Voronoi tessellations in large random trees and random maps of finite excess

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joint work with Louigi Addario-Berry, Omer Angel, Guillaume Chapuy and Christina Goldschmidt

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Voronoi tesselations in measured metric spaces

Let \( X = (E, d, \mu) \) be a measured metric space (with \( \mu(E) = 1 \))

Consider \( k \) points \( p_1, \ldots, p_k \) in \( E \)

The space \( E \) is ‘partitioned’ into cells \( C_1, \ldots, C_k \) where

\[
C_i = \{ p \in X, \quad d(p, p_i) = \min d(p, p_j)_{j \in [1..k]} \}
\]

The corresponding Voronoi vector is

\[
\text{Vor}^{(k)} := (\mu(C_1), \ldots, \mu(C_k))
\]

(Rk: \( \mu(C_1) + \cdots + \mu(C_k) = 1 \) when cell intersections have zero measure)
The discrete case

Graph $G = \text{discrete metric space}$

$E = \text{vertex-set}$ \hspace{1cm} $d = \text{graph distance}$

$\mu = \frac{1}{|V|} \sum_{v \in V} \delta_v$

uniform distribution on vertex-set

possibly with edge-lengths
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$\mu = \frac{1}{|V|} \sum_{v \in V} \delta_v$

uniform distribution

$\text{Vor}^{(3)} = \left( \frac{8}{17}, \frac{6}{17}, \frac{5}{17} \right)$
Voronoi vector for random points in a metric space

Let $X = (E, d, \mu)$ be a fixed measured metric space.

Consider $k$ random points $p_1, \ldots, p_k$ in $E$ (chosen under $\mu$).

What is the distribution of the corresponding (random) vector $\text{Vor}^{(k)}$?

**Examples:** (for $k = 2$ and $n \to \infty$)

- **$n$-cycle**
  
  \[
  \sim \delta_{(1/2, 1/2)}
  \]

- **$n$-star**
  
  \[
  \sim \frac{1}{2} \delta_{(1, 0)} + \frac{1}{2} \delta_{(0, 1)}
  \]

  (closest to center wins all)
Let $X = (E, d, \mu)$ be a random metric space.

For $k \geq 2$ fixed, let $p_1, \ldots, p_k$ be random points of $X$.

Consider the associated Voronoi vector $\text{Vor}^{(k)} = (\mu(C_1), \ldots, \mu(C_k))$.

Which distribution can we have for the (doubly) random vector $\text{Vor}^{(k)}$?
Voronoi vector for a random metric space

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Which distribution can we have for the (doubly) random vector $\text{Vor}^{(k)}$?

The model is called **Voronoi-uniform** if $\text{Vor}^{(k)}$ is uniformly distributed on

$$\Delta_k := \{(x_1, \ldots, x_k), \quad x_i \geq 0, \quad \sum_{i=1}^{k} x_i = 1\}$$

(for $k = 2$ each component of $\text{Vor}^{(2)}$ has uniform law on $[0, 1]$)
Voronoi vector for a random metric space

- Let $X = (E, d, \mu)$ be a random metric space.
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(for $k = 2$ each component of $\text{Vor}^{(2)}$ has uniform law on $[0,1]$)

- Similarly a sequence $X_n$ of random discrete metric spaces is said to be **Voronoi-uniform** as $n \to \infty$ if the Voronoi vector $V_n^{(k)}$ satisfies

  $$\xrightarrow{\text{proba}} \quad \text{Uniform law on } \Delta_k$$

As it turns out, several models of random graphs have this behaviour.
Example for the complete graph

Consider the complete graph $K_n$ with $\text{Exp}(1)$ edge-lengths,

$$\forall e \in K_n, \quad P(\ell(e) \geq t) = e^{-t}$$

Then this model is Voronoi-uniform as $n \to \infty$

cf Pólya urn model
starting with one ball in each bag

- Grow the cells $C_1, \ldots, C_k$ (at unit speed) from $p_1, \ldots, p_k$
- at each time $t$ where a new vertex $v$ gets absorbed,
  it gets absorbed by cell $C_i$ with probability
  \[
  \frac{|C_i|}{|C_1| + \cdots + |C_k|}
  \]
- convergence of urn composition (as $n \to \infty$) to uniform law on $\Delta_k$
Results for random maps and trees

• For **random maps**:

