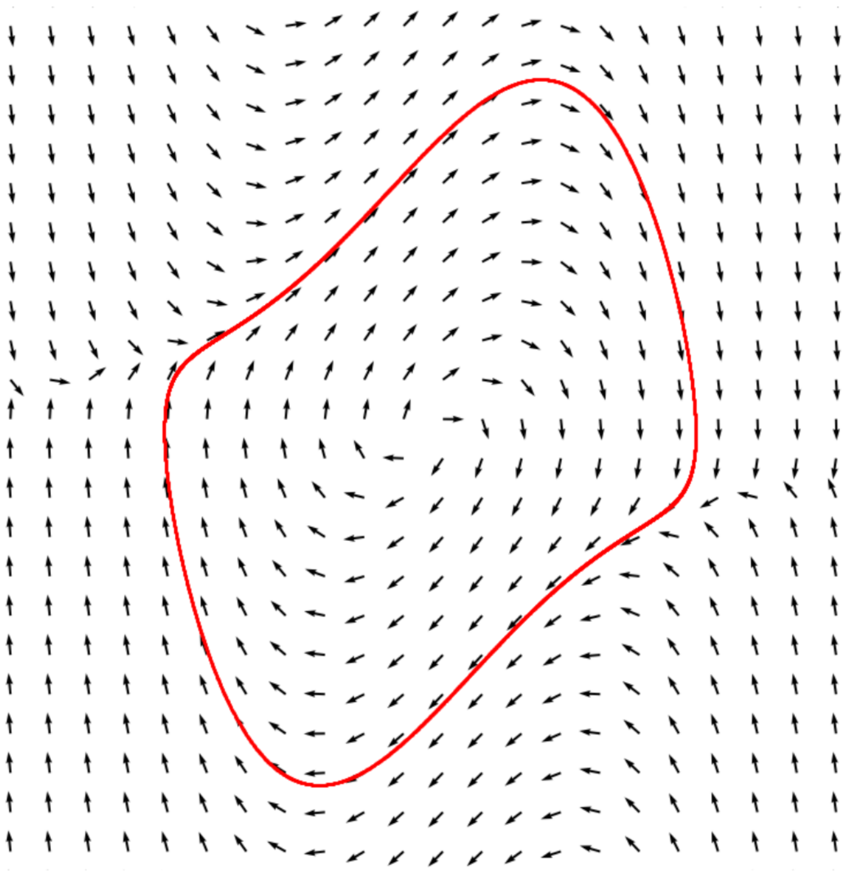
we can find $\mathbf{P} \succ 0$ such that $V(\mathbf{x}) = \mathbf{x}^\mathsf{T} \cdot \mathbf{P} \cdot \mathbf{x}$ is a Lyapunov function

$$\begin{aligned}
& V(\mathbf{x}_{k+1}) = V(\mathbf{x}_k) - \mathbf{x}_k^\mathsf{T}\mathbf{x}_k \\
\Leftrightarrow \quad & \mathbf{x}_{k+1}^\mathsf{T} \cdot \mathbf{P} \cdot \mathbf{x}_{k+1} = \mathbf{x}_k^\mathsf{T} \cdot \mathbf{P} \cdot \mathbf{x}_k - \mathbf{x}_k^\mathsf{T}\mathbf{x}_k \\
\Leftrightarrow \quad & \mathbf{x}_k^\mathsf{T} \cdot \mathbf{A}^\mathsf{T} \cdot \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{x}_k - \mathbf{x}_k^\mathsf{T} \cdot \mathbf{P} \cdot \mathbf{x}_k = -\mathbf{x}_k^\mathsf{T}\mathbf{x}_k
\end{aligned}$$

We have to solve the Lyapunov equation

$$\mathbf{A}^\mathsf{T} \cdot \mathbf{P} \cdot \mathbf{A} - \mathbf{P} = -\mathbf{I}$$

Stability of cycles



System: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
Poincaré section $\mathscr{G}$: $g(\mathbf{x}) = 0$

