Cartes de genre non fixé et arbres bourgeonnants

Éric Fusy (CNRS/LIX)

travaux en commun avec Emmanuel Guitter

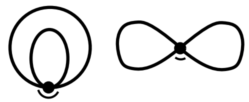
Séminaire IRIF, 4 juin 2020

Original question

Tree-bijections ensure that the generating function of 4-regular planar maps is $R_1(g)$, where $R_1(g), R_2(g), \ldots$ are solution to the recursive system (with $R_0 = 0$)

$$\boxed{R_i(g) = 1 + g\, R_i(g)(R_{i-1}(g) + R_i(g) + R_{i+1}(g)) \qquad i \geq 1} \qquad (\star)$$

$$R_1(g) = 1 + 2g + 9g^2 + 54g^3 + 378g^4 + \cdots$$



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$(R_i(g) =$ 2-point function of 4-regular planar maps$)$

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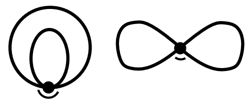

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($R_i(g)$ = 2-point function of 4-regular planar maps)

On the other hand, using orthogonal polynomials, one shows that the GF of 4-regular maps of unfixed genus is $r_1(g)$, where $r_1(g), r_2(g), \ldots$ are solution to the recursive system (with $r_0 = 0$)

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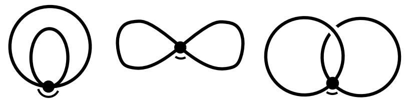

$[g^1]r_1(g) = 3$

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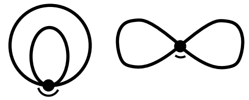

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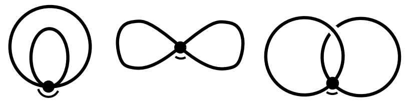
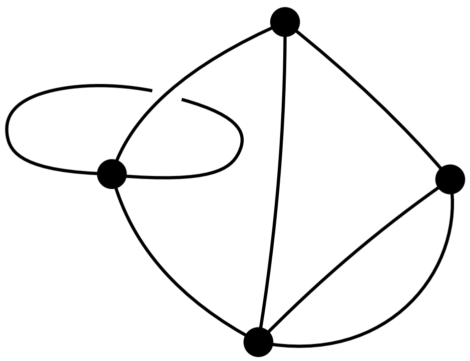


$[g^1]r_1(g) = 3$

Question: bijective interpretation of $(\star\star)$? (unified with $(\star)$?)

Maps

map $=$ multigraph $+$ rotation-system



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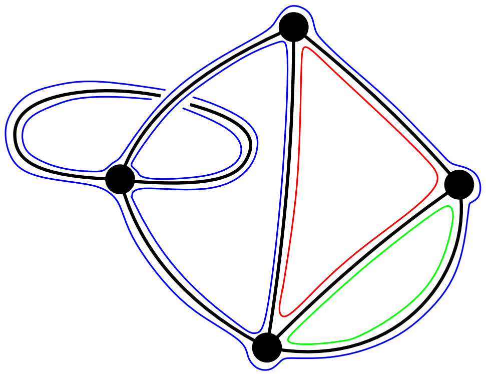

faces $\leftrightarrow$ facial walks

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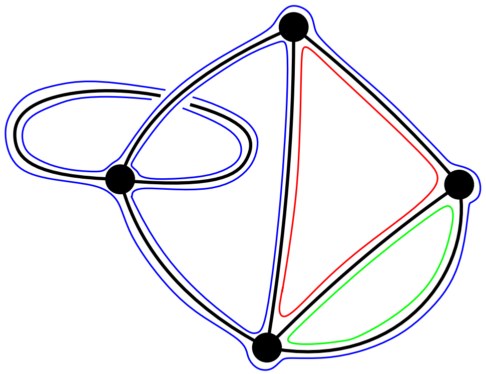

$|V| = 4$, $|E| = 7$, $|F| = 3$
$\Rightarrow \text{genus} = 1$

faces $\leftrightarrow$ facial walks

Euler relation:

$$\boxed{|V| - |E| + |F| = 2 - 2 \cdot \text{genus}}$$

with V, E, F the sets of vertices, edges, faces

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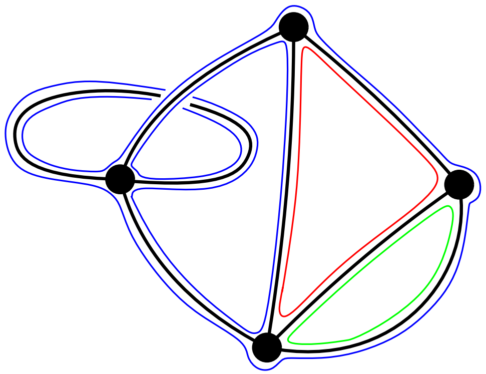

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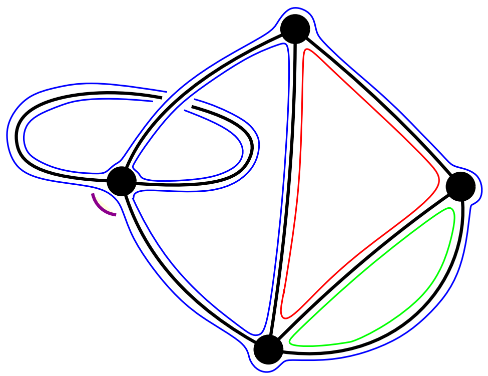

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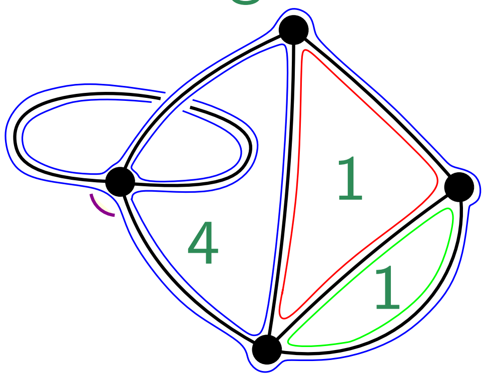
4-face-coloring



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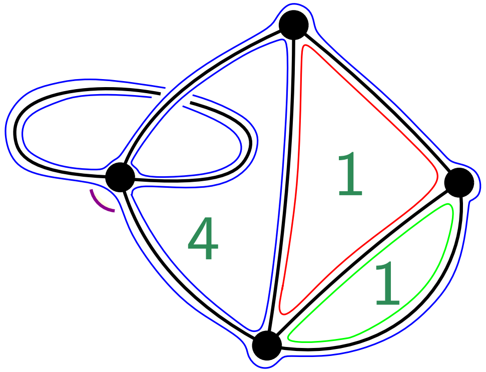

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Eulerian map: all vertex-degrees are even

Outline

- planar case $(R_i = 1 + gR_i(R_{i-1} + R_i + R_{i+1}))$

  recall bijective approach based on blossoming trees

- unfixed genus $(r_i = i + gr_i(r_{i-1} + r_i + r_{i+1}))$

  standard counting methods & orthogonal polynomials
  adaptation of the planar case bijection

- N-face-colored maps (formula in terms of $(r_i)_{i\leq N}$)

  bijective conjecture

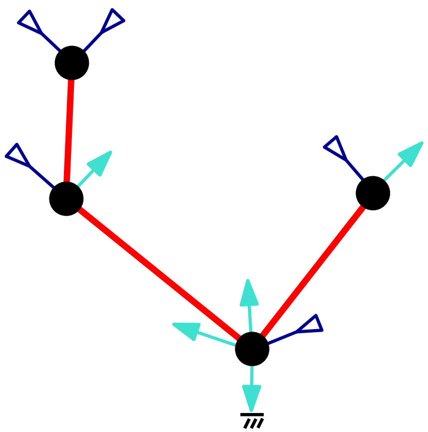
We focus on Eulerian maps with controlled vertex-degrees
(approach also applies for maps with controlled vertex-degrees
& for bipartite m-regular maps)

Planar case

Blossoming trees

blossoming tree = plane tree with 2 kinds of leaves (opening/closing)
such that #(opening leaves) = #(closing leaves)
& rooted at an opening leaf

T



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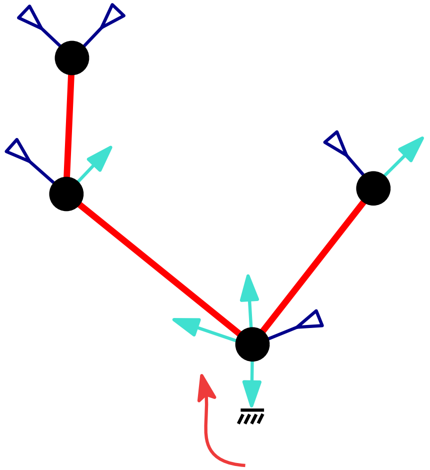
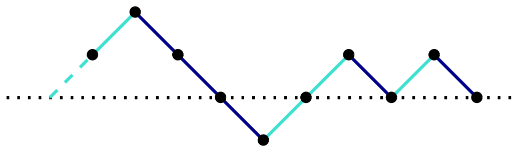


leaf-path $w(T)$



T is **balanced** if $w(T)$ is above x-axis

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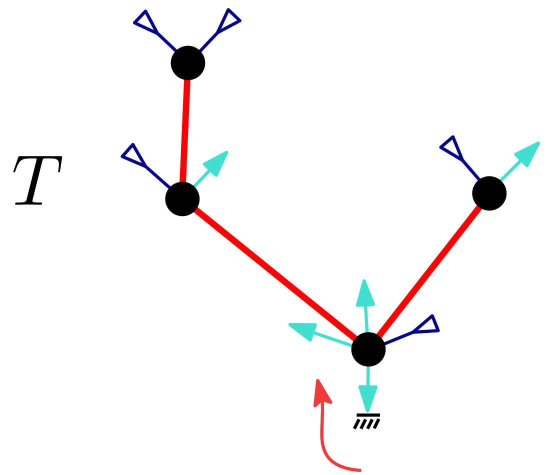
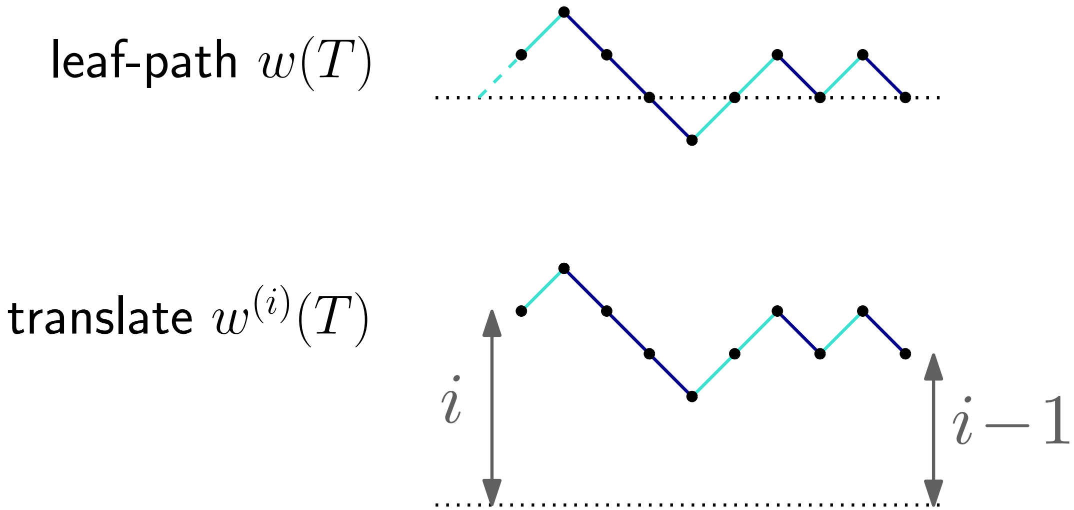




T is **balanced** if $w(T)$ is above x-axis

For $i \geq 1$, T is ***i*-balanced** if $w^{(i)}(T)$ is above x-axis

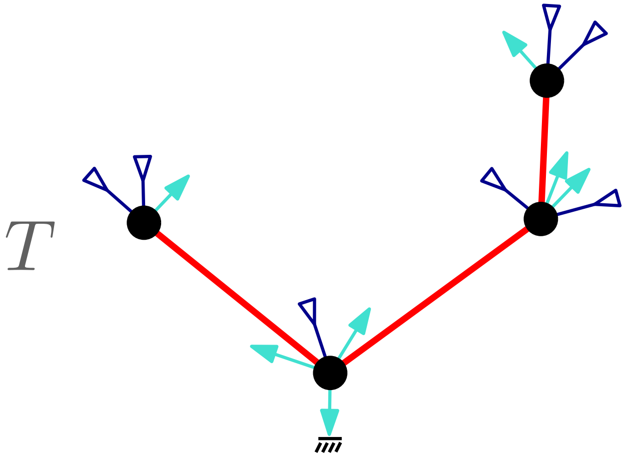
(**Rk**: balanced $\Leftrightarrow$ 1-balanced)

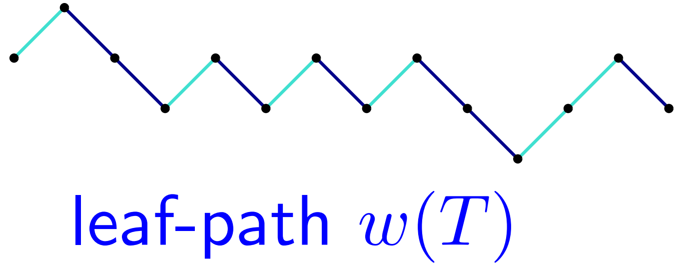
Eulerian trees

[Schaeffer'97]

Eulerian tree = blossoming tree where nodes have even degree

each node v has $\frac{1}{2}\deg(v) - 1$ children that are opening leaves





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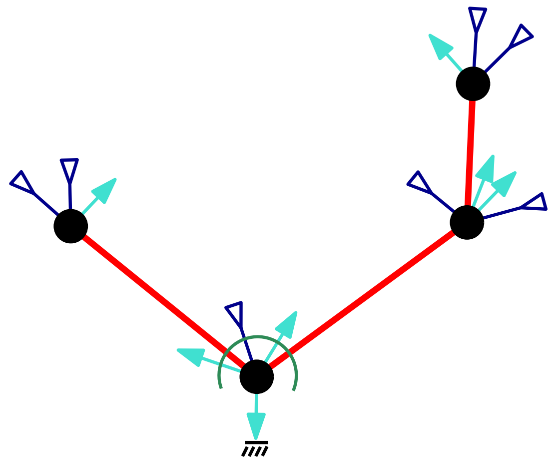
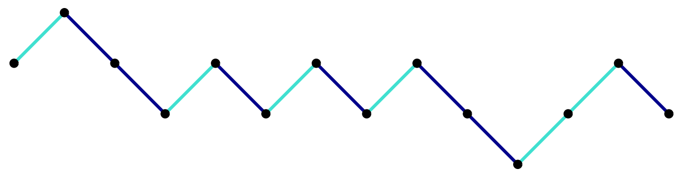
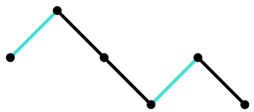




leaf-path $w(T)$

root-path $\wp(T)$

(up-steps for)



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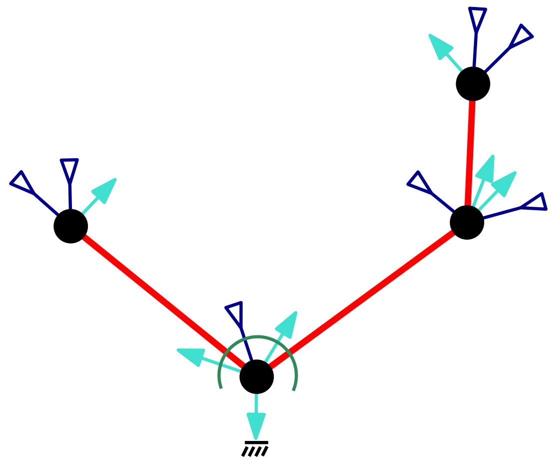

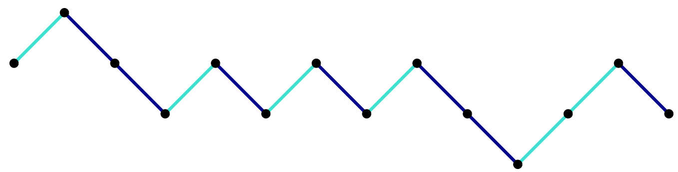

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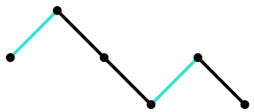
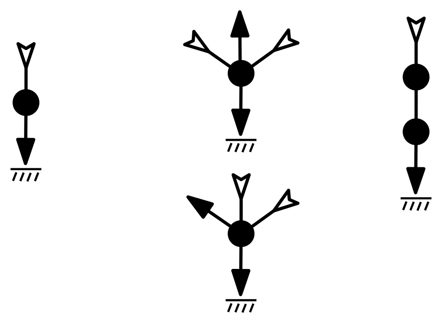


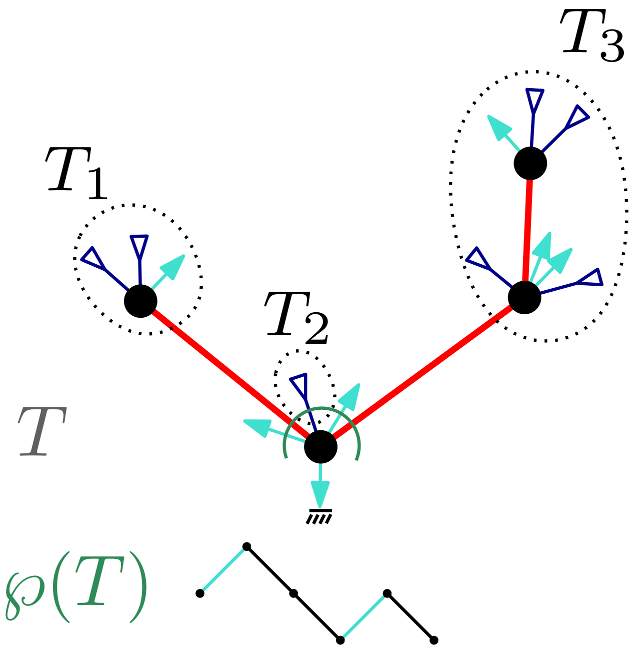
For $i \geq 1$ let $R_i(t)$ be the GF of i-balanced Eulerian trees
(with weight $t^k g_k$ per node of degree $2k$)

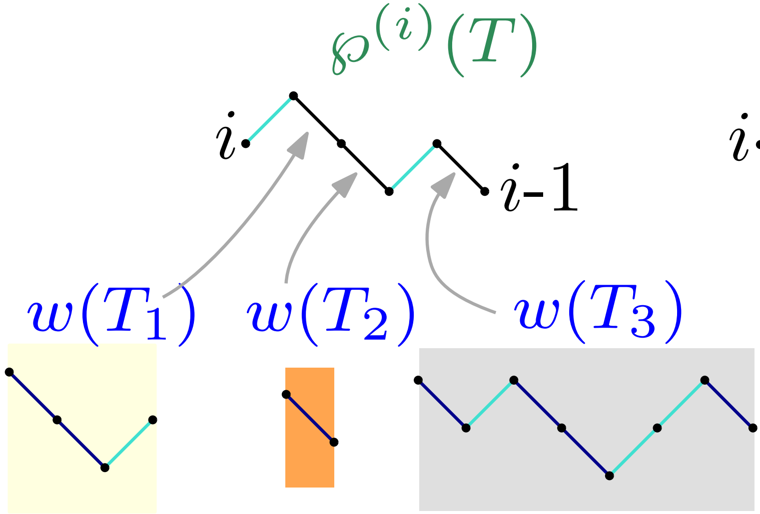
$$R_1(t) = 1 + g_2 t + (2g_4 + g_2^2)t^2 + \cdots$$

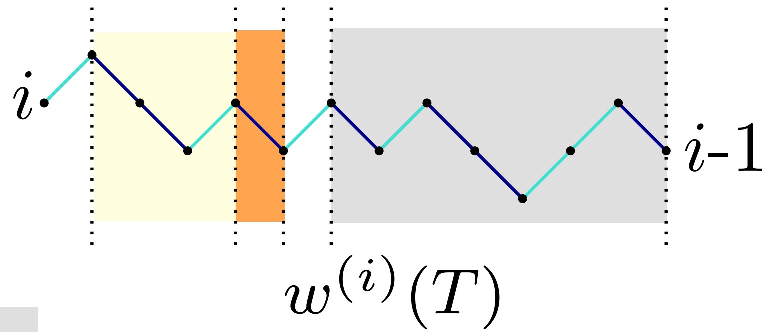


Recursive system for Eulerian trees

[Bouttier-Di Francesco-Guitter'03]







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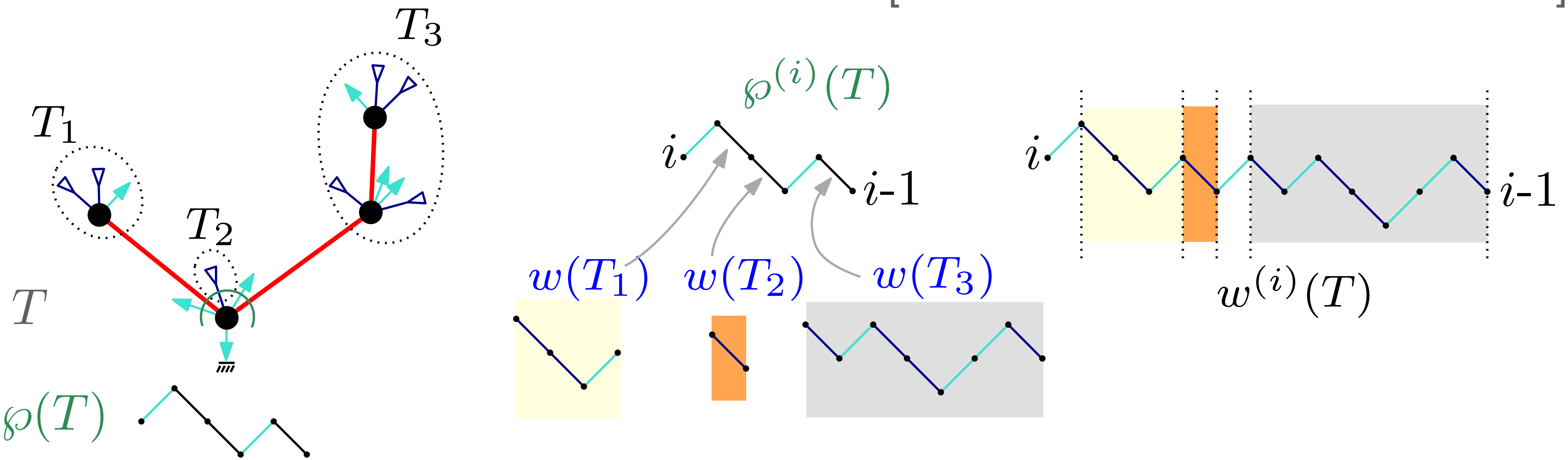

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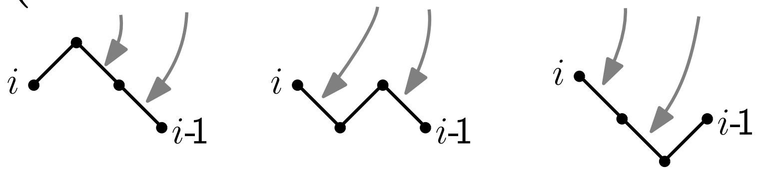
Special case 4-regular maps $(g_k = \delta_{k=2})$:

$$R_i = 1 + t^2\left(R_{i+1}R_i + R_iR_i + R_iR_{i-1}\right) = 1 + t^2 R_i(R_{i-1} + R_i + R_{i+1})$$

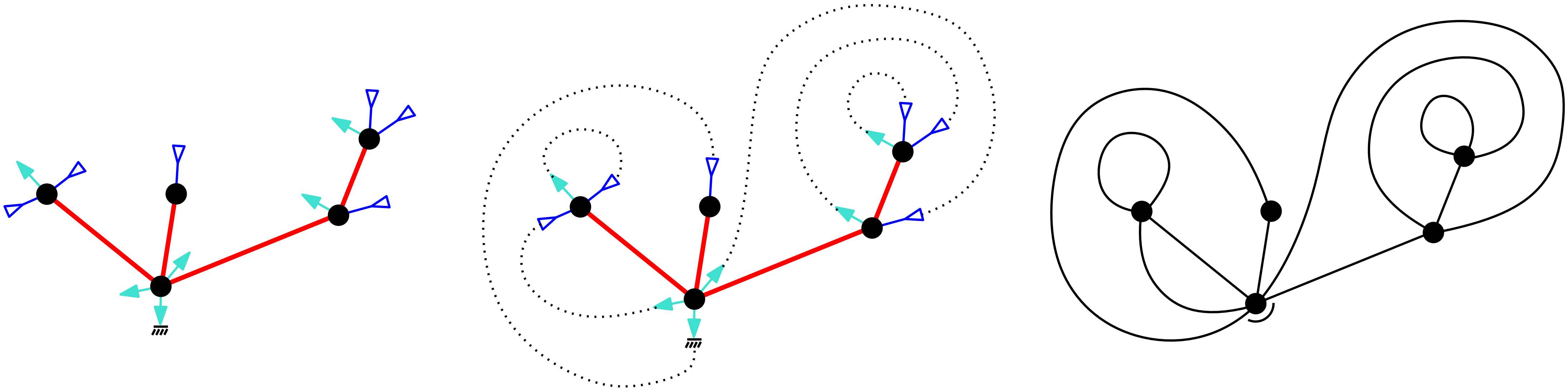


Bijection for planar Eulerian maps

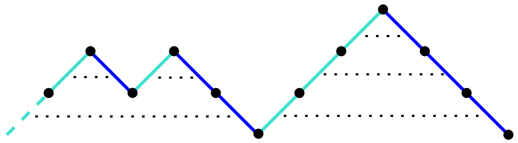
[Schaeffer'97]

Balanced Eulerian tree $\rightarrow$ Eulerian planar map

match (forward and planarly) the opening leaves with the closing leaves



leaf-path



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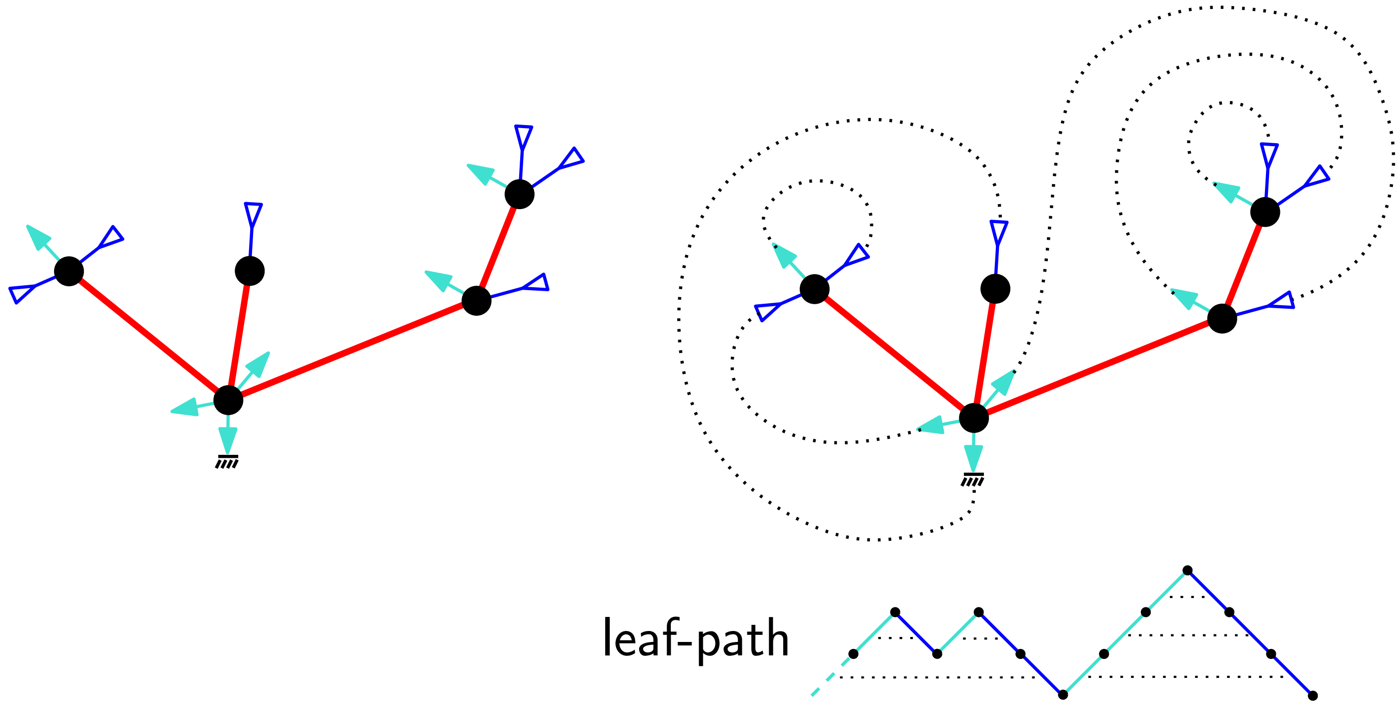

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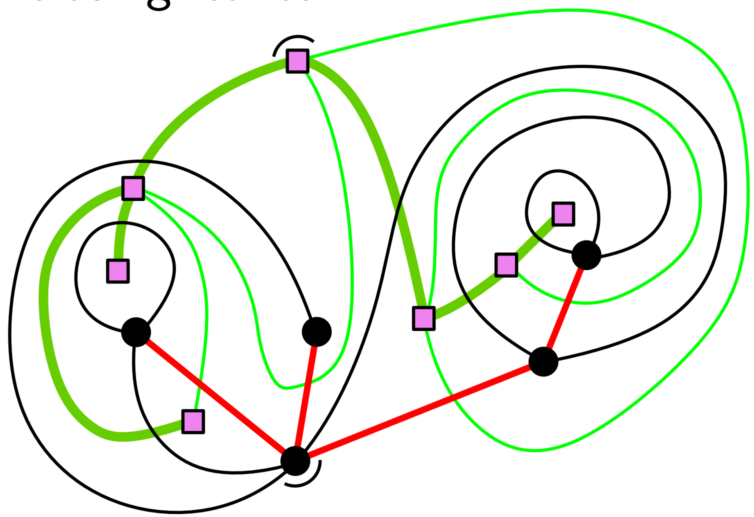

Inverse mapping

cut the edges dual to those in the leftmost BST of the dual map

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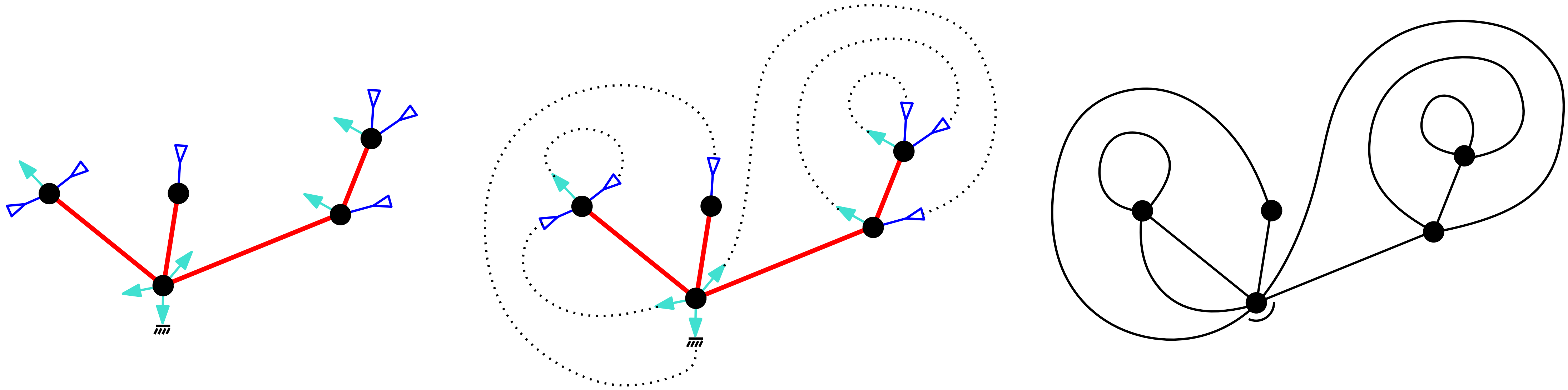

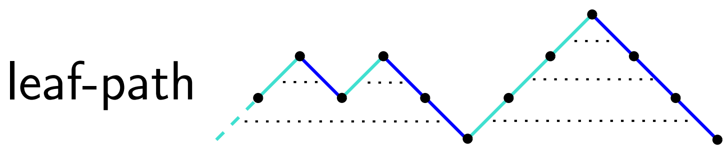

$\Rightarrow$ $R_1(t)$ is the GF of Eulerian planar maps (weight t per edge, g_k per vertex of degree $2k$)

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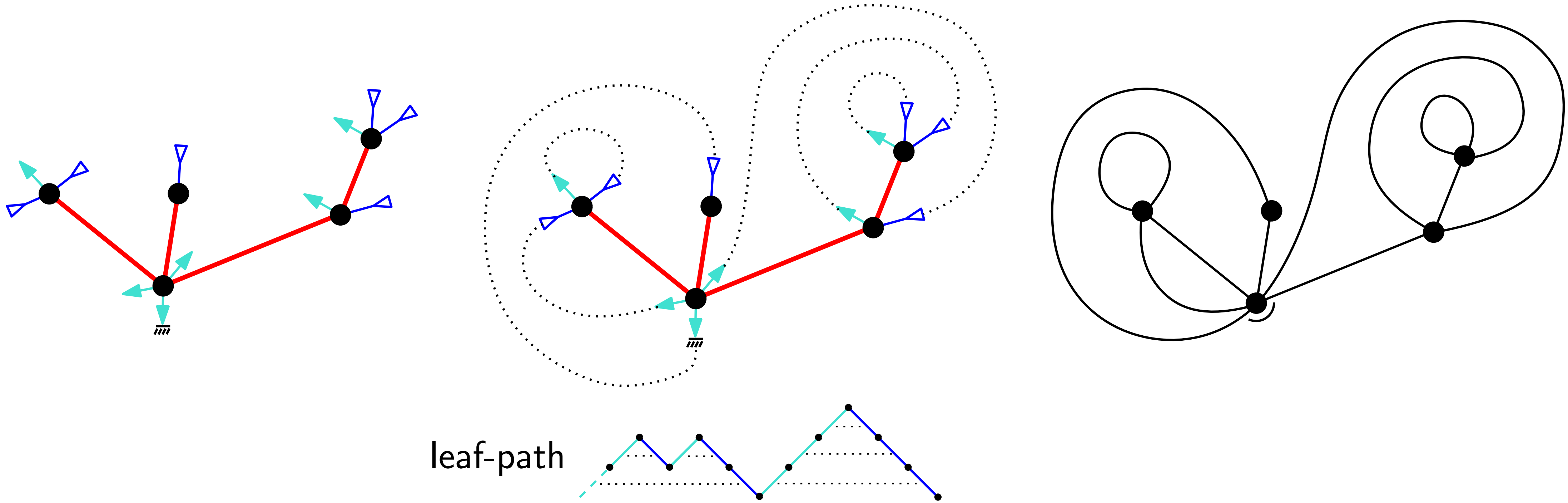