**Conjecture:** [Chapuy’16]

For \( g \geq 0 \) let \( Q_n^{(g)} \) be the random bipartite quadrangulation of genus \( g \) with \( n \) faces. Then \( Q_n^{(g)} \) is Voronoi-uniform when \( n \to \infty \)

\[ \iff \] continuum limit (Brownian map in genus \( g \)) is Voronoi-uniform
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supported by proof (using the “\( t_g \)-recurrence”) that for all \( g \geq 0 \)

\[ \mathbb{E}(\text{Vor}_1^{(2)} \cdot \text{Vor}_2^{(2)}) = \frac{1}{6} \quad \mathbb{E}(\text{Vor}_1^{(3)} \cdot \text{Vor}_2^{(3)} \cdot \text{Vor}_3^{(3)}) = \frac{1}{60} \]
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Recent proof for \( g = 0 \) and \( k = 2 \) [Guitter’17]
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• For random trees and random unicellular maps:

Theorem: [Addario-Berry,Angel,Chapuy,F,Goldschmidt’18]

For $g \geq 0$ let $U_n^{(g)}$ be the random unicellular map of genus $g$ with $n$ edges. Then $U_n^{(g)}$ is Voronoi-uniform when $n \to \infty$

\[ \Leftrightarrow \text{the continuum limit is Voronoi-uniform (CRT for genus } 0) \]
Results for random maps and trees

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**Conjecture:** [Chapuy’16]

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\[ \Leftrightarrow \text{continuum limit (Brownian map in genus } g \text{) is Voronoi-uniform} \]

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• For random trees and random unicellular maps:

**Theorem:** [Addario-Berry,Angel,Chapuy,F,Goldschmidt’18]

For \( g \geq 0 \) let \( U^{(g)}_n \) be the random unicellular map of genus \( g \) with \( n \) edges

Then \( U^{(g)}_n \) is Voronoi-uniform when \( n \to \infty \)

\[ \Leftrightarrow \text{the continuum limit is Voronoi-uniform (CRT for genus } 0 \text{)} \]

(also holds for random unicellular maps on non-orientable surfaces)
Distance between 2 random points in random trees

Consider a random plane tree on \( n \) edges with two marked corners.

Distance \( D_n = 3 \)
Consider a random plane tree on \( n \) edges with two marked corners

\[
\text{distance } D_n = 3
\]

- Distribution of \( D_n \):
  \[
P(D_n = \ell) = \frac{a_{n, \ell}}{a_n}
\]

where \( a_{n, \ell} = \# \text{ trees on } n \text{ edges with 2 marked corners at distance } \ell \)

\( a_n = \# \text{ trees on } n \text{ edges with 2 marked corners} \)
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\[
\begin{align*}
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  \( a_n = \# \) trees on \( n \) edges with 2 marked corners

Then \( a_n = \text{Cat}_n \cdot (2n - 1) \sim \frac{2 \cdot 4^n}{\sqrt{\pi n}} \)

And Lagrange inversion \( \Rightarrow \) \( a_{n,\ell} = \frac{\ell + 1}{n} \binom{2n + 2}{n + \ell} \)
Consider a random plane tree on $n$ edges with two marked corners.

- Distribution of $D_n$: $P(D_n = \ell) = \frac{a_{n,\ell}}{a_n}$

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  $\sim \frac{4^{n+1}}{n \sqrt{\pi}} xe^{-x^2}$ for $\frac{\ell}{\sqrt{n}} \to x$
Distance between 2 random points in random trees

Consider a random plane tree on $n$ edges with two marked corners

![Diagram of a random tree with two marked corners](image)

- Distance $D_n = 3$

- Distribution of $D_n$: $P(D_n = \ell) = \frac{a_{n,\ell}}{a_n}$

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  $a_n = \#$ trees on $n$ edges with 2 marked corners

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  $\sim \frac{4^{n+1}}{n \sqrt{\pi}} xe^{-x^2}$ for $\frac{\ell}{\sqrt{n}} \to x$

  $\Rightarrow \frac{a_{n,\ell}}{a_n} \sim \frac{1}{\sqrt{n}} 2xe^{-x^2}$ hence $\frac{D_n}{\sqrt{n}} \xrightarrow{\text{proba}} \text{law of density } 2xe^{-x^2}$ (Rayleigh law)
Joint law for the distance and Voronoi masses