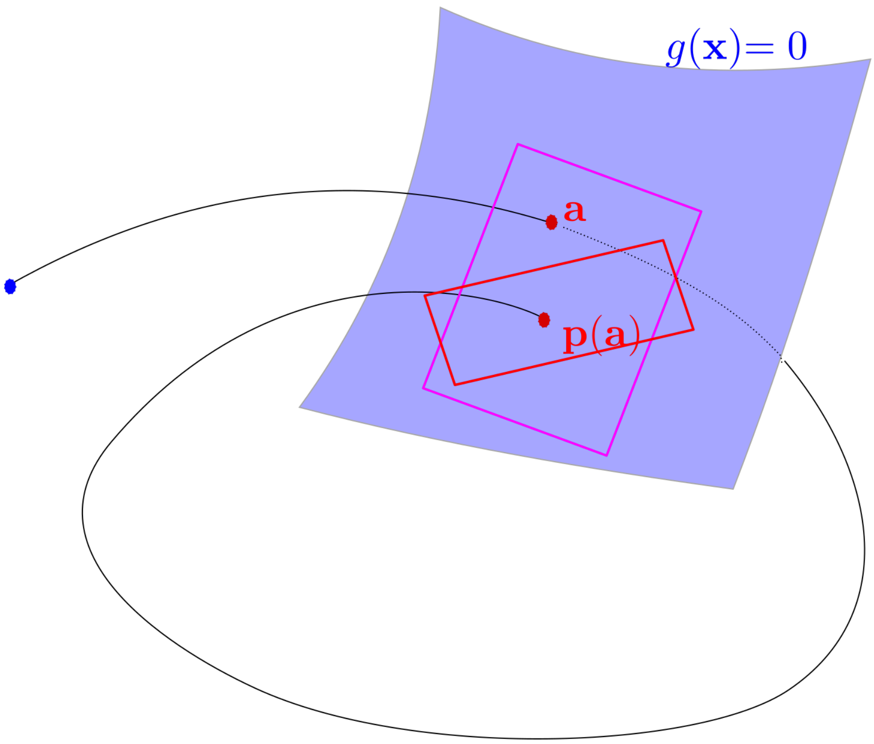
We define

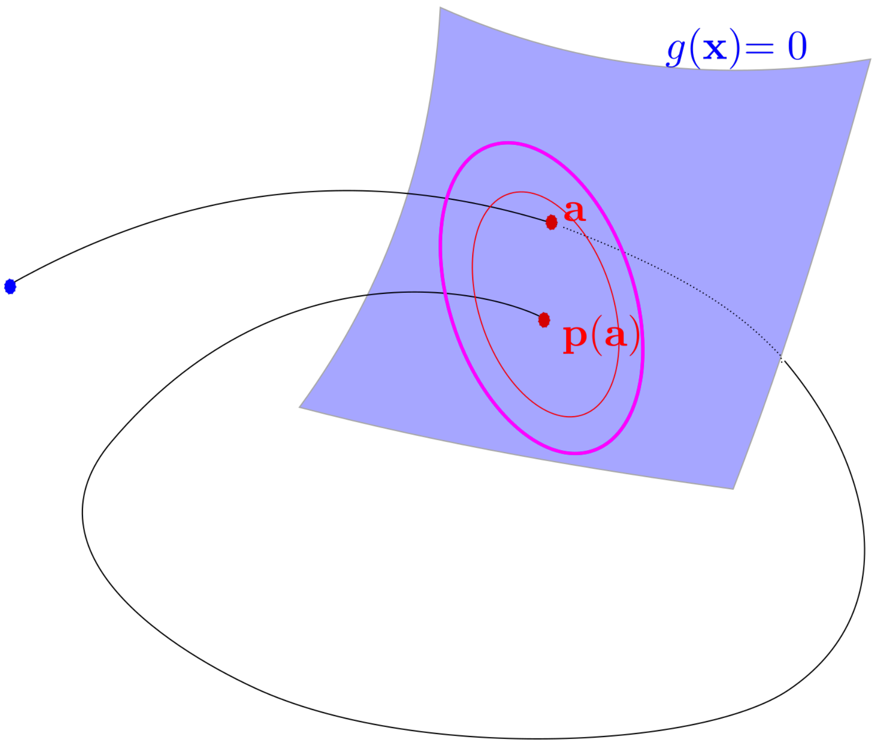
$$\mathbf{p}: \begin{array}{ccc} \mathscr{G} & \rightarrow & \mathscr{G} \\ \mathbf{a} & \mapsto & \mathbf{p}(\mathbf{a}) \end{array}$$