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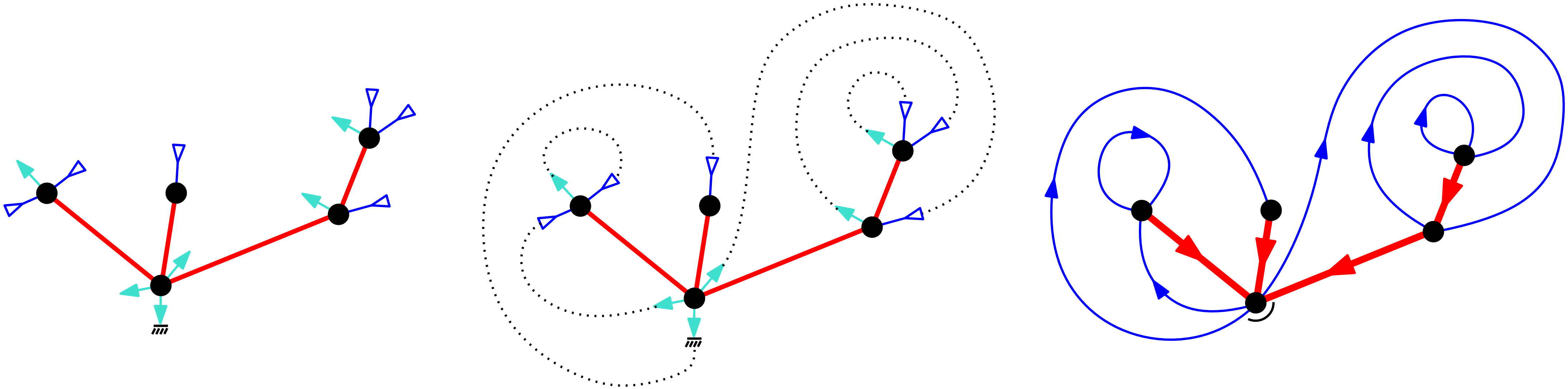
Rk: 'Cyclic lemma' argument on Eulerian trees ensures also that

$$R_1(t) = 2\int_0^1 R(x,t)\mathrm{d}x \quad \text{with } R(x,t) = x + \sum_{k\geq 1} t^k g_k \binom{2k-1}{k-1} R(x,t)^k$$

$\Rightarrow$ bijective proof of Tutte's slicings formula

Recovering the tree via Eulerian orientations

[F'07, Albenque-Poulalhon'15]



Rk: via the closure mapping, the Eulerian map is endowed with

- a spanning tree T
- an orientation O

such that edges $\in T$ are directed toward the root

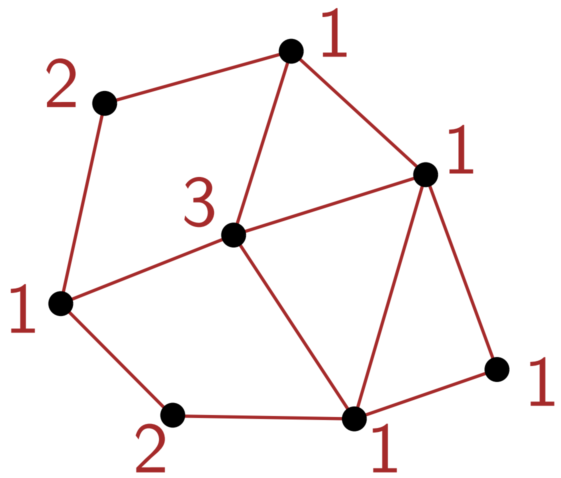
edges $\notin T$ 'turn clockwise' around T

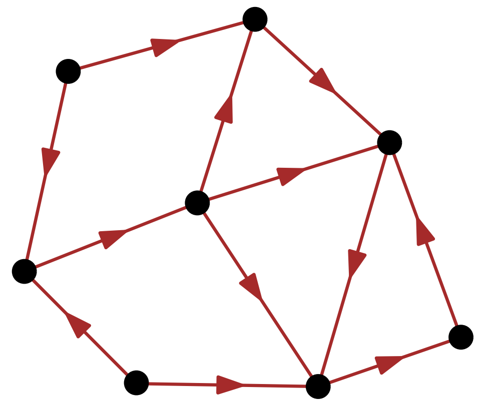
α-orientations

[Propp'02], [Felsner'03]

For $G = (V, E)$ a graph and $\alpha : V \to \mathbb{N}$

α-orientation of G = orientation where every vertex v has outdegree $\alpha(v)$





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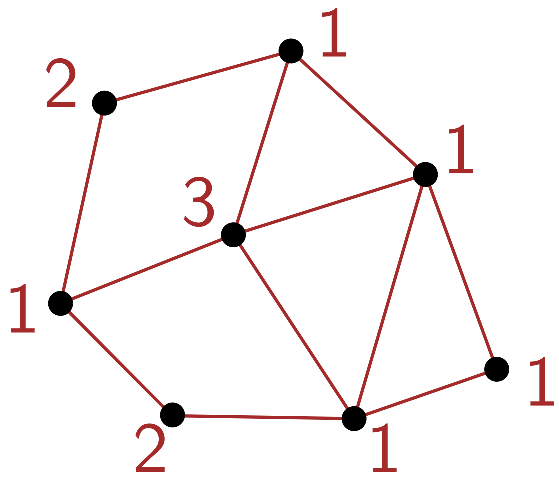

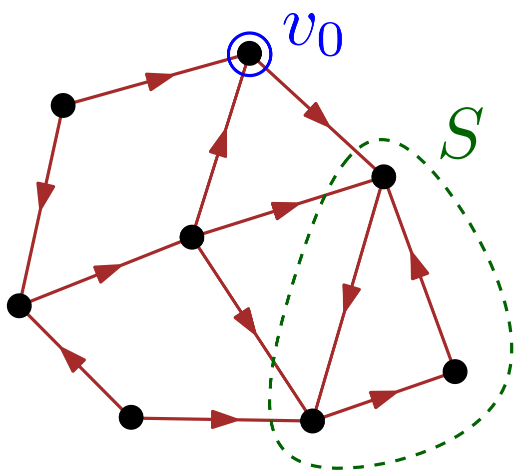

Let $v_0 \in V$ be a marked vertex

an orientation is **v_0-accessible** if $\forall v \in V$ there is a path from v to v_0

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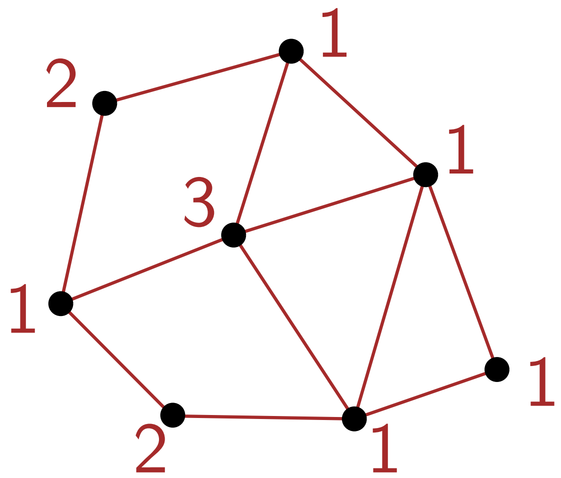
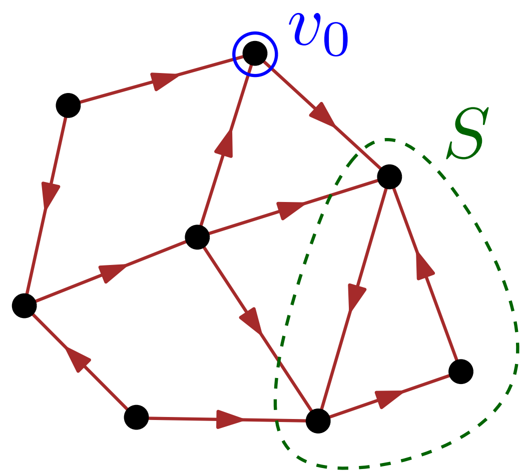
α-orientation of G = orientation where every vertex v has outdegree $\alpha(v)$





Let $v_0 \in V$ be a marked vertex

an orientation is **v_0-accessible** if $\forall v \in V$ there is a path from v to v_0

Property: either all α-orientations are v_0-accessible or none

In the first case (and non-emptiness), α is called **root-accessible**

Bernardi's bijection (planar case)

[Bernardi'07]

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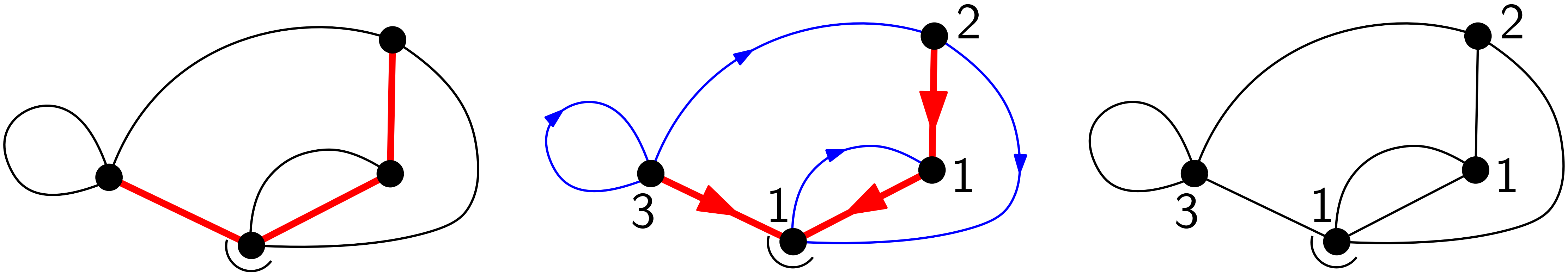

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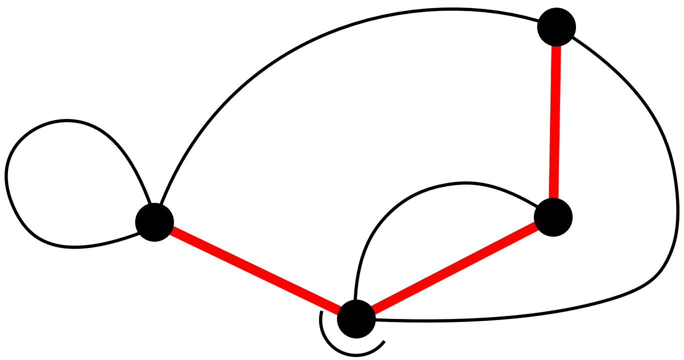

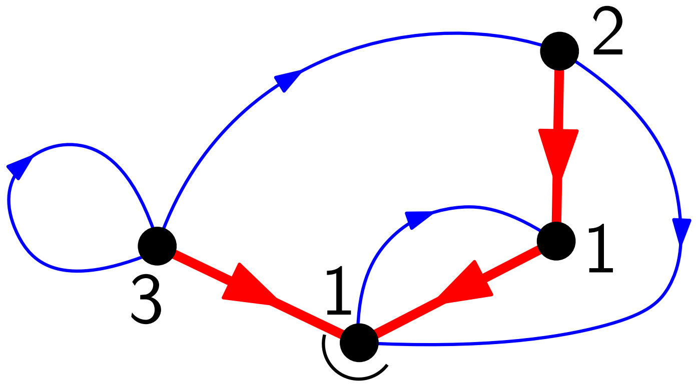

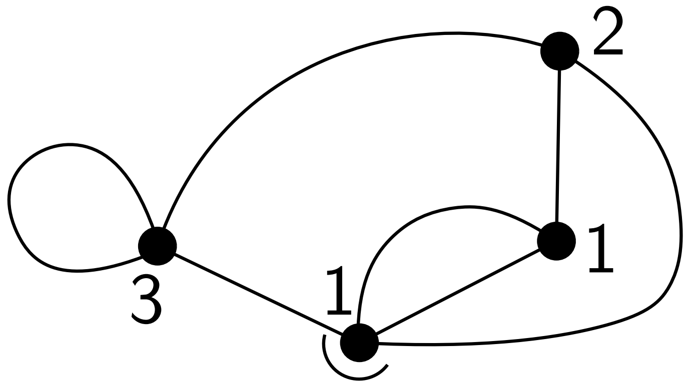

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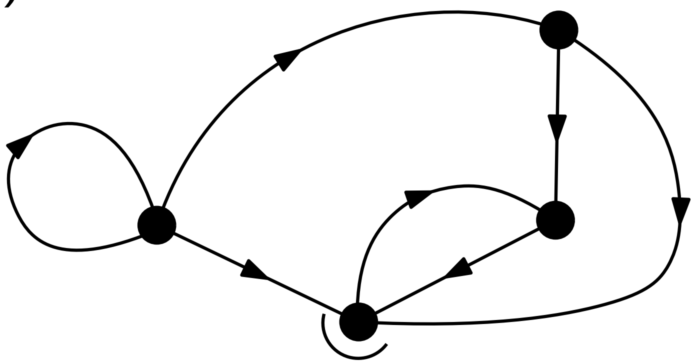

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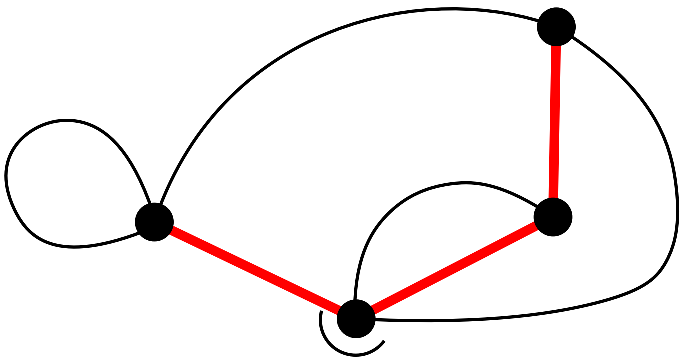

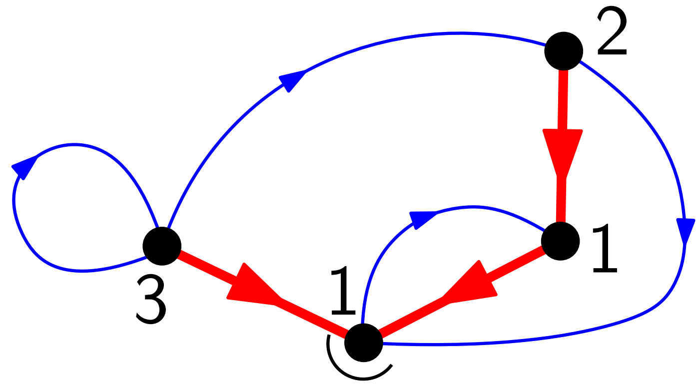

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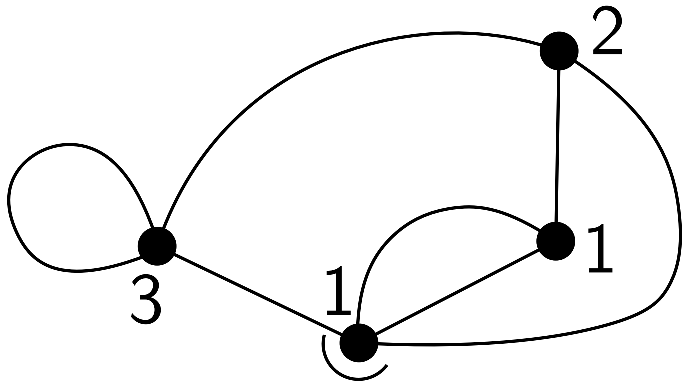

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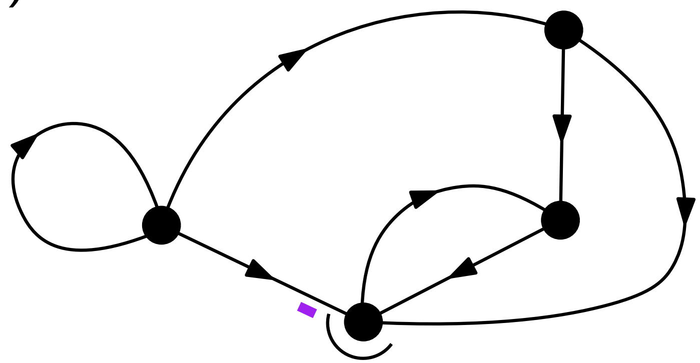

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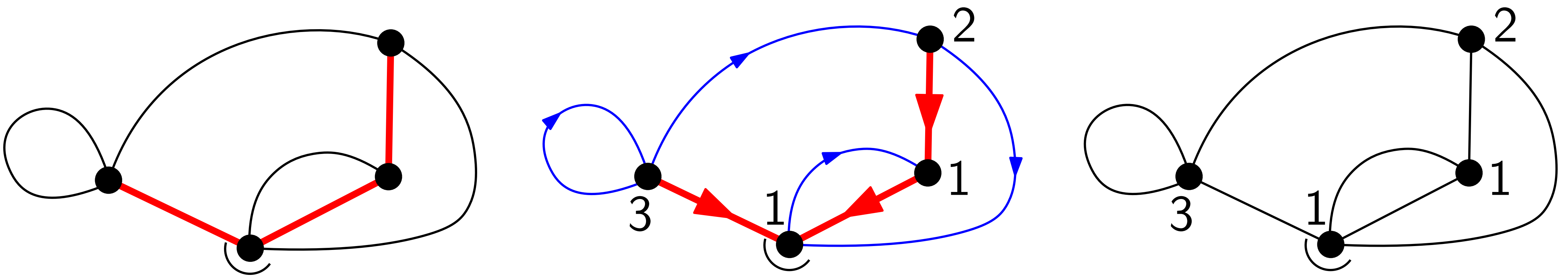

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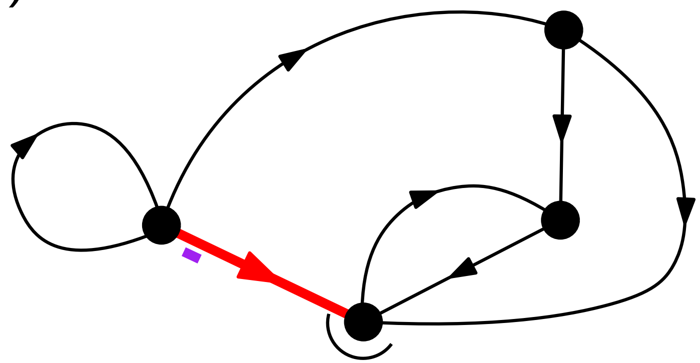

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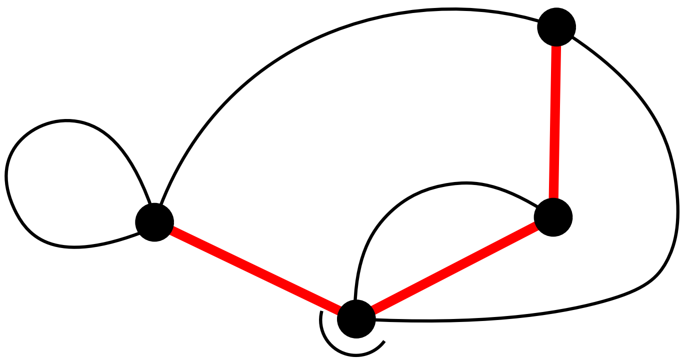

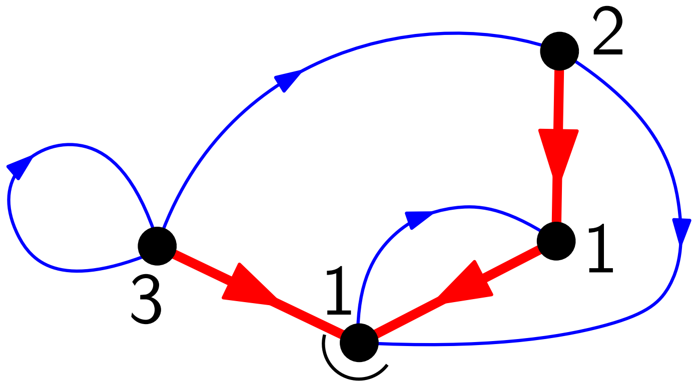

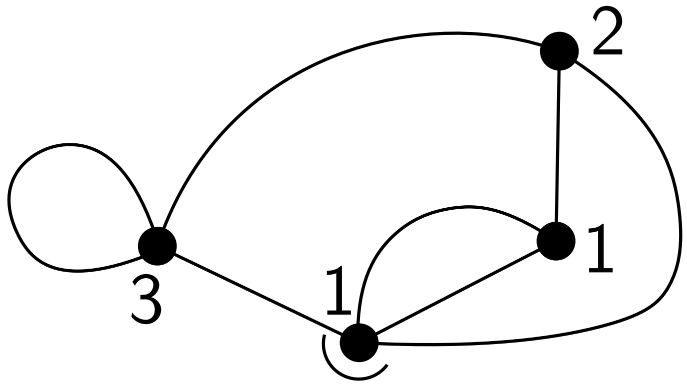

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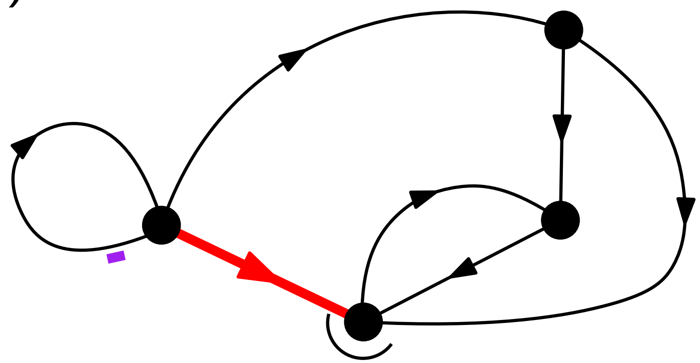

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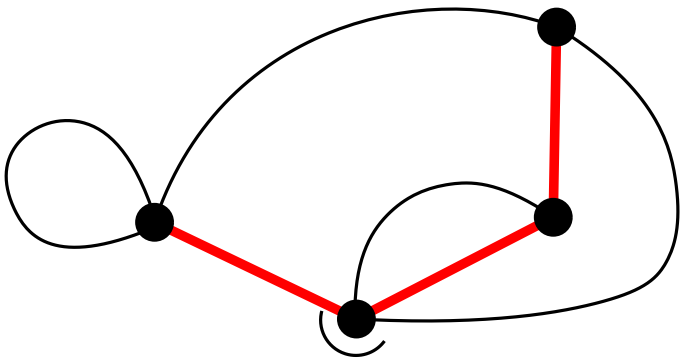

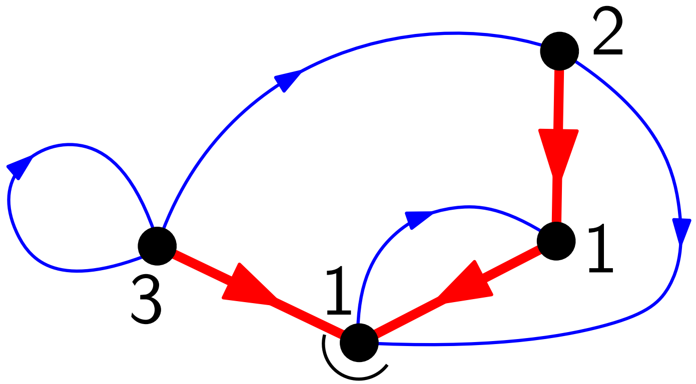

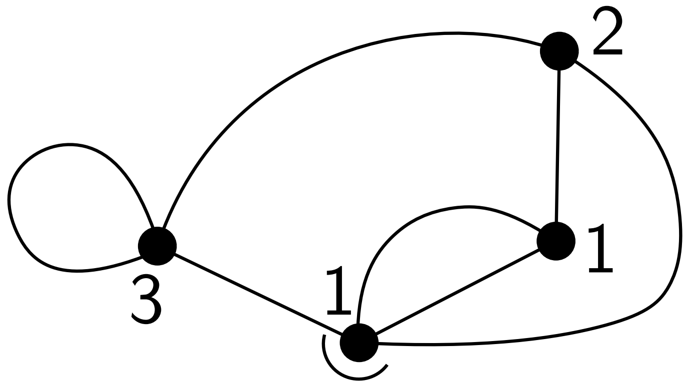

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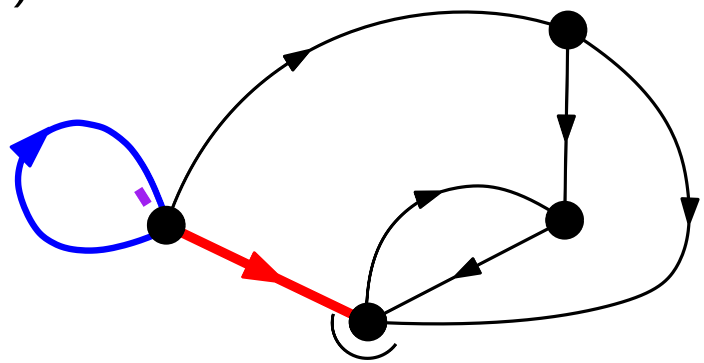

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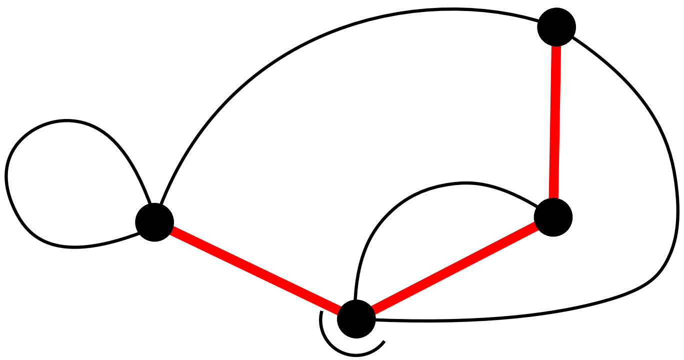

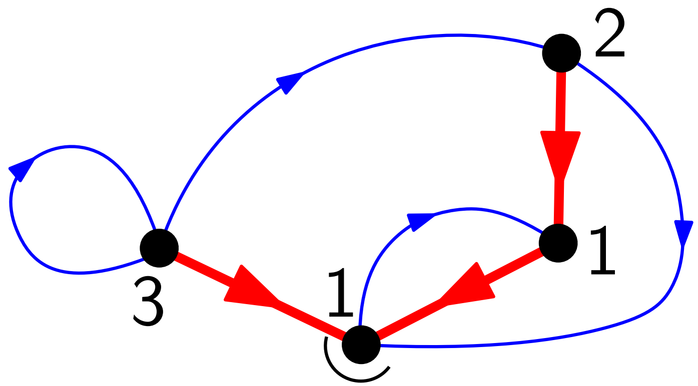

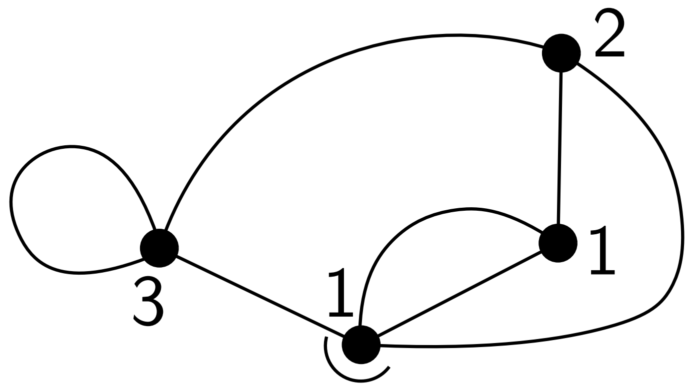

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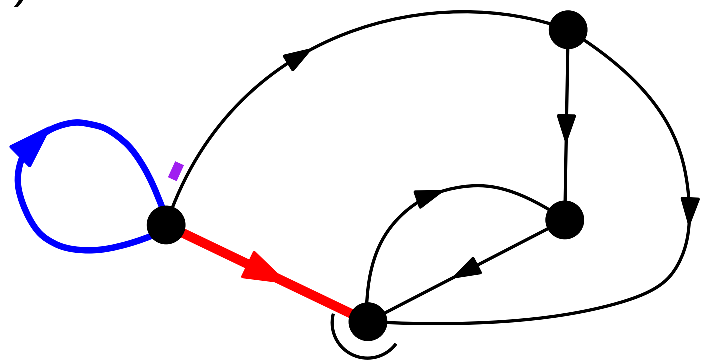

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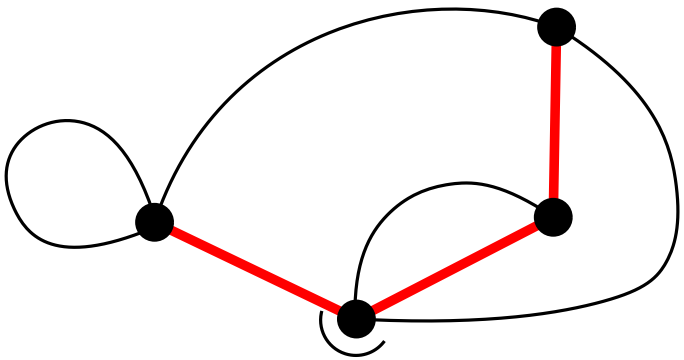

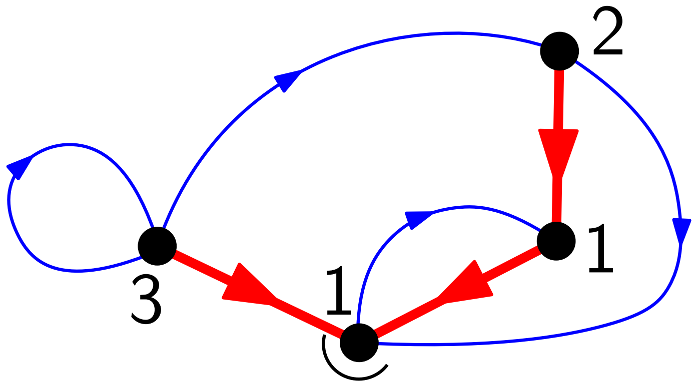

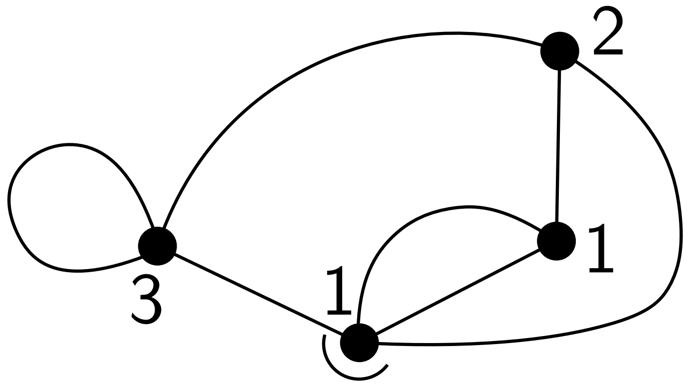

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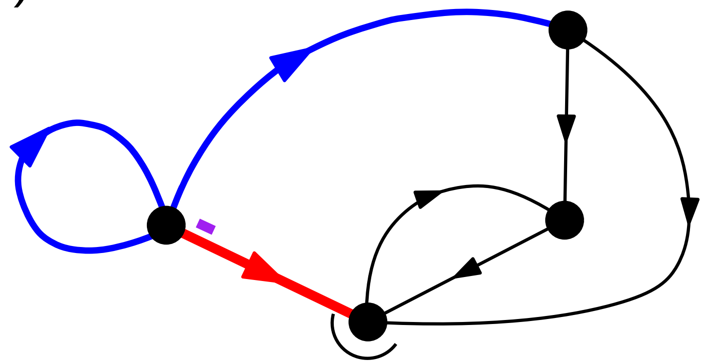

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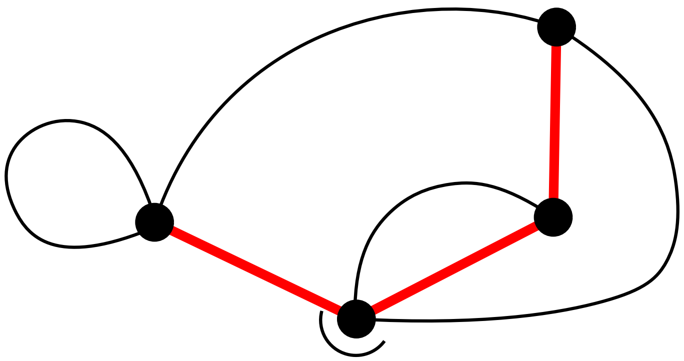

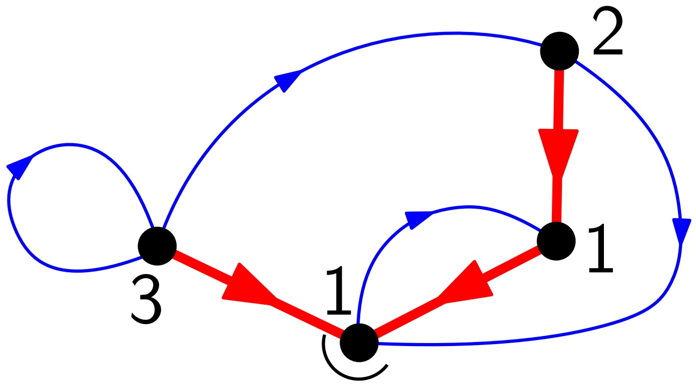

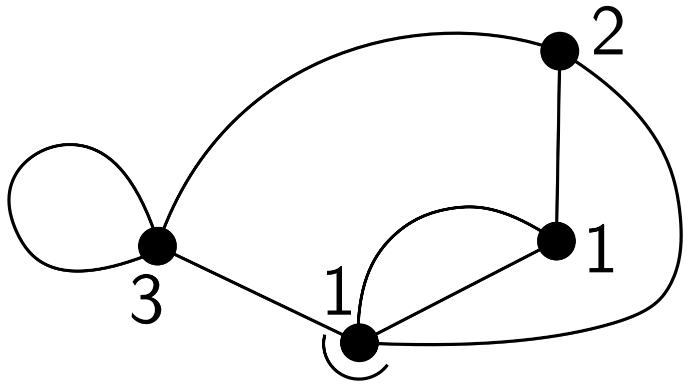

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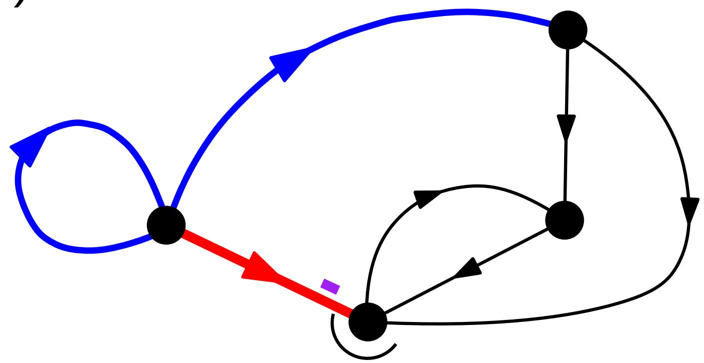