$n_1$ vertices

$n_2$ vertices

$n_1 + n_2 = n$

$n_1 = \alpha \cdot n$

$\ell = x \cdot \sqrt{n}$
Joint law for the distance and Voronoi masses

\[ \text{\# such configurations} = a_{n_1,\ell} \cdot a_{n_2,\ell} \]

\[ \sim \frac{4^n}{\pi n_1 n_2} x_1 x_2 e^{-x_1^2 - x_2^2} \]

\[ \text{with} \left\{ \begin{array}{l}
 x_1 = \frac{\ell}{\sqrt{n_1}} = \frac{x}{\sqrt{\alpha}} \\
 x_2 = \frac{\ell}{\sqrt{n_2}} = \frac{x}{\sqrt{(1-\alpha)}}
\end{array} \right. \]

\[ n_1 + n_2 = n \]
\[ n_1 = \alpha \cdot n \]
\[ \ell = x \cdot \sqrt{n} \]
Joint law for the distance and Voronoi masses

\[ n_1 \text{ vertices} \quad \begin{array}{c}
1 \\
\ell \\
2
\end{array} \quad \begin{array}{c}
\n_2 \text{ vertices} \\
\ell
\end{array} \quad n_1 + n_2 = n \]

\[ n_1 = \alpha \cdot n \quad \ell = x \cdot \sqrt{n} \]

\# such configurations = \( a_{n_1, \ell} \cdot a_{n_2, \ell} \)

\[ \sim \frac{4^n}{\pi n_1 n_2} x_1 x_2 e^{-x_1^2 - x_2^2} \quad \text{with} \quad \begin{cases} 
\frac{x_1}{\sqrt{n_1}} = \frac{\ell}{\sqrt{\alpha}} = \frac{x}{\sqrt{\alpha}} \\
\frac{x_2}{\sqrt{n_2}} = \frac{\ell}{\sqrt{1-\alpha}} = \frac{x}{\sqrt{(1-\alpha)}} 
\end{cases} \]

\[ \sim \frac{4^n}{n^2 \pi} \cdot \frac{x^2}{(\alpha(1-\alpha))^{3/2}} \exp \left( - \frac{x^2}{\alpha(1-\alpha)} \right) \]
Joint law for the distance and Voronoi masses

\( n_1 \) vertices \hspace{2cm} \( n_2 \) vertices

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\[ \Rightarrow \text{convergence to joint density} \quad f(\alpha, x) = \frac{4x^2}{\sqrt{\pi}(\alpha(1-\alpha))^{3/2}} \exp \left( - \frac{x^2}{\alpha(1-\alpha)} \right) \]
Joint law for the distance and Voronoi masses

\[ n_1 \text{ vertices} \quad | \quad n_2 \text{ vertices} \]

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\[ \# \text{ such configurations} = a_{n_1,\ell} \cdot a_{n_2,\ell} \]
\[ \sim \frac{4^n}{\pi n_1 n_2} x_1 x_2 \ e^{-x_1^2 - x_2^2} \]
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\[ \text{Rk: } \forall \alpha \in (0, 1), \quad \int_{-\infty}^{+\infty} f(\alpha, x)dx = 1 \quad \text{cf change of variable} \quad u = \frac{x}{\sqrt{\alpha(1-\alpha)}} \]

\[ \Rightarrow \text{marginal law in } \alpha \text{ is uniform on } [0, 1] \Rightarrow \text{uniformity for random trees} \]

\[ \text{case } k = 2 \]
Bijective approach

Let $A^{(k)}_n := \text{set of trees on } n \text{ edges with } k \text{ marked corners}$

**Voronoi partition**

$\text{Vor}^{(k)} = (\mu(C_1), \ldots, \mu(C_k))$

**Contour partition**

$\text{Int}^{(k)} = \frac{1}{2n} (\text{length}(I_1), \ldots, \text{length}(I_k))$

**Idea:** Find a bijection $\Phi$ from $A^{(k)}_n$ to itself such that

for $T' = \phi(T)$ one has $\text{Int}^{(k)}(T') = \text{Vor}^{(k)}(T)$ (up to $o(1)$ corrections)

This will prove uniformity, since clearly for $T'$ taken at random in $A^{(k)}_n$

$\text{Int}^{(k)}(T') \xrightarrow{\text{proba}} \text{Uniform law on } \Delta_k$
The bijection $\phi$ permutes the attached subtrees