where $\mathbf{p}(\mathbf{a})$ is the point of $\mathscr{G}$ such that the trajectory initialized at $\mathbf{a}$ intersects $\mathscr{G}$ for the first time.

The Poincaré first recurrence map is defined by

$$\mathbf{a}(k+1) = \mathbf{p}(\mathbf{a}(k))$$


$g(\mathbf{x})=0$
$\mathbf{a}$
$\mathbf{p}(\mathbf{a})$



See [2]

Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

The flow is $\mathbf{\Phi}(\mathbf{x}_0, t)$.
Define $\mathbf{A}(\mathbf{x}_0, t) = \frac{\partial \mathbf{\Phi}(\mathbf{x}_0, t)}{\partial \mathbf{x}_0}$. We have the *variational equation*

$$\dot{\mathbf{A}} = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \cdot \mathbf{A}$$

with $\mathbf{A}(0) = \mathbf{I}$.

Example : Van der Pol system.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ (1 - x_1^2)x_2 - x_1 \end{pmatrix}.$$

We have

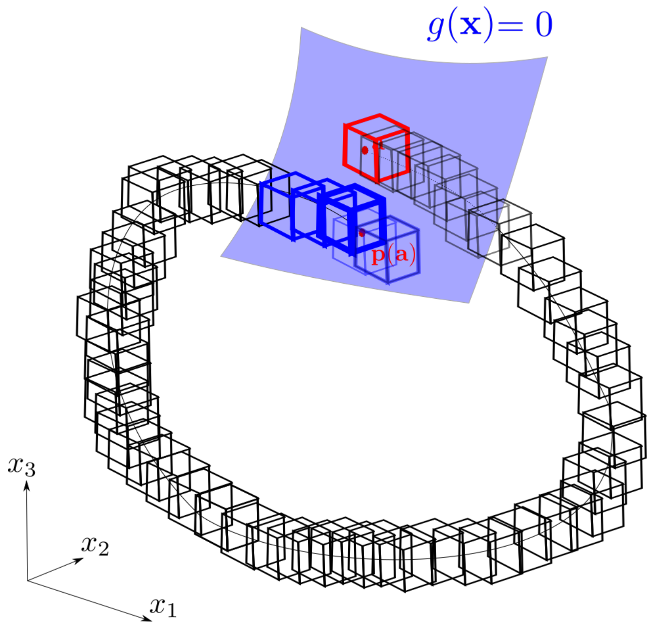
$$\begin{pmatrix} \dot{a}_{11} & \dot{a}_{12} \\ \dot{a}_{21} & \dot{a}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2x_1x_2 - 1 & 1 - x_1^2 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

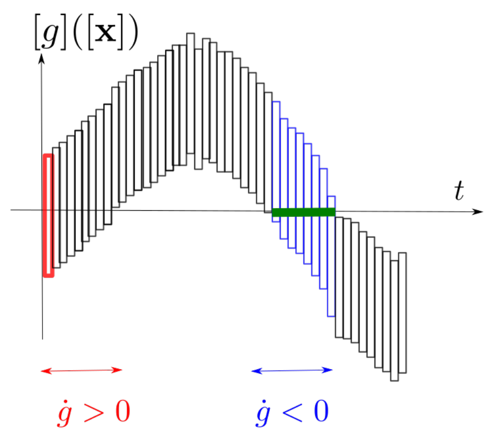
with

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{a}_{11} \\ \dot{a}_{12} \\ \dot{a}_{21} \\ \dot{a}_{22} \end{pmatrix} = \begin{pmatrix} x_2 \\ (1 - x_1^2)x_2 - x_1 \\ a_{21} \\ a_{22} \\ (-2x_1 x_2 - 1)\, a_{11} + \left(1 - x_1^2\right) a_{21} \\ (-2x_1 x_2 - 1)\, a_{12} + \left(1 - x_1^2\right) a_{22} \end{pmatrix}$$

