

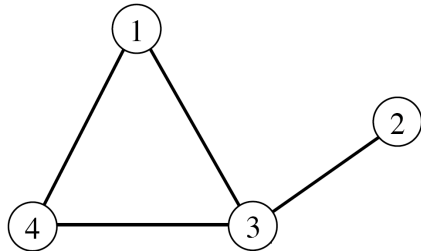
Asymptotic enumeration in unlabelled subcritical graph families

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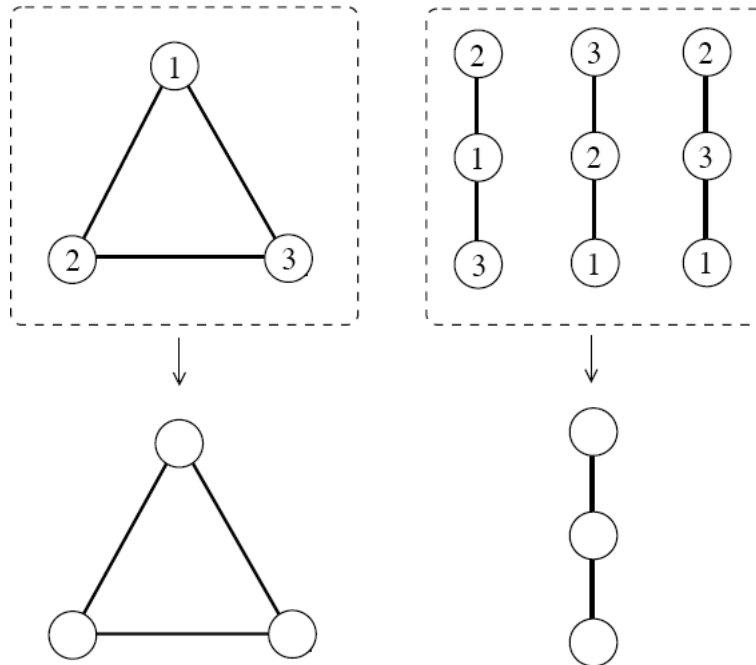
Graphs: labelled/unlabelled

- Graphs are classically labelled at vertices



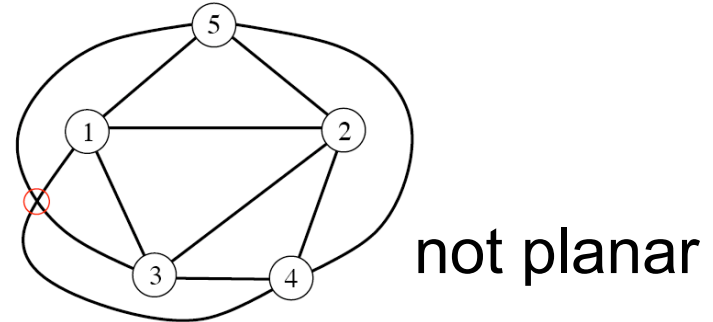
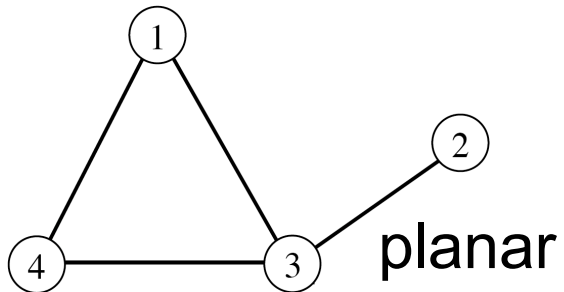
a graph on 4 vertices

- Unlabelled graphs = graphs up to isomorphisms
- Example: connected graphs on 3 vertices:

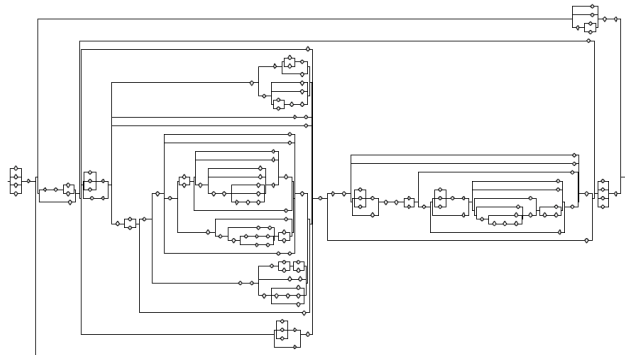


Families of graphs

- Planar graphs: can be embedded in the plane



- Series-parallel graphs: no minor K_4



- Outerplanar graphs: there exists an outerplanar embedding

Recent asymptotic results

- **Definition:** a sequence c_n is of subexponential order α if

$$c_n \sim c \gamma^n n^\alpha \text{ as } n \rightarrow \infty$$

for some positive constants c, γ

- **Subexp. orders:**

	Labelled $c_n = \mathcal{G}_n /n!$	Unlabelled $c_n = \tilde{\mathcal{G}}_n $
Planar	-7/2	?
Series-parallel	-5/2	-5/2
Outerplanar	-5/2	-5/2

Labelled: [Gimenez, Noy'05], [Bodirsky, Gimenez, Kang, Noy'05]

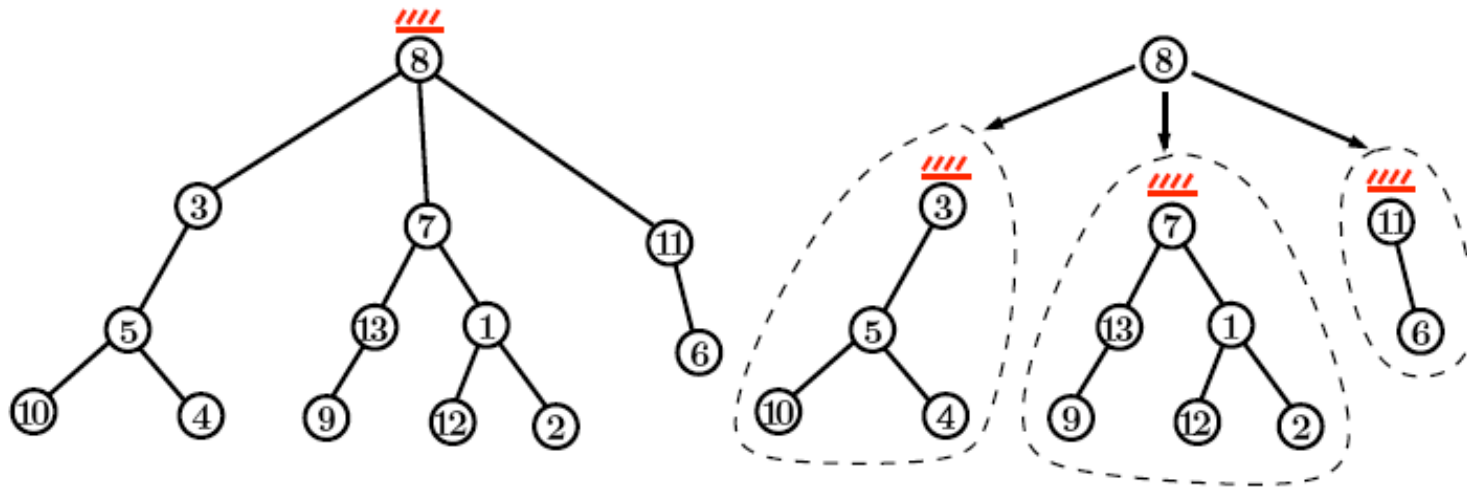
Unlabelled: [Bodirsky, F, Kang, Vigerske'07] + work in progress

- **Remark:** subexp. order **-7/2** is typical of (unrooted) **maps**
subexp. order **-5/2** is typical of (unrooted) **trees**

Part I: Asymptotic enumeration of labelled graph families

Trees

- (unrooted) tree = acyclic connected graph
- Rooted tree = tree pointed at a vertex: $r_n = n t_n$
- Decomposition at the root into subtrees:



$$\mathcal{R} = \mathcal{Z} \star \text{Set} \mathcal{R} \Rightarrow R(z) = z \exp(R(z))$$

General methodology

- Assume $y=g(z)$ is **solution** of an equation of the form

$$y = F(z, y)$$

with $F(z,y)$ **nonlinear** in y and $F(0,y)=0$

Example: for rooted trees, $g(z)=z \exp(g(z))$, so $F(z,y)=z \exp(y)$

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- If $F(z,y)$ is analytic at (ρ, τ) , then $F_y(\rho, \tau)=1$ and

$$g(z) = \tau - c\sqrt{1 - z/\rho} + \underset{z \rightarrow \rho}{O} (1 - z/\rho)$$

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$$g(z) = \tau - c\sqrt{1 - z/\rho} + O_{z \rightarrow \rho}(1 - z/\rho)$$

- Transfer theorems $\xrightarrow{+ \text{aperiodicity}}$

$$[z^n]g(z) \underset{n \rightarrow \infty}{\sim} c' \rho^{-n} n^{-3/2}$$

Methodology applied to trees

- The series $y=R(z)$ counting rooted trees is solution of

$$y = F(z, y)$$

where $F(z,y) = z \exp(y)$

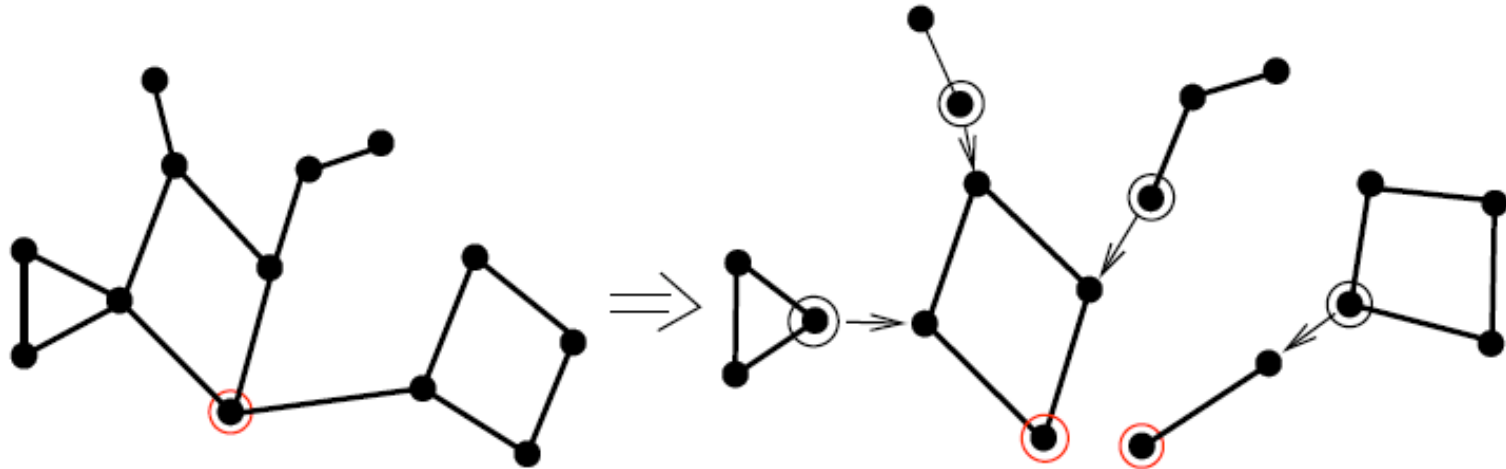
- $R(z)$ has positive radius of convergence and is aperiodic
- $F(z,y)$ is analytic everywhere

Hence,

$$r_n \sim c \rho^{-n} n^{-3/2} \Rightarrow t_n = \frac{r_n}{n} \sim c \rho^{-n} n^{-5/2}$$

Connected from 2-connected graphs

Pointed connected graph = set of pointed 2-connected graphs where each non-pointed vertex is substituted by a pointed connected graph



$$\mathcal{C}^\bullet = \mathcal{Z} \star \text{Set}(\mathcal{B}' \circ_v \mathcal{C}^\bullet) \quad \Rightarrow \quad \mathcal{C}^\bullet(z) = z \exp(B'(C^\bullet(z)))$$

Methodology for graphs

- $y = C^\bullet(z)$ is solution of
$$y = F(z, y), \quad \text{with } F(z, y) := z \exp(B'(y))$$
- $\rho := \mathbf{RadiusConv}(C^\bullet(z)), \tau := C^\bullet(\rho), R := \mathbf{RadiusConv}(B'(y))$

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(otherwise $\exists x_0 < \rho$ such that $\frac{\partial F}{\partial y}(x_0, C^\bullet(x_0)) = 1$, a contradiction)

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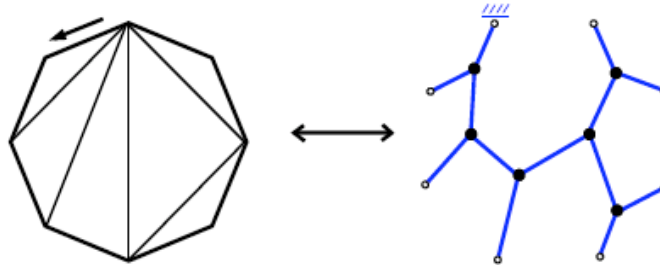
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 (otherwise $\exists x_0 < \rho$ such that $\frac{\partial F}{\partial y}(x_0, C^\bullet(x_0)) = 1$, a contradiction)

$$\Rightarrow F(z, y) \text{ analytic at } (\rho, \tau) \Rightarrow C^\bullet(z) = * - * \sqrt{1 - z/\rho} + \dots$$

$$\underset{\text{aperiodic}}{\Rightarrow} [z^n]C^\bullet(z) \sim c\rho^{-n}n^{-3/2} \Rightarrow \boxed{[z^n]C(z) = \frac{1}{n}[z^n]C^\bullet(z) \sim c\rho^{-n}n^{-5/2}}$$

Applied to two graph families

- Outerplanar: $\mathcal{B}' \simeq$ Rooted dissections \simeq Rooted plane trees

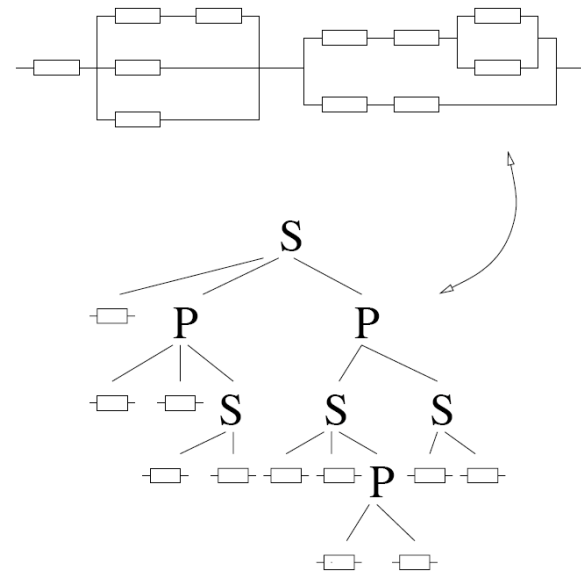


- Series-parallel:

$$\vec{\mathcal{B}} \supseteq \mathcal{B}' \supseteq \mathbb{Z}^3 \star \vec{\mathcal{B}}$$