For $i \in \{1, 2\}$ each attached tree in $C_i^<$ gets moved to $I_i$

$\Rightarrow$ In proba we have $\mu(C_i) \sim \mu(I_i) \sim \frac{1}{2n} \text{length}(I_i) \rightarrow \text{Unif}(0, 1)$
Bijection for $k \geq 3$ (induction on $k$)
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if min=0 or min is not unique, the bijection fails
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Summary of the results for trees

Let $\mathcal{A}_n^k$ be the set of trees with $n$ edges and $k$ marked corners. Then there is a subfamily $\mathcal{B}_n^k \subset \mathcal{A}_n^k$ (no-failure case) such that

$$|\mathcal{B}_n^k| = |\mathcal{A}_n^k| \cdot (1 - O(n^{-1/2}))$$

and a bijection $\phi$ from $\mathcal{B}_n^k$ to itself that permutes the attached subtrees.

For $T$ random in $\mathcal{A}_n^k$, $\text{Vor}^{(k)}(T) \sim \text{Int}^{(k)}(T) \sim \text{Uniform law on } \Delta_k$

so that for $i \in [1..k]$ each attached subtree in $C_i <$ gets moved to $I_i$.
Induced results

The CRT is the continuum limit of random trees (with edge lengths $\sqrt{n}$)

So the **CRT is Voronoi-uniform**
Induced results

The CRT is the continuum limit of random trees (with edge lengths $/\sqrt{n}$)

So the **CRT is Voronoi-uniform**

⇒ any model of random graphs converging to the CRT is Voronoi-uniform as $n \to \infty$

This includes

- random dissections of an $n$-gon
  - [Curien, Haas, Kortchemski’14] [Bettinelli’17]

- random outerplanar maps with $n$ edges
  - [Caraceni’16] [Stufler’17]

- random stacked triangulations of $n$ vertices
  - [Albenque, Marckert’08]

- random graphs of size $n$ from a subcritical family (outerplanar graphs, series-parallel graphs)
  - [Panagiotou, Stufler, Weller’14] [Curien, Haas, Kortchemski’14] [Bettinelli’17] [Stufler’17]
Proof of uniformity directly on the CRT

Let $T$ be a CRT with $k$ random points $p_1, \ldots, p_k$

To prove that $T$ is Voronoi-uniform, we have to prove that

(i) for every $k \geq 2$, $\text{Vor}^{(k)}(T)$ and $\text{Int}^{(k)}(T)$ are equidistributed
Proof of uniformity directly on the CRT

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To prove that $T$ is Voronoi-uniform, we have to prove that

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Let $S$ be the skeleton of $T$ ($k$-leaf binary tree with random edge-lengths)
Proof of uniformity directly on the CRT

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To prove that $T$ is Voronoi-uniform, we have to prove that

(i) for every $k \geq 2$, $\text{Vor}^{(k)}(T)$ and $\text{Int}^{(k)}(T)$ are equidistributed

Let $S$ be the skeleton of $T$ ($k$-leaf binary tree with random edge-lengths)

To prove (i), it is enough to prove

(ii) for every $k \geq 2$, $2\text{Vor}^{(k)}(S)$ and $\text{Int}^{(k)}(S)$ are equidistributed
Proof of uniformity directly on the CRT

Let $T$ be a CRT with $k$ random points $p_1, \ldots, p_k$.

To prove that $T$ is Voronoi-uniform, we have to prove that

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To prove (i), it is enough to prove

(ii) for every $k \geq 2$, $2\operatorname{Vor}^{(k)}(S)$ and $\operatorname{Int}^{(k)}(S)$ are equidistributed

Proof for $k = 2$
Proof of uniformity directly on the CRT

Proof (by induction on $k$) that $2\text{Vor}^{(k)}(S)$ and $\text{Int}^{(k)}(S)$ are equidistributed.

$R_k$: can rescale $S$ so that the edge-lengths are independent $\text{Exp}(1)$-laws.