Using an interval ODE solver, we get an enclosure for $\mathbf{x}(t), \mathbf{A}(t)$, for a given initial box $[\mathbf{x}_0]$.
We can also get the time at which the system crosses the surface $g(\mathbf{x}) = 0$.


$g(\mathbf{x})= 0$
$\mathbf{p}(\mathbf{a})$
x_3
x_2
x_1

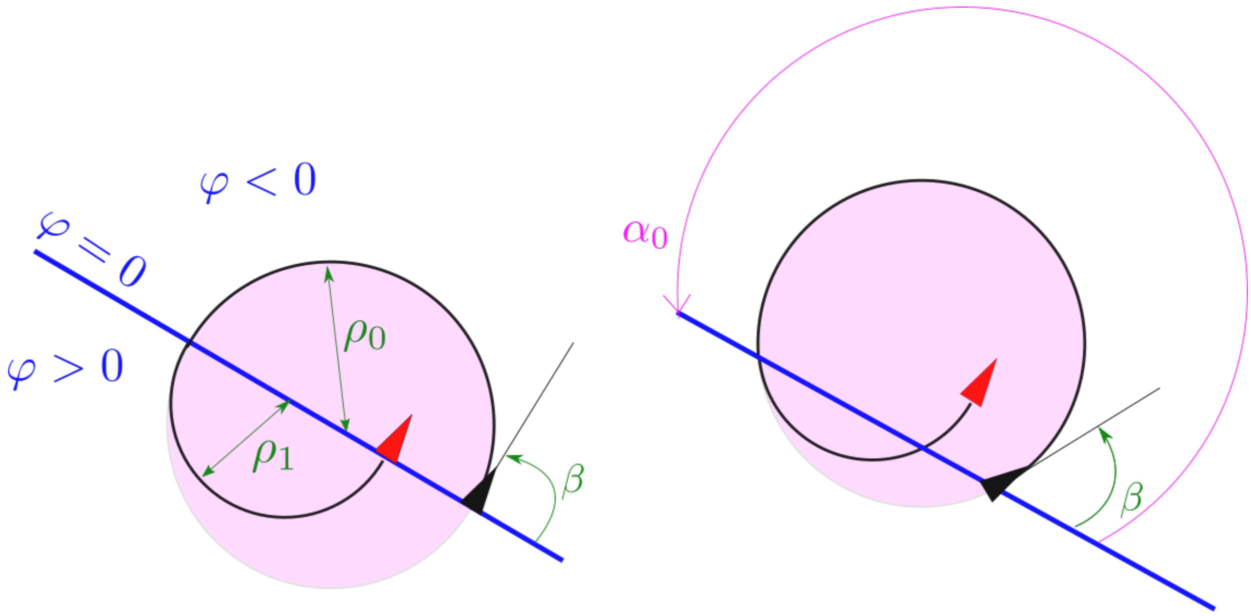

$[g]([\mathbf{x}])$
t
$\dot{g} > 0$
$\dot{g} < 0$

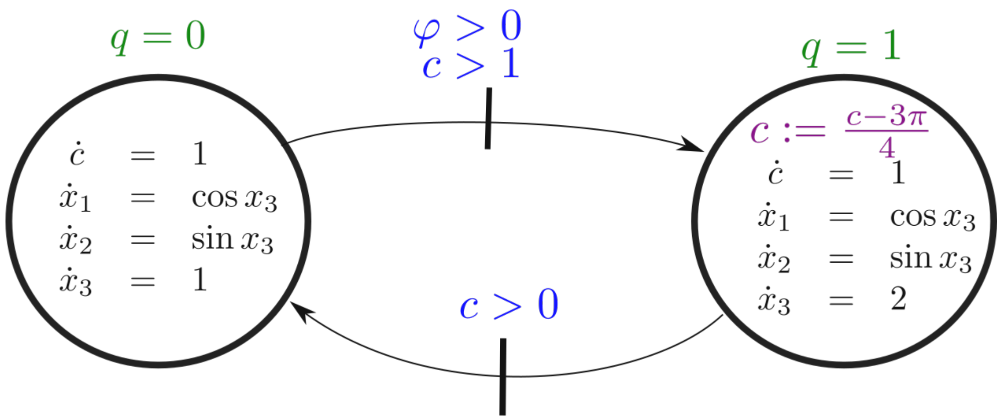