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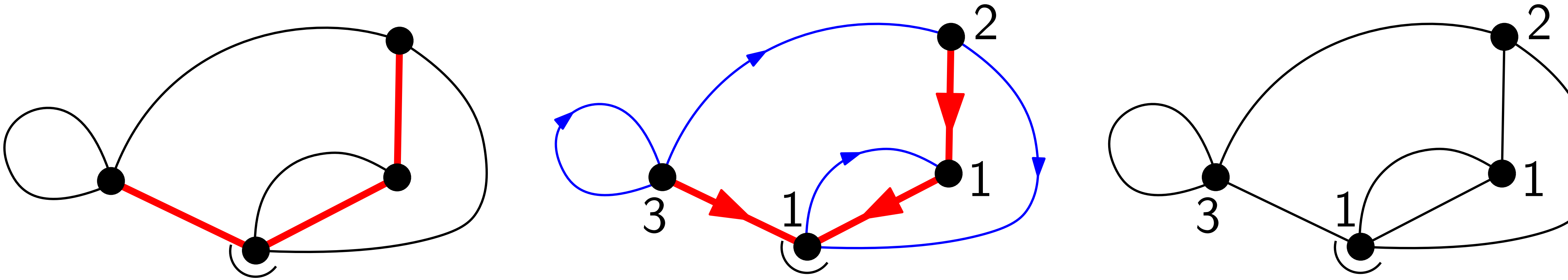

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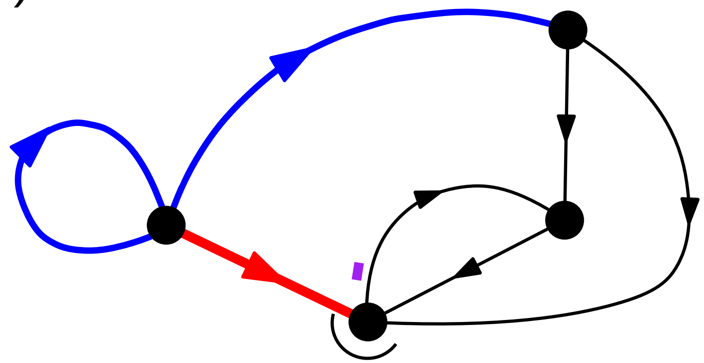

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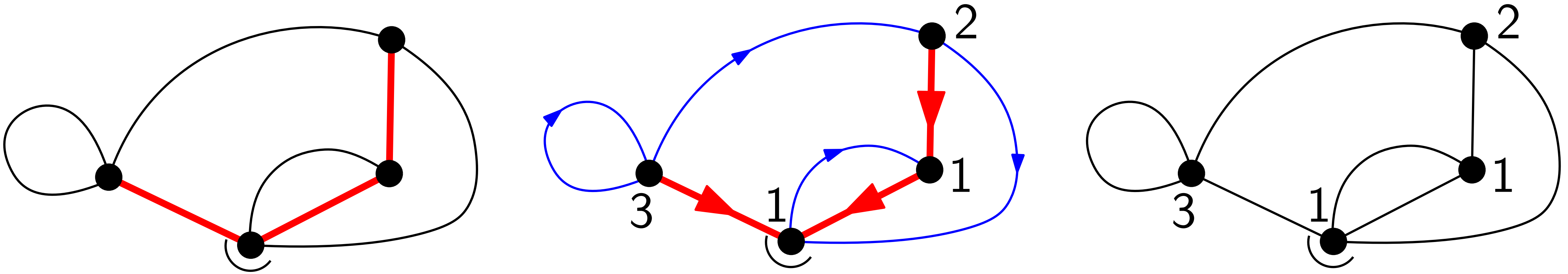

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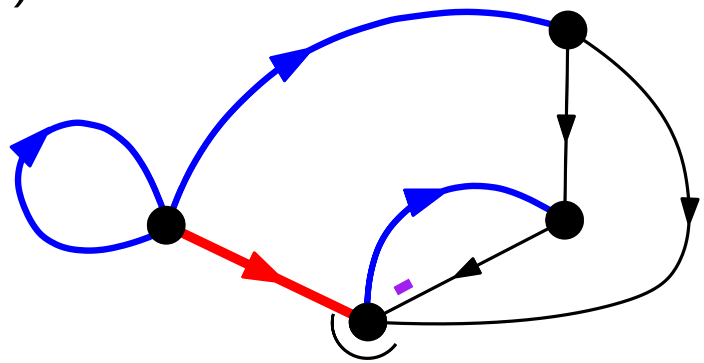

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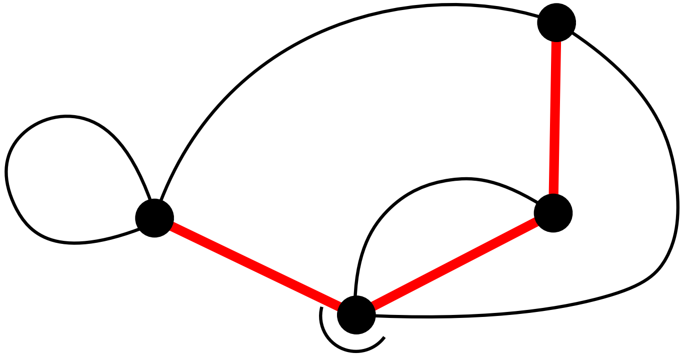

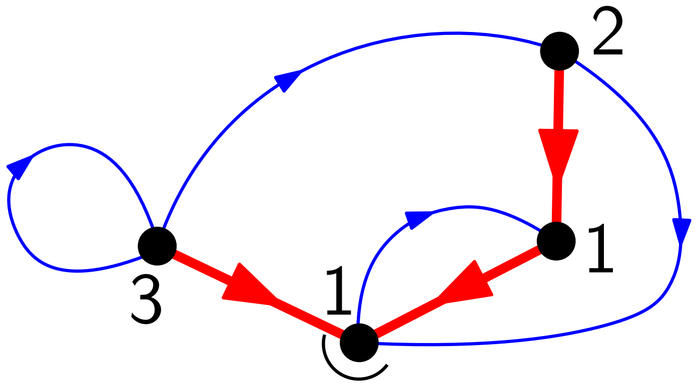

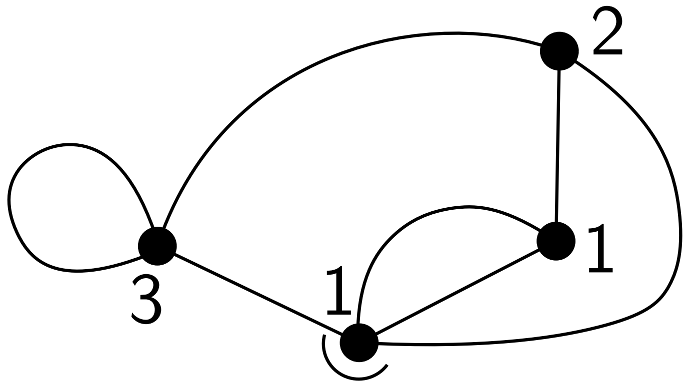

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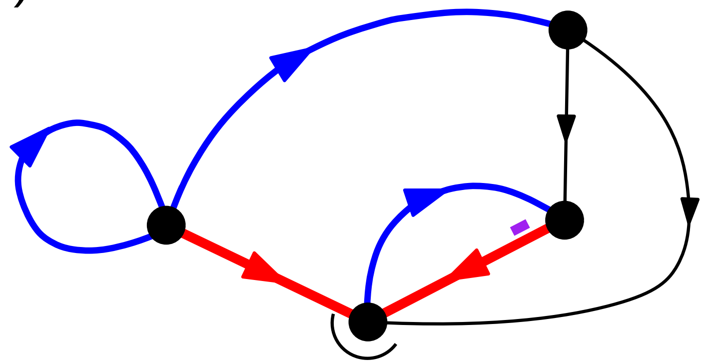

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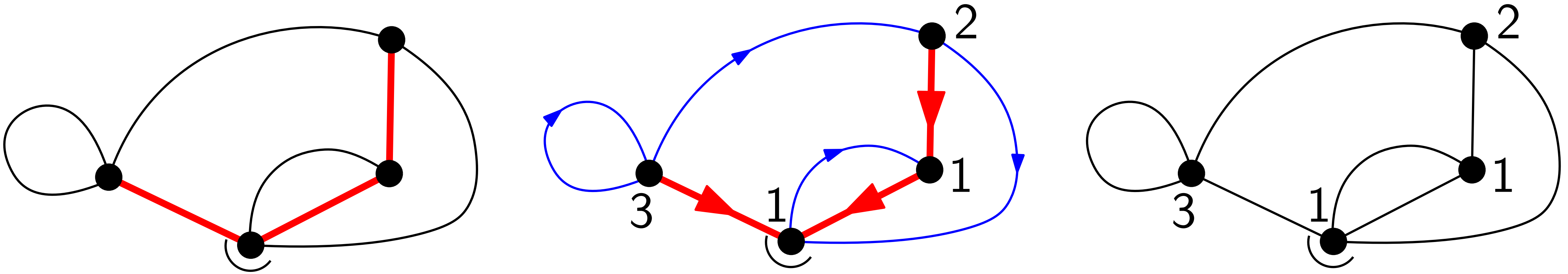

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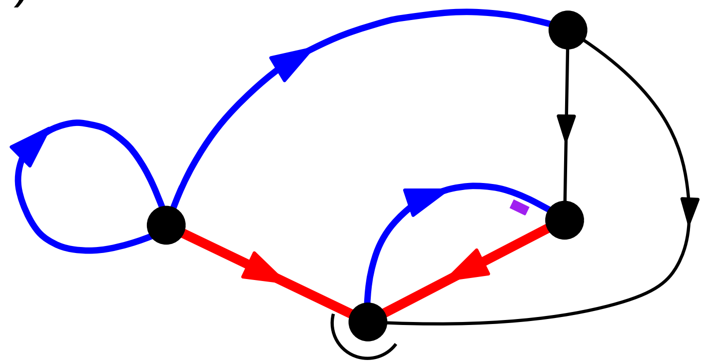

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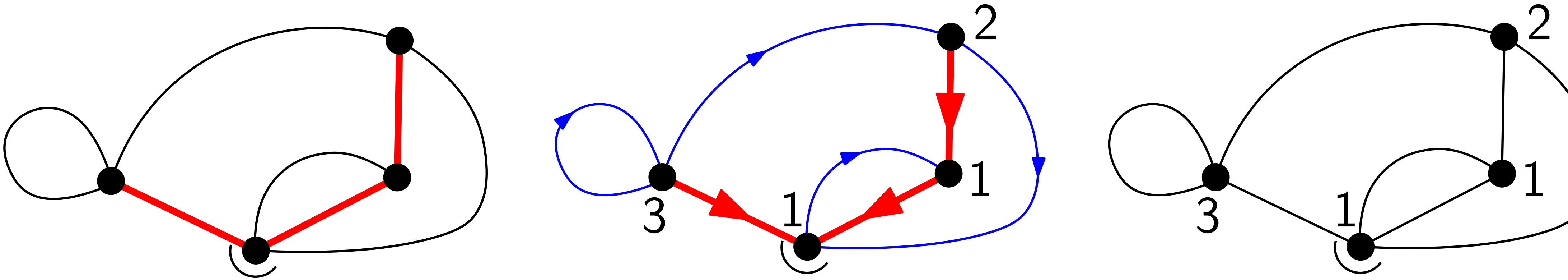

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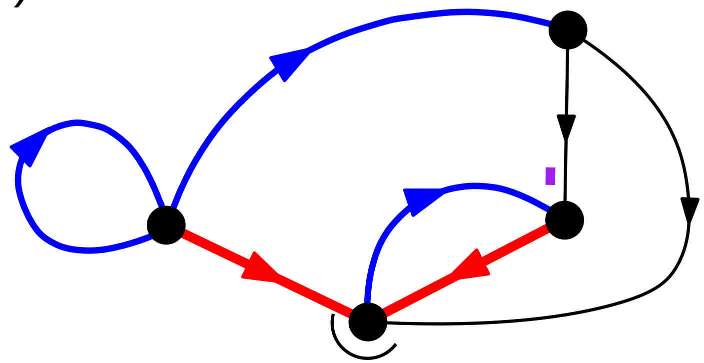

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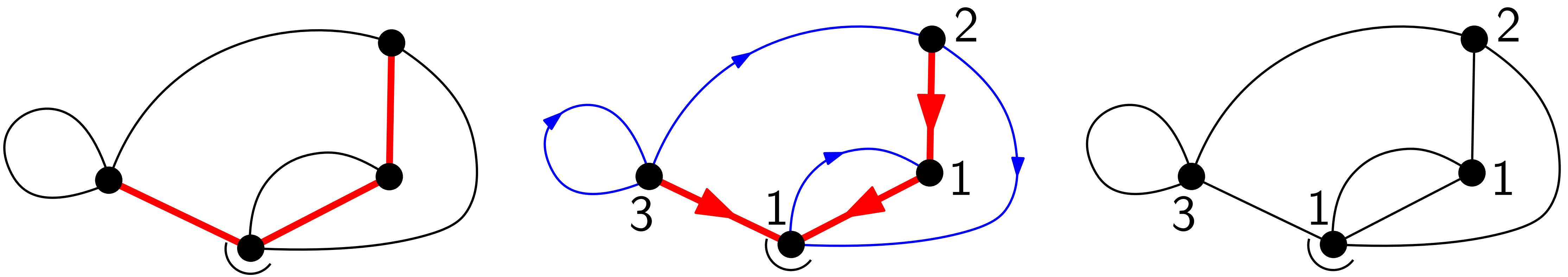

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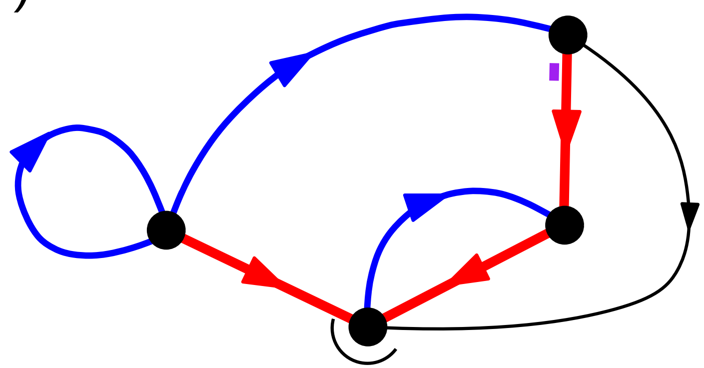

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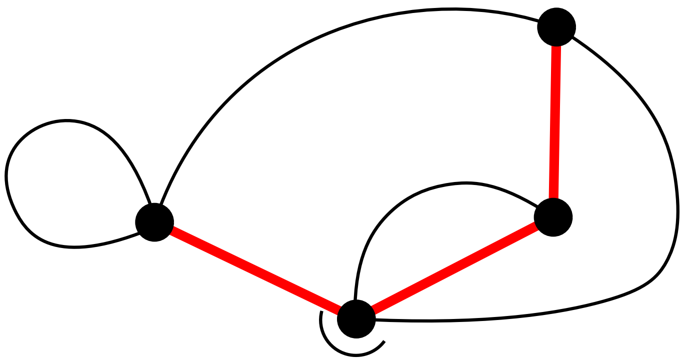

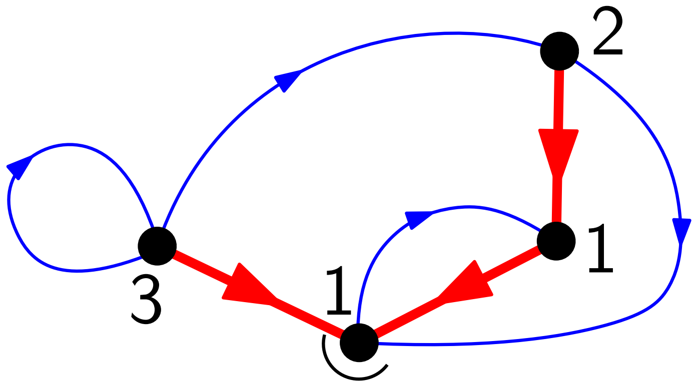

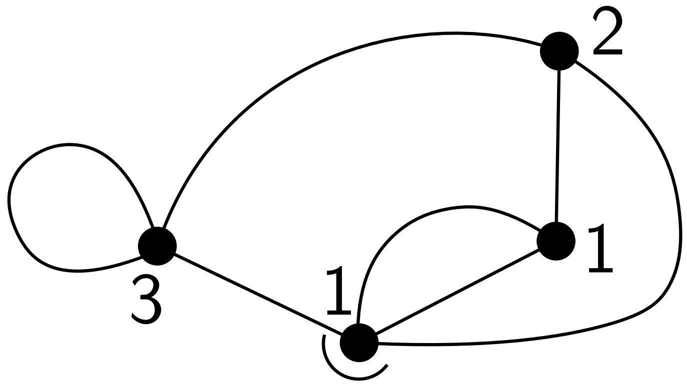

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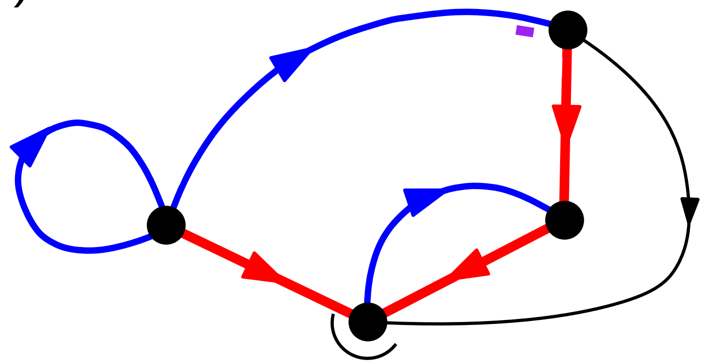

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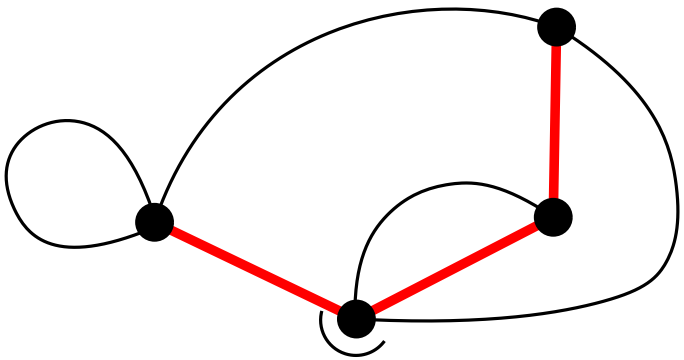

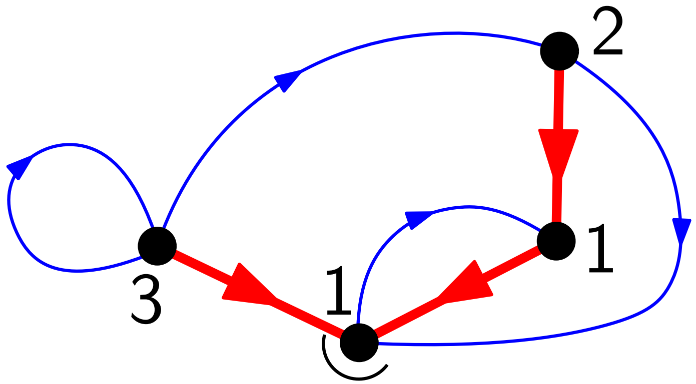

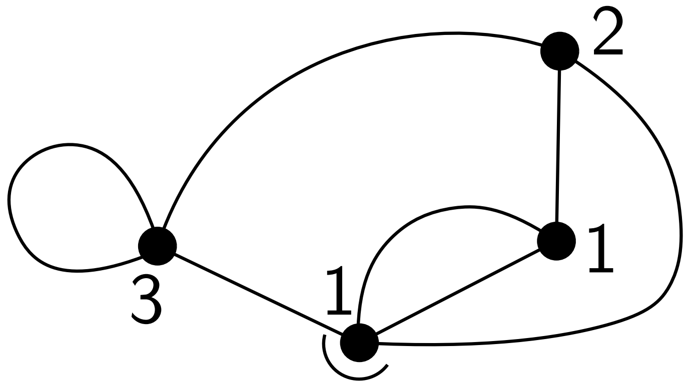

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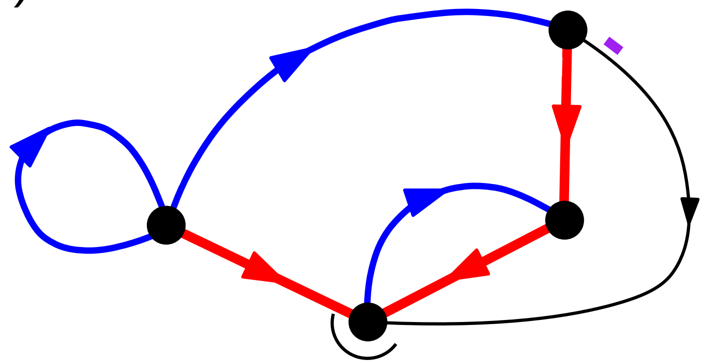

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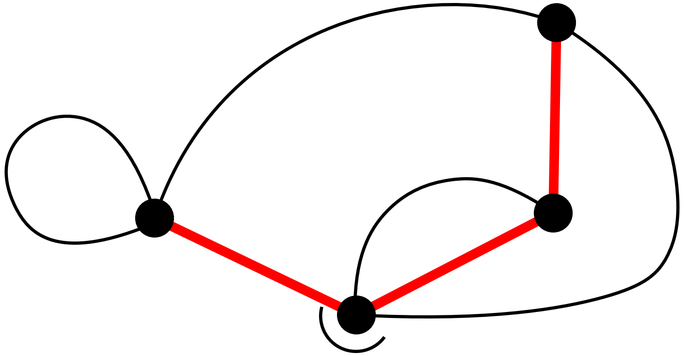

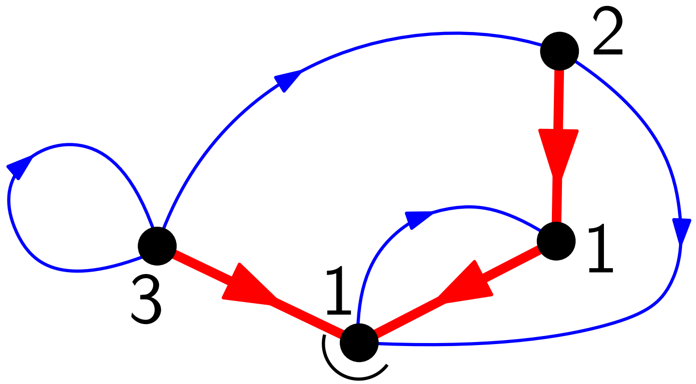

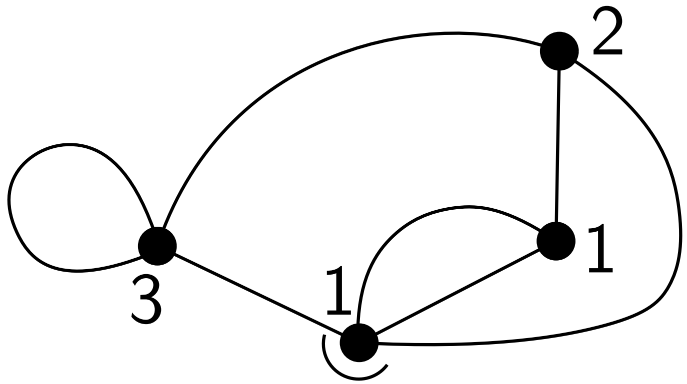

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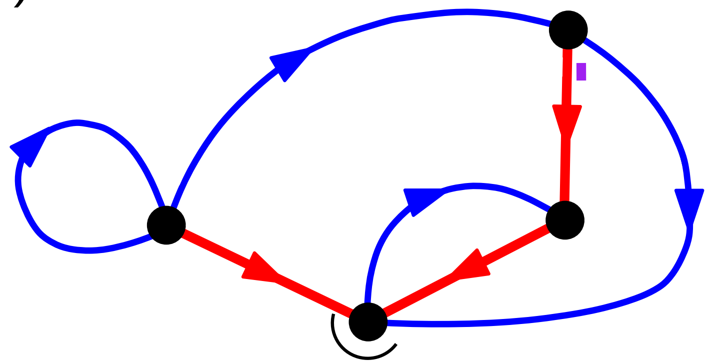

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[Bernardi'07]

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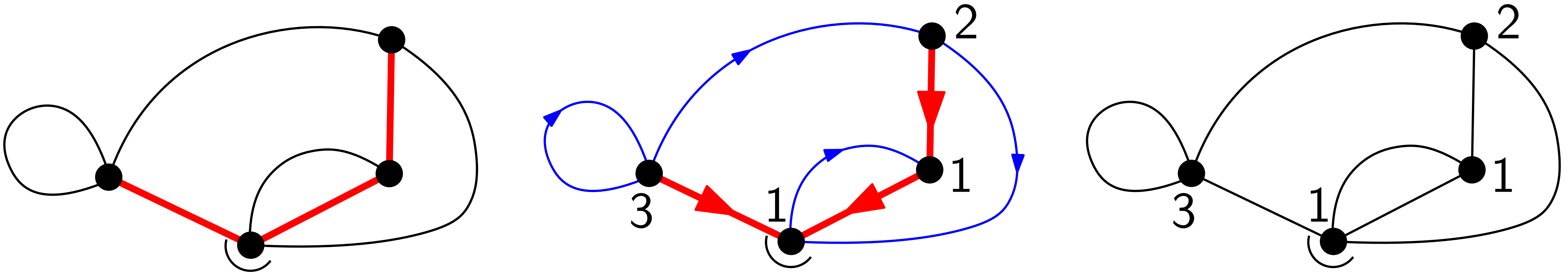

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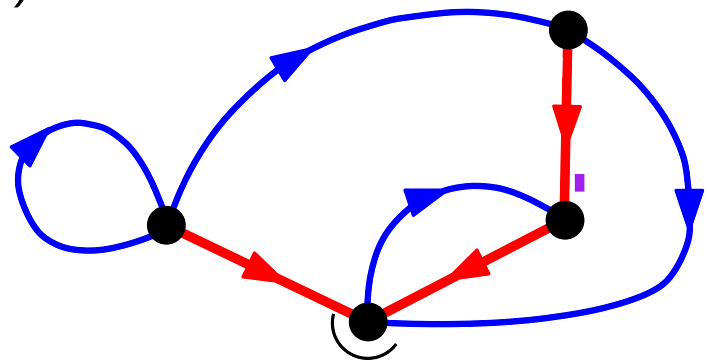

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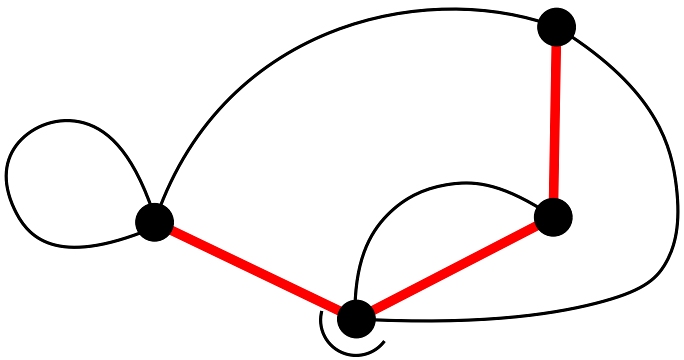

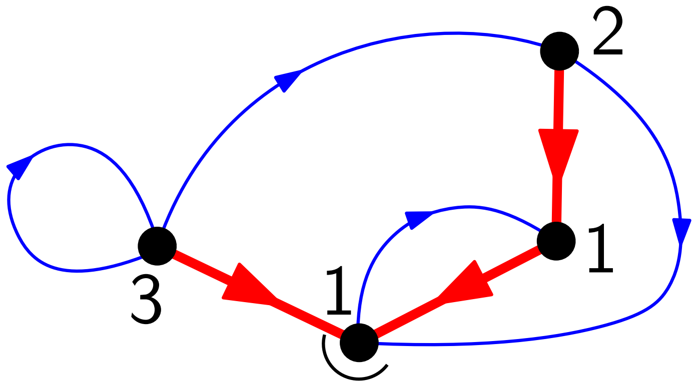

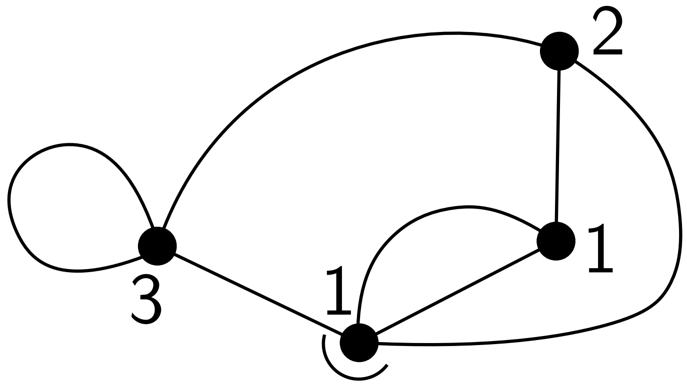

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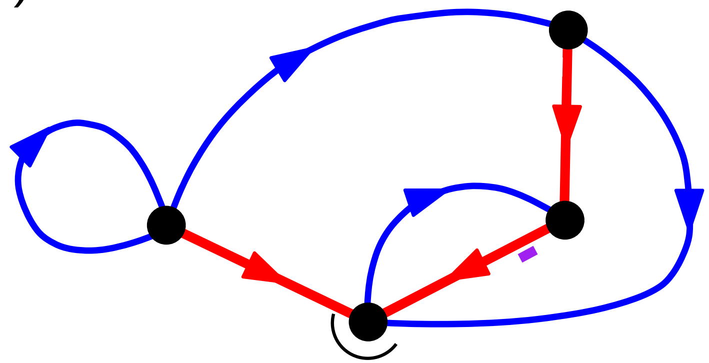

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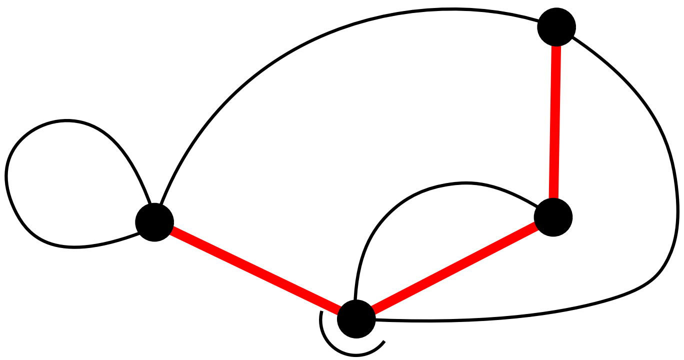

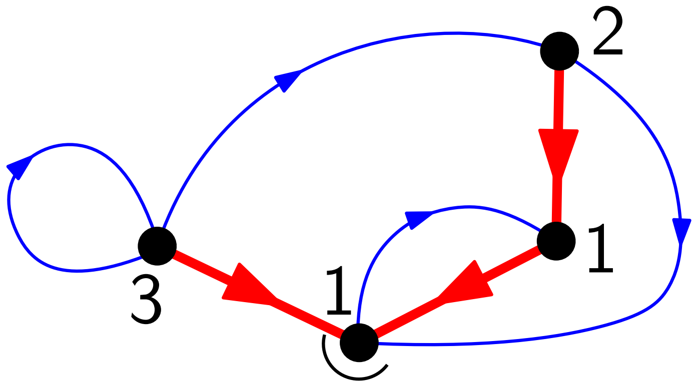

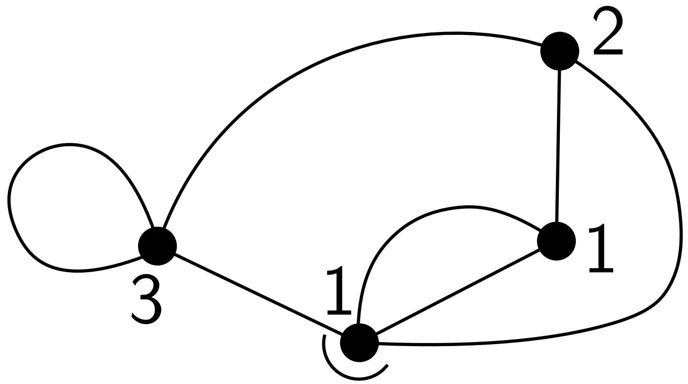

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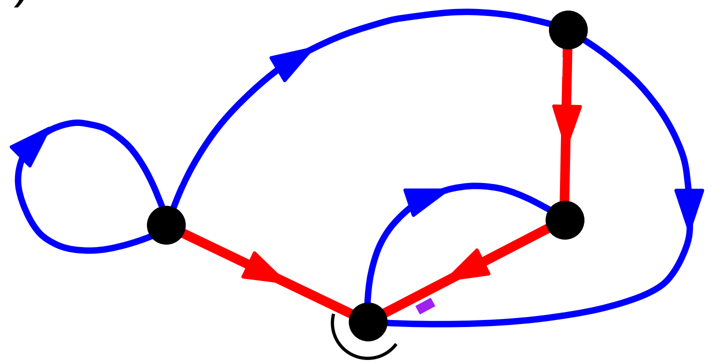

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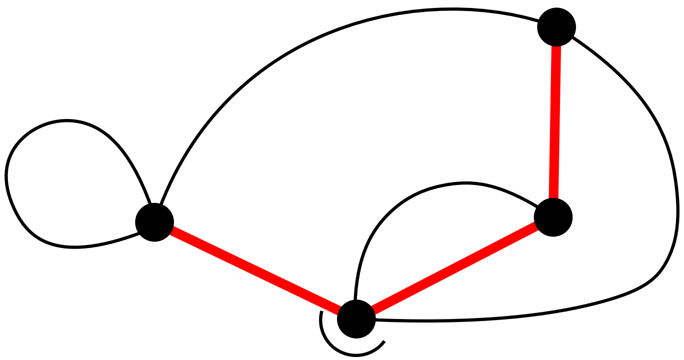

