Bijections for \(d\)-angulated dissections

Éric Fusy (CNRS/LIX)
Joint work with Olivier Bernardi (Brandeis)
Planar maps

Def. Planar map = connected graph embedded in the plane up to isotopy

A map is rooted by marking a corner incident to the outer face
**$(k, d)$-dissections. Irreducibility**

$(k, d)$-dissection = map with simple outer boundary of length $k$
inner faces of degree $d$, girth $d$

$k = 5 \ d = 3$

$k = 6 \ d = 4$
\[(k, d)\text{-dissections. Irreducibility}\]

\[(k, d)\text{-dissection} = \text{map with simple outer boundary of length } k\]
\innerfacesofdegree d, \text{girth } d\]

\[
k = 5 \quad d = 3\]
\[
k = 6 \quad d = 4\]

A \((k, d)\text{-dissection}\) is \textbf{irreducible} if all non-facial cycles have length \(> d\)
(k, d)-dissections. Irreducibility

(k, d)-dissection = map with simple outer boundary of length k
inner faces of degree d, girth d

A (k, d)-dissection is **irreducible** if all non-facial cycles have length > d
Counting formulas for dissections ($d = 3, 4$)

- # rooted $k$-outer triangulated dissections with $n$ inner vertices
  - simple
  \[
  \frac{2(2k-3)!}{(k-1)!(k-3)!} \frac{(4n+2k-5)!}{n!(3n+2k-3)!}
  \]
  [Brown’64]
  [Poulalhon, Schaeffer’06], [Albenque, Poulalhon’13]
  [Bernardi, F’10]
  
  - irreducible ($k \geq 4$)
  \[
  \frac{(2k-4)!}{(k-4)!(k-1)!} \frac{(3n+k-4)!}{n!(2n+k-2)!}
  \]
  [Tutte’62]
  [F’05] $k = 4$
  [Bouttier, Guitter’13]

- # rooted $k$-outer ($k = 2p$) quadrangulated dissections with $n$ inner vertices
  - simple
  \[
  \frac{3(3p-2)!}{(p-2)!(2p-1)!} \frac{(3n+3p-4)!}{n!(2n+3p-2)!}
  \]
  [Brown’65]
  [Albenque, Poulalhon’13]
  [Bernardi, F’10]
  
  - irreducible ($p \geq 3$)
  \[
  \frac{(3p-3)!}{(p-3)!(2p-1)!} \frac{(2n+p-3)!}{n!(n+p-1)!}
  \]
  [Mullin, Schellenberg’68]
  [F, Poulalhon, Schaeffer’05] $p = 3$
  [Bouttier, Guitter’13]
1. **Master bijection** between a class of **oriented maps** and a class of bicolored **decorated trees** (which are called mobiles).

![Diagram of oriented map and mobile]

2. Application to $d$-angulations of girth $d$ (starting with $d = 3$)

3. Application to $d$-angulated irreducible dissections
Master bijection between oriented maps and mobiles
Minimal accessible orientations

An orientation of a rooted plane map is called:

- **accessible** if every vertex can be reached from the root-vertex
- **minimal** if there is no counterclockwise cycle
Minimal accessible orientations

An orientation of a rooted plane map is called

- **accessible** if every vertex can be reached from the root-vertex
- **minimal** if there is no counterclockwise cycle
Families of orientations and mobiles

Let $\mathcal{O}$ be the set of orientations on planar maps such that:

- there is no ccw circuit
- Each inner vertex is accessible from the outer (unoriented simple) cycle
- the outer cycle is a source

Let $\mathcal{M}$ be the set of mobiles, i.e., bipartite plane trees with arrows (called buds) at black vertices and with more buds than edges.
From oriented maps to mobiles

Local rules
Theorem [Bernardi-F’10]: $\Phi$ is a bijection between $\mathcal{O}$ and $\mathcal{M}$. Moreover,