$\text{induction on } k$
Extension to maps of finite excess

For $g \geq 0$ and $k = (k_1, \ldots, k_r)$ with $k_i \geq 1$

$\mathcal{M}_{n}^{(k,g)} := \text{set of maps of genus } g \text{ with } n \text{ edges and } r \text{ faces } f_1, \ldots, f_r$

where in each face $f_i$ there are $k_i$ marked corners $c_{i,1}, \ldots, c_{i,k_i}$

$g = 0, \ r = 2, \ k = (2, 3)$
Extension to maps of finite excess

For $g \geq 0$ and $k = (k_1, \ldots, k_r)$ with $k_i \geq 1$

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where in each face $f_i$ there are $k_i$ marked corners $c_{i,1}, \ldots, c_{i,k_i}$

Voronoi partition

Contour partition

$N = k_1 + \cdots + k_r$ (total number of marked corners)

$\{p_i\}_{1 \leq i \leq N} = \{\text{vertices at marked corners in } f_1, \ldots, f_r\}$

Voronoi vector $\text{Vor} := (\mu(C_1), \ldots, \mu(C_N))$

$I_{i,j} := j\text{th interval in } f_i$

vector $U_i := \frac{1}{2n}(\text{length}(I_{i,1}), \ldots, \text{length}(I_{i,k_i}))$

Interval vector $\text{Int} := \text{concatenate } U_1; U_2; \ldots; U_r$
Result: For $g \geq 0$ and $k = (k_1, \ldots, k_r)$
there is a subfamily $\mathcal{B}_n^{(g,k)} \subset \mathcal{M}_n^{(g,k)}$ with $|\mathcal{B}_n^{(g,k)}| \sim |\mathcal{M}_n^{(g,k)}|$
and a bijection $\phi$ from $\mathcal{B}_n^{(g,k)}$ to itself such that for $M' = \phi(M)$
we have $\text{Int}(M') = \text{Vor}(M)$ up to $o(1)$ error terms.
Extension to maps of finite excess

Result: For $g \geq 0$ and $k = (k_1, \ldots, k_r)$, there is a subfamily $B_{n}^{(g, k)} \subset \mathcal{M}_{n}^{(g, k)}$ with $|B_{n}^{(g, k)}| \sim |\mathcal{M}_{n}^{(g, k)}|$ and a bijection $\phi$ from $B_{n}^{(g, k)}$ to itself such that for $M' = \phi(M)$ we have $\text{Int}(M') = \text{Vor}(M)$ up to $o(1)$ error terms.

- Random maps in $\mathcal{M}_{n}^{(g, k)}$ have a scaling limit called the CRM$(g, k)$.

  The bijection implies that Vor and Int are equidistributed in the CRM$(g, k)$.
Extension to maps of finite excess

Result: For \( g \geq 0 \) and \( k = (k_1, \ldots, k_r) \), there is a subfamily \( B_n^{(g,k)} \subset M_n^{(g,k)} \) with \( |B_n^{(g,k)}| \sim |M_n^{(g,k)}| \) and a bijection \( \phi \) from \( B_n^{(g,k)} \) to itself such that for \( M' = \phi(M) \) we have \( \text{Int}(M') = \text{Vor}(M) \) up to \( o(1) \) error terms.

- Random maps in \( M_n^{(g,k)} \) have a scaling limit called the \( \text{CRM}^{(g,k)} \).
  The bijection implies that \( \text{Vor} \) and \( \text{Int} \) are equidistributed in the \( \text{CRM}^{(g,k)} \).

- This can be proved directly, on the associated skeleton and using induction.

3 cases for the skeleton (surrounded the leaf with shortest incident edge): cut, split, merge.
Induced results

For \( g \geq 0 \) and \( k = (k_1, \ldots, k_r) \)
the 2 vectors \( \text{Vor} \) and \( \text{Int} \) are equidistributed in the \( \text{CRM}^{(g,k)} \)

- **Case \( r = 1 \) (unicellular maps)**
  \( \text{Int} \) is uniformly distributed on \( \Delta_k \), hence so is \( \text{Vor} \)
  \( \Rightarrow \) the \( \text{CRUM}_g \) is Voronoi-uniform

- **Case \( k_1 = 1, \ldots, k_r = 1 \) (one marked corner in each face)**
  for a random map in \( \mathcal{M}^{(g,k)} \), \( \text{Vor} \sim \frac{1}{2n} \cdot (\text{deg}(f_1), \ldots, \text{deg}(f_r)) \)
  \( g = 0 \): Tutte’s slicings formula gives \( \text{Vor} \sim \text{density} \propto x_1^{1/2} \cdots x_r^{1/2} \) on \( \Delta_r \)
  (Dirichlet \( \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \))