Asymptotic enumeration in unlabelled subcritical graph families

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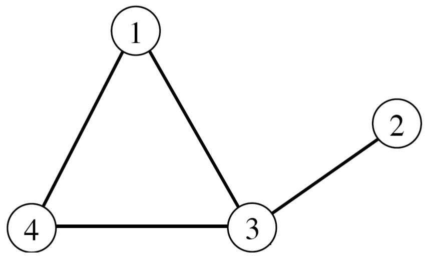
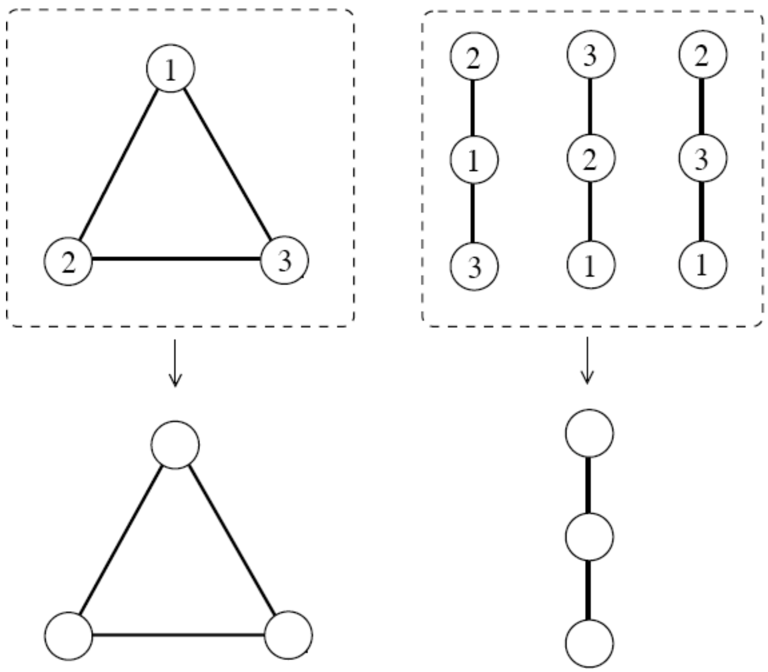
Graphs: labelled/unlabelled

- Graphs are classically labelled at vertices



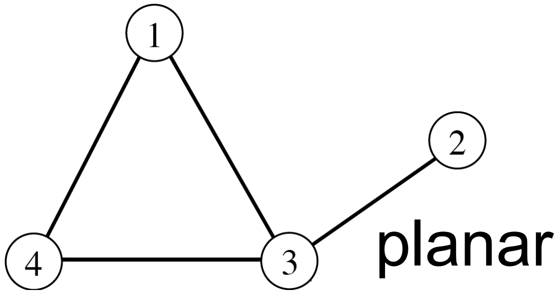
a graph on 4 vertices

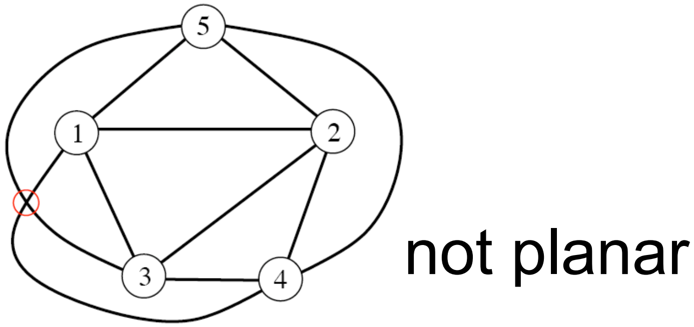
- Unlabelled graphs = graphs up to isomorphisms
- Example: connected graphs on 3 vertices:



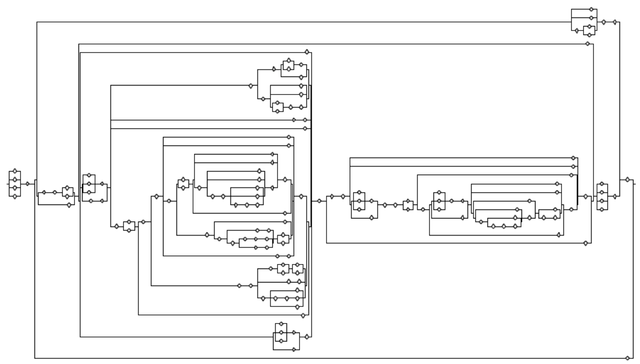
Families of graphs

- Planar graphs: can be embedded in the plane





- Series-parallel graphs: no minor K_4



- Outerplanar graphs: there exists an outerplanar embedding

Recent asymptotic results

- **Definition:** a sequence c_n is of subexponential order α if

$$c_n \sim c\,\gamma^n\,n^\alpha \quad \text{as } n \to \infty$$

for some positive constants c, γ

- **Subexp. orders:**

	Labelled $c_n = \lvert\mathcal{G}_n\rvert/n!$	Unlabelled $c_n = \lvert\widetilde{\mathcal{G}}_n\rvert$
Planar	-7/2	?
Series-parallel	-5/2	-5/2
Outerplanar	-5/2	-5/2

Labelled: [Gimenez, Noy'05], [Bodirsky, Gimenez, Kang, Noy'05]

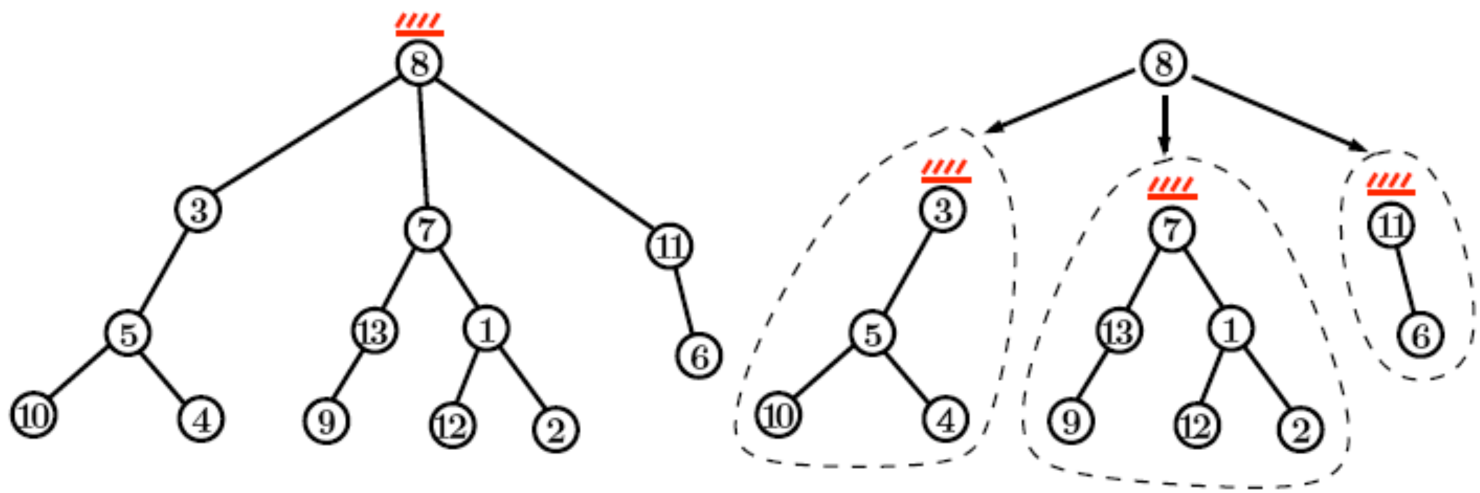
Unlabelled: [Bodirsky, F, Kang, Vigerske'07] + work in progress

- **Remark:** subexp. order -7/2 is typical of (unrooted) maps
  subexp. order -5/2 is typical of (unrooted) trees

Part I: Asymptotic enumeration of labelled graph families

Trees

- (unrooted) tree = acyclic connected graph
- Rooted tree = tree pointed at a vertex: $r_n = n\, t_n$
- Decomposition at the root into subtrees:



$$\mathcal{R} = \mathcal{Z} \star \mathrm{Set}\,\mathcal{R} \Rightarrow R(z) = z\exp(R(z))$$