- degrees of internal faces $\leftrightarrow$ degrees of black vertices
- indegrees of internal vertices $\leftrightarrow$ degrees of white vertices
From oriented maps to mobiles

**Theorem [Bernardi-F’10]:** \( \Phi \) is a **bijection** between \( \mathcal{O} \) and \( \mathcal{M} \). Moreover,

- degrees of internal faces \( \leftrightarrow \) degrees of black vertices
- indegrees of internal vertices \( \leftrightarrow \) degrees of white vertices

**cf [Bernardi’07], [Bernardi-Chapuy’10]**
And the inverse (closure) mapping
Using the master bijection for map enumeration
Scheme for the strategy
(1) Map family $\mathcal{C}$ identifies with a subfamily $O_C$ of $O$ with conditions on:

- Face degrees
- Vertex indegrees
Scheme for the strategy

(1) Map family $\mathcal{C}$ identifies with a subfamily $\mathcal{O}_\mathcal{C}$ of $\mathcal{O}$ with conditions on:

- Face degrees
- Vertex indegrees

**Example:** $\mathcal{C} =$ Family of *simple triangulations*

$\mathcal{C} \simeq \text{subfamily } \mathcal{O}_\mathcal{C} \text{ of } \mathcal{O} \text{ with}$

- Face-degree $= 3$
- Vertex-indegree $= 3$
Scheme for the strategy

(1) Map family $\mathcal{C}$ identifies with a subfamily $\mathcal{O}_C$ of $\mathcal{O}$ with conditions on:
- Face degrees
- Vertex indegrees

Example: $\mathcal{C} =$ Family of simple triangulations

(2) Specialize the master bijection to the subfamily $\mathcal{O}_C$

degrees of internal faces $\longleftrightarrow$ degrees of black vertices
indegrees of internal vertices $\longleftrightarrow$ degrees of white vertices
\( \alpha \)-orientations

Let \( G = (V, E) \) be a graph

Let \( \alpha \) be a function from \( V \) to \( \mathbb{N} \)

\[
\begin{align*}
\alpha : & \quad a \to 2 \\
& \quad b \to 1 \\
& \quad c \to 2 \\
& \quad d \to 0 \\
& \quad e \to 2
\end{align*}
\]
α-orientations

Let \( G = (V, E) \) be a graph.
Let \( \alpha \) be a function from \( V \) to \( \mathbb{N} \).

Def: An \textbf{α-orientation} is an orientation of \( G \) where for each \( v \in V \):

\[
\text{indegree}(v) = \alpha(v)
\]
**α-orientations**

Let $G = (V, E)$ be a graph
Let $\alpha$ be a function from $V$ to $\mathbb{N}$

Def: An $\alpha$-orientation is an orientation of $G$ where for each $v \in V$

$$\text{indegree}(v) = \alpha(v)$$
\( \alpha \)-orientations: existence criterion

- If an \( \alpha \)-orientation exists, then

\[
\begin{align*}
(\text{i}) & \quad \sum_{v \in V} \alpha(v) = |E| \\
(\text{ii}) & \quad \forall S \subseteq V, \, \sum_{v \in S} \alpha(v) \geq |E_S|
\end{align*}
\]
\( \alpha \)-orientations: existence criterion

- If an \( \alpha \)-orientation \textbf{exists}, then

\[
\begin{align*}
\text{(i)} & \quad \sum_{v \in V} \alpha(v) = |E| \\
\text{(ii)} & \quad \forall S \subseteq V, \quad \sum_{v \in S} \alpha(v) \geq |E_S|
\end{align*}
\]

- If the \( \alpha \)-orientation is \textbf{accessible} from a vertex \( u \in V \) then

\[
\sum_{v \in S} \alpha(v) > |E_S| \quad \text{whenever } u \notin S \text{ and } S \neq \emptyset
\]
**α-orientations: existence criterion**

- If an α-orientation exists, then

\[
\begin{align*}
(i) & \quad \sum_{v \in V} \alpha(v) = |E| \\
(ii) & \quad \forall S \subseteq V, \sum_{v \in S} \alpha(v) \geq |E_S|
\end{align*}
\]

- If the α-orientation is accessible from a vertex \( u \in V \) then

\[
(iii) \quad \sum_{v \in S} \alpha(v) > |E_S| \quad \text{whenever} \quad u \notin S \quad \text{and} \quad S \neq \emptyset
\]

**Lemma (folklore):** The conditions are necessary and sufficient
α-orientations: existence criterion

- If an α-orientation exists, then

\[ \sum_{v \in V} \alpha(v) = |E| \]

\[ \forall S \subseteq V, \sum_{v \in S} \alpha(v) \geq |E_S| \]

- If the α-orientation is accessible from a vertex \( u \in V \) then

\[ \sum_{v \in S} \alpha(v) > |E_S| \text{ whenever } u \notin S \text{ and } S \neq \emptyset \]

Lemma (folklore): The conditions are necessary and sufficient

⇒ accessibility from \( u \in V \) just depends on \( \alpha \) (not on which α-orientation)
α-orientations for plane maps

**Fundamental lemma:** If a plane map admits an α-orientation, then it admits a **unique** α-orientation **without ccw circuit**, called **minimal**

More precisely, the set of α-orientations is a **distributive lattice** [Khueller et al.’93], [Propp’93], [O. de Mendez’94], [Felsner’03]
Example: simple triangulations
Fact: A triangulation with $n$ internal vertices has $3n$ internal edges.

Proof: The numbers $v, e, f$ of vertices edges and faces satisfy:
• Incidence relation: $3f = 2e$.
• Euler relation: $v - e + f = 2$.

Call 3-orientation such an $\alpha$-orientation.
**Fact:** A triangulation with $n$ internal vertices has $3n$ internal edges.

**Natural candidate for indegree function:**

$$\alpha : v \mapsto 3 \text{ for each internal vertex } v.$$ 

call **3-orientation** such an $\alpha$-orientation.
**Fact:** A triangulation with $n$ internal vertices has $3n$ internal edges.

**Natural candidate for indegree function:**

$$\alpha : v \mapsto 3 \text{ for each internal vertex } v.$$  
$$v \mapsto 1 \text{ for each external vertex } v.$$  

Call **3-orientation** such an $\alpha$-orientation.
**Fact:** A triangulation admitting a 3-orientation is simple

- $k$ internal vertices
- $3k + 1$ internal edges
Thm [Schnyder 89]: A simple triangulation admits a 3-orientation, and any 3-orientation is accessible from the outer boundary

New (easier) proof: Any simple planar graph $G = (V, E)$ satisfies

$$|E| \leq 3|V| - 6 \quad \text{(Euler relation)}$$

Hence $\forall S \subseteq V$, $|E_S| \leq \alpha(S)$,

with strict inequality when $S$ misses at least one outer vertex

hence the existence/accessibility conditions are satisfied.$\square$
Triangulations

• From the lattice property (taking the min) we have
  family of simple triangulations ↔ subfamily $\mathcal{F}$ of $\mathcal{O}$ where:
  - faces have degree 3
  - inner vertices have outdegree 3

• From the master bijection specialized to $\mathcal{F}$, we have
  $\mathcal{F} \leftrightarrow$ subfamily of mobiles where all vertices have degree 3

[F, Poulalhon, Schaeffer’08], other bijection in [Poulalhon, Schaeffer’03]
Counting: The generating function of mobiles with vertices of degree 3 rooted on a white corner is $T(x) = U(x)^3$, where $U(x) = 1 + xU(x)^4$.

Consequently, the number of (rooted) simple triangulations with $2n$ faces is $\frac{1}{n(2n-1)} \binom{4n-2}{n-1}$.
Triangulations: two constructions

- Mobiles
  - [FuPoSc’08], [Bernardi-F’10]

- Blossoming trees
  - [PoSc’03], [AIPo’13]
Generalization to $d$-angulations of girth $d$
Fact: A $d$-angulation with $(d-2)n$ internal vertices has $dn$ internal edges.
Fact: A $d$-angulation with $(d-2)n$ internal vertices has $dn$ internal edges.