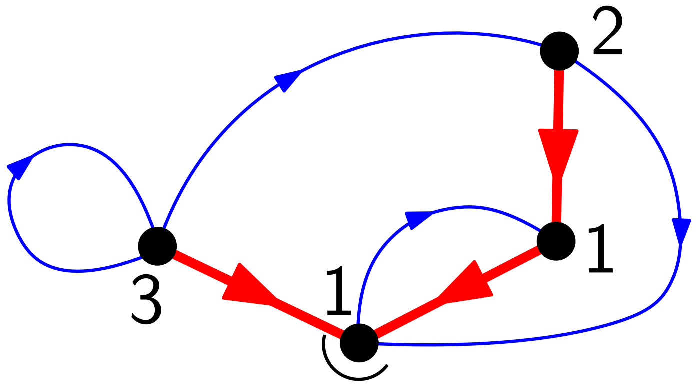

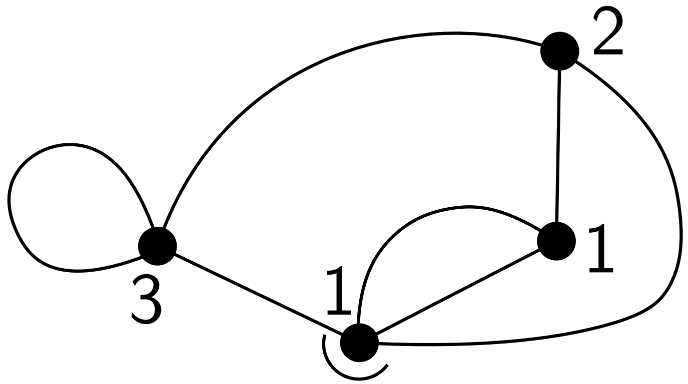

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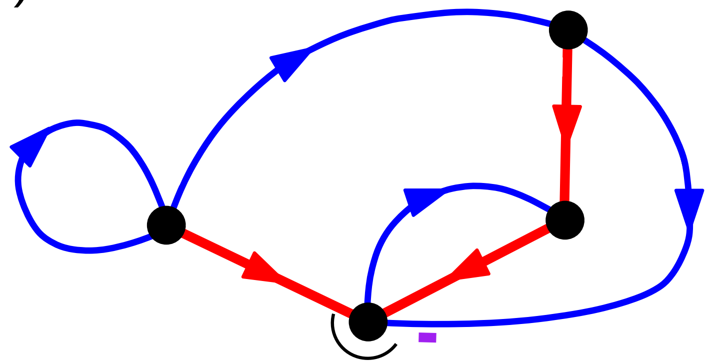

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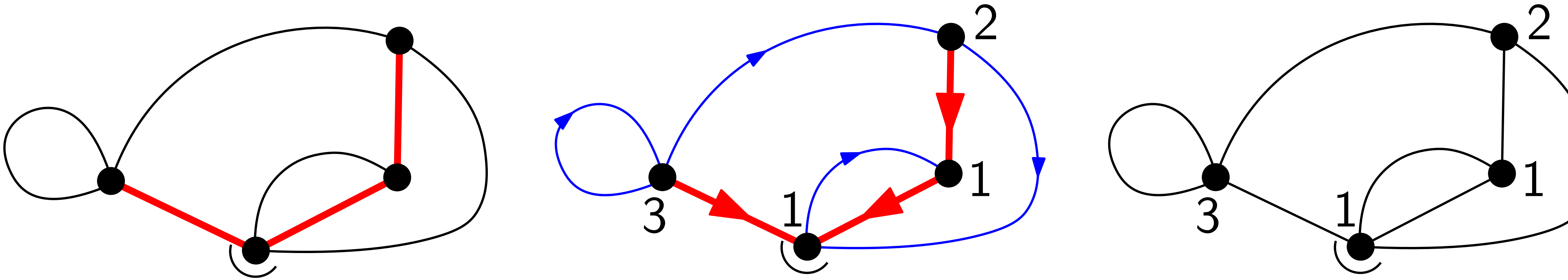

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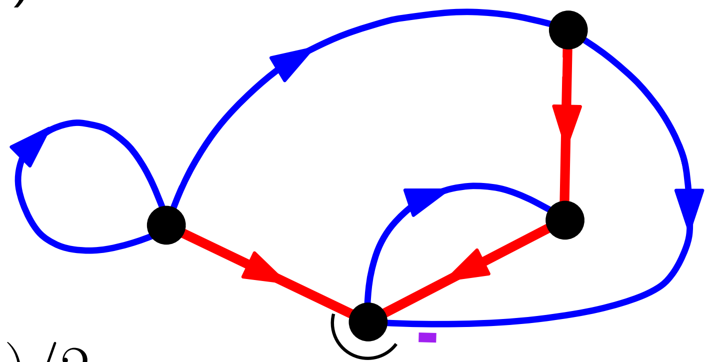
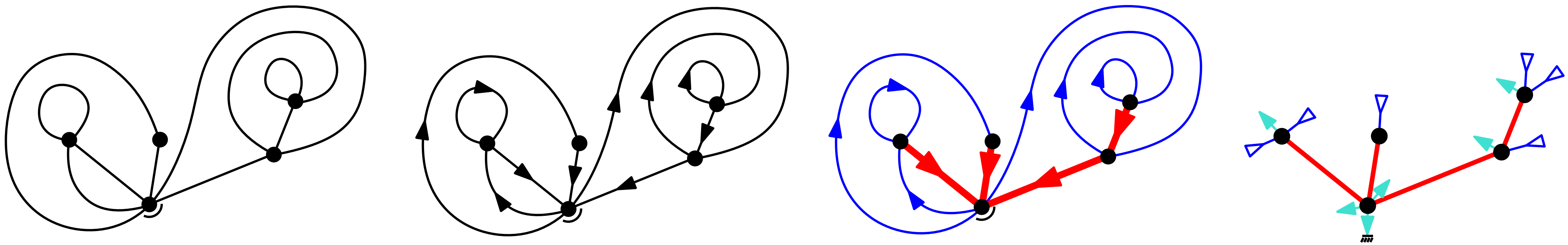


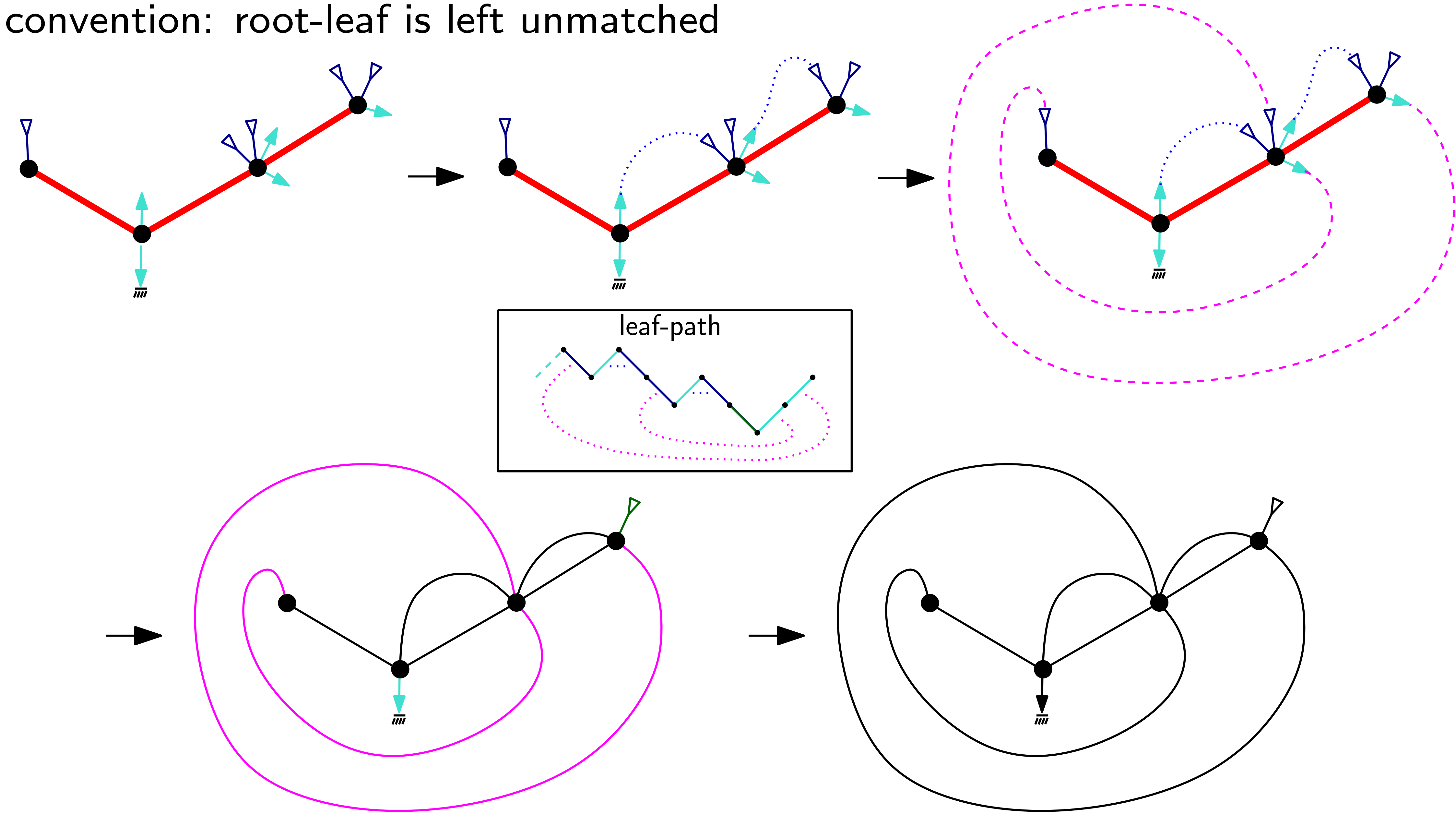
We apply it to Eulerian planar maps, with $\alpha(v) = \deg(v)/2$



Extended bijection: Eulerian trees → 2-leg maps

[Bouttier-Di Francesco-Guitter'03]

convention: root-leaf is left unmatched

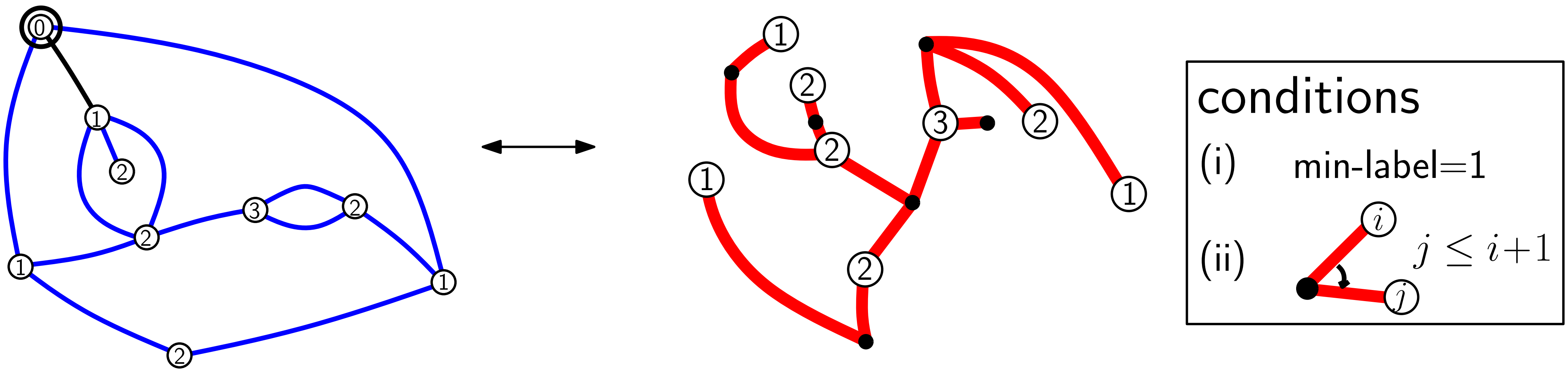


Rk: For $i \geq 1$, the tree is i-balanced iff two legs are at (dual) distance $\leq i - 1$

Other bijection for $R_i(t)$, explicit expression

- Bijection with labeled mobiles

[Bouttier-Di Francesco-Guitter'04]



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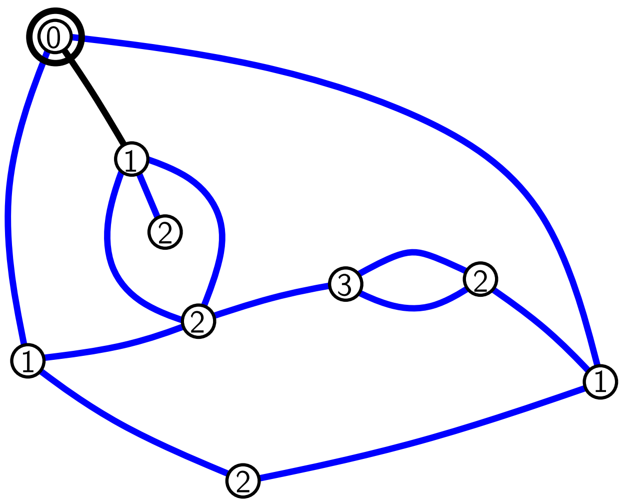
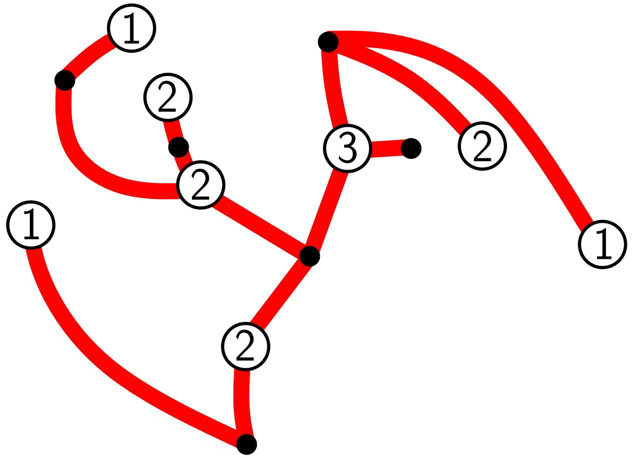
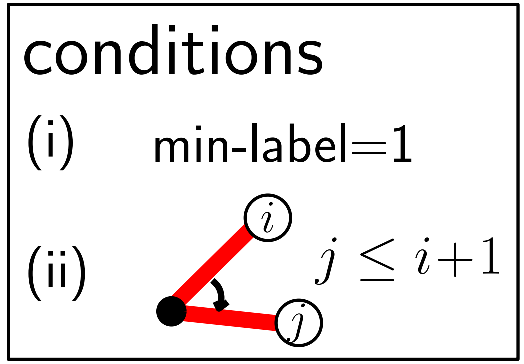
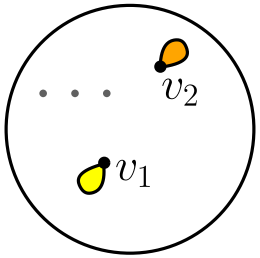


$\longleftrightarrow$

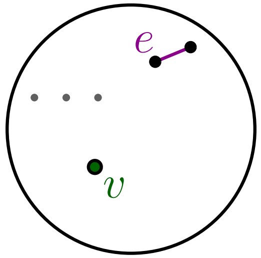




$\mathrm{dist}(v_1, v_2) \leq i-1$



$\xleftarrow{\text{blossoming trees}}$ $R_i(t)$ $\xrightarrow[\text{blossoming trees (root-leaf matched)}]{\text{labeled mobiles}}$



$\mathrm{dist}(v, e) \leq i-1$

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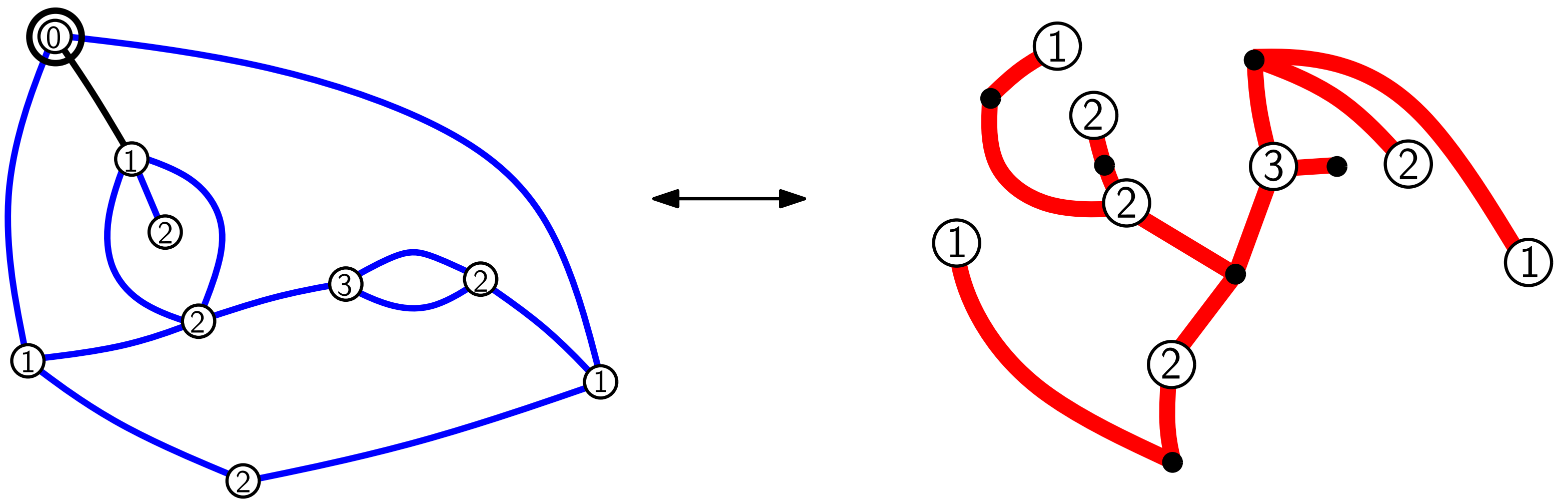
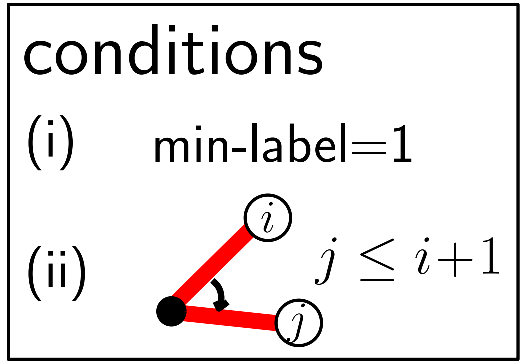
- Bijection with labeled mobiles

[Bouttier-Di Francesco-Guitter'04]





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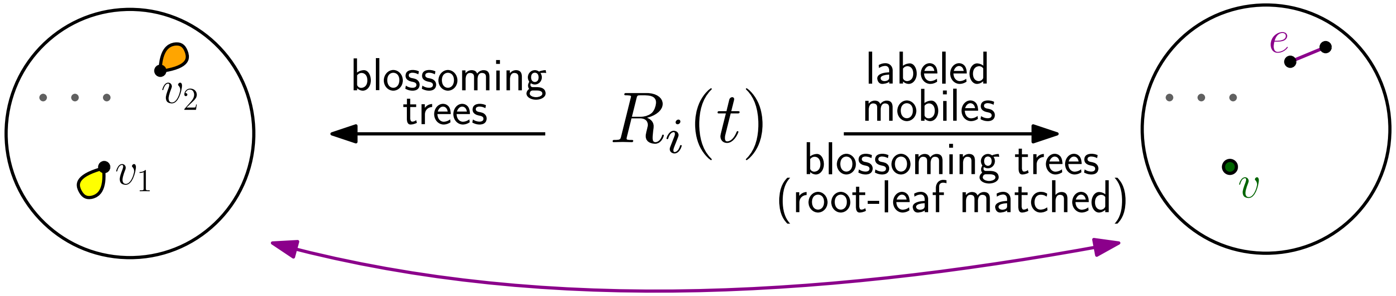
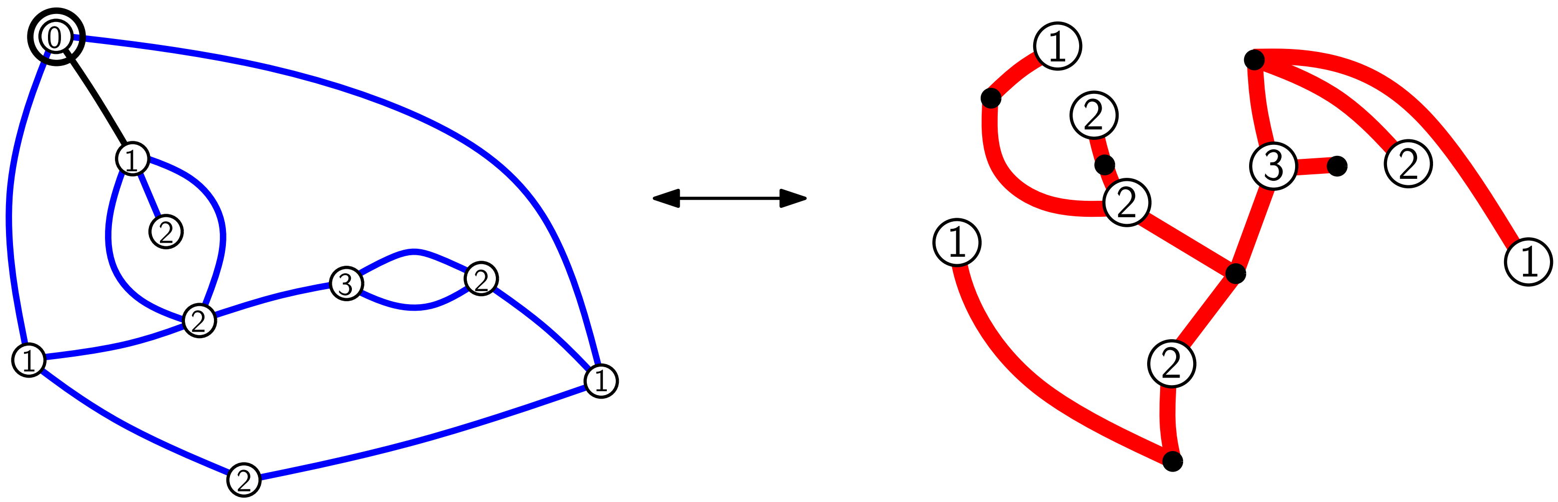


$\mathsf{dist}(v, e) \leq i-1$

slit-slide-sew bijection [Bettinelli'18]

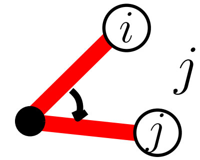
Other bijection for $R_i(t)$, explicit expression

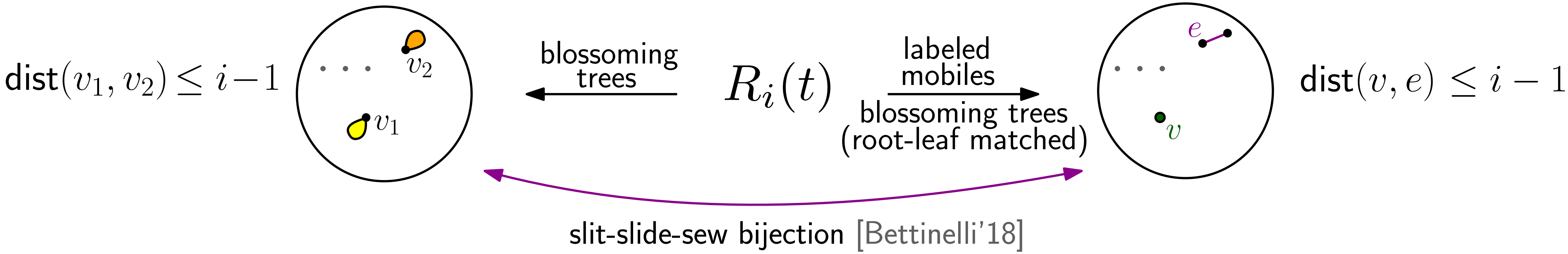
- Bijection with labeled mobiles [Bouttier-Di Francesco-Guitter'04]



conditions

(i) min-label=1

(ii)  $j \leq i+1$



- Explicit expression: [Bouttier-Di Francesco-Guitter'03] [Bouttier-Guitter'12]

$$R_n(t) = \frac{H_n H_{n-2}}{H_{n-1}^2} \quad \text{with} \quad H_n = \det_{0 \leq i,j \leq n} F_{i+j}$$

$F_a :=$ GF eulerian planar maps root-vertex degree a

for vertex-degrees $\leq 2p+2$, expression simplifies as biratio involving $(p \times p)$-determinants

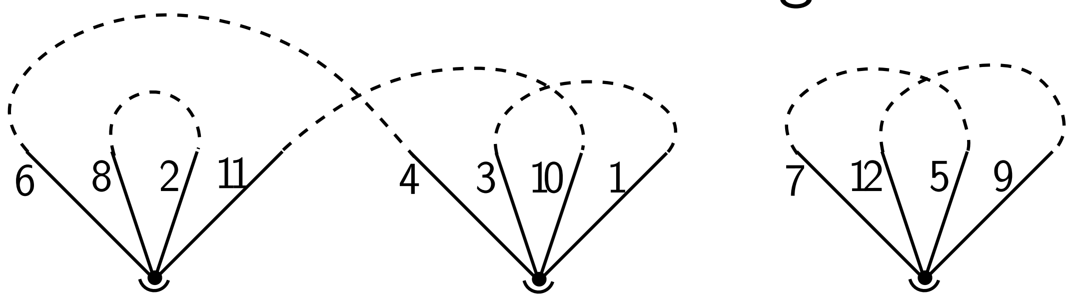
Maps of unfixed genus

- standard counting approaches
- approach based on orthogonal polynomials
- bijective interpretation

4-regular maps

- **1st approach: configuration model**

Let $\mathcal{U}_n :=$ family of 4-regular maps on n vertices that are unrooted & half-edge-labeled & not necessarily connected



$$|\mathcal{U}_n| = \frac{1}{4^n n!}(4n)!(4n-1)!!$$

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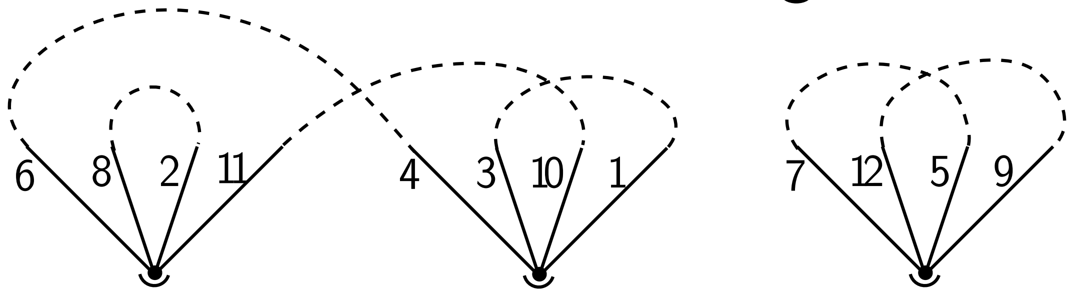

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($|\mathcal{U}_0| = 1$ with convention $(-1)!! = 1$)

$\Rightarrow$ EGF of $\mathcal{U} = \cup_n \mathcal{U}_n$ is $U(g) = \sum_{n\geq 0} \frac{|\mathcal{U}_n|}{(4n)!} g^n = \sum_{n\geq 0} \frac{(4n-1)!!}{4^n n!} g^n$

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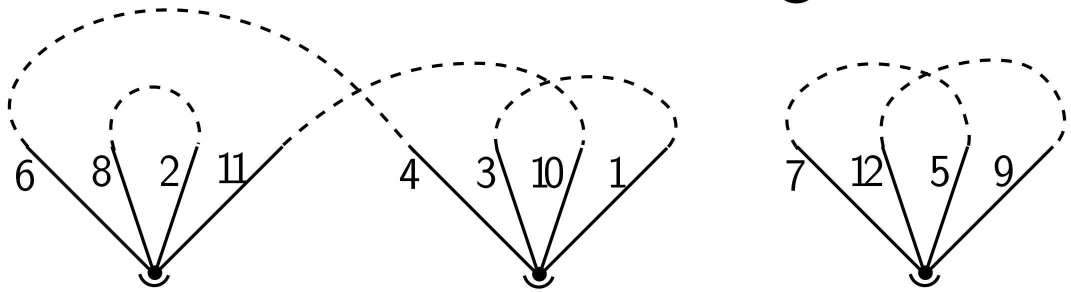

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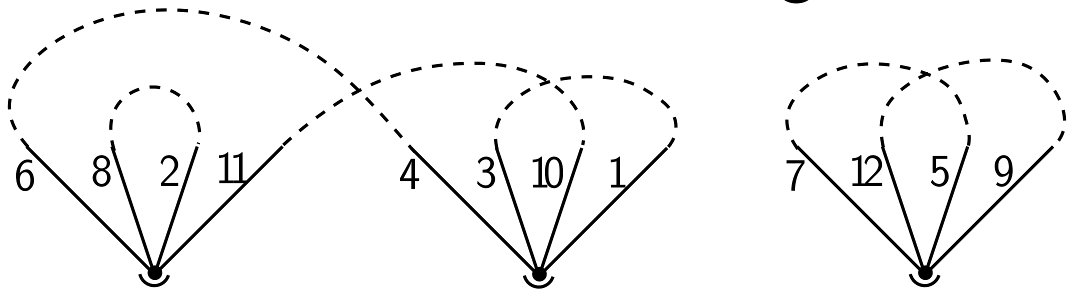

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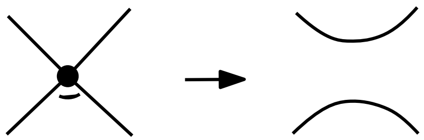
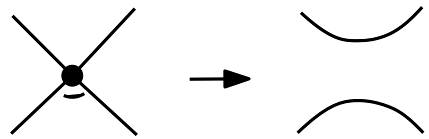
cf [Arquès-Béraud'00, Vidal-Petitot'10, Courtiel-Yeats-Zeilberger'17]

$$M(g) = 3g \ + \ 6g\ M(g) \ + \ (4g^2 \tfrac{\mathrm{d}}{\mathrm{d}g} M(g) - 2gM(g)) \ + \ g\ M(g)^2$$

2 loops at v_0 1 loop at v_0



still connected



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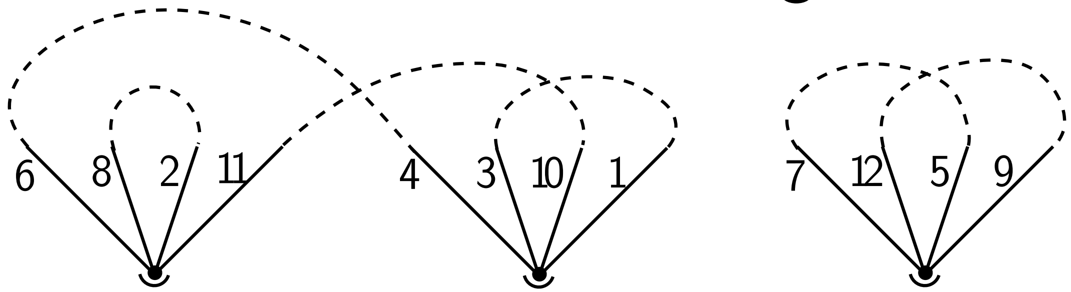

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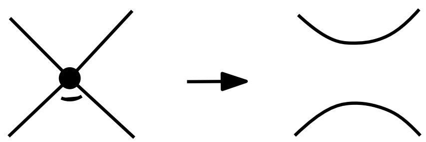

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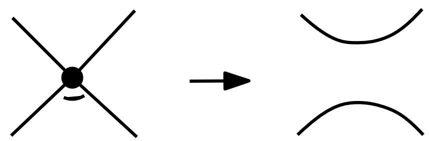

disconnected

(differential equation of Riccati type $\Rightarrow$ nice continued fraction expansion)

Extension to Eulerian maps

- **1st approach: configuration model**

The EGF of Eulerian maps that are unrooted, half-edge-labeled, and not necessarily connected is $\boxed{U(t) = \Lambda\Big(\exp\big(\sum_{k\geq 1} \frac{1}{2k} t^k g_k \big)\Big)}$

with Λ the operator: $\Lambda\big(\sum_{n\geq 0} c_n t^n \big) := \sum_{n\geq 0} (2n-1)!! c_n t^n$

$\Rightarrow$ the GF of rooted Eulerian maps is $\boxed{M(t) = 2t \frac{\mathrm{d}}{\mathrm{d}t} \log(U(t))}$

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where $\widetilde{M}_i(t) :=$ subseries of $M_i(t)$ with no loop at v_0

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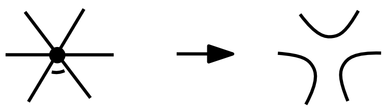

$e^{C(t)} \cdot \widetilde{M}_i(t) = 2^i \frac{\mathrm{d}^i}{\mathrm{d}t^i} e^{C(t)}$ with $2t\frac{\mathrm{d}}{\mathrm{d}t}C(t) = M(t)$

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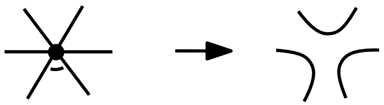
$\Rightarrow$ the GF of rooted Eulerian maps is $M(t) = 2t \frac{\mathrm{d}}{\mathrm{d}t} \log(U(t))$

- **2nd approach: deletion of root-vertex v_0 (non-linear DE for $M(t)$)**

$M(t) = \sum_{k\geq 1} t^k g_k M_k(t)$ where $M_k(t)$ subseries of $M(t)$ with $\deg(v_0) = 2k$

$$M_k(t) = (2k-1)!! + \sum_{i=1}^{k} \frac{(2k)!}{(2i)!(k-i)!2^{k-i}} \widetilde{M}_i(t)$$

where $\widetilde{M}_i(t) :=$ subseries of $M_i(t)$ with no loop at v_0



$$e^{C(t)} \cdot \widetilde{M}_i(t) = 2^i \frac{\mathrm{d}^i}{\mathrm{d}t^i} e^{C(t)}$$ with $2t \frac{\mathrm{d}}{\mathrm{d}t} C(t) = M(t)$

$\Rightarrow \widetilde{M}_i(t) = \mathrm{polynomial}$ in $\frac{1}{t}, M(t), \ldots, \frac{\mathrm{d}^{i-1}}{\mathrm{d}t^{i-1}} M(t)$

$\Rightarrow$ differential equation of order $r-1$ for $M(t)$ when max-degree $\leq 2r$

Orthogonal polynomials (preparation)