General methodology

- Assume y=g(z) is solution of an equation of the form

$$y = F(z, y)$$

with F(z,y) nonlinear in y and F(0,y)=0

Example: for rooted trees, g(z)=z exp(g(z)), so F(z,y)=z exp(y)

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- If g(z) has positive radius of convergence ρ,
  then y=g(z) converges to a constant τ as z tends to ρ

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- If F(z,y) is analytic at (ρ, τ), then F_y(ρ, τ)=1 and

$$g(z) = \tau - c\sqrt{1 - z/\rho} + \underset{z \to \rho}{O}(1 - z/\rho)$$

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- If F(z,y) is analytic at (ρ, τ), then $F_y(\rho, \tau)=1$ and

$$g(z) = \tau - c\sqrt{1 - z/\rho} + \underset{z \to \rho}{O}(1 - z/\rho)$$

- Transfer theorems $\xrightarrow[\text{+ aperiodicity}]{}$ $\boxed{[z^n]g(z) \underset{n\to\infty}{\sim} c'\,\rho^{-n} n^{-3/2}}$

Methodology applied to trees

- The series y=R(z) counting rooted trees is solution of

$$y = F(z, y)$$

  where F(z,y) = z exp(y)

- R(z) has positive radius of convergence and is aperiodic

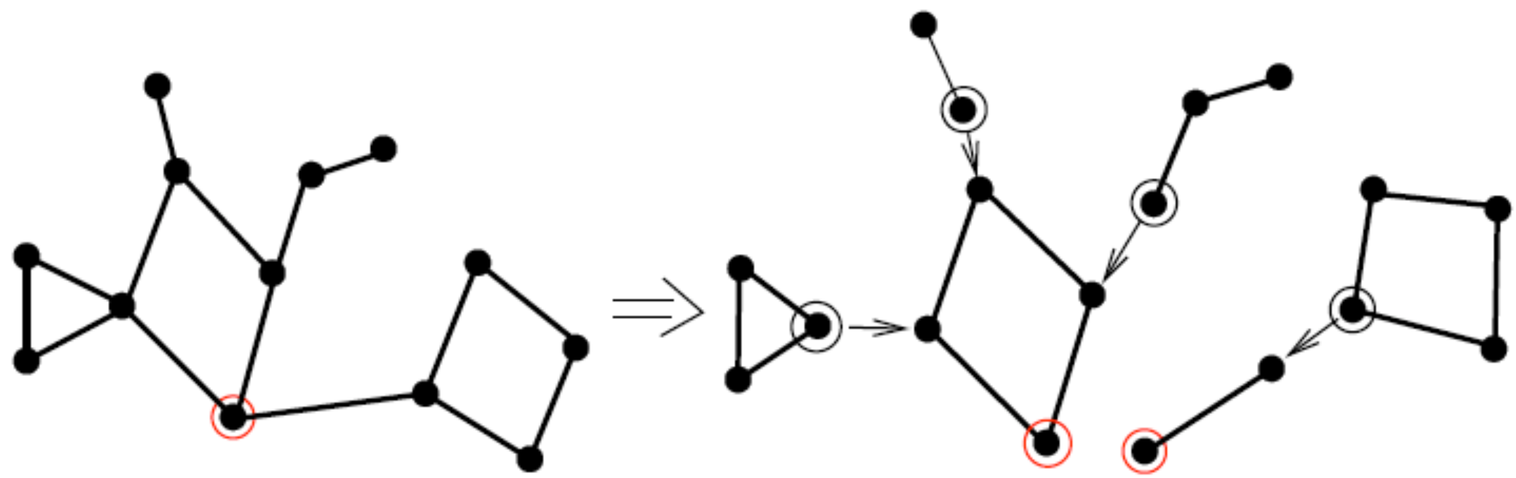
- F(z,y) is analytic everywhere

Hence,

$$r_n \sim \; c\,\rho^{-n} n^{-3/2} \;\Rightarrow\; t_n = \frac{r_n}{n} \sim c\,\rho^{-n} n^{-5/2}$$