and

$$\vec{\mathcal{B}} \simeq \text{Rooted tree family}$$



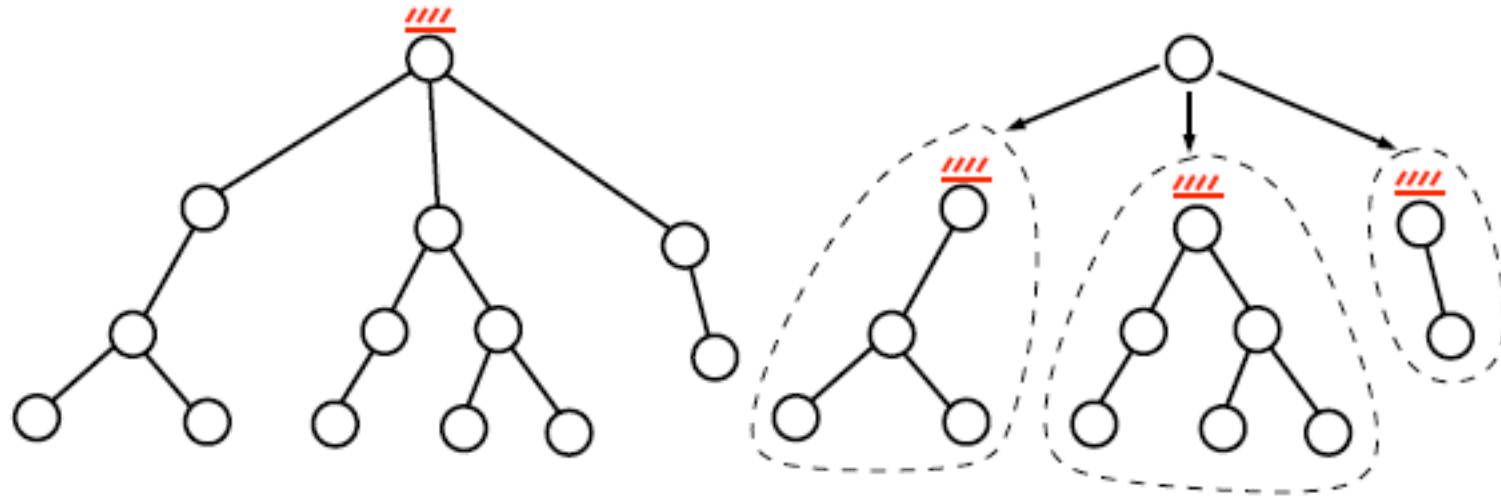
Remark: other proof in [Bodirsky, Gimenez, Kang, Noy'05]

less adaptable to the unlabelled case

Part II: Asymptotic enumeration of unlabelled graph families

Unlabelled rooted trees (1)

- Rooted unlabelled tree = **multiset** of subtrees



$$\mathcal{R} = \mathcal{Z} \times \text{MultiSet}(\mathcal{R}) \Rightarrow R(z) = z \exp \left(\sum_{i \geq 1} \frac{1}{i} R(z^i) \right)$$

(to be compared with $R(z) = z \exp(R(z))$ in the **labelled case**)

Unlabelled rooted trees (2)

- $y = R(z)$ is solution of the equation

$$y = z \exp(y) \exp \left(\underbrace{\sum_{i \geq 2} \frac{1}{i} R(z^i)}_{S(z)} \right)$$

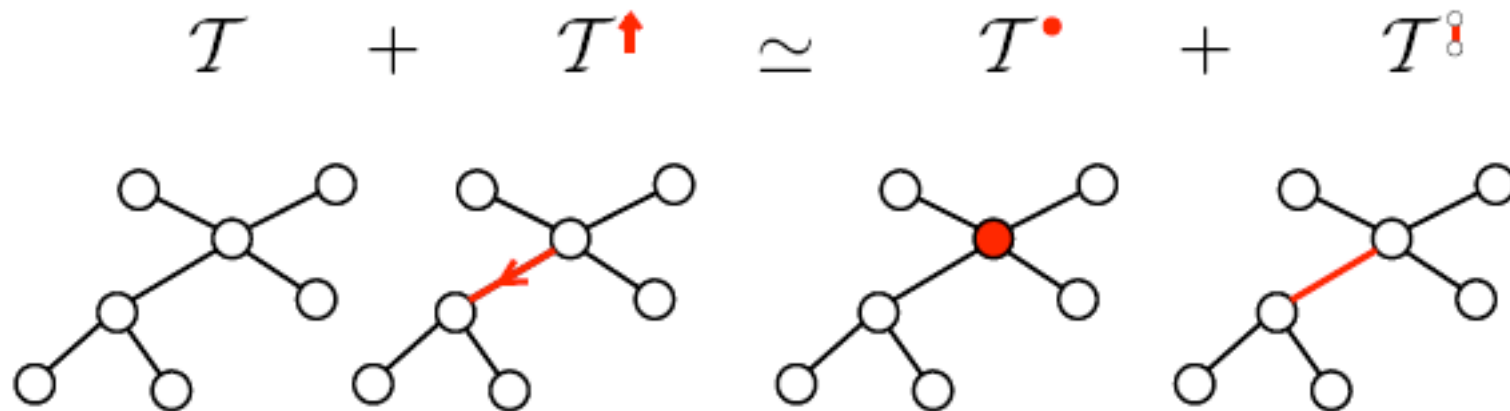
- Call $\rho := \text{RadiusConv}(C^\bullet(z))$, $\tau := C^\bullet(\rho)$
- $\rho < 1 \Rightarrow S(z)$ analytic at $\rho \Rightarrow F(z, y)$ analytic at (ρ, τ)

Hence $R(z) = \tau - c\sqrt{1 - z/\rho} + \dots$

$$\Rightarrow r_n \sim c' \rho^{-n} n^{-3/2}$$

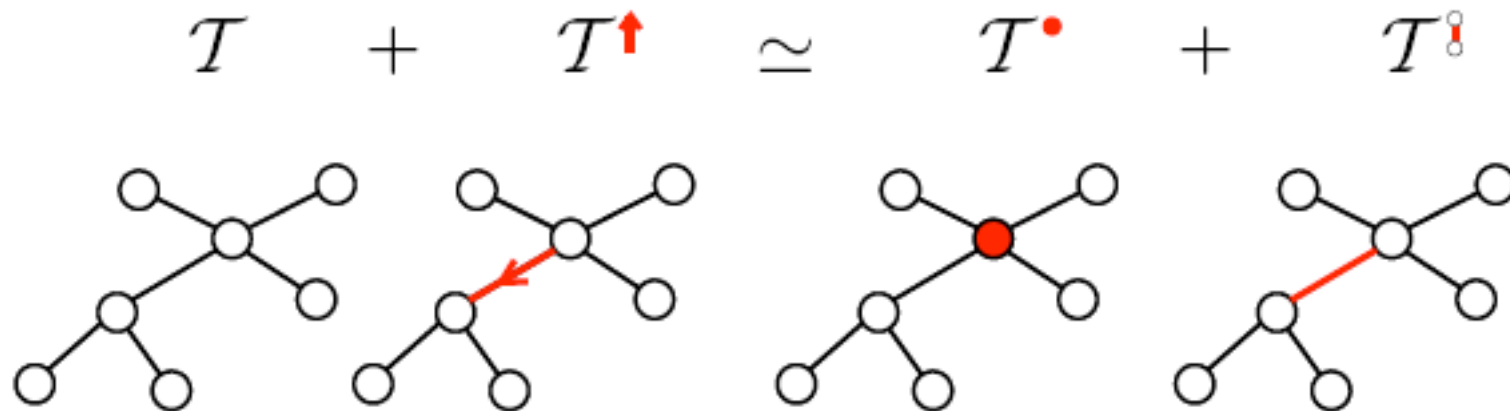
Unlabelled unrooted trees

- **Problem:** we do not have $t_n = n r_n$ as in the labelled case
- Instead, use the dissymmetry theorem [Robinson, Leroux]



Unlabelled unrooted trees

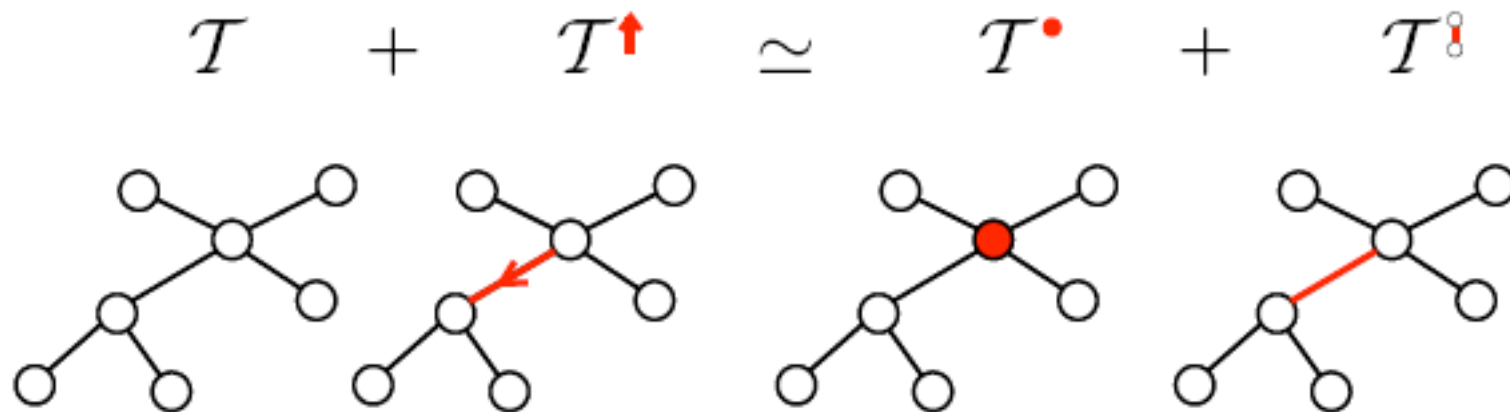
- **Problem:** we do not have $t_n = n r_n$ as in the labelled case
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$$\Rightarrow t(x) = R(x) + \frac{1}{2}R(x^2) - \frac{1}{2}R(x)^2$$