Rk: $(2n-1)!! = \frac{1}{\sqrt{2\pi}} \int x^{2n} e^{-x^2/2} \mathrm{d}x$

Hence $\Lambda W(t) = \frac{1}{\sqrt{2\pi}} \int W(tx^2) e^{-x^2/2} \mathrm{d}x$

$$\Rightarrow U(t) = \Lambda\Big(\exp\big(\sum_{k \geq 1} \frac{1}{2k} t^k g_k \big)\Big) = \frac{1}{\sqrt{2\pi}} \int e^{V(t,x) - x^2/2} \mathrm{d}x$$

$$\boxed{\begin{array}{c} V(t,x) \\ \shortparallel \\ \sum_{k\geq 1} \frac{1}{2k} g_k t^k x^{2k} \end{array}}$$

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Then $$M(t) = 2t\frac{\mathrm{d}}{\mathrm{d}t}\log(U(t)) = \frac{2tU'(t)}{U(t)} = \frac{\frac{1}{\sqrt{2\pi}}\int x^2 e^{V(t,x)-x^2/2}\mathrm{d}x}{\frac{1}{\sqrt{2\pi}}\int e^{V(t,x)-x^2/2}\mathrm{d}x} - 1$$

proved either by integration by part

or noticing that numerator = GF maps (not necess. connected)

rooted at vertex of degree 2

Orthogonal polynomials

[Bessis-Itzykson-Zuber'80]

Consider the "scalar product"

$$< F, G > := \frac{1}{\sqrt{2\pi}} \int F(t,x) G(t,x) e^{V(t,x) - x^2/2} \mathrm{d}x$$

Orthogonal polynomials

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Let $(p_i(t,x))_{i\geq 1}$ be the unique family of orthogonal polynomials (in x) such that $p_i(t,x) = x^i + \text{lower degrees}$ (Hermite polynomials for $t = 0$)

Orthogonal polynomials

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Rk: $V(t,x) = V(t,-x) \;\Rightarrow\; p_i(t,-x) = (-1)^i p_i(t,x)$ (so $p_1(t,x) = x$)

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General properties:

$< xp_i, p_{i-1} > / < p_{i-1}, p_{i-1} >$

3-term recurrence $\boxed{xp_i(t,x) = p_{i+1}(t,x) + r_i(t) p_{i-1}(t,x)}$

(no component p_a with $a \leq i-2$ because $< xp_i, p_a > = < p_i, xp_a > = 0$)

Orthogonal polynomials

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Let $h_i(t) := < p_i, p_i >$

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In particular $r_1(t) = \dfrac{h_1(t)}{h_0(t)} = \dfrac{< x, x >}{< 1, 1 >} = M(t) + 1$

Orthogonal polynomials

[Bessis-Itzykson-Zuber'80]

Recall $(p_i(t,x))_{\geq 1}$ orthogonal polynomials with $p_i(t,x) = x^i +$ lower degrees

$$h_i := < p_i, p_i > \qquad\qquad r_i = h_i/h_{i-1}$$

Recurrence for $(p_i(t,x))_{i\geq 1}$: $\qquad xp_i(t,x) = p_{i+1}(t,x) + r_i(t)p_{i-1}(t,x)$

Recursive system for $(r_i(t))_{i\geq 1}$ (4-regular, $g_k = \delta_{k=2}$)

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$$||$$

$$< p_i, x p_{i-1} > - t^2 < {\color{blue} x^3 p_i}, {\color{red} p_{i-1}} >$$

Orthogonal polynomials

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$$||$$

$$h_i$$

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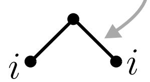
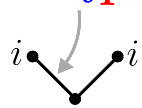
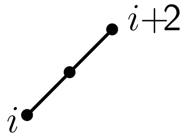
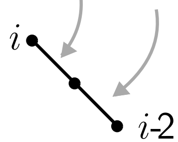
$$x^2 p_i = r_{i+1} p_i + r_i p_i + p_{i+2} + r_i r_{i-1} p_{i-2}$$









Recursive system for $(r_i(t))_{i\geq 1}$ (4-regular, $g_k = \delta_{k=2}$)

$$< \frac{\partial}{\partial x} p_i, p_{i-1} > = i\ h_{i-1}$$

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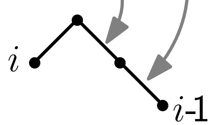
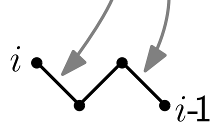
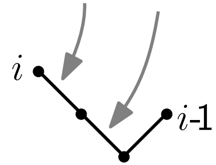
$$< p_i, x p_{i-1} > - t^2 < x^3 p_i, p_{i-1} >$$

$$|| \qquad\qquad ||$$

$$h_i \qquad - t^2 h_{i-1} \big(r_{i+1} r_i + r_i r_i + r_i r_{i-1} \big)$$







Orthogonal polynomials

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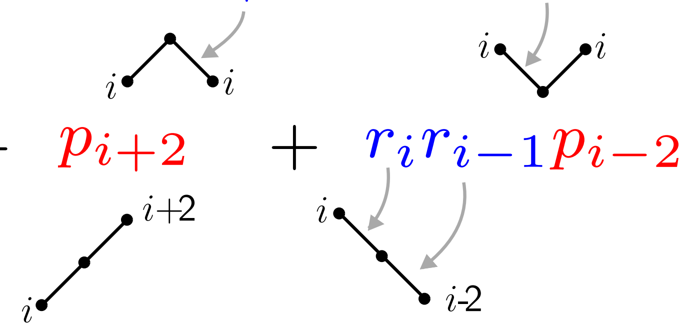
Recursive system for $(r_i(t))_{i\geq 1}$ (4-regular, $g_k = \delta_{k=2}$)

$$x^2 p_i = r_{i+1} p_i + r_i p_i + p_{i+2} + r_i r_{i-1} p_{i-2}$$



$$< \frac{\partial}{\partial x} p_i, p_{i-1} > = i\ h_{i-1}$$

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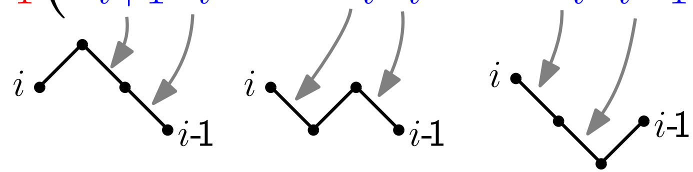
$$h_i \quad - t^2 h_{i-1} \big(r_{i+1} r_i + r_i r_i + r_i r_{i-1} \big)$$



Dividing by h_{i-1} yields

$$\boxed{r_i = i + t^2 r_i (r_{i-1} + r_i + r_{i+1})}$$

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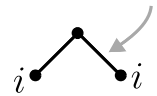
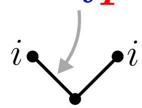
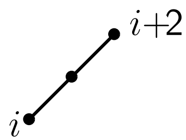
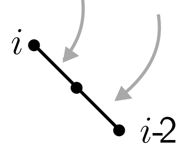
Recursive system for $(r_i(t))_{i\geq 1}$ (general case)

$$x^2 p_i = r_{i+1} p_i + r_i p_i + p_{i+2} + r_i r_{i-1} p_{i-2}$$









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$$||$$

$$\frac{1}{\sqrt{2\pi}} \int \frac{\partial}{\partial x} p_i(t,x) p_{i-1}(t,x) e^{V(t,x) - x^2/2} \mathrm{d}x$$

$||$ integ. parts

$$< p_i, x p_{i-1} > - \sum_{k\geq 1} t^k g_k < x^{2k-1} p_i, p_{i-1} >$$

$$||\qquad\qquad\qquad ||$$

$$h_i - h_{i-1} \sum_{k\geq 1} t^k g_k \sum_{\wp \in \mathrm{Dyck}_{2k-1}^{(i \to i-1)}} \prod_{\substack{\text{descending steps} \\ h \to h-1 \text{ of } \wp}} r_h(t)$$

$\Rightarrow$ $/h_{i-1}$

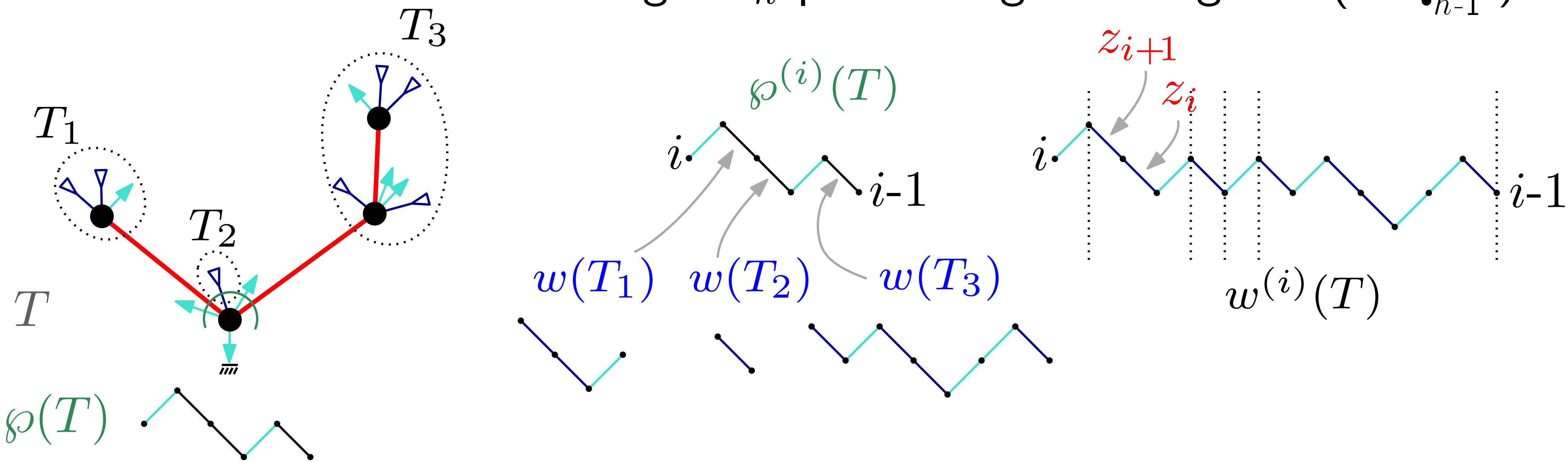
$$\boxed{r_i(t) = i + \sum_{k\geq 1} g_k t^k \sum_{\wp \in \mathrm{Dyck}_{2k-1}^{(i \to i-1)}} \prod_{\substack{\text{descending steps} \\ h \to h-1 \text{ of } \wp}} r_h(t)}$$

Model of trees for $r_i(t)$

Let $\hat{r}_i(t) :=$ GF i-balanced Eulerian trees with:

-weight $t^k g_k$ per node of degree $2k$

-weight z_h per closing of i-height h (h h-1)



$$\boxed{\hat{r}_i(t) = z_i + \sum_{k\geq 1} g_k t^k \sum_{\wp\in \mathrm{Dyck}_{2k-1}^{(i\to i-1)}} \prod_{\substack{\text{descending steps}\\ h\to h-1 \text{ of } \wp}} \hat{r}_h(t)}$$

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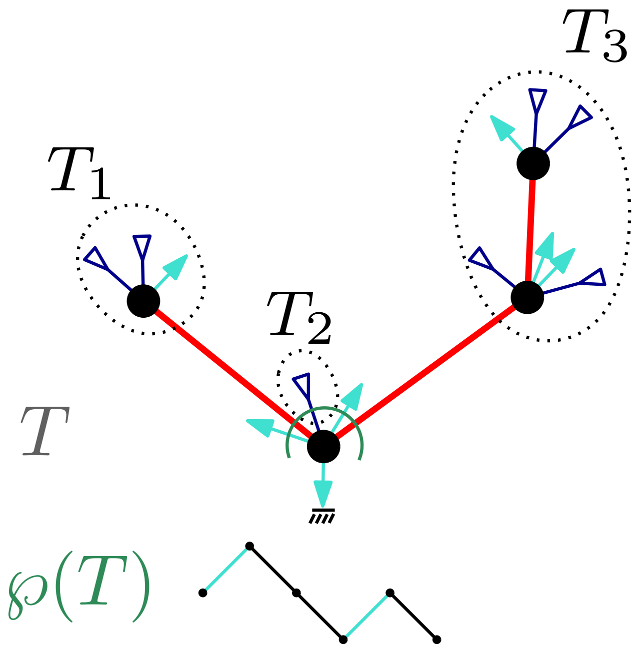
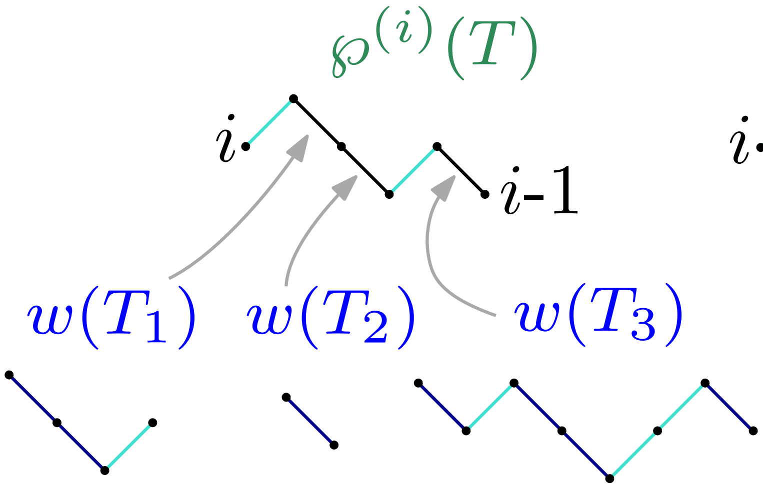
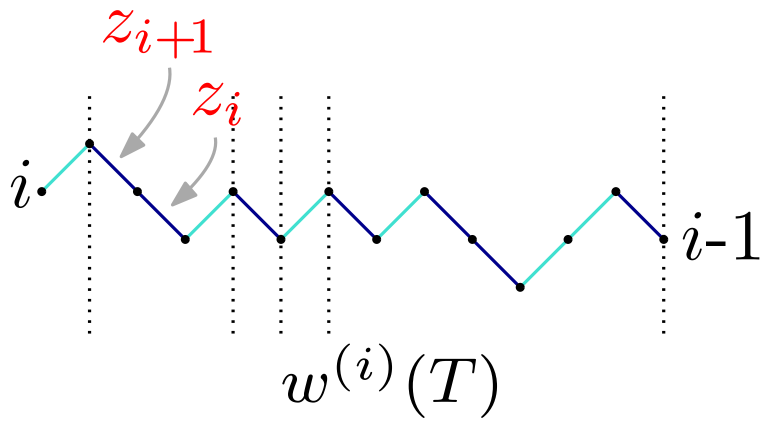
Let $\hat{r}_i(t) :=$ GF i-balanced Eulerian trees with:

-weight $t^k g_k$ per node of degree $2k$

-weight z_h per closing of i-height h ($h \to h\text{-}1$)







$$\hat{r}_i(t) = z_i + \sum_{k\geq 1} g_k t^k \sum_{\wp \in \mathrm{Dyck}_{2k-1}^{(i\to i-1)}} \prod_{\substack{\text{descending steps} \\ h\to h-1 \text{ of } \wp}} \hat{r}_h(t)$$

i-enriched Eulerian tree := i-balanced Eulerian tree

+ assign index $\iota \in [0..h-1]$ to each closing leaf of i-height h

The GF of i-enriched Eulerian trees is $\hat{r}_i(t)|_{z_j = j} = r_i(t)$

From 1-enriched trees to Eulerian maps

closing leaves $c_1, \ldots, c_k$ are treated in clockwise order around the tree,
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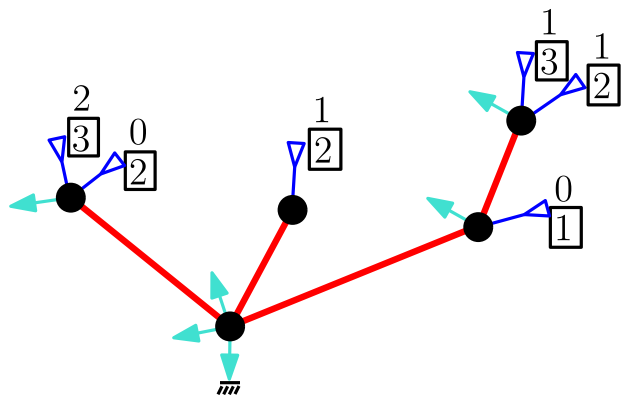
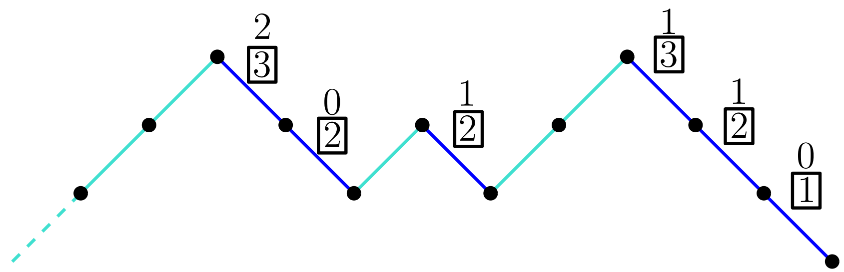


leaf-path



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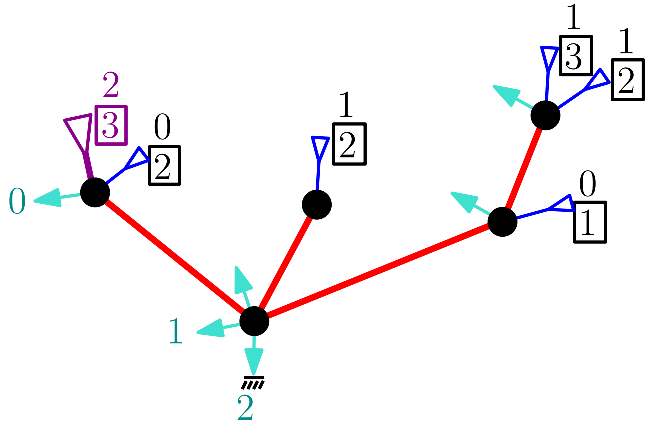

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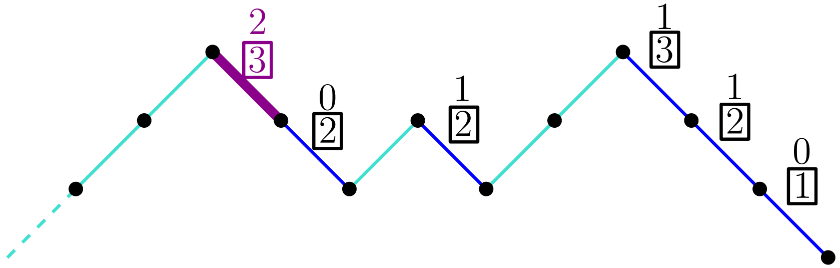
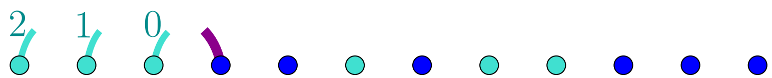


leaf-matching



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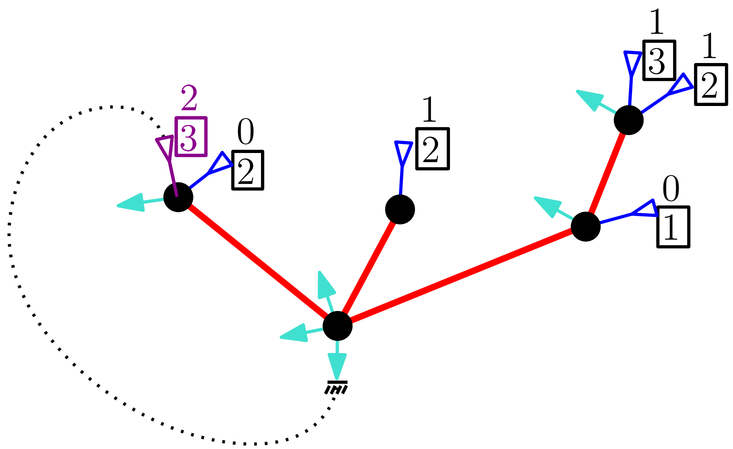

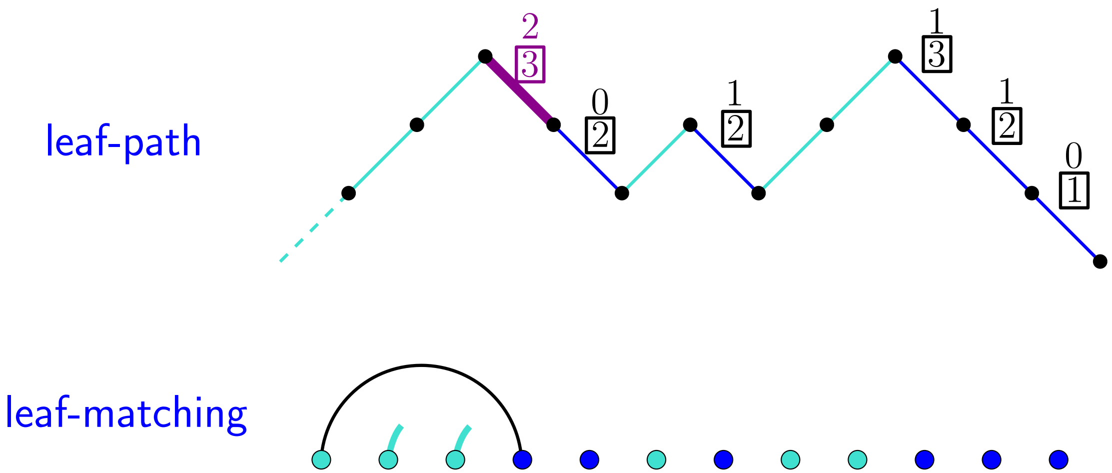

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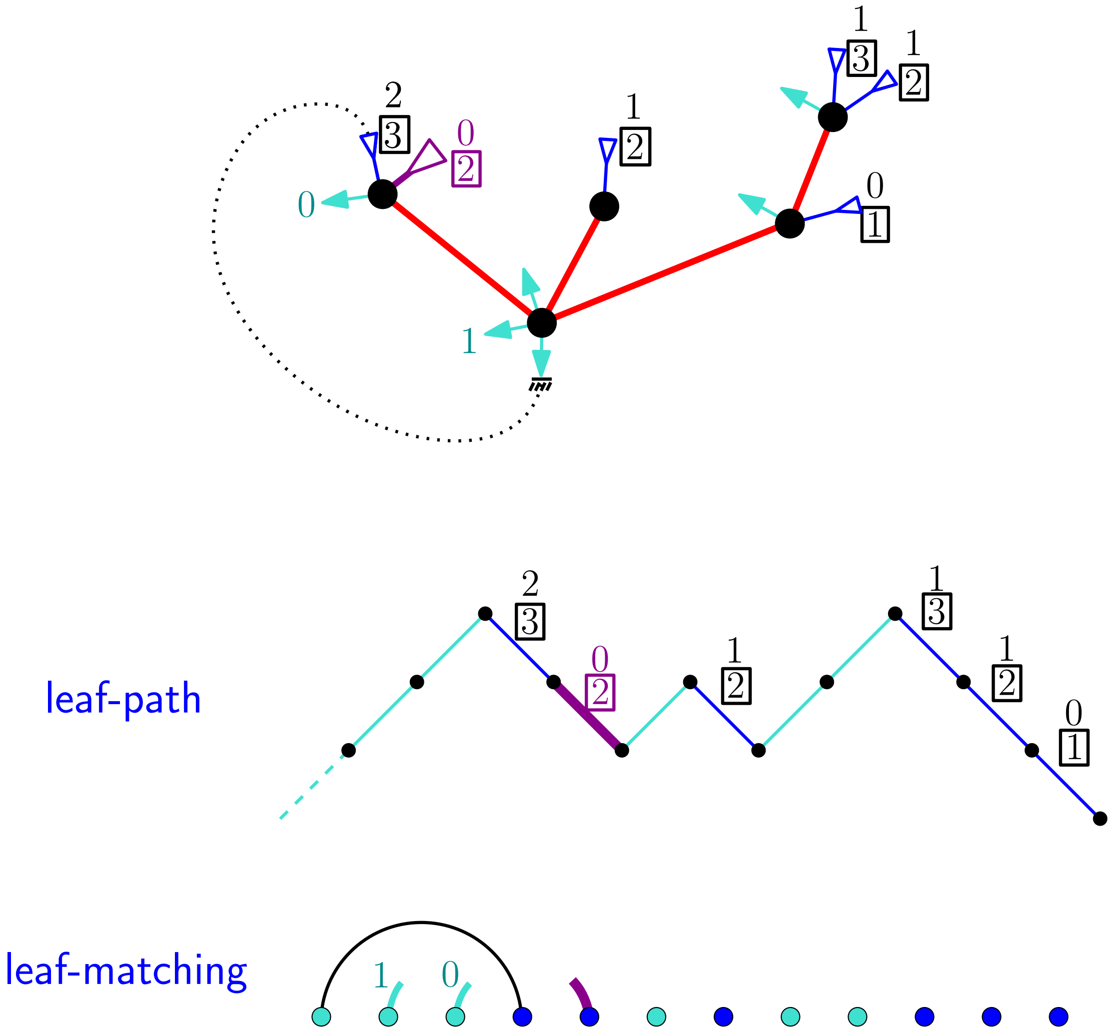

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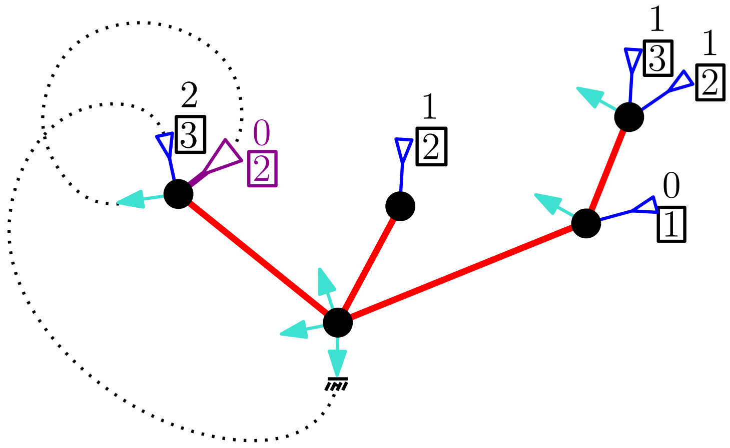

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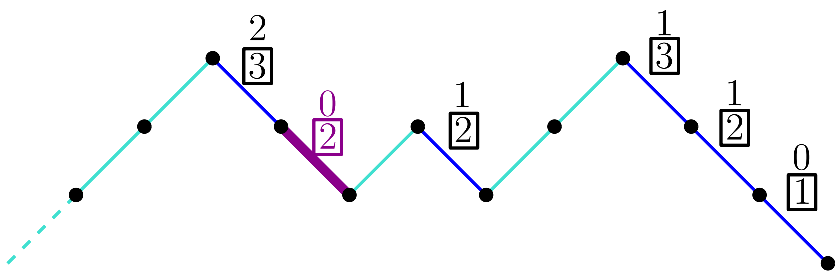

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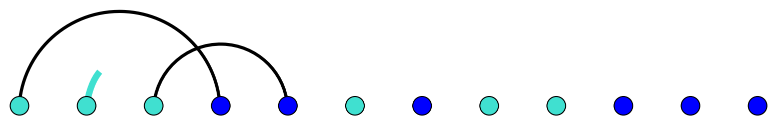

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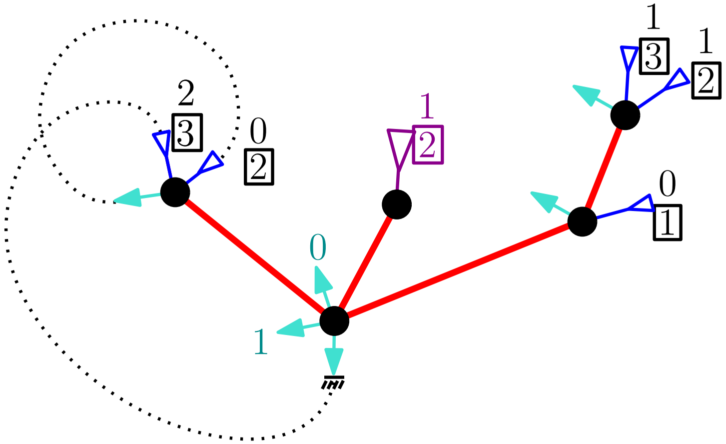

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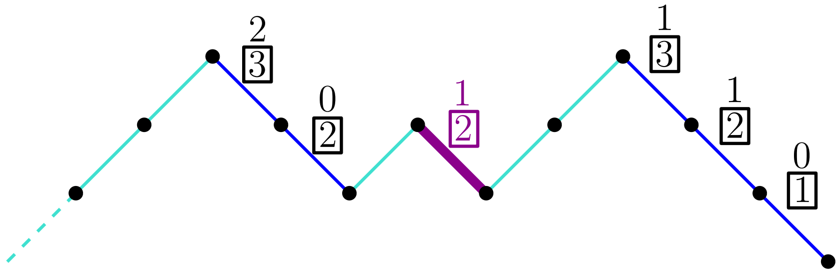

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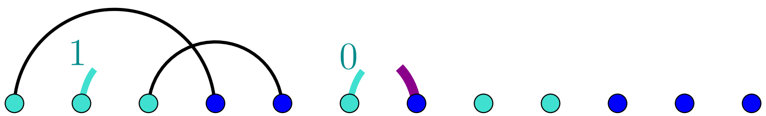

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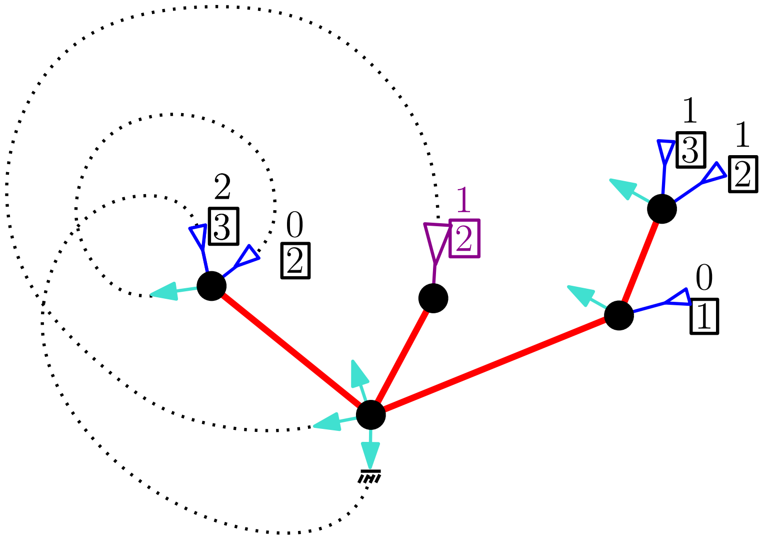

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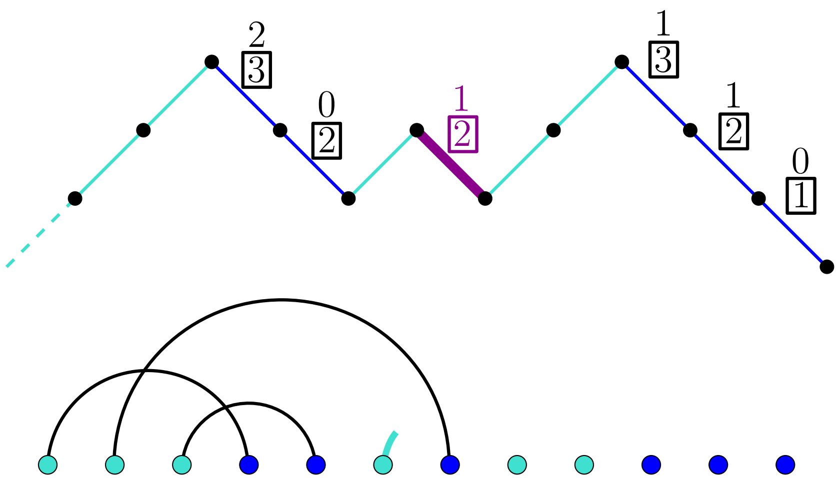

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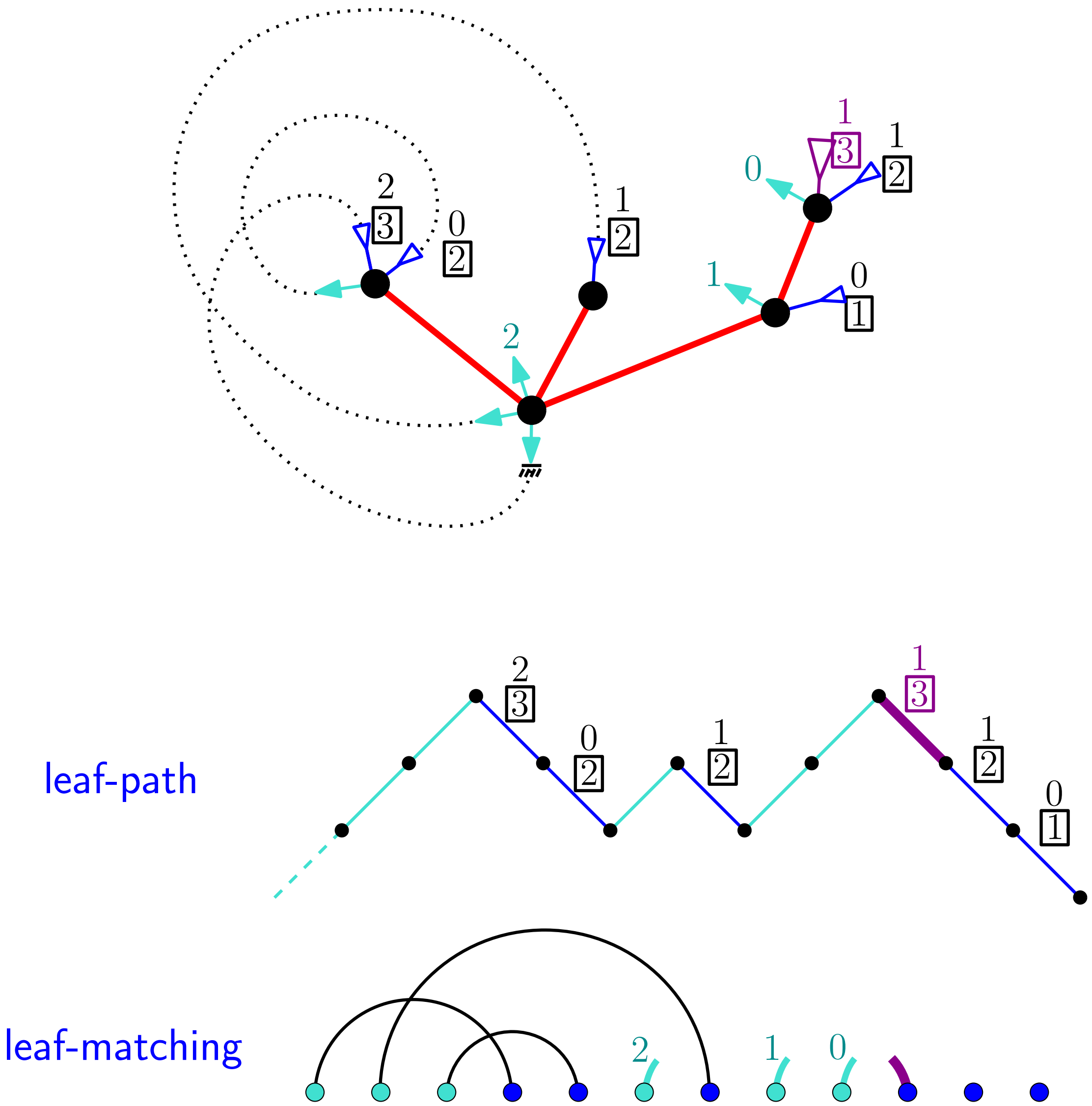