Connected from 2-connected graphs

Pointed connected graph = set of pointed 2-connected graphs where each non-pointed vertex is substituted by a pointed connected graph



$$\mathcal{C}^\bullet = \mathcal{Z} \star \mathrm{Set}(\mathcal{B}' \circ_v \mathcal{C}^\bullet) \quad \Rightarrow \quad C^\bullet(z) = z\exp(B'(C^\bullet(z)))$$

Methodology for graphs

- $y = C^\bullet(z)$ **is solution of**
$$y = F(z, y), \quad \text{with } F(z, y) := z \exp(B'(y))$$

- $\rho := \mathbf{RadiusConv}(C^\bullet(z))$, $\tau := C^\bullet(\rho)$, $R := \mathbf{RadiusConv}(B'(y))$

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- **Two possibilities:** $\underbrace{\tau < R}_{\textbf{subcritical}}$ **or** $\underbrace{\tau = R}_{\textbf{critical}}$

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- **Key lemma: if** $B'(y) = * - * \sqrt{1 - y/R} + \cdots$, **then** $\tau < R$
(otherwise $\exists x_0 < \rho$ **such that** $\frac{\partial F}{\partial y}(x_0, C^{\bullet}(x_0)) = 1$, **a contradiction)**

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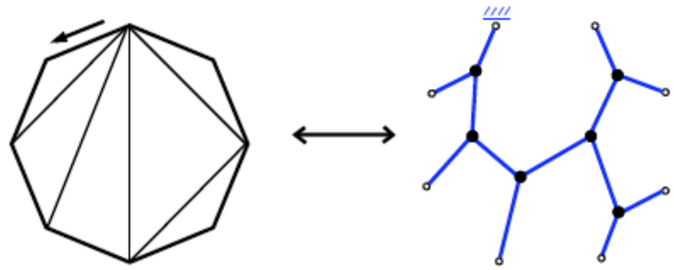
$$\Rightarrow F(z,y) \text{ analytic at } (\rho,\tau) \Rightarrow C^\bullet(z) = * - *\sqrt{1 - z/\rho} + \cdots$$

$$\underset{aperiodic}{\Longrightarrow} [z^n]C^\bullet(z) \sim c\rho^{-n}n^{-3/2} \Rightarrow \boxed{[z^n]C(z) = \frac{1}{n}[z^n]C^\bullet(z) \sim c\rho^{-n}n^{-5/2}}$$

Applied to two graph families

- Outerplanar: $\mathcal{B}' \simeq \text{Rooted dissections} \simeq \text{Rooted plane trees}$

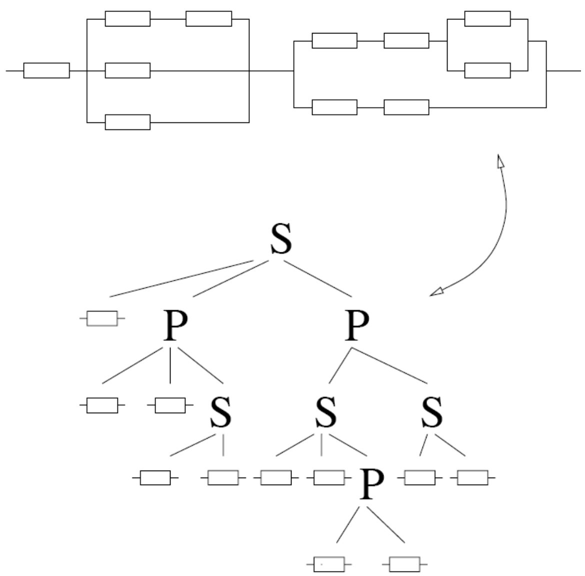


- Series-parallel:

$$\overrightarrow{\mathcal{B}} \supseteq \mathcal{B}' \supseteq \mathcal{Z}^3 \star \overrightarrow{\mathcal{B}}$$