Rolling stability problem

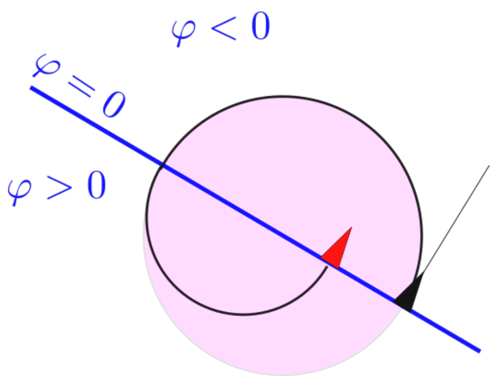
Robot moving on a plane described by

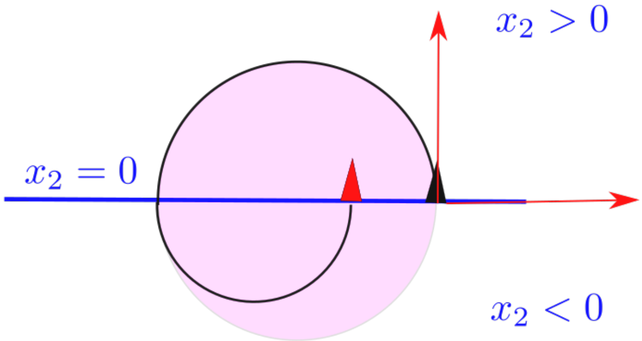
$$\begin{cases} \dot{x}_1 & = & \cos x_3 \\ \dot{x}_2 & = & \sin x_3 \\ \dot{x}_3 & = & u \end{cases}$$

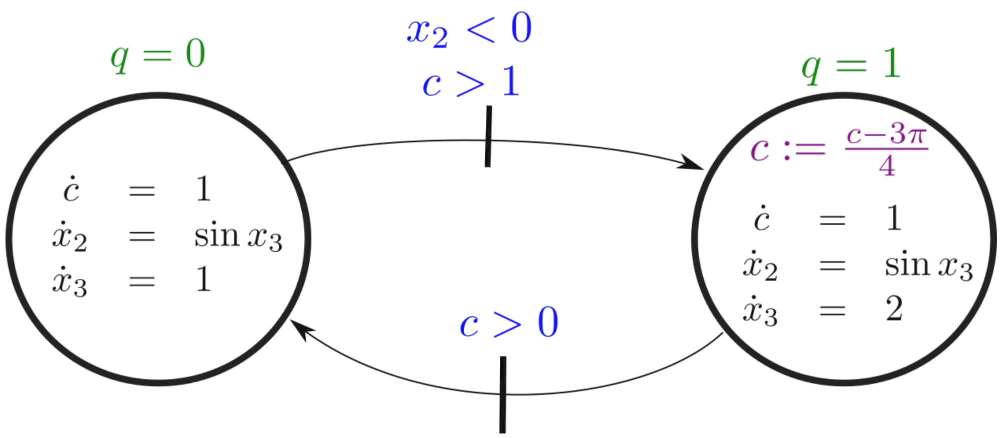
The robot is able to measure a function $\varphi(x_1, x_2)$ has to moves along $\varphi(x_1, x_2) = 0$. [1]


$\varphi < 0$
$\varphi = 0$
$\varphi > 0$
ρ_0
ρ_1
β
α_0
β


$q = 0$
$\varphi > 0$
$c > 1$
$q = 1$
$\dot{c} = 1$
$\dot{x}_1 = \cos x_3$
$\dot{x}_2 = \sin x_3$
$\dot{x}_3 = 1$
$c := \frac{c-3\pi}{4}$
$\dot{c} = 1$
$\dot{x}_1 = \cos x_3$
$\dot{x}_2 = \sin x_3$
$\dot{x}_3 = 2$
$c > 0$

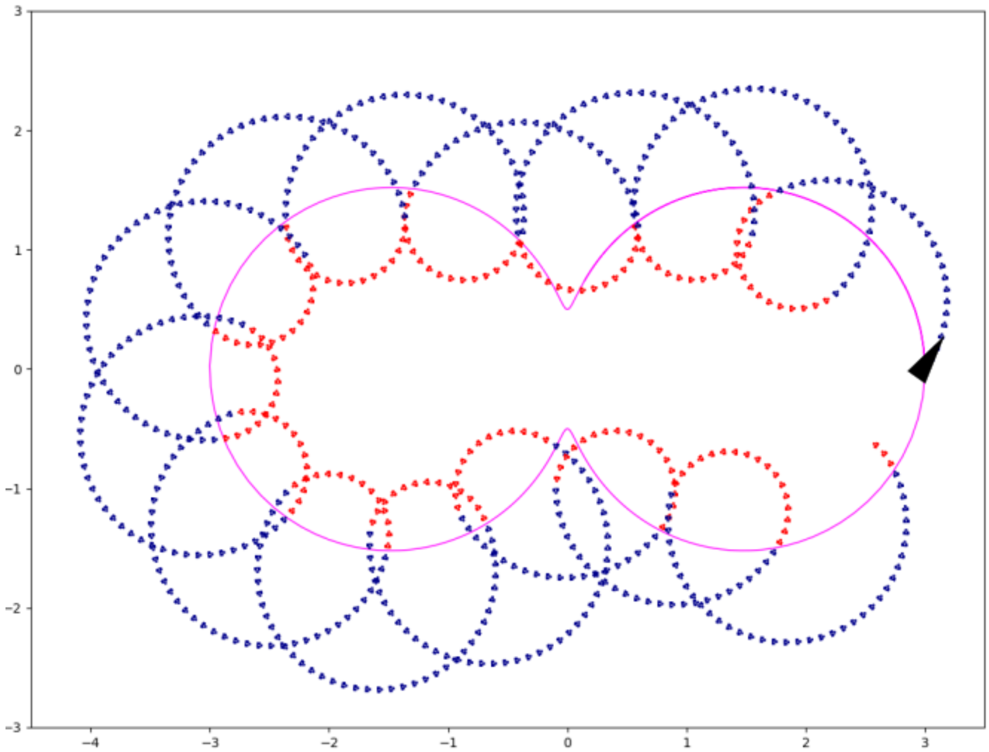

$\varphi < 0$
$\varphi = 0$
$\varphi > 0$


$x_2 > 0$
$x_2 = 0$
$x_2 < 0$


$q = 0$
$x_2 < 0$
$c > 1$
$q = 1$
$\dot{c} = 1$
$\dot{x}_2 = \sin x_3$
$\dot{x}_3 = 1$
$c := \frac{c - 3\pi}{4}$
$\dot{c} = 1$
$\dot{x}_2 = \sin x_3$
$\dot{x}_3 = 2$
$c > 0$

We consider the Hippopede of Proclus given by $\varphi(x_1, x_2) = 0$ where

$$\varphi(x_1, x_2) = 9x_1^2 + x_2^2 - \left(x_1^2 + y_2^2\right)^2.$$



The online Python program can be found here:
https://replit.com/@aulin/rolling

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