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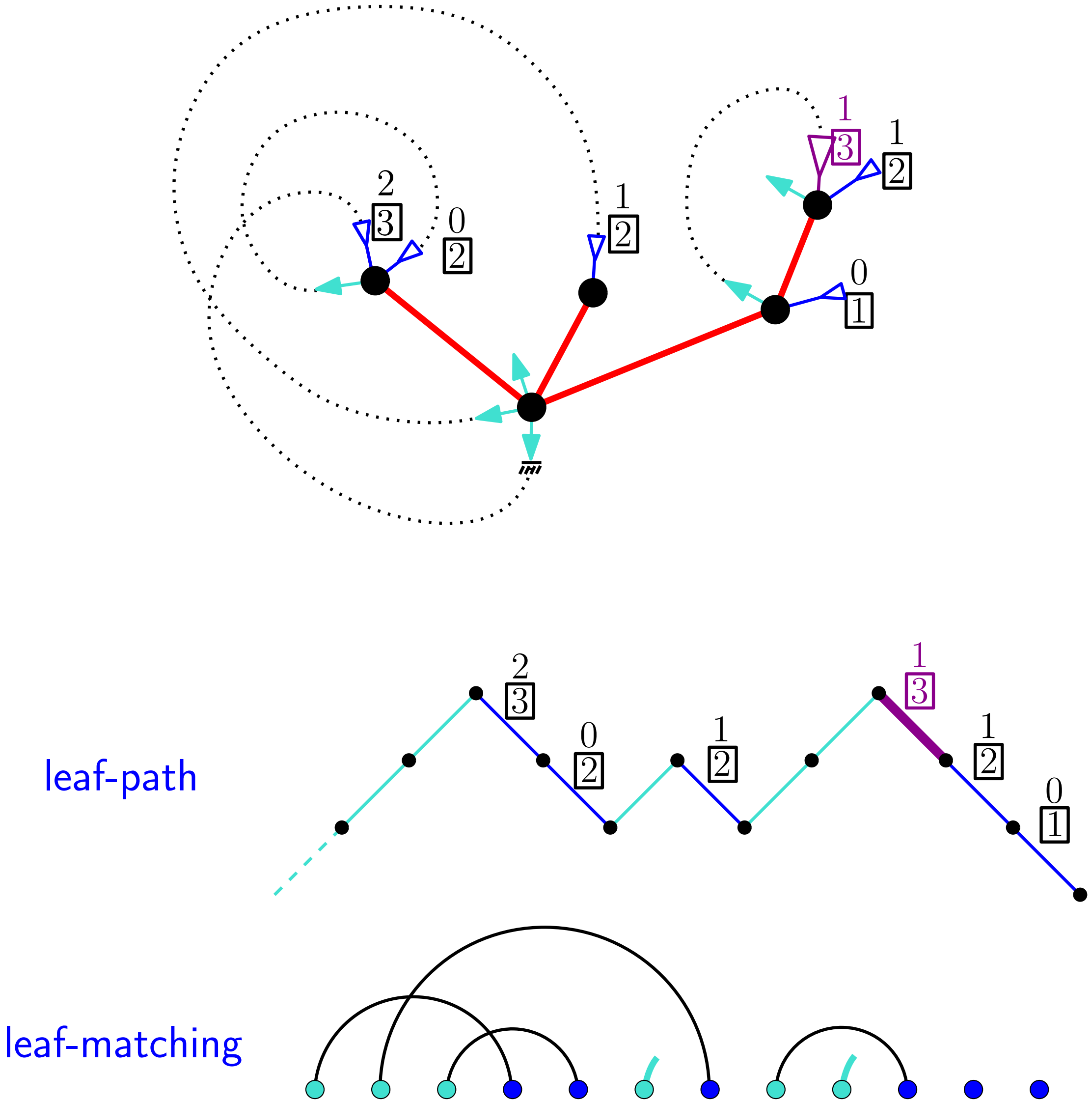

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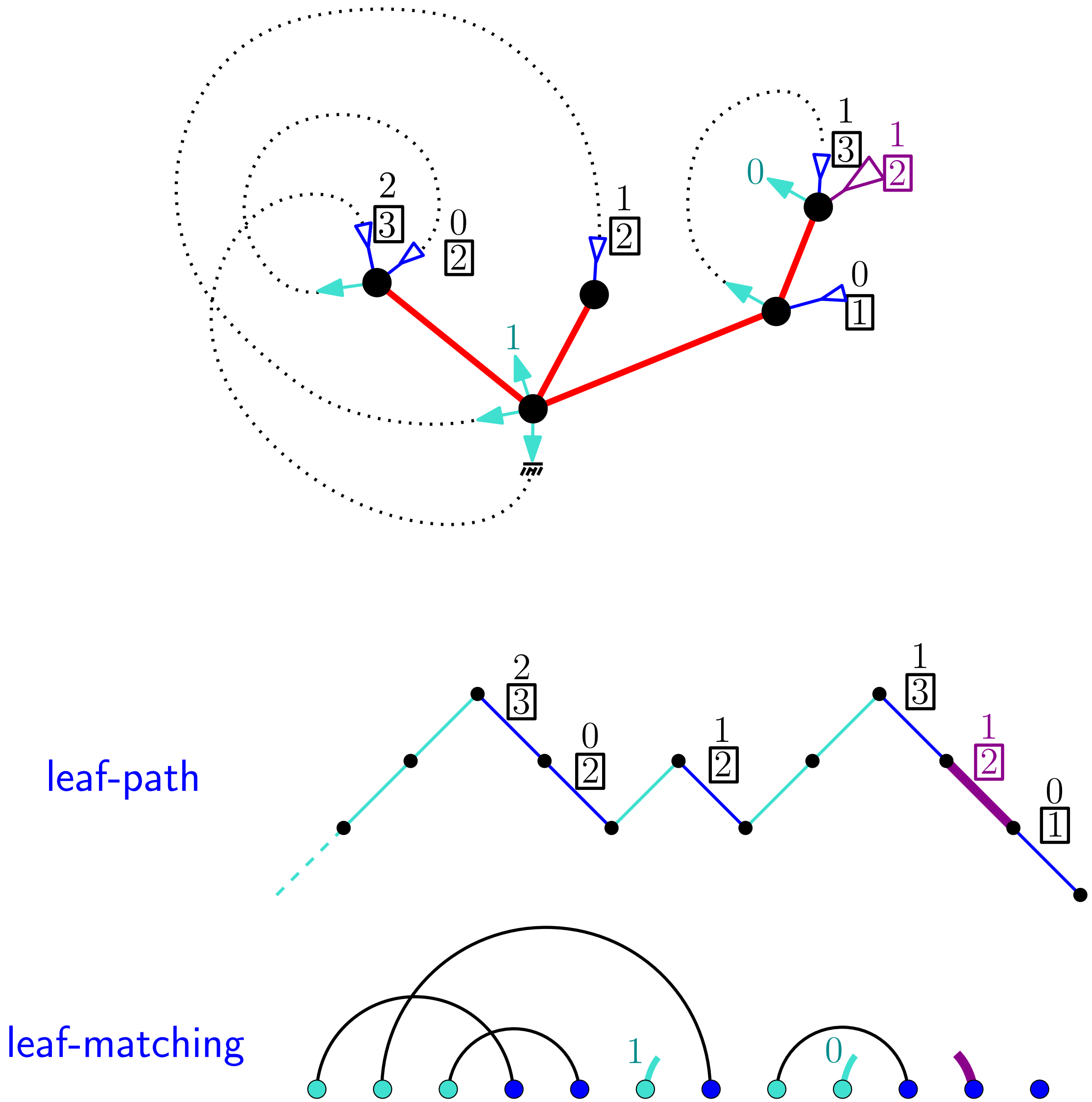

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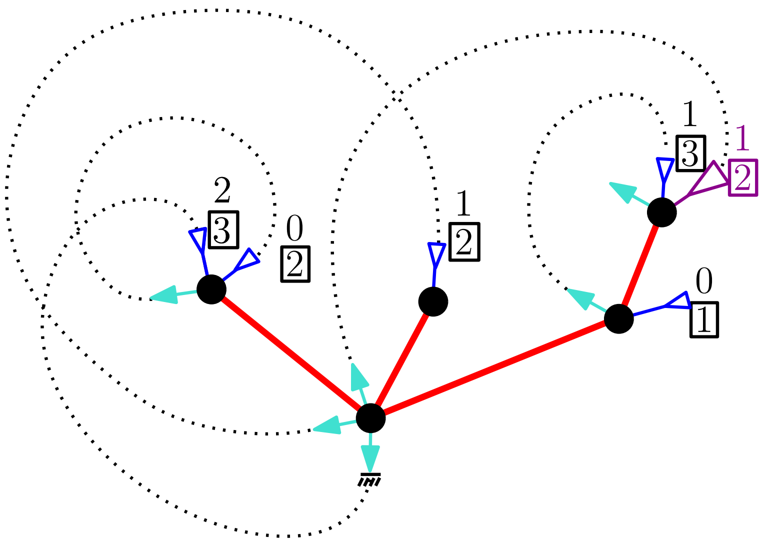

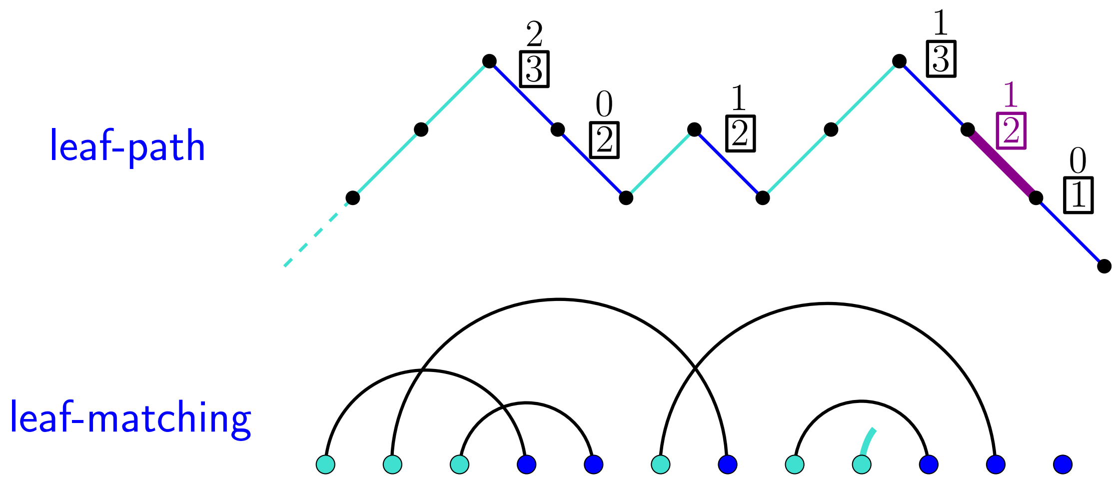

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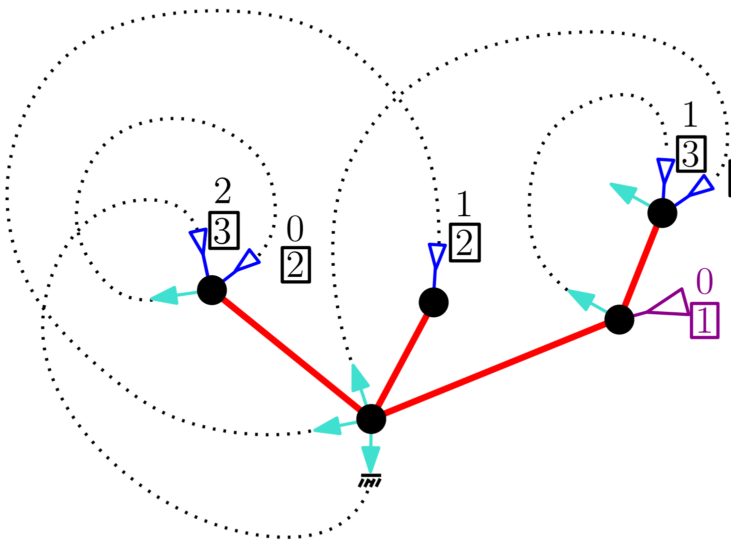

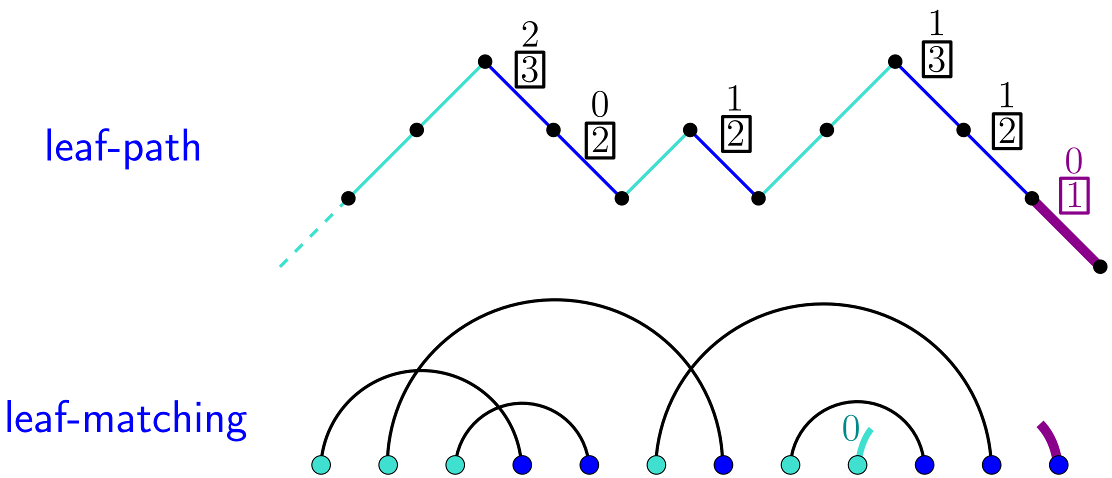

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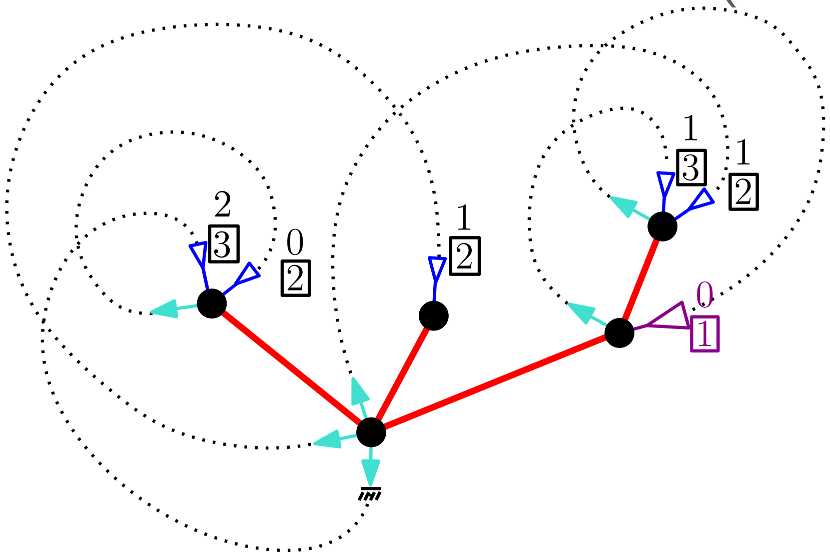

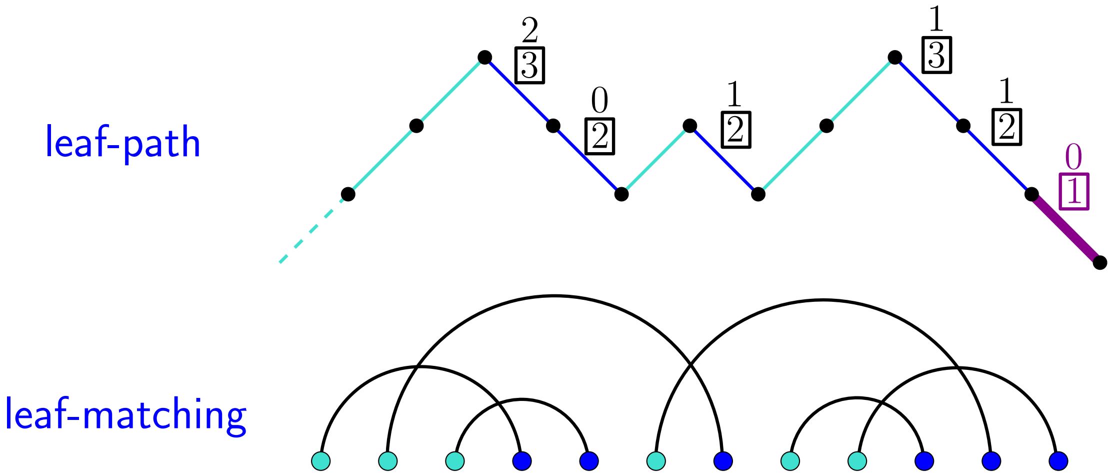

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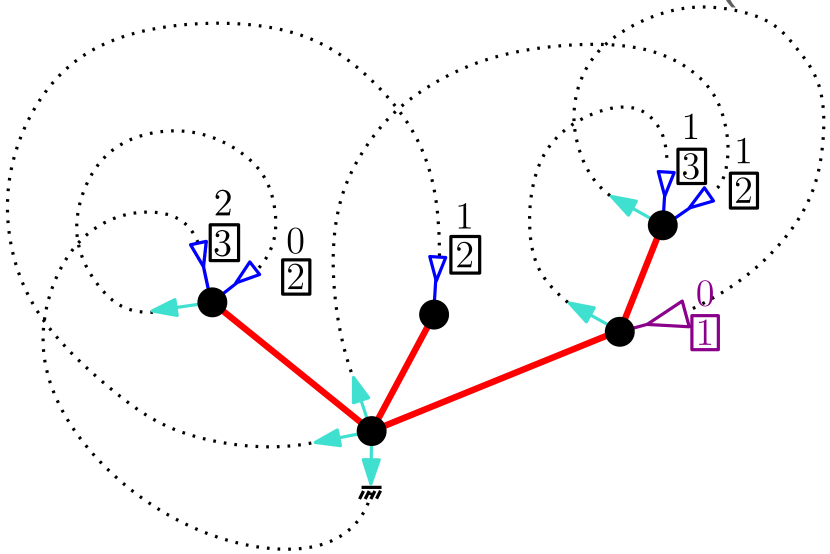

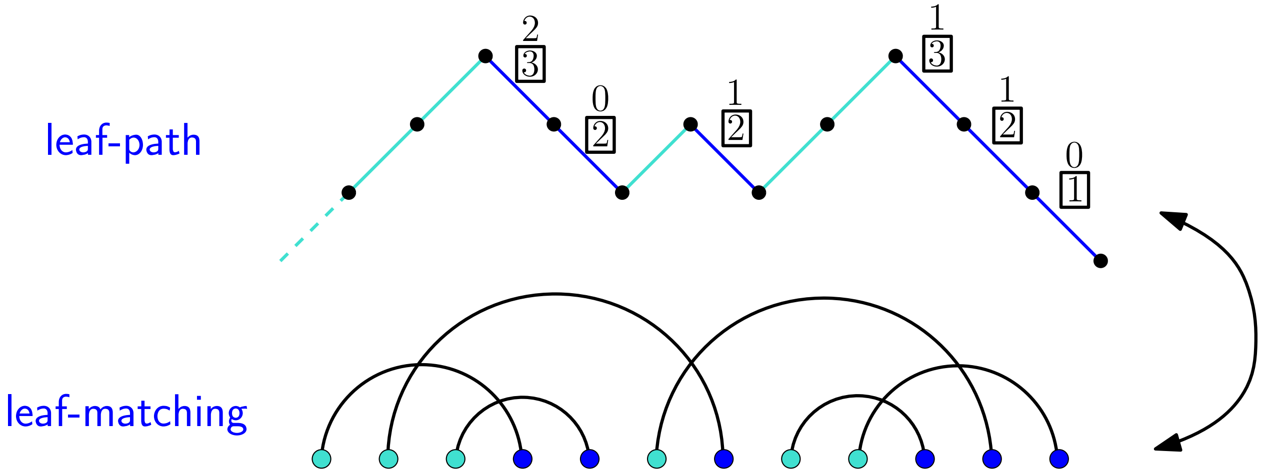

[Flajolet'80]

[de Médicis-Viennot'94]

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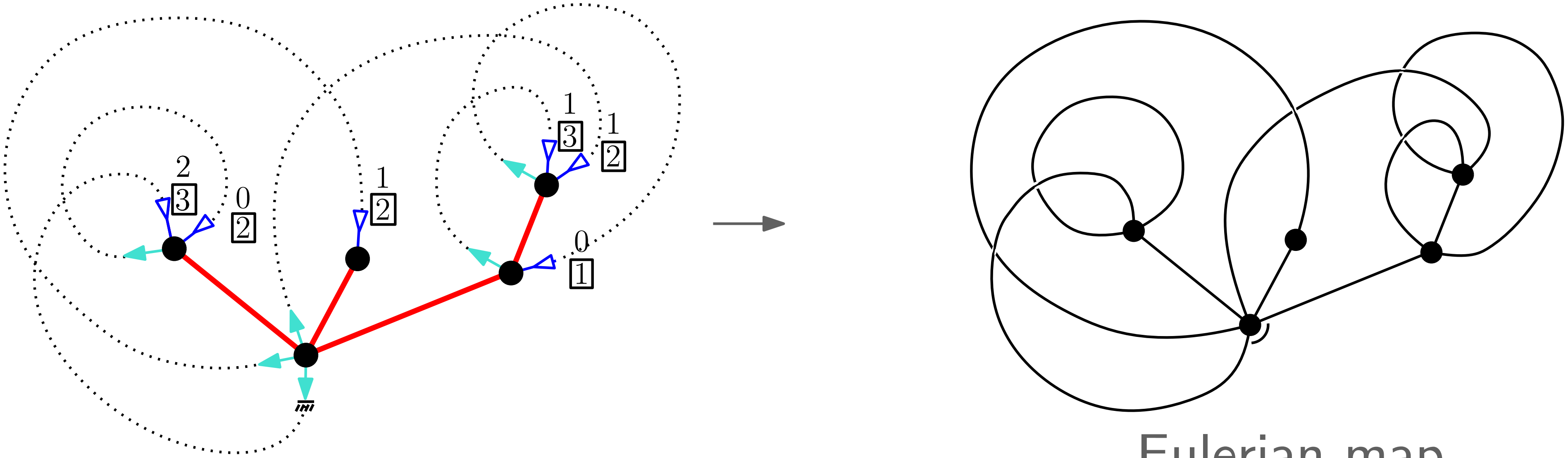
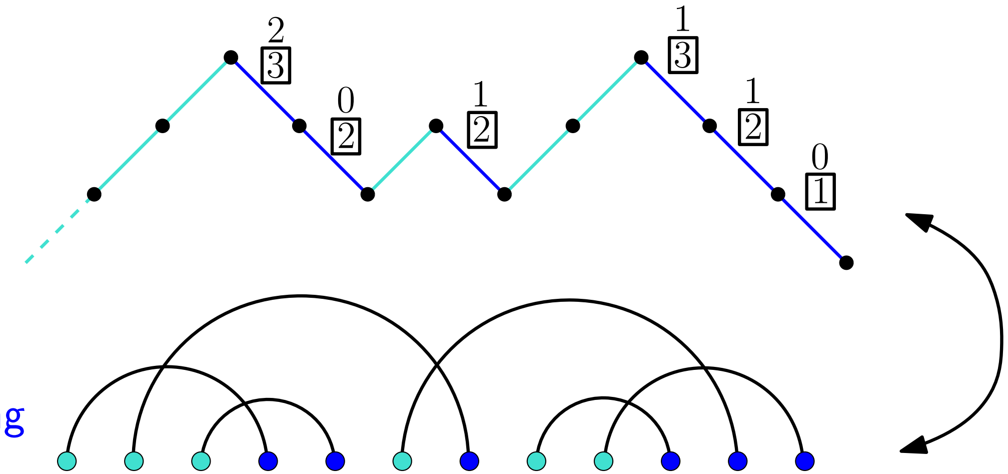


Eulerian map

leaf-path

leaf-matching



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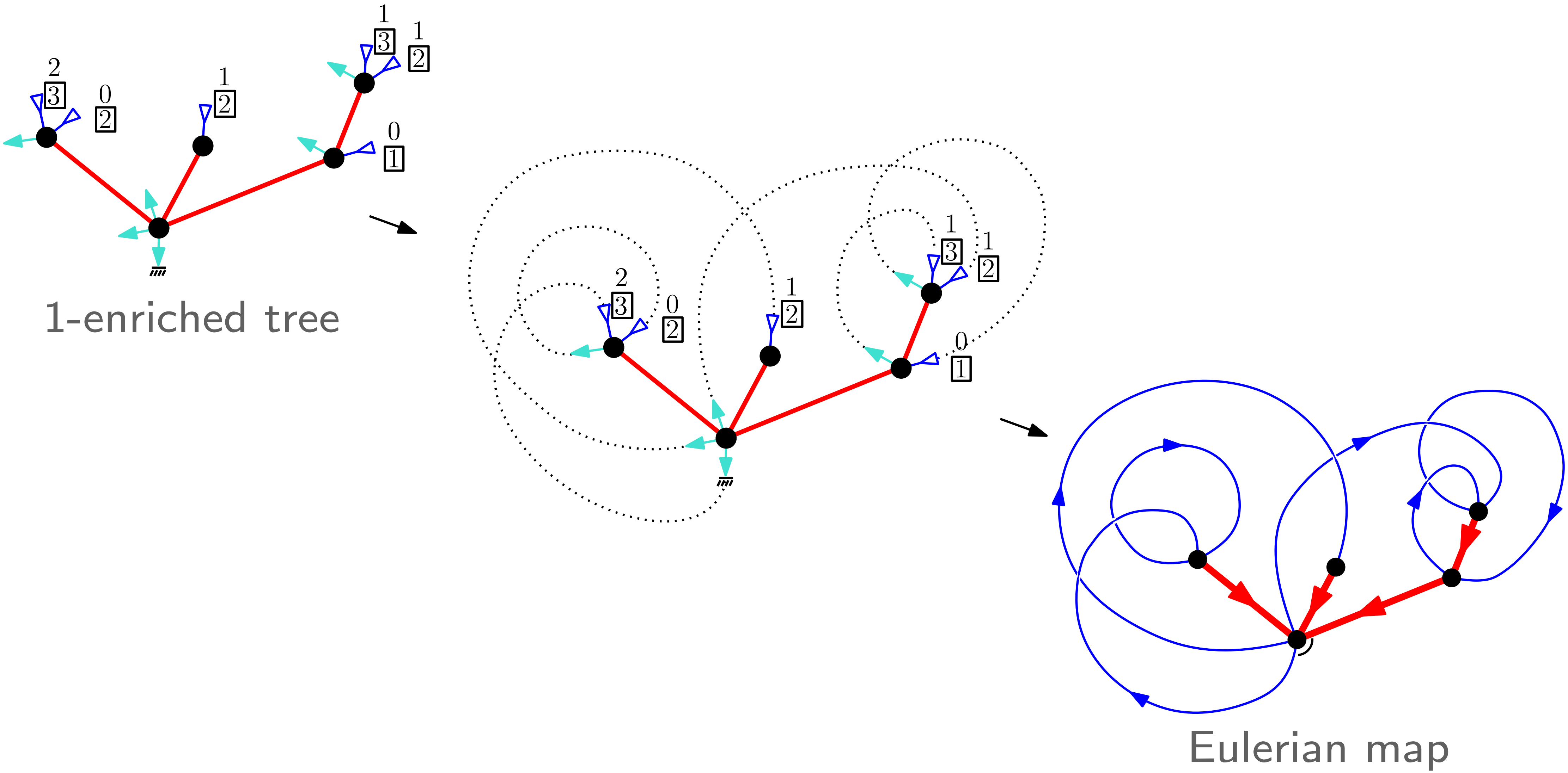


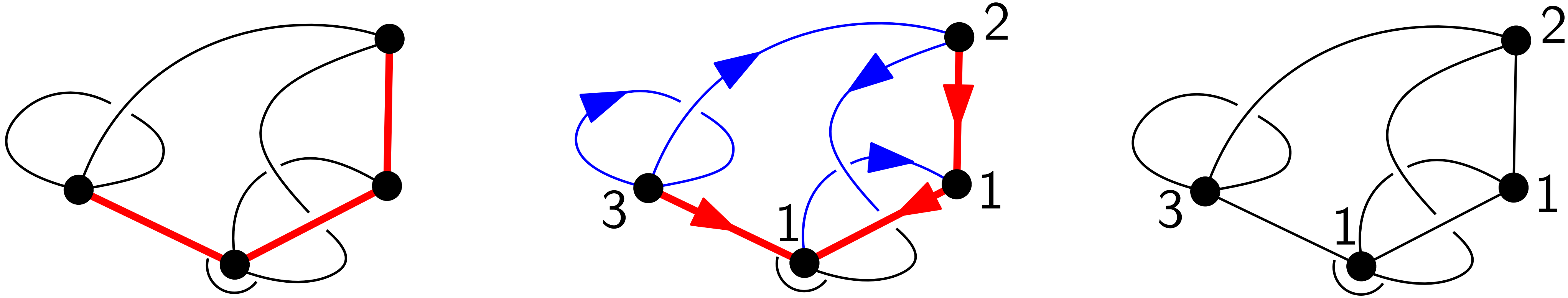
Rk: Via the mapping, the Eulerian map is naturally endowed with a spanning tree T and an Eulerian orientation such that edges $\in T$ are toward the root, edges $\notin T$ 'turn clockwise' around T

Bernardi's bijection (any genus)

[Bernardi'08]

Let M be a fixed rooted map, with vertex-set V

spanning trees of M $\longleftrightarrow$ root-accessible $\alpha : V \to \mathbb{N}$



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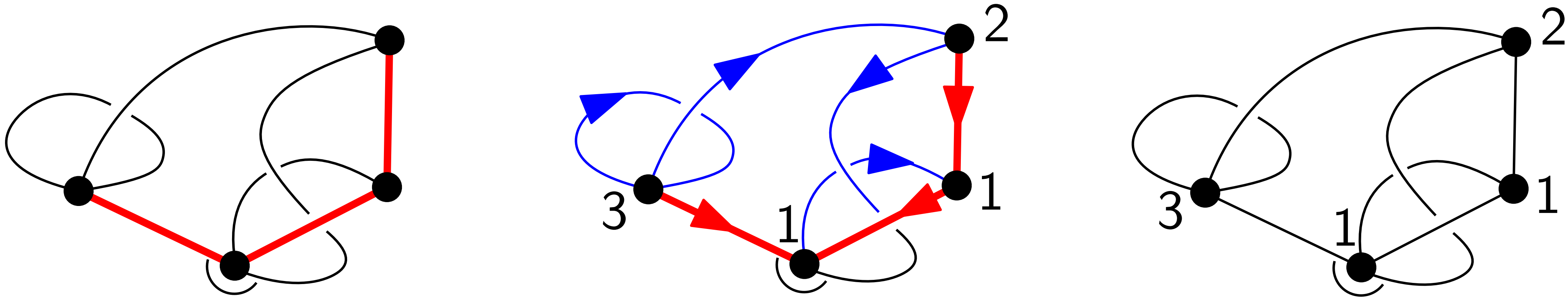

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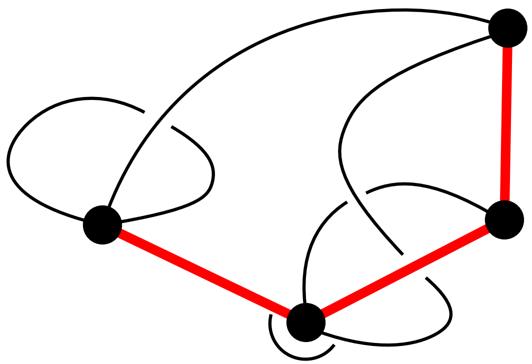

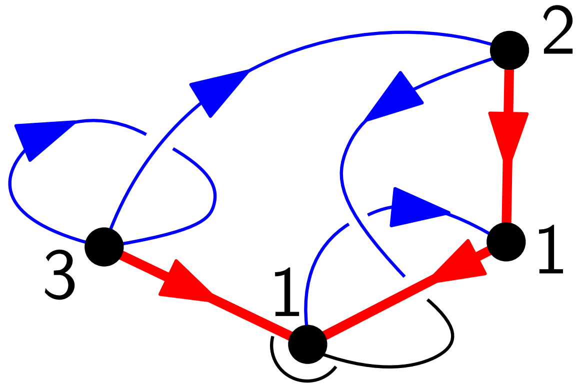

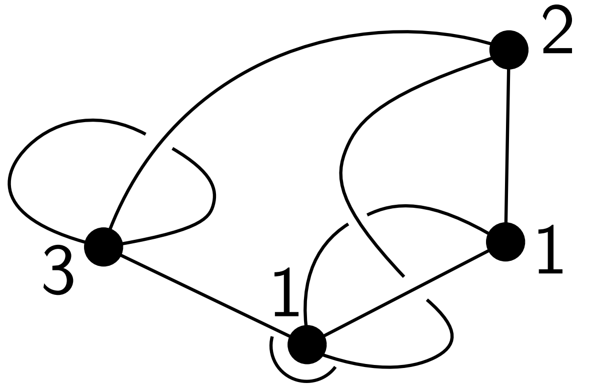

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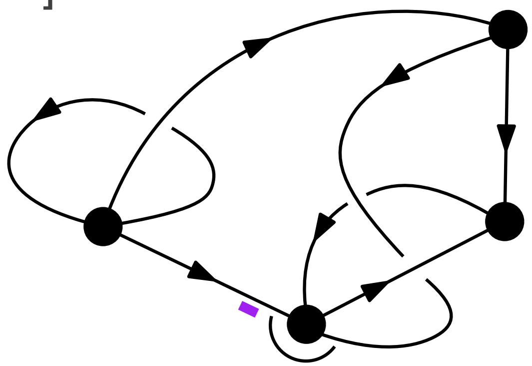

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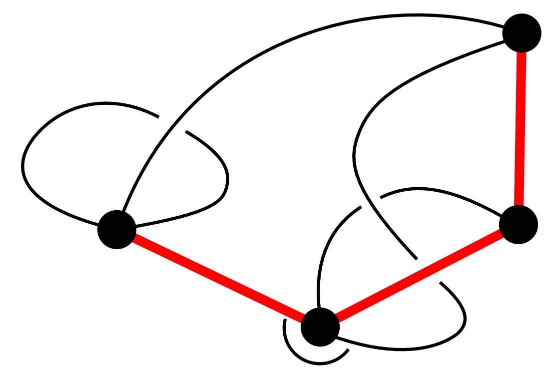

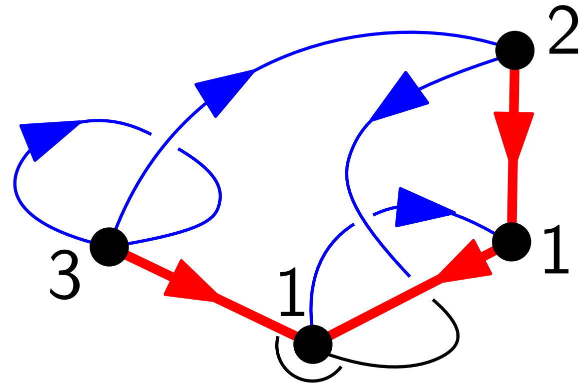

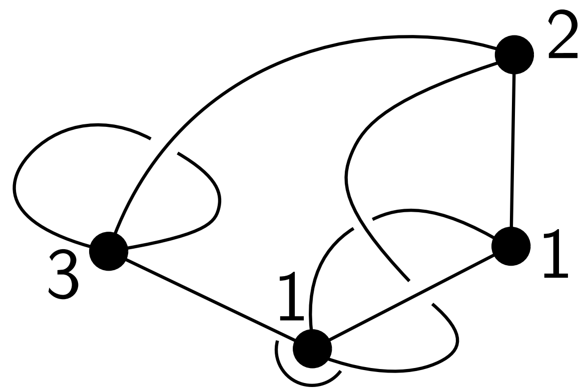

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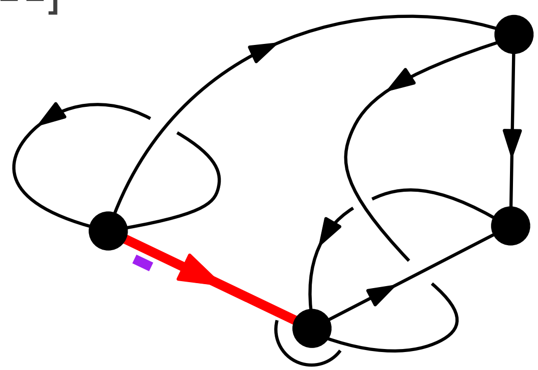

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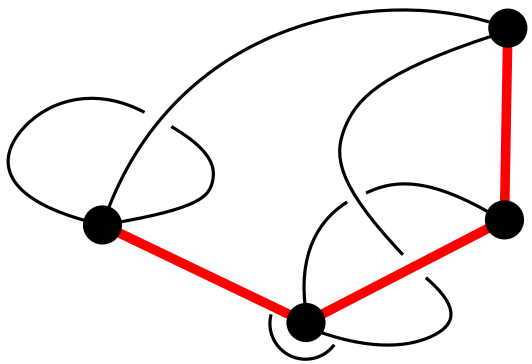

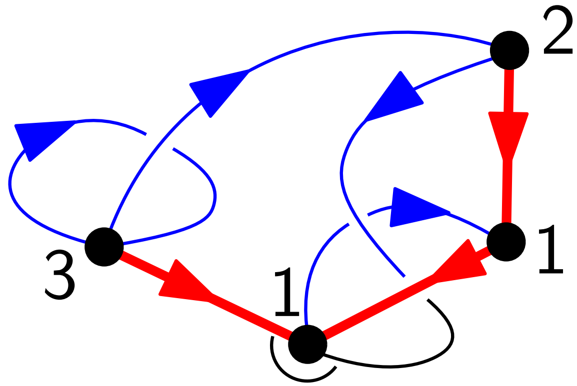

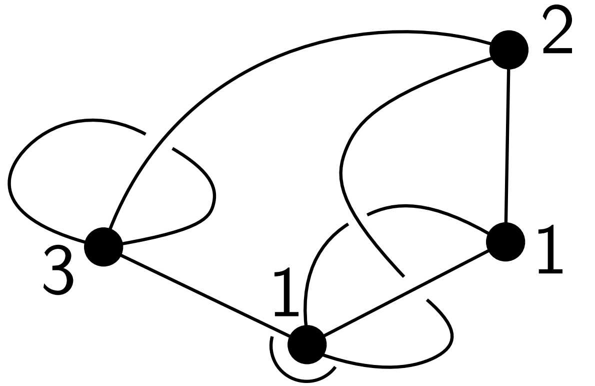

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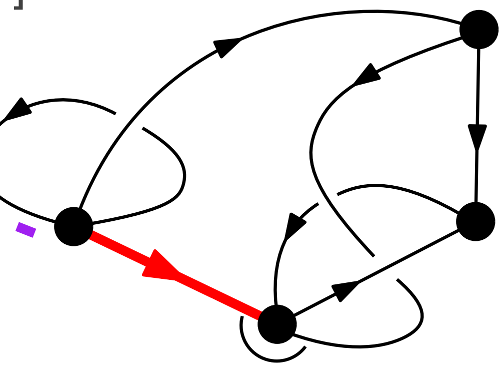

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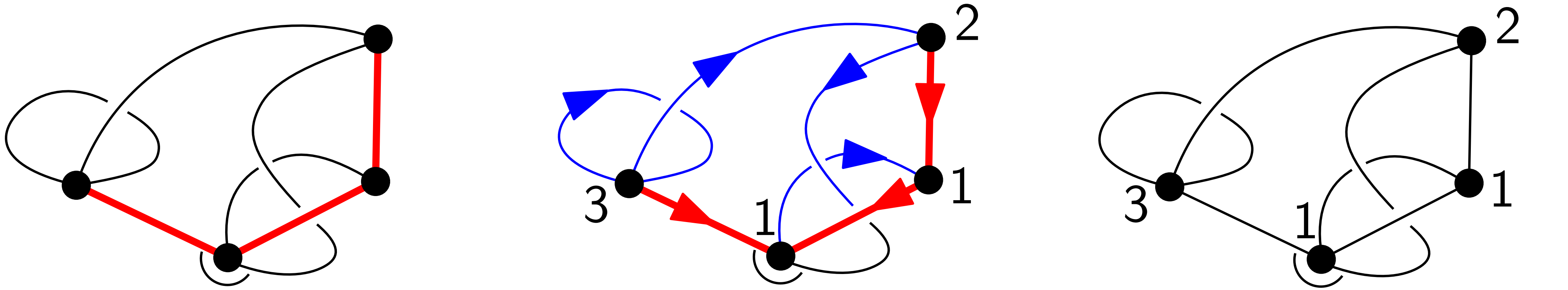

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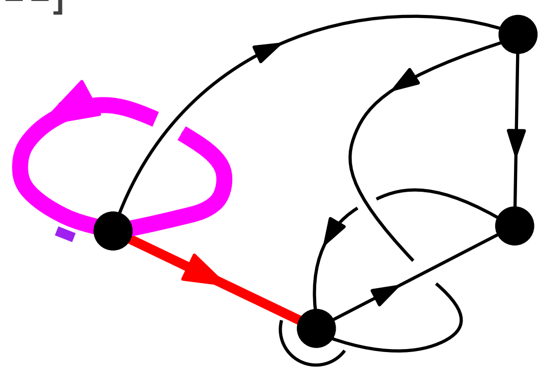

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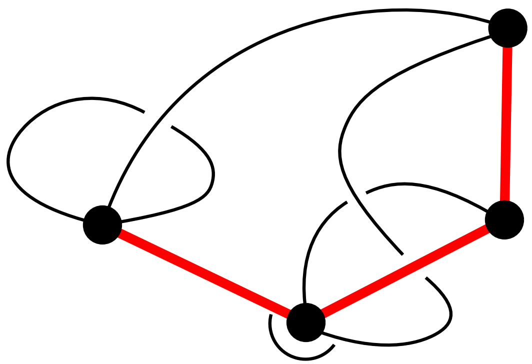

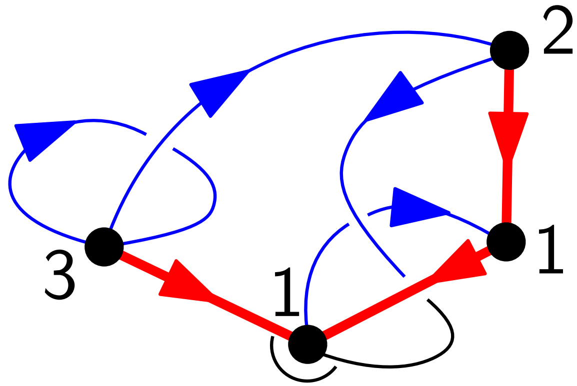

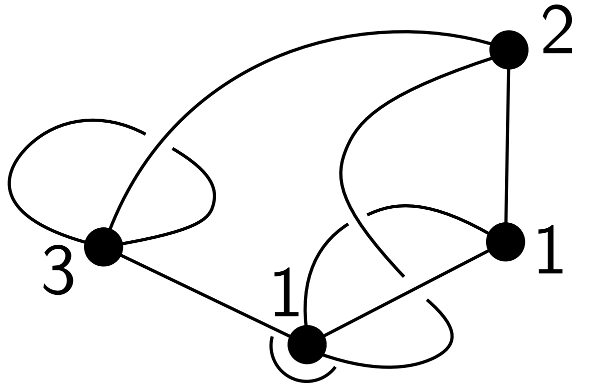

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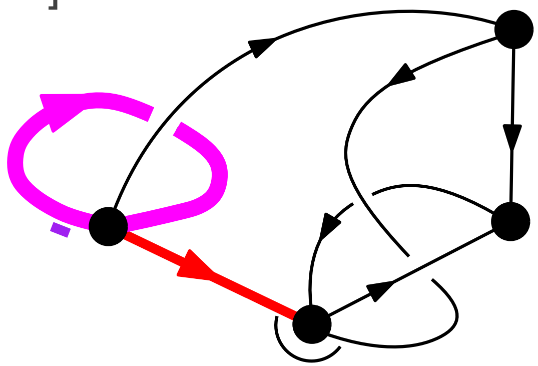

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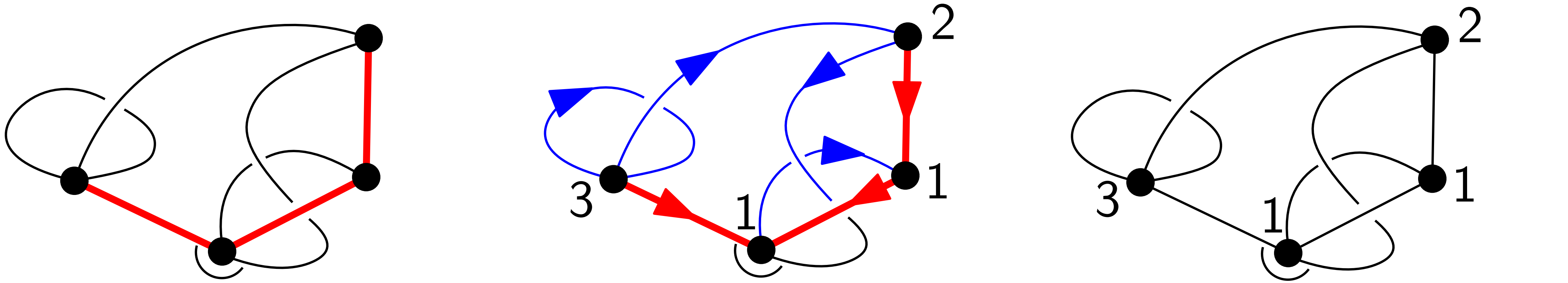

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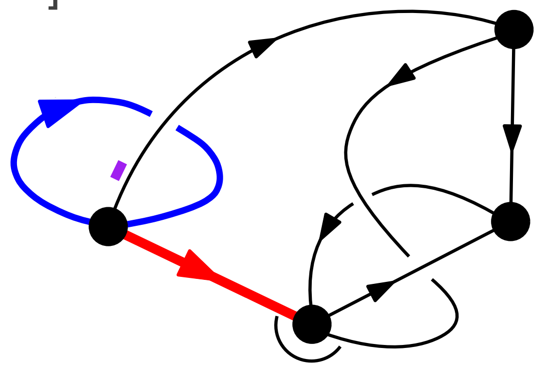

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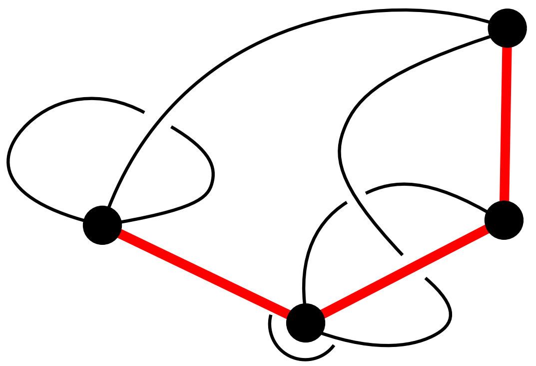

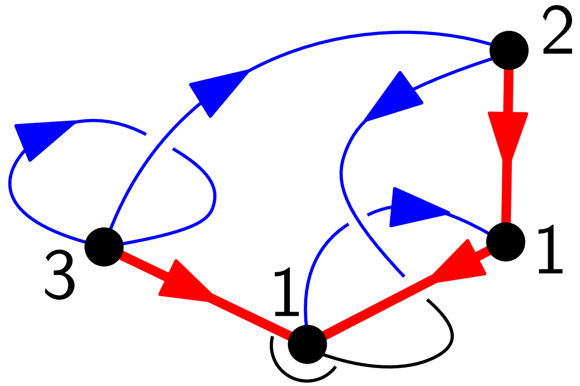

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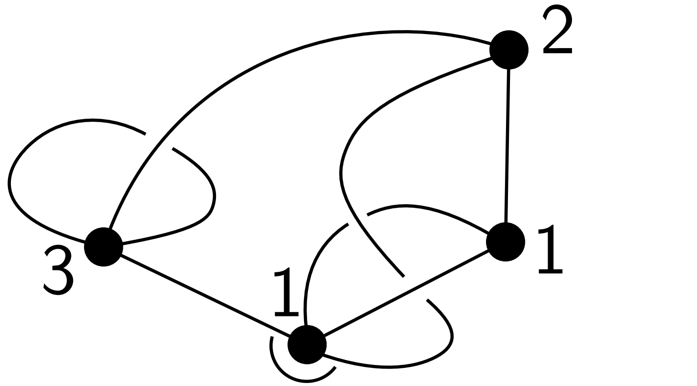

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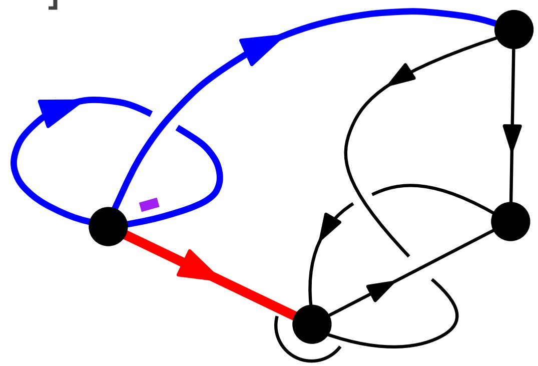

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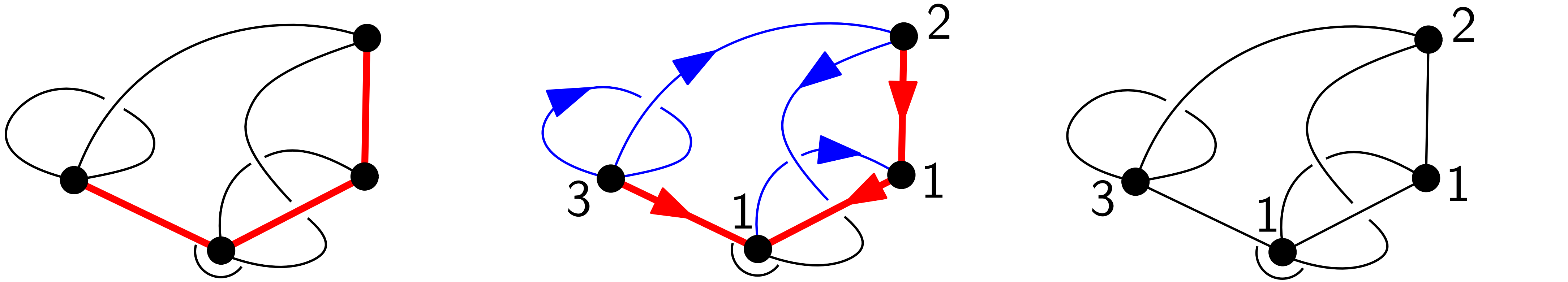

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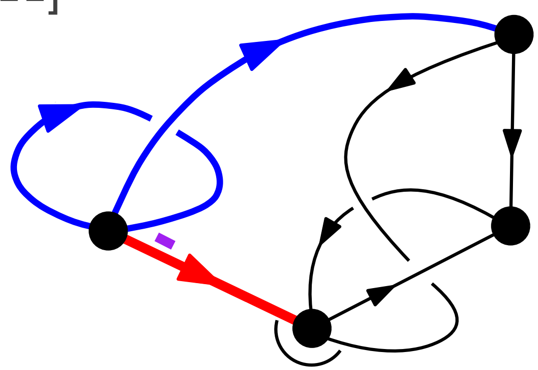

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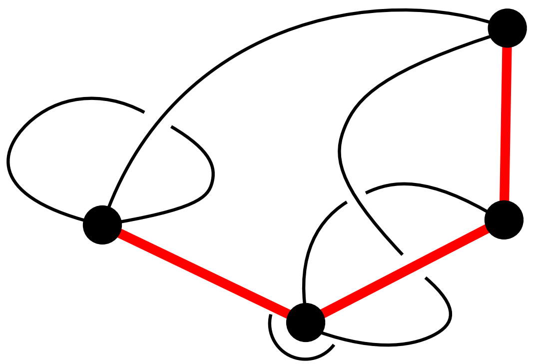

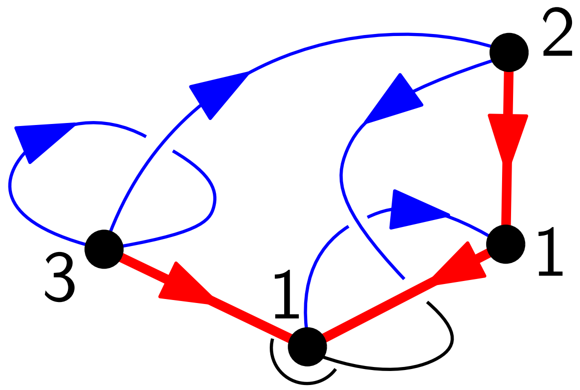

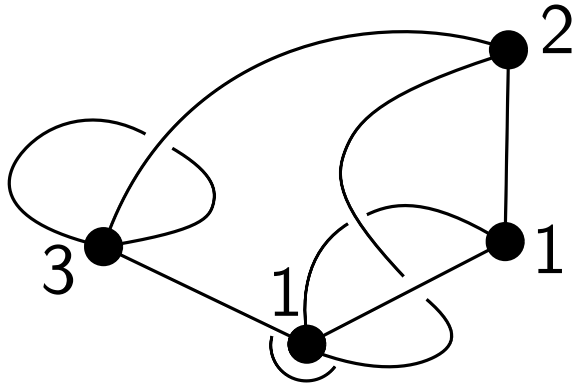

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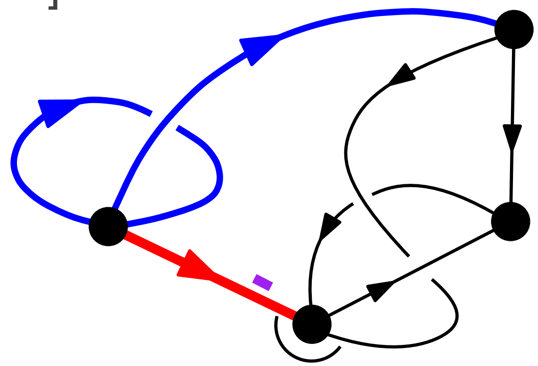

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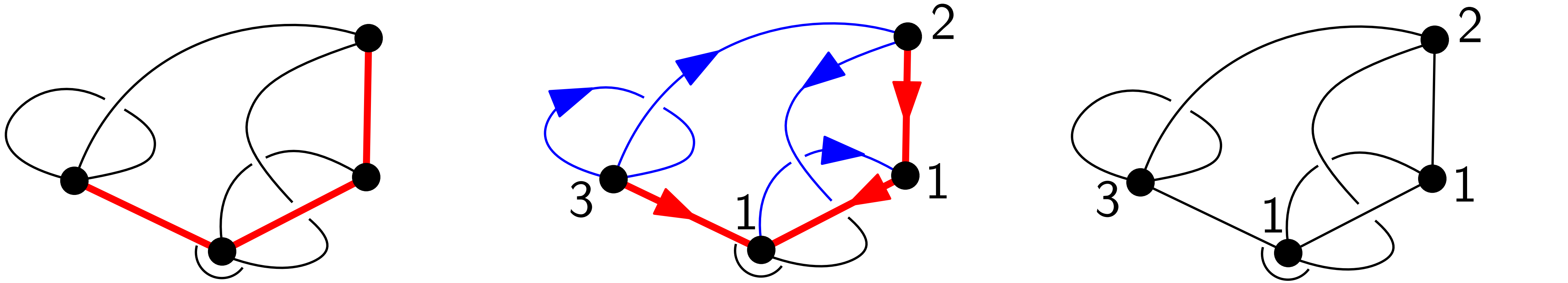

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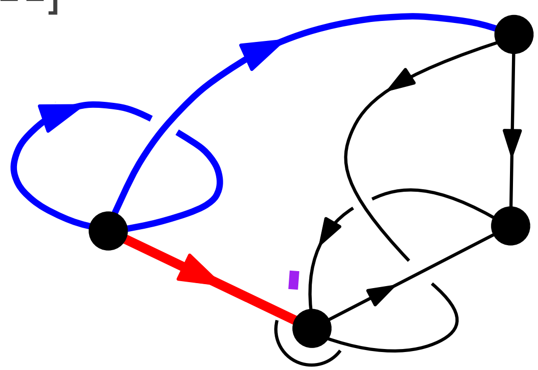

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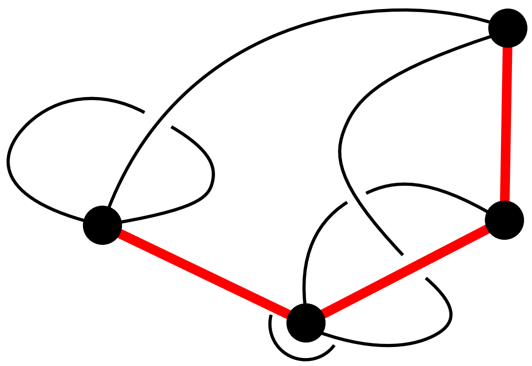

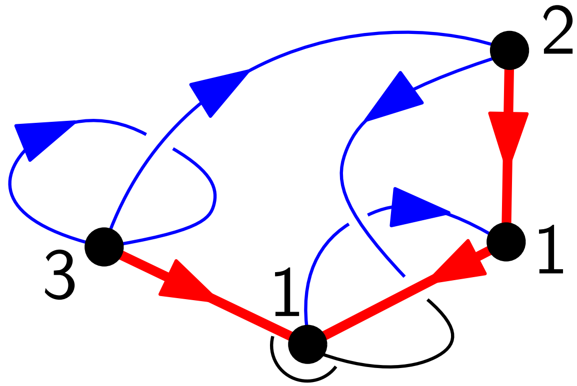

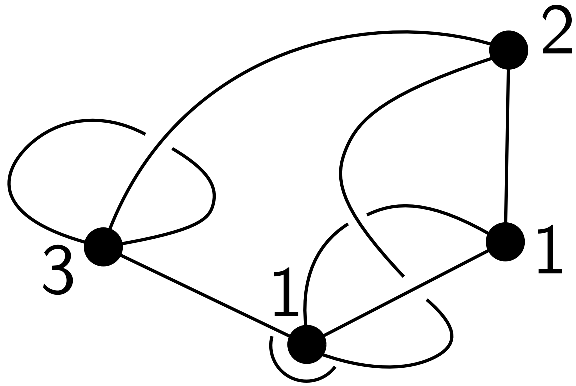

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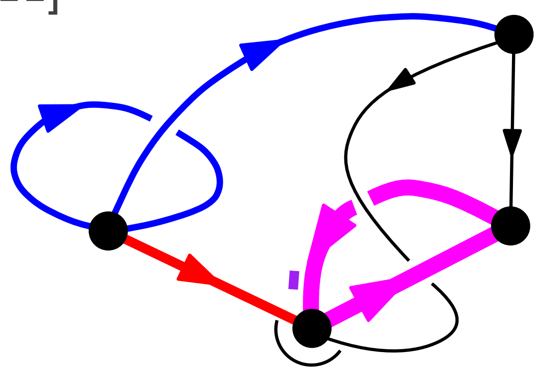

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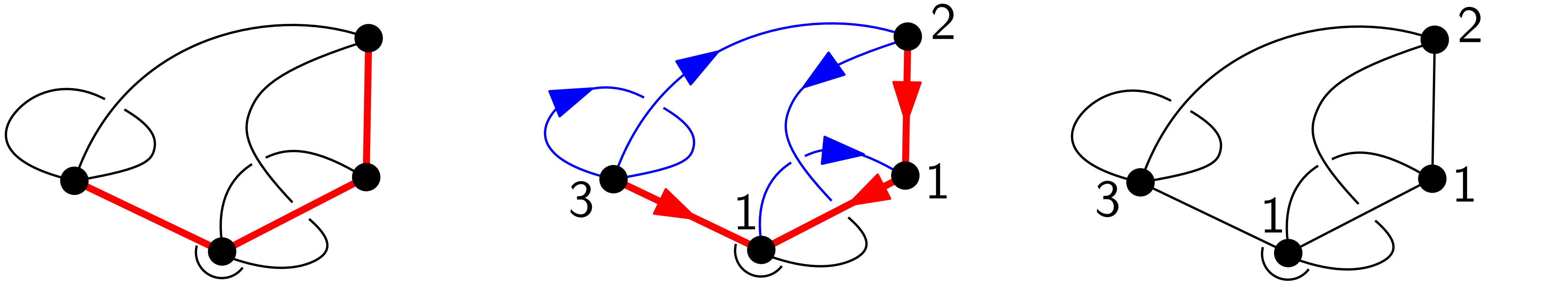

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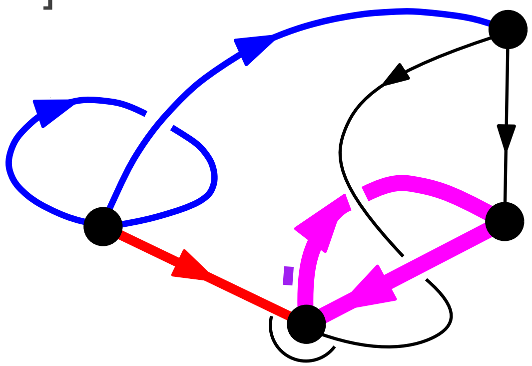

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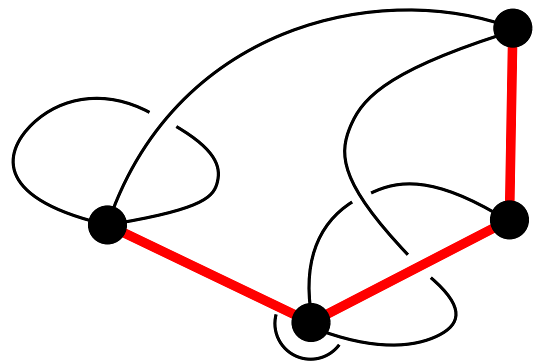

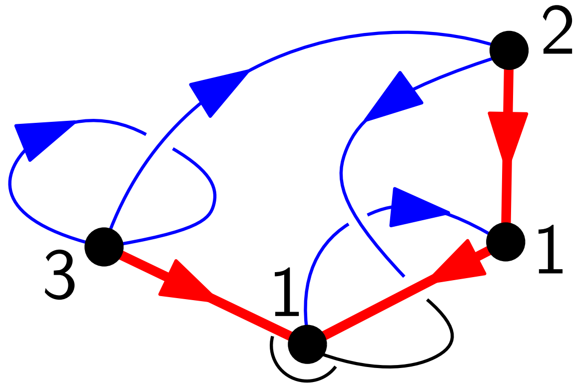

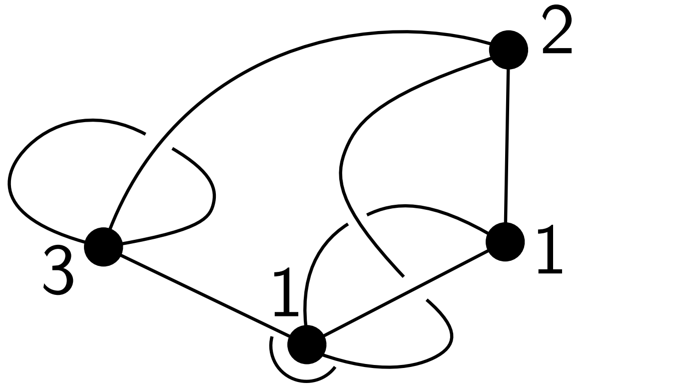

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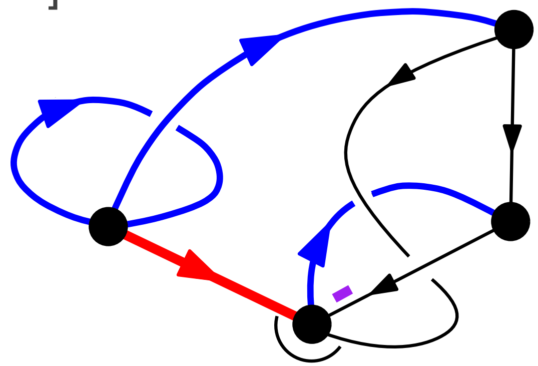

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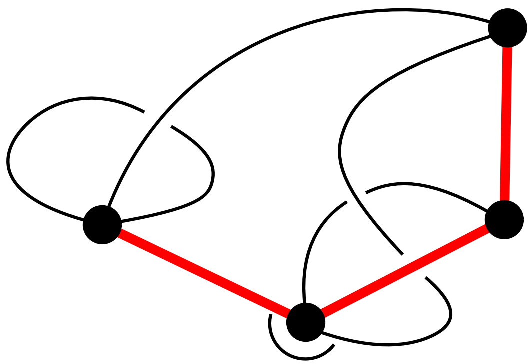

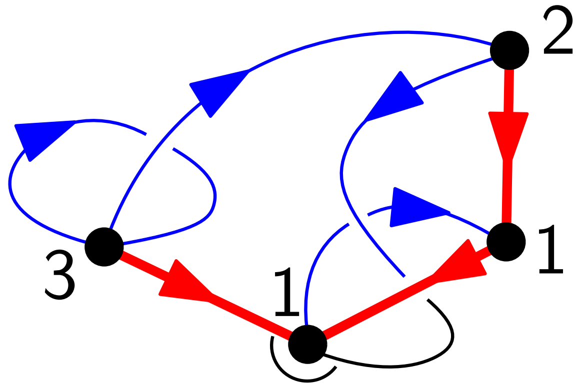

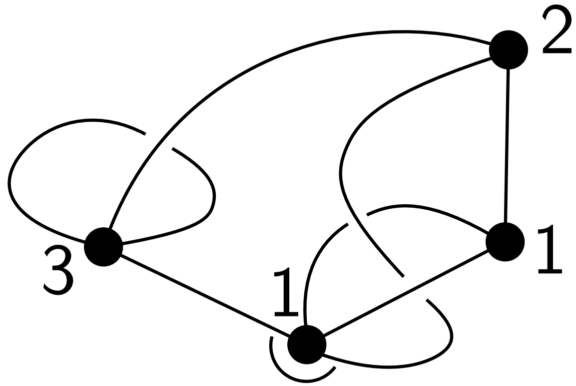

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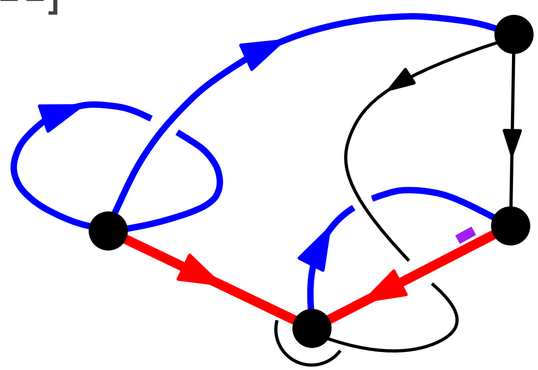

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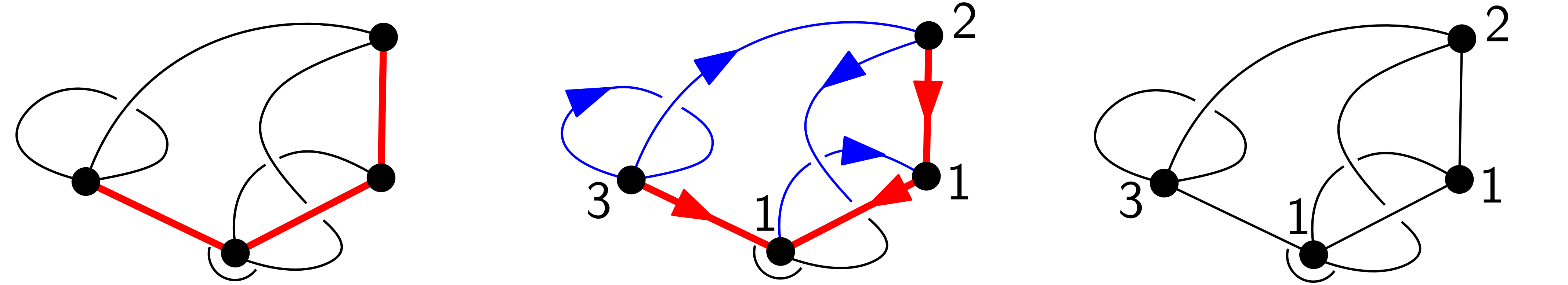