and

$$\overrightarrow{\mathcal{B}} \simeq \text{Rooted tree family}$$

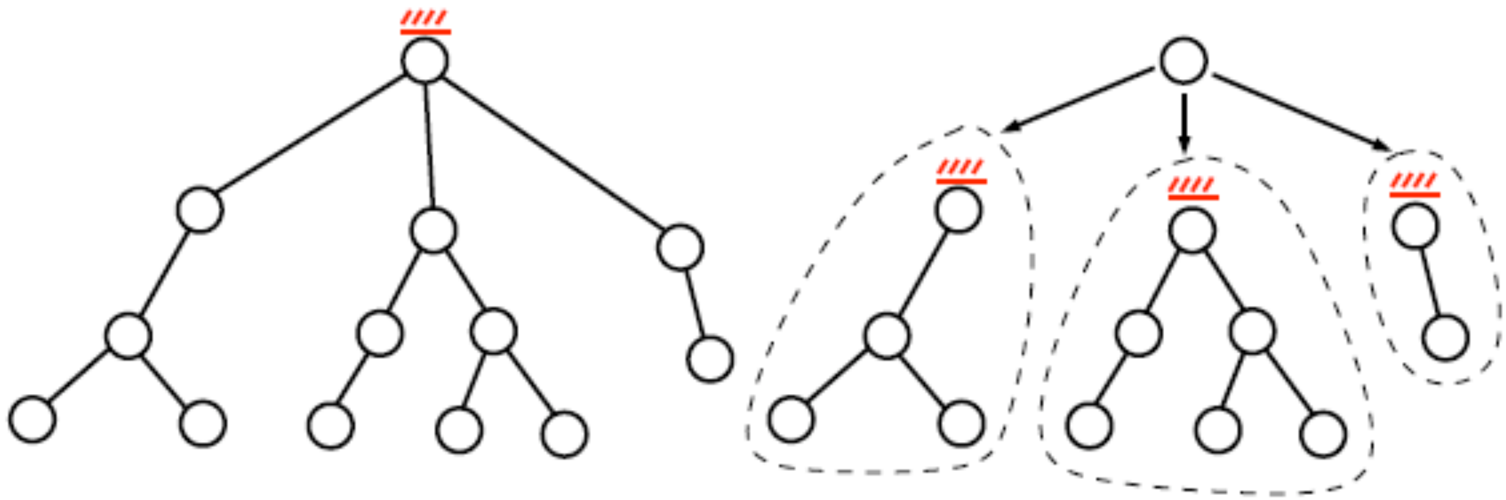


Remark: other proof in [Bodirsky, Gimenez, Kang, Noy'05]
less adaptable to the unlabelled case

Part II: Asymptotic enumeration of unlabelled graph families

Unlabelled rooted trees (1)

- Rooted unlabelled tree = multiset of subtrees



$$\mathcal{R} = \mathcal{Z} \times \mathrm{MultiSet}(\mathcal{R}) \Rightarrow R(z) = z \exp\left(\sum_{i \geq 1} \frac{1}{i} R(z^i)\right)$$

(to be compared with $R(z) = z\exp(R(z))$ in the labelled case)

Unlabelled rooted trees (2)

- $y = R(z)$ **is solution of the equation**

$$y = z \exp(y) \exp\Big(\underbrace{\sum_{i \geq 2} \frac{1}{i} R(z^i)}_{S(z)} \Big)$$

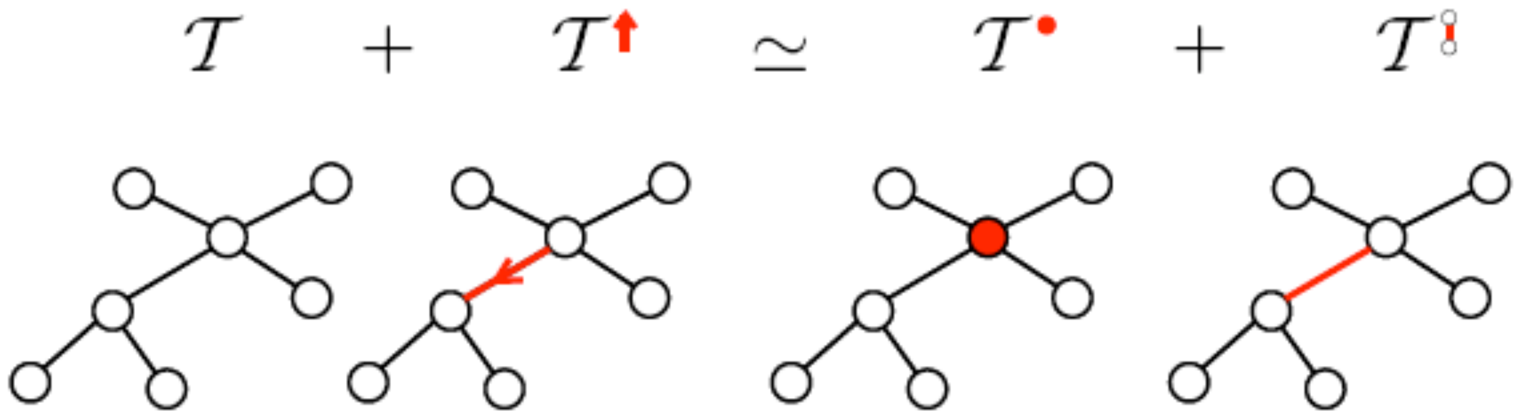
- **Call** $\rho := \textbf{RadiusConv}(C^\bullet(z))$, $\tau := C^\bullet(\rho)$
- $\rho < 1 \Rightarrow S(z)$ **analytic at** $\rho \Rightarrow \; F(z,y)$ **analytic at** (ρ, τ)

Hence $R(z) = \tau - c\sqrt{1 - z/\rho} + \cdots$

$$\Rightarrow \quad r_n \sim c' \rho^{-n} n^{-3/2}$$

Unlabelled unrooted trees

- **Problem**: we do not have $t_n = n\, r_n$ as in the labelled case
- Instead, use the dissymmetry theorem [Robinson, Leroux]



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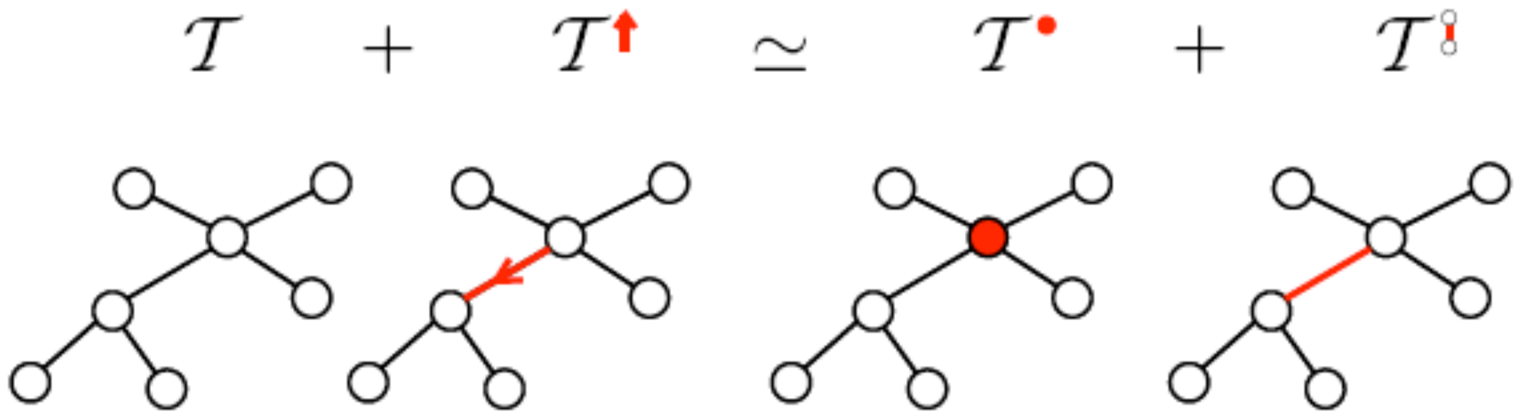

$$\Rightarrow \quad \boxed{t(x) = R(x) + \frac{1}{2}R(x^2) - \frac{1}{2}R(x)^2}$$

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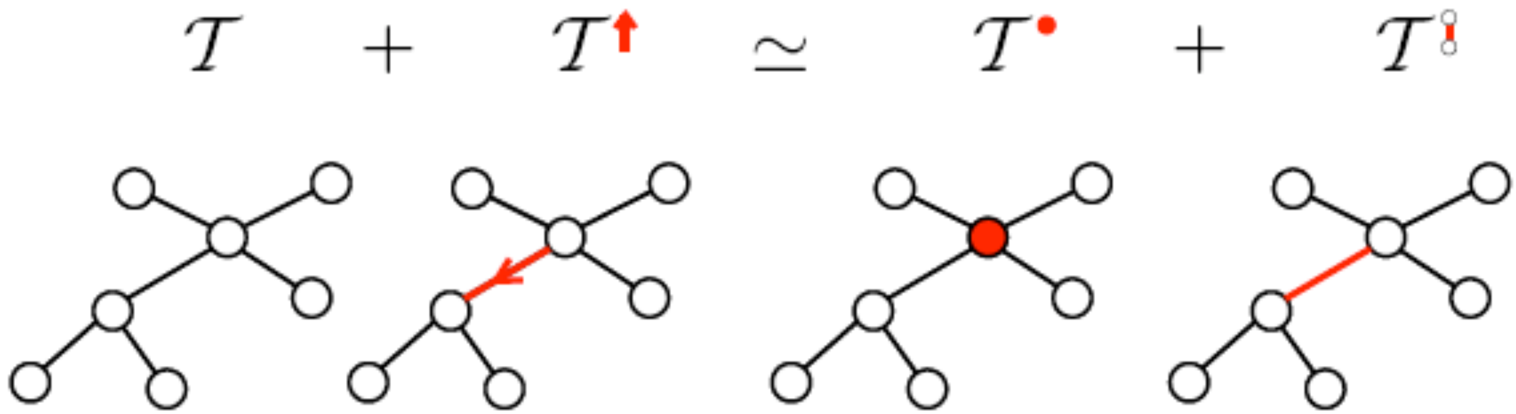

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$$\Rightarrow \quad t(x) = c_0 + c_2(1 - x/\rho) + c_3(1 - x/\rho)^{3/2} + \cdots$$

$$\Rightarrow \quad t_n \sim c' \rho^{-n} n^{-5/2}$$

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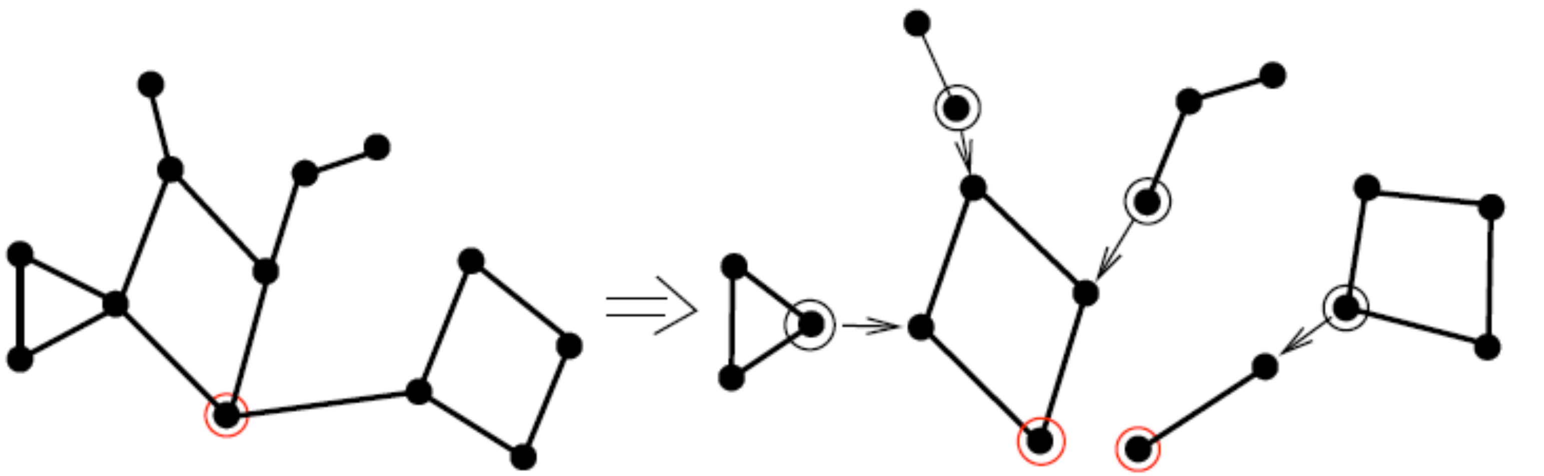

$$\mathcal{C}^\bullet = \mathcal{Z} \star \mathrm{MultiSet}(\mathcal{B}' \circ_v \mathcal{C}^\bullet)$$

$$\Rightarrow \mathcal{C}^\bullet(z) = z \exp\left(\sum_{i \geq 1} \frac{1}{i} g(z^i)\right)$$

$$\textbf{with } g(z) = Z_{\mathcal{B}'}(C^\bullet(z), C^\bullet(z^2), C^\bullet(z^3), \ldots)$$

($Z_{\mathcal{A}}(s_1, s_2, \ldots)$ **denotes the** ***cycle index sum*** **of a class** $\mathcal{A}$)

Methodology for unlabelled graphs

- $y = C^\bullet(z)$ **is solution of** $y = F(z, y)$**, where**

$$F(z,y) = z \exp\underbrace{\left(\sum_{i\geq 2}\frac{1}{i}g(z^i)\right)}_{S(z)} \cdot \exp\left(Z_{\mathcal{B}'}(y, C^\bullet(z^2), C^\bullet(z^3), \ldots)\right)$$

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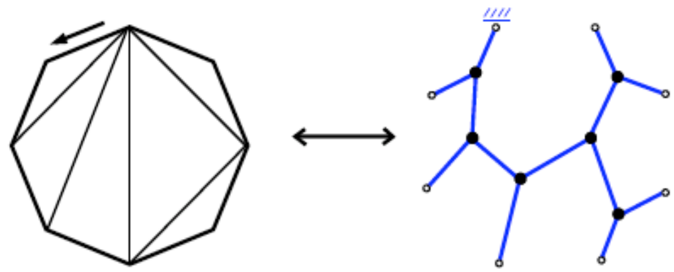
$$\underset{dissym.\ theo.}{\Longrightarrow} \quad C(z) = c_0 + c_2(1 - z/\rho) + c_3(1 - z/\rho)^{3/2} + \cdots$$

$$\Rightarrow \quad [z^n]C(z) \sim c\rho^{-n}n^{-5/2}$$

Applied to two graph families

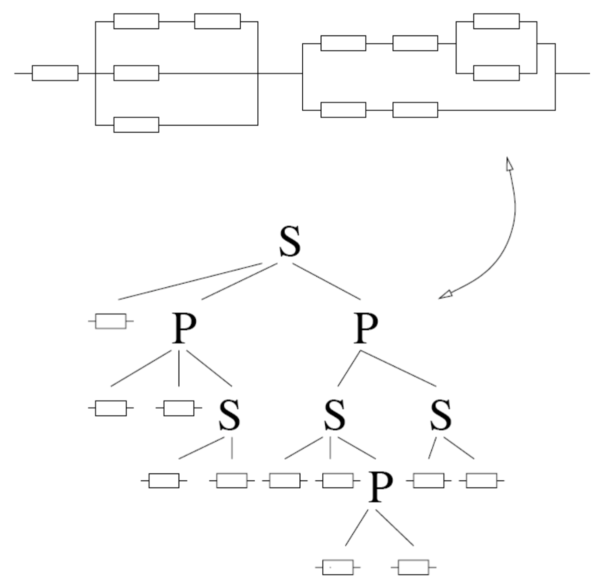
Using the key lemma, we show $[z^n]C(z) \sim c\rho^{-n} n^{-5/2}$ for:

- Unlabelled outerplanar graphs



[Bodirsky, F, Kang, Vigerske'07]

- Unlabelled series-parallel graphs



work in progress with
Drmota, Kang, Kraus, Rue

Conclusion

- We have a quite general criterion to prove that graph families (labelled or not) have subexponential order -5/2
- These families are ``subcritical''
- Is there a robust criterion to prove the subexponential order of critical graph families (like planar graphs) ?

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