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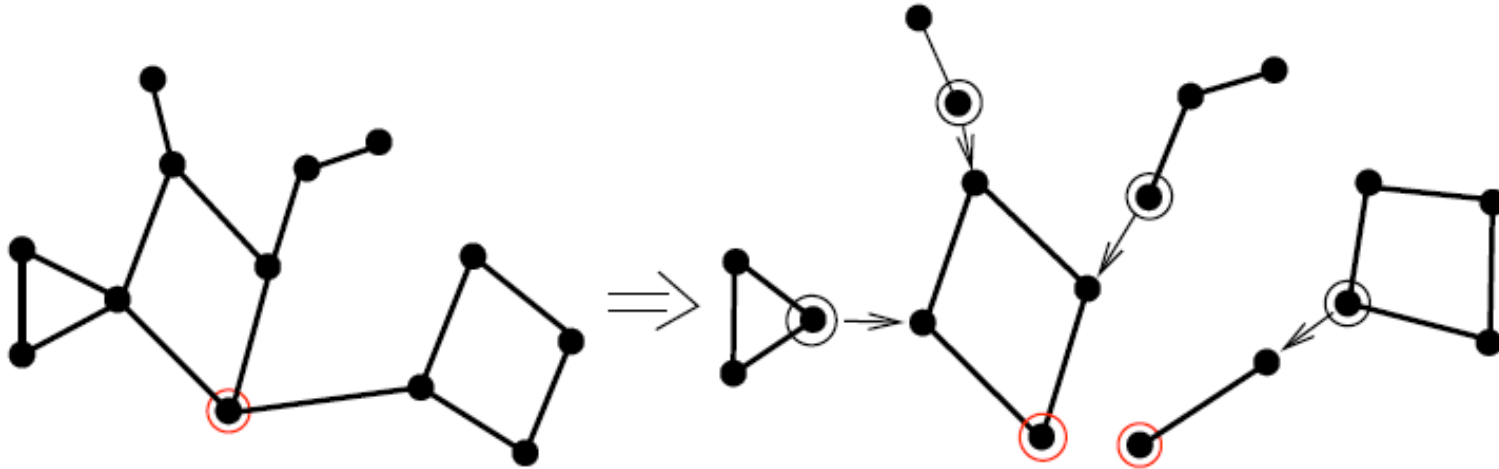
$$\Rightarrow t(x) = R(x) + \frac{1}{2}R(x^2) - \frac{1}{2}R(x)^2$$

$$\Rightarrow t(x) = c_0 + c_2(1 - x/\rho) + c_3(1 - x/\rho)^{3/2} + \dots$$

$$\Rightarrow t_n \sim c' \rho^{-n} n^{-5/2}$$

Connected from 2-connected graphs

Pointed connected graph = **Multiset** of pointed 2-connected graphs where each non-pointed vertex is substituted by a pointed connected graph



$$\mathcal{C}^\bullet = \mathcal{Z} \star \text{MultiSet}(\mathcal{B}' \circ_v \mathcal{C}^\bullet)$$

$$\Rightarrow \mathcal{C}^\bullet(z) = z \exp \left(\sum_{i \geq 1} \frac{1}{i} g(z^i) \right)$$

$$\text{with } g(z) = Z_{\mathcal{B}'}(\mathcal{C}^\bullet(z), \mathcal{C}^\bullet(z^2), \mathcal{C}^\bullet(z^3), \dots)$$