Natural candidate for indegree function:

$$\alpha : v \mapsto \frac{d}{d-2} \text{ for each internal vertex } v \ldots$$
**Fact:** A $d$-angulation with $(d-2)n$ internal vertices has $dn$ internal edges.

**Idea:** We can look for an orientation of $(d-2)G$ with indegree function

$$\alpha : v \mapsto d$$

for each internal vertex $v$.  

\[\text{d-angulations of girth } d\]
**Fact:** A $d$-angulation with $(d-2)n$ internal vertices has $dn$ internal edges.

**Idea:** We can look for an orientation of $(d-2)G$ with indegree function $\alpha: v \mapsto d$ for each internal vertex $v$. Call $d/(d-2)$-orientation such an orientation.
**d-angulations of girth \(d\)**

**Thm [Bernardi-F’10]:** Let \(G\) be a \(d\)-angulation. Then \((d-2)G\) admits a \(d/(d-2)\)-orientation if and only if \(G\) has girth \(d\).
**d-angulations of girth d**

**Thm [Bernardi-F’10]:** Let $G$ be a $d$-angulation. Then $(d-2)G$ admits a $d/(d-2)$-orientation if and only if $G$ has girth $d$.

**Proof:** Similar to $d = 3$. Uses the fact that a planar graph $G = (V, E)$ of girth at least $d$ satisfies $(d-2)|E| \leq d|V| - 2d$.
Master bijection in the flow-formulation

Local rules

- degrees of inner faces
- total flows at inner vertices
- total flows at inner edges
- degrees of black vertices
- total weights at white vertices
- total weights at edges
Specialization to $d$-angulations of girth $d$:

Bijection $d$-angulations of girth $d \leftrightarrow$ weighted mobiles such that:
- each black vertex has degree $d$
- each white vertex has total weight $d$
- each edge has total weight $d - 2$ (weight $> 0$ at $\bigcirc$, weight $= 0$ at $\bullet$)

[Albenque, Poulalhon’13]: other bijection (with blossoming trees)
Generating function expression

For $i \in [0..d]$, $\mathcal{L}_i :=$ family of such mobiles with a root-leg of weight $i$

Let $L_i(x)$ be the GF of $\mathcal{L}_i$ where $x$ marks black nodes

Examples:

$d = 5$

mobile in $\mathcal{L}_2$

For $d \geq 3$, $F_d(x) :=$ GF of (rooted) $d$-angulations of girth $d$ by inner faces

- **Bijection** when an inner face is marked
  \[ F'(x) = (1 + L_{d-2})^d \]

- **Root-decomposition** of mobiles in $\mathcal{L}_i \Rightarrow (L_0, L_1, \ldots, L_d)$ are given by

\[
\begin{cases}
  L_0 &= x \cdot (1 + L_{d-2})^{d-1}, \\
  L_d &= 1, \\
  L_i &= \sum_{j>0} L_{d-2-j} L_{i+j} \quad \text{for } i = 1 \ldots d - 1
\end{cases}
\]
Simplification in the bipartite case

- For $d$ even, $d = 2b$, we have $\frac{d}{d-2} = \frac{b}{b-1}$

- Can work with $b/(b-1)$-orientations:
  - edges have weight $b-1$
  - vertices have indegree $b$

Example: $b = 2$, simple quadrangulations

recover a bijection of Schaeffer (1999)
Bijections for irreducible $(4, 3)$-dissections (triangulated case)
3-orientations for triangulated dissections

**Def:** A 3-orientation of a \((k, 3)\)-dissection is an orientation of the inner edges where all inner vertices have indegree 3.

**Rk:** Euler relation \(\Rightarrow k - 2\) inner edges point to the boundary.

Again minimal means “no ccw cycle” (not unique for \(k \geq 4\)).
Characterizing irreducibility on orientations

Co-accessibility: from every inner vertex one can reach the outer boundary.

For a \((k, 3)\) dissection \(D\) endowed with a (any) 3-orientation \(O\),

\[
\text{\(D\) is irreducible iff \(O\) is co-accessible}
\]
Characterizing irreducibility on orientations

Co-accessibility: from every inner vertex one can reach the outer boundary.

For a \((k, 3)\) dissection \(D\) endowed with a (any) 3-orientation \(O\),

\[
\text{\(D\) is irreducible iff \(O\) is co-accessible}
\]

Proof:

\(D\) not irreducible \(\Rightarrow\) \(O\) not co-accessible

In orange the complex induced by vertices that can reach the outer boundary

The holes have to be triangular
Characterizing irreducibility on orientations

Co-accessibility: from every inner vertex one can reach the outer boundary.

For a \((k, 3)\) dissection \(D\) endowed with a (any) 3-orientation \(O\),

\[ D \text{ is irreducible iff } O \text{ is co-accessible} \]

\(R_k:\) also gives a simple algorithm to extract the irreducible core

in gray the vertices (and incident edges) that can not reach the outer boundary
Irreducible 4-outer triangulation $\rightarrow$ ternary tree

Let $T$ be an irreducible 4-outer triangulation
Let $T$ be an irreducible 4-outer triangulation. Endow $T$ with a minimal 3-orientation.
Irreducible 4-outer triangulation $\rightarrow$ ternary tree

Let $T$ be an irreducible 4-outer triangulation
Endow $T$ with a minimal 3-orientation

In red the canonical spanning tree (spanning inner vertices)
Let $T$ be an irreducible 4-outer triangulation
Endow $T$ with a minimal 3-orientation

In red the canonical spanning tree (spanning inner vertices)
In green the other ingoing half-edges
Irreducible 4-outer triangulation → ternary tree

Let $T$ be an irreducible 4-outer triangulation
Endow $T$ with a minimal 3-orientation

In red the canonical spanning tree (spanning inner vertices)
In green the other ingoing half-edges

The red-green graph is a rooted ternary tree
Let $T$ be an irreducible 4-outer triangulation. Endow $T$ with a minimal 3-orientation.

- **In red** the canonical spanning tree.
- **In green** the other ingoing half-edges.

Return the edge to the root in the canonical spanning tree and bi-orient the other red edges.

The red-green graph is a rooted ternary tree.
Seeing the mapping as a mobile construction
Uniqueness of the orientation
We have proved the existence of a minimal orientation such that
• Inner edges are directed or bidirected
• Inner (resp. outer) vertices have indegree 4 (resp. 0)
We have proved the existence of a minimal orientation such that
- Inner edges are directed or bidirected
- Inner (resp. outer) vertices have indegree 4 (resp. 0)

To prove uniqueness, transfer to an $\alpha$-orientation (on star-graph)

Moreover, the transfer rules preserve minimality
Uniqueness of the orientation

We have proved the existence of a minimal orientation such that

- Inner edges are directed or bidirected
- Inner (resp. outer) vertices have indegree 4 (resp. 0)

To prove uniqueness, transfer to an $\alpha$-orientation (on star-graph)

Moreover, the transfer rules preserve minimality

Byproduct: new algo to find the minimal transversal structure from min. 3-ori
Bijection for irreducible \((6, 4)\)-dissections (quadrangulated case)
Irreducible \((6, 4)\)-dissections

1. minimal 2-orientation

2. canonical spanning tree

3. in red-green a rooted binary tree

4. return root-edge bi-orient tree-edges

5. binary tree (unrooted)

recover [F, Poulalhon, Schaeffer’05]
Uniqueness of the orientation

Byproduct: new algo to compute minimal Schnyder wood of a 3-connected map from min. 2-ori

\[ \alpha = 3 \text{ at inner } \circ \]
\[ \alpha = 1 \text{ other vert. } \]
Bijection for irreducible $d$-angulated dissections
Extension to any $d$ of results for $d \in \{3, 4\}$

- Case $d$ odd, irreducible $(d + 1, d)$-dissections

  1. $d-2$ legs
  2. $\begin{array}{c}i \quad j \end{array}$ edges
     $i, j$ positive
     add up to $d-1$
  3. Total weight $d+1$ at each vertex

- Case $d$ even, $d = 2b$, irreducible $(d + 2, d)$-dissections

  1. $b-1$ legs
  2. $\begin{array}{c}i \quad j \end{array}$ edges
     $i, j$ positive
     add up to $b$
  3. Total weight $b+1$ at each vertex
Extensions of the bijections obtained so far
Combining both bijections (bipartite case)

2b-angulations girth 2b

irreducible (2b, 2b − 2)-dissections
Combining both bijections (bipartite case)

2b-angulations girth 2b

irreducible (2b, 2b − 2)-dissections

(2b)-outer dissections, inner face degrees ∈ \{2b − 2, 2b\}
cycles of length ≥ 2b except contours of (2b − 2)-faces
Allowing for higher face-degrees (bipartite case)

$(2b)$-outer dissections, inner face degrees $\in \{2b-2, 2b, 2b+2, 2b+4, \ldots\}$
cycles of length $\geq 2b$ except contours of $(2b-2)$-faces

$b = 3$

face of degree $2b-2$

face of degree $2b+2i$

either or
degree $2b + 2i$, with $i$ orange legs
Allowing for higher face-degrees (bipartite case)

$(2b)$-outer dissections, inner face degrees $\in \{2b - 2, 2b, 2b + 2, 2b + 4, \ldots\}$
cycles of length $\geq 2b$ except contours of $(2b - 2)$-faces

$b = 3$

- can be done also in the non-bipartite setting
- extends GF expressions for annular maps from [Bernardi,F’11]
  this time allowing for “small faces”
- recover annular maps GF expressions in [Bouttier,Guitter’13]
Irreducible \((k, d)\) dissections, for any \(k\)

Case \(d = 3\), take an irreducible \((k, 3)\)-dissection, mark an inner face
Irreducible \((k, d)\) dissections, for any \(k\)

Case \(d = 3\), take an irreducible \((k, 3)\)-dissection, mark an inner face

endow it with a minimal 3-orientation
Irreducible \((k, d)\) dissections, for any \(k\)

Case \(d = 3\), take an irreducible \((k, 3)\)-dissection, mark an inner face endow it with a minimal 3-orientation compute the canonical spanning tree
Irreducible \((k, d)\) dissections, for any \(k\)

Case \(d = 3\), take an irreducible \((k, 3)\)-dissection, mark an inner face endow it with a minimal 3-orientation compute the canonical spanning tree and associated corner bicoloration
Irreducible \((k, d)\) dissections, for any \(k\)

take the marked inner face as outer face
Irreducible \( (k, d) \) dissections, for any \( k \)

- take the marked inner face as outer face
- make all corners in the outer face magenta (by returning a path)
Irreducible \((k, d)\) dissections, for any \(k\)

take the marked inner face as outer face
make all corners in the outer face magenta (by returning a path)
Irreducible $(k, d)$ dissections, for any $k$

then make the underlying $\alpha$-orientation minimal (considering the boundary face as a big vertex)
Irreducible \((k, d)\) dissections, for any \(k\)

then make the underlying \(\alpha\)-orientation minimal (considering the boundary face as a big vertex)
Irreducible \((k, d)\) dissections, for any \(k\)

apply the transfer rules in the other direction
Irreducible \((k, d)\) dissections, for any \(k\)

\[\frac{(2k-4)!}{(k-4)!(k-1)!} \frac{(3n+k-4)!}{n!(2n+k-2)!}\]

\(\Rightarrow\) bijective proof of Tutte’s formula

same approach works for any \(d \geq 3\)