($Z_{\mathcal{A}}(s_1, s_2, \dots)$ denotes the *cycle index sum* of a class \mathcal{A})

Methodology for unlabelled graphs

- $y = C^\bullet(z)$ is solution of $y = F(z, y)$, where

$$F(z, y) = z \exp \left(\underbrace{\sum_{i \geq 2} \frac{1}{i} g(z^i)}_{S(z)} \right) \cdot \exp (Z_{\mathcal{B}'}(y, C^\bullet(z^2), C^\bullet(z^3), \dots))$$

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- Call $\rho := \text{RadiusConv}(C^\bullet(z))$. Assume $\rho < 1$. Call $\tau := C^\bullet(\rho)$
Call $R := \text{RadiusConv}(h(y))$,
with $h(y) := Z_{\mathcal{B}'}(y, C^\bullet(\rho^2), C^\bullet(\rho^3), \dots)$

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- **Key lemma:** if $h(y) = * - * \sqrt{1 - y/R} + \dots$, then $\tau < R$

$$\Rightarrow C^\bullet(z) = * - * \sqrt{1 - z/\rho} + \dots$$

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$$\Rightarrow C^\bullet(z) = * - * \sqrt{1 - z/\rho} + \dots$$

$$\xRightarrow{\text{dissym. theo.}} C(z) = c_0 + c_2(1 - z/\rho) + c_3(1 - z/\rho)^{3/2} + \dots$$

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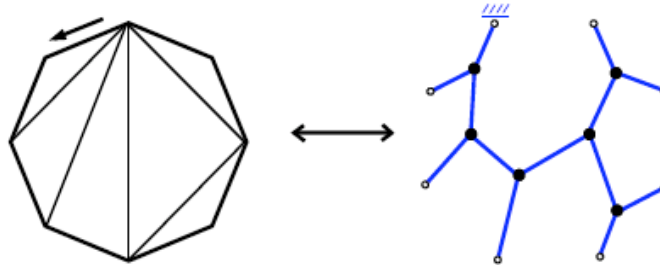
$$\xRightarrow{\text{dissym. theo.}} C(z) = c_0 + c_2(1 - z/\rho) + c_3(1 - z/\rho)^{3/2} + \dots$$

$$\Rightarrow [z^n]C(z) \sim c\rho^{-n} n^{-5/2}$$

Applied to two graph families

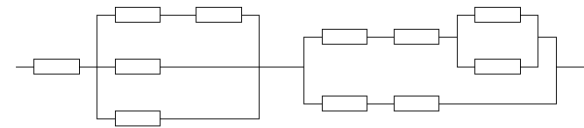
Using the key lemma, we show $[z^n]C(z) \sim c\rho^{-n}n^{-5/2}$ for:

- Unlabelled outerplanar graphs

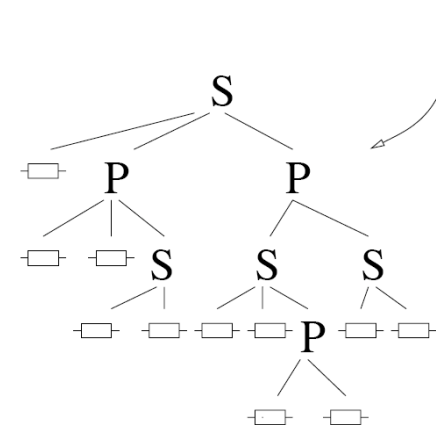


[Bodirsky, F, Kang, Vigerske'07]

- Unlabelled series-parallel graphs



work in progress with
Drmota, Kang, Kraus, Rue



Conclusion

- We have a quite general criterion to prove that graph families (labelled or not) have subexponential order $-5/2$
- These families are “subcritical”
- Is there a robust criterion to prove the subexponential order of **critical** graph families (like planar graphs) ?

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Planar	$-7/2$?
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Outerplanar	$-5/2$	$-5/2$