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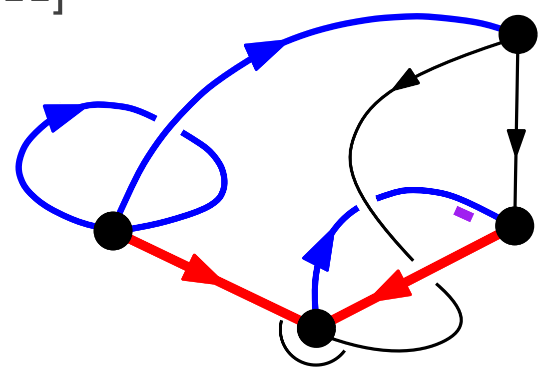

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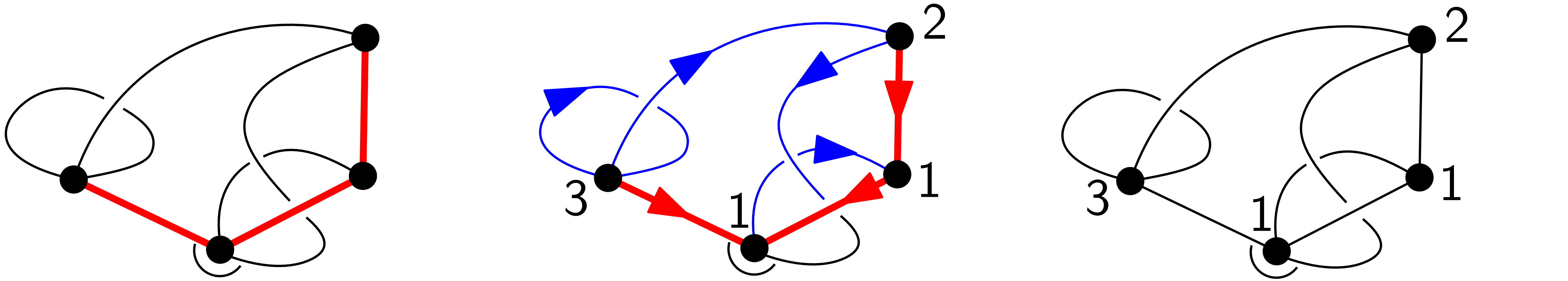

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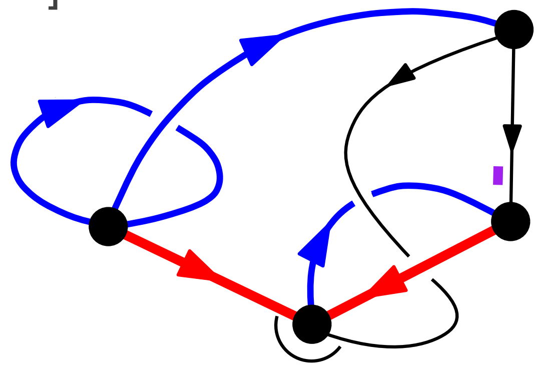

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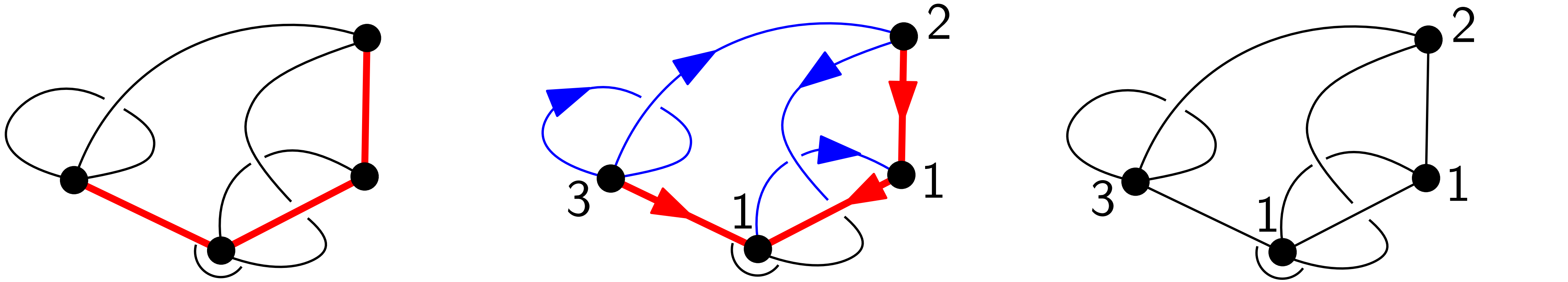

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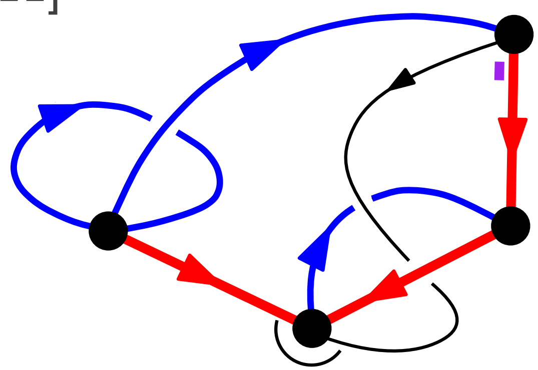

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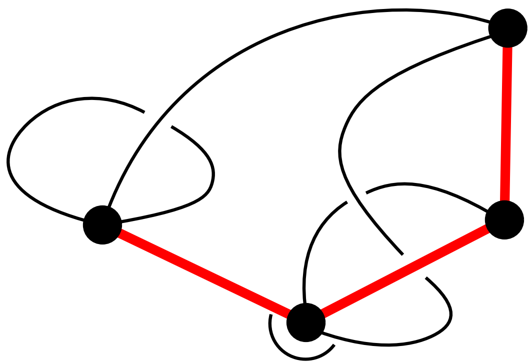

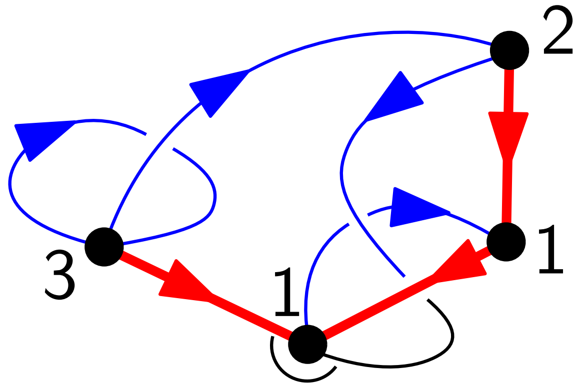

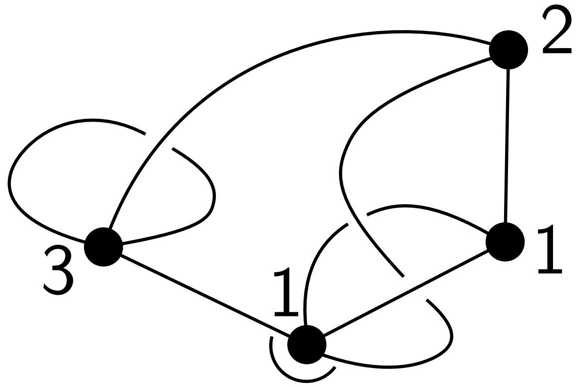

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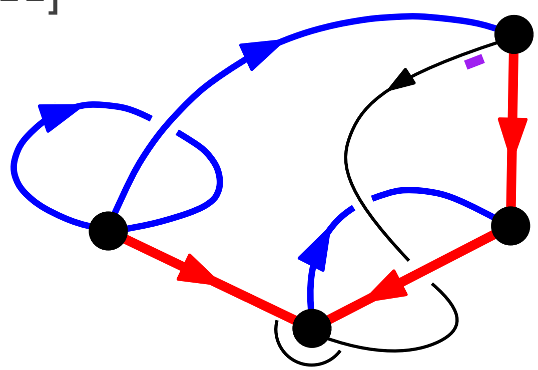

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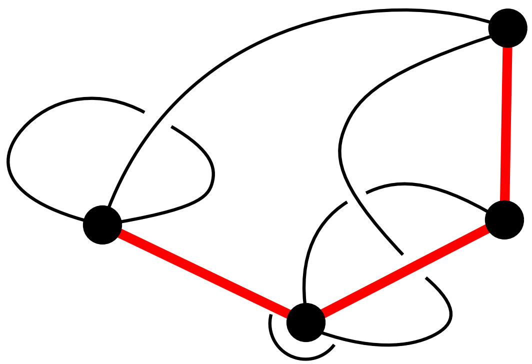

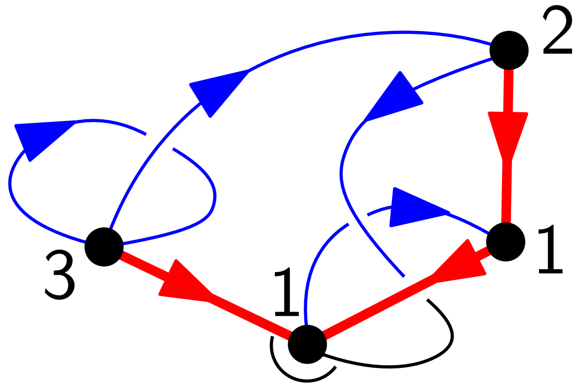

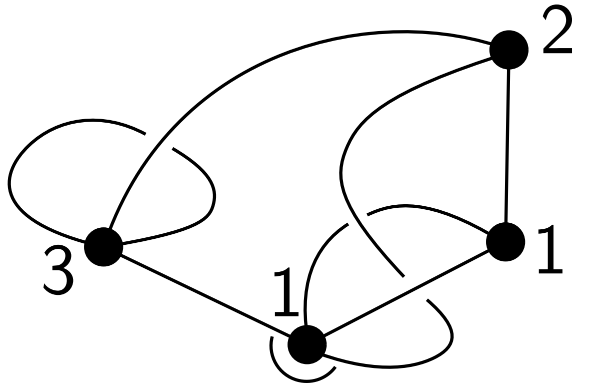

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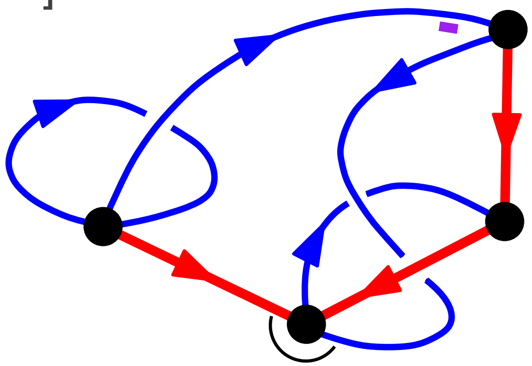

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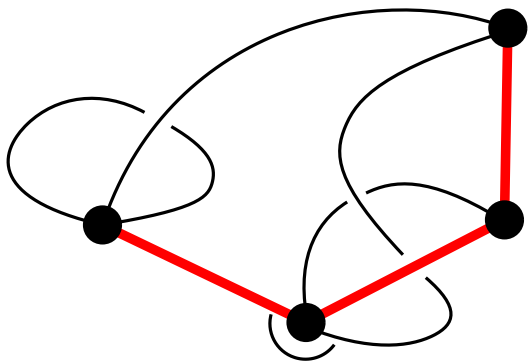

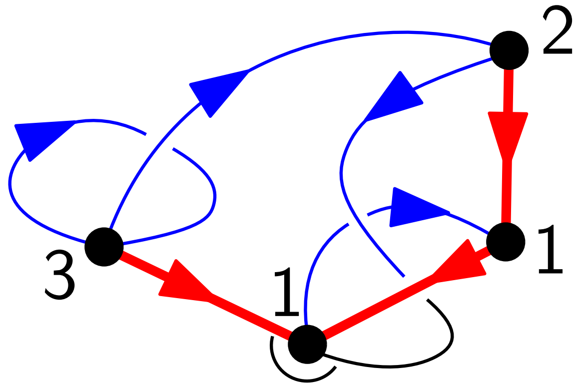

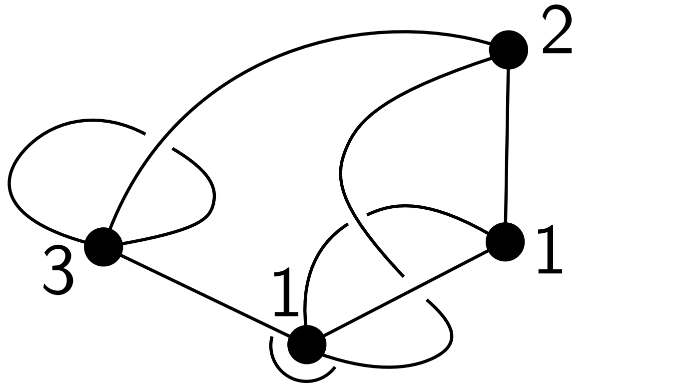

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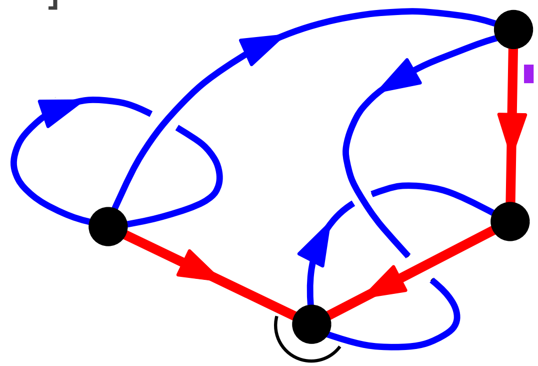

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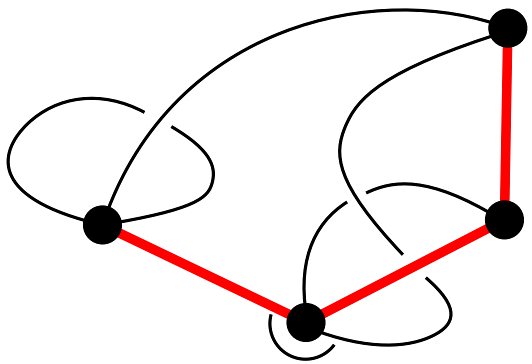

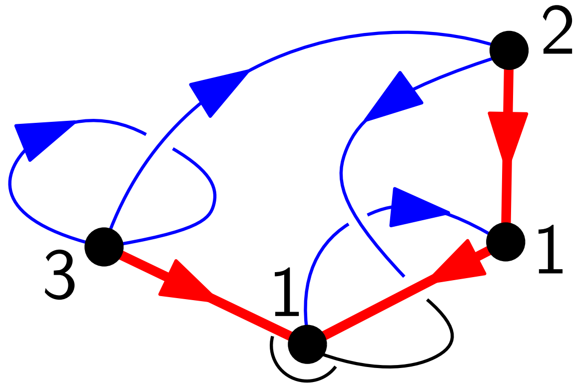

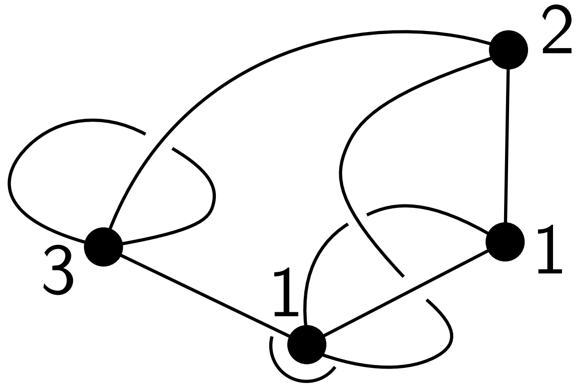

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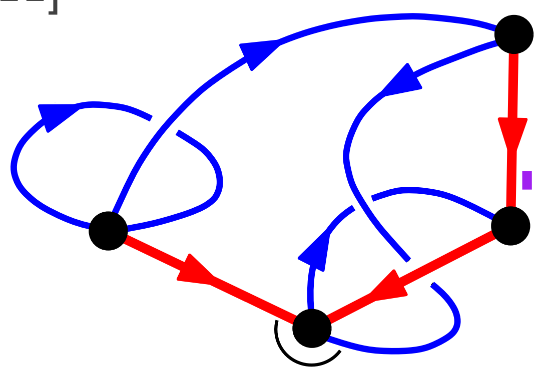

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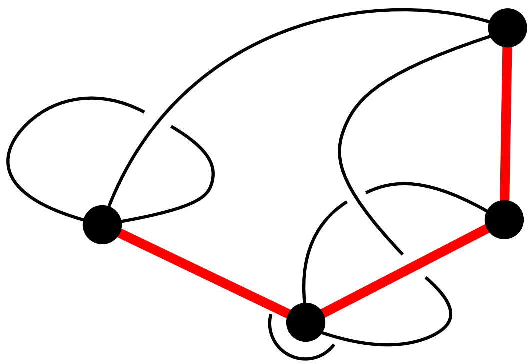

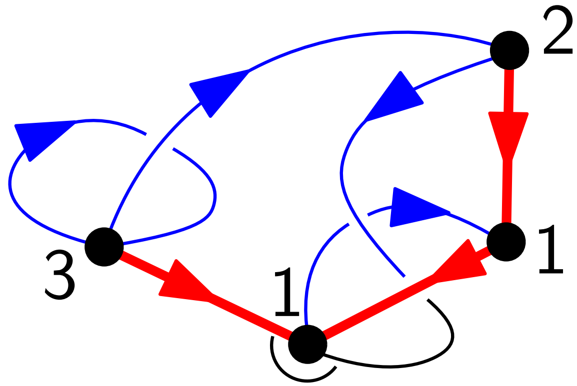

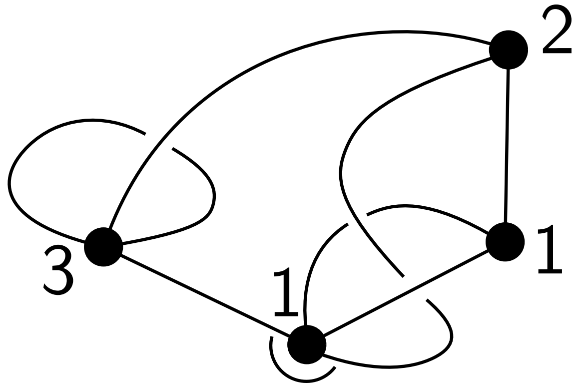

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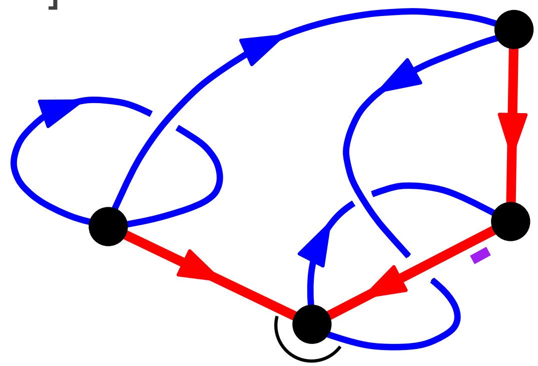

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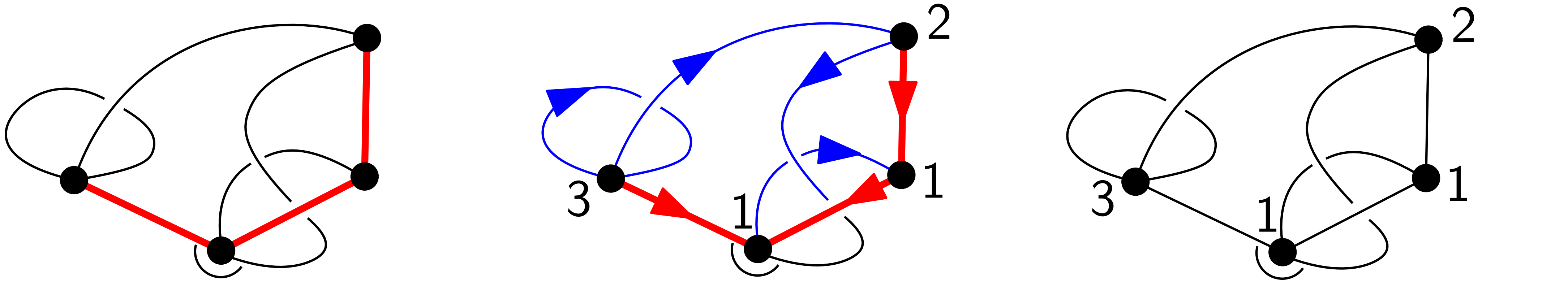

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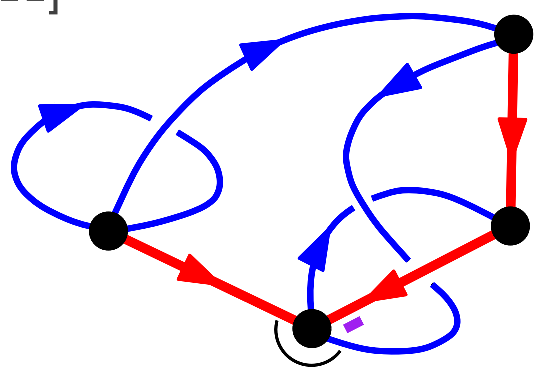

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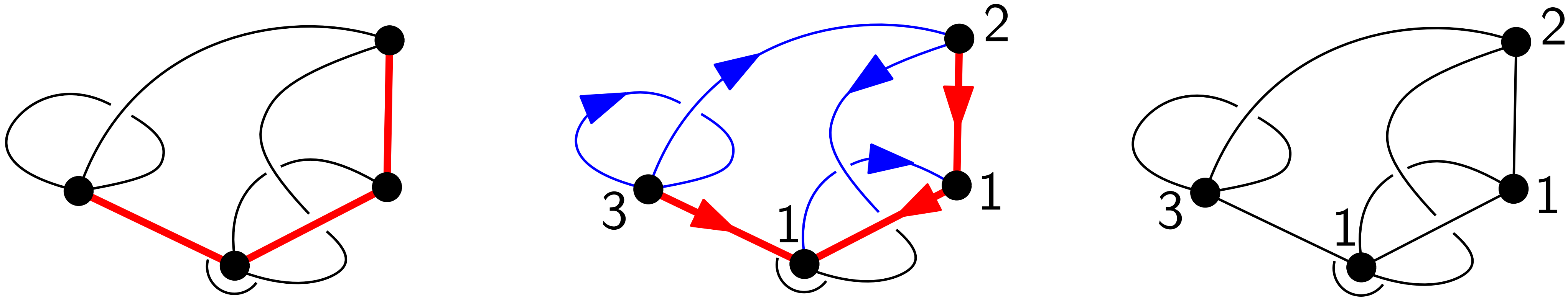

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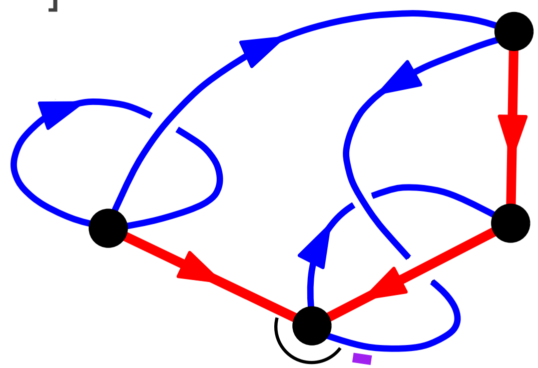
(orientations for maps endowed with spanning unicellular map)

From a given α-orientation, the spanning tree and the minimal α-orientation are computed jointly by an adapted traversal procedure

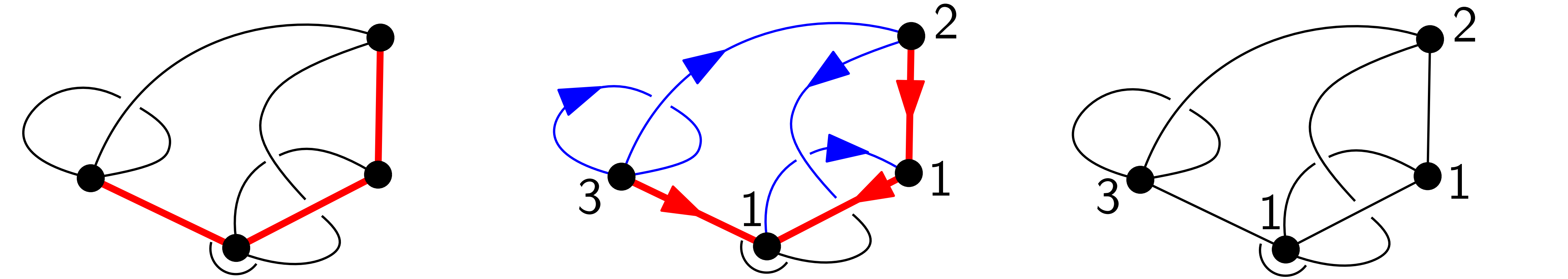


Bernardi's bijection (any genus)

[Bernardi'08]

Let M be a fixed rooted map, with vertex-set V

spanning trees of M $\longleftrightarrow$ root-accessible $\alpha : V \to \mathbb{N}$



'minimal' α-orientation

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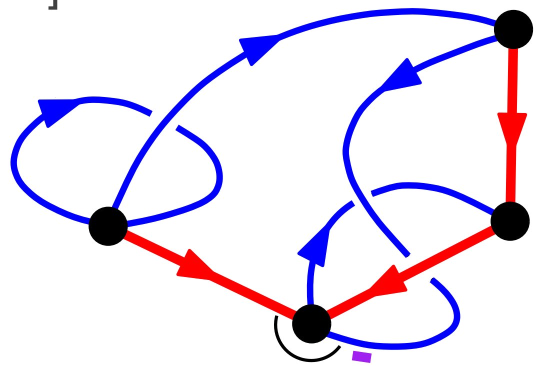

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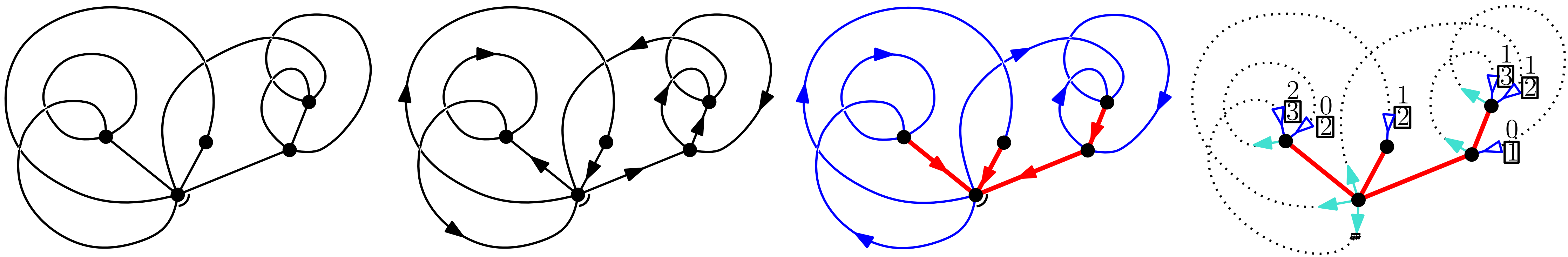
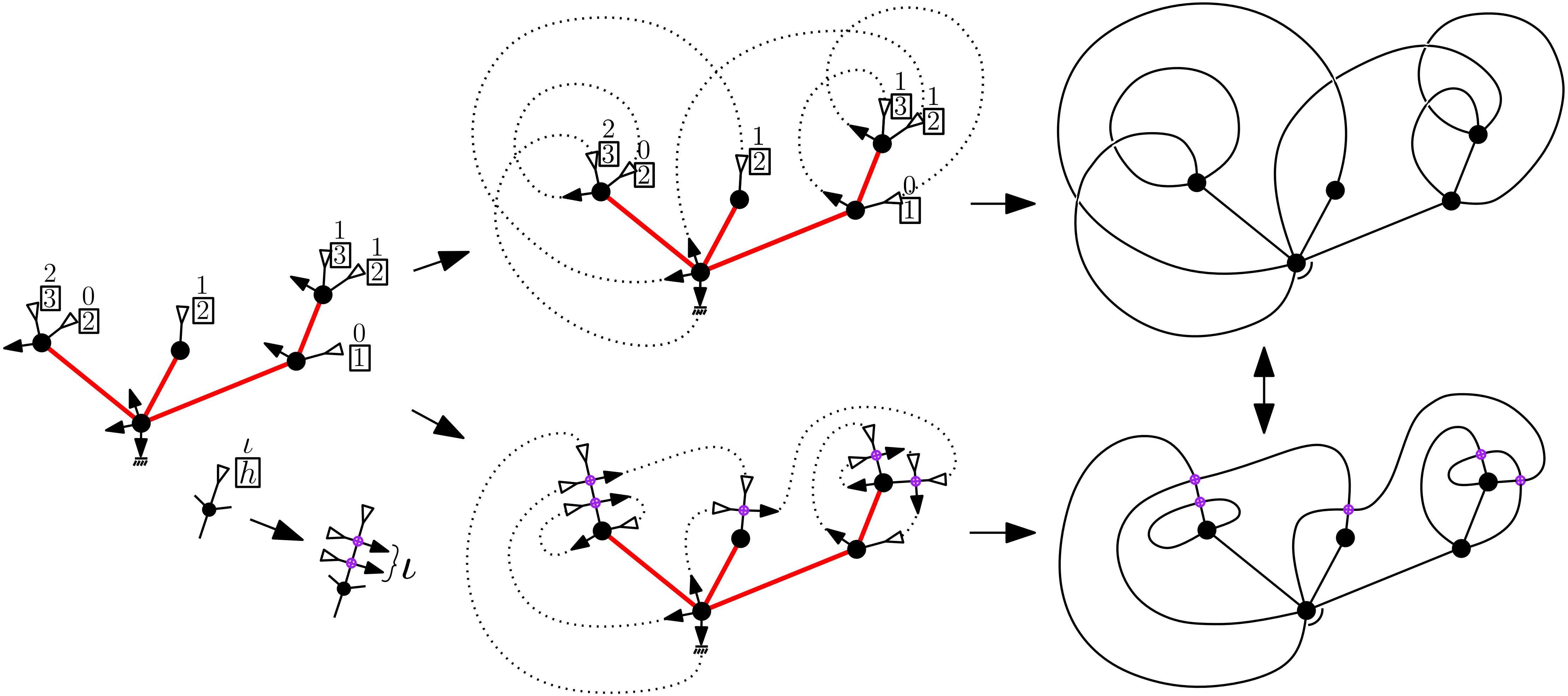
We apply it to Eulerian maps, with $\alpha(v) = \deg(v)/2$



Planarized version of the bijection

Bijection ⇔ leaf-extensions + Schaeffer's planar construction



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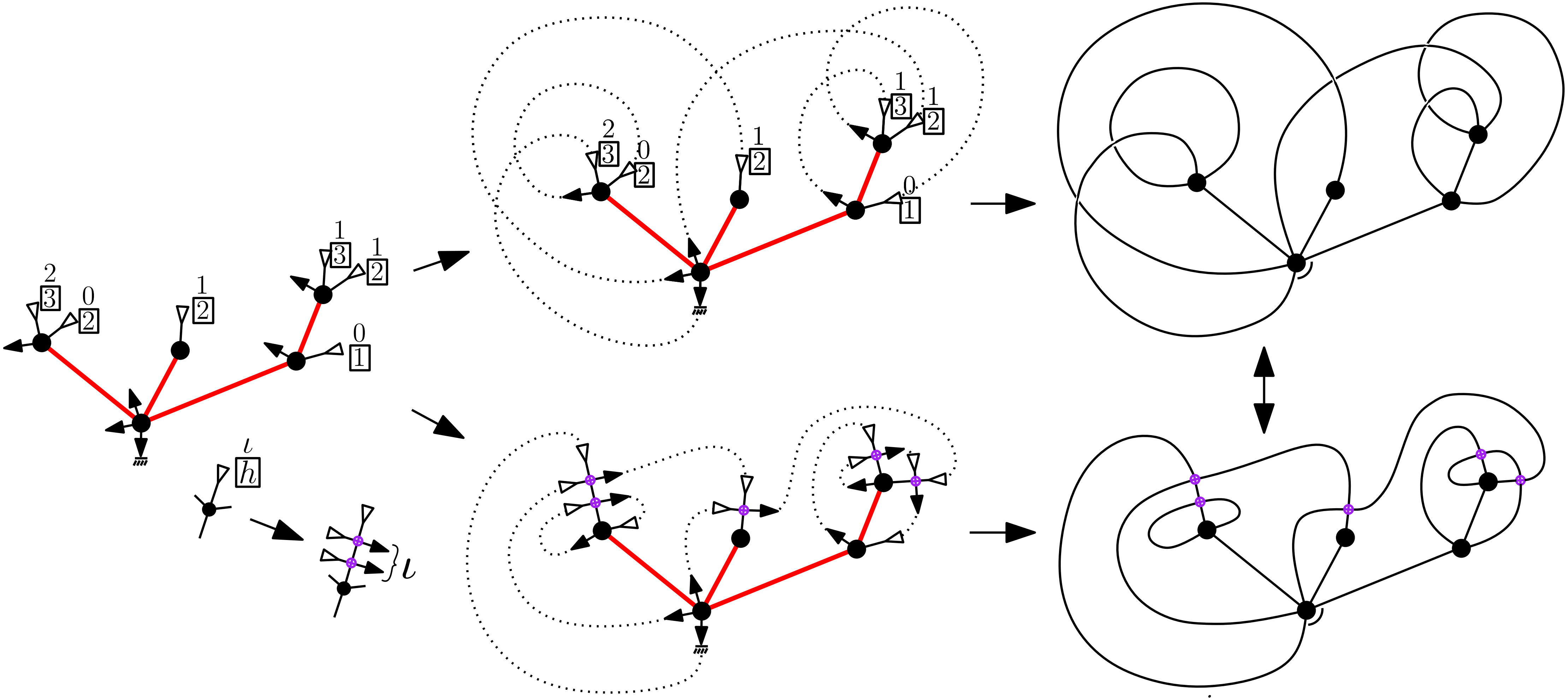

Rk: Let $r_i(t,q) := \hat{r}_i|_{z_j=[j]}$, where $[j] := 1 + \cdots + q^{j-1} = \dfrac{1-q^j}{1-q}$

Then $r_1(t,q)$ = GF of Eulerian maps with q conjugate to crossing-number

$R_i(t) = r_i(t,0)$ $\qquad$ $r_i(t) = r_i(t,1)$ $\qquad$ w.r.t canonical spanning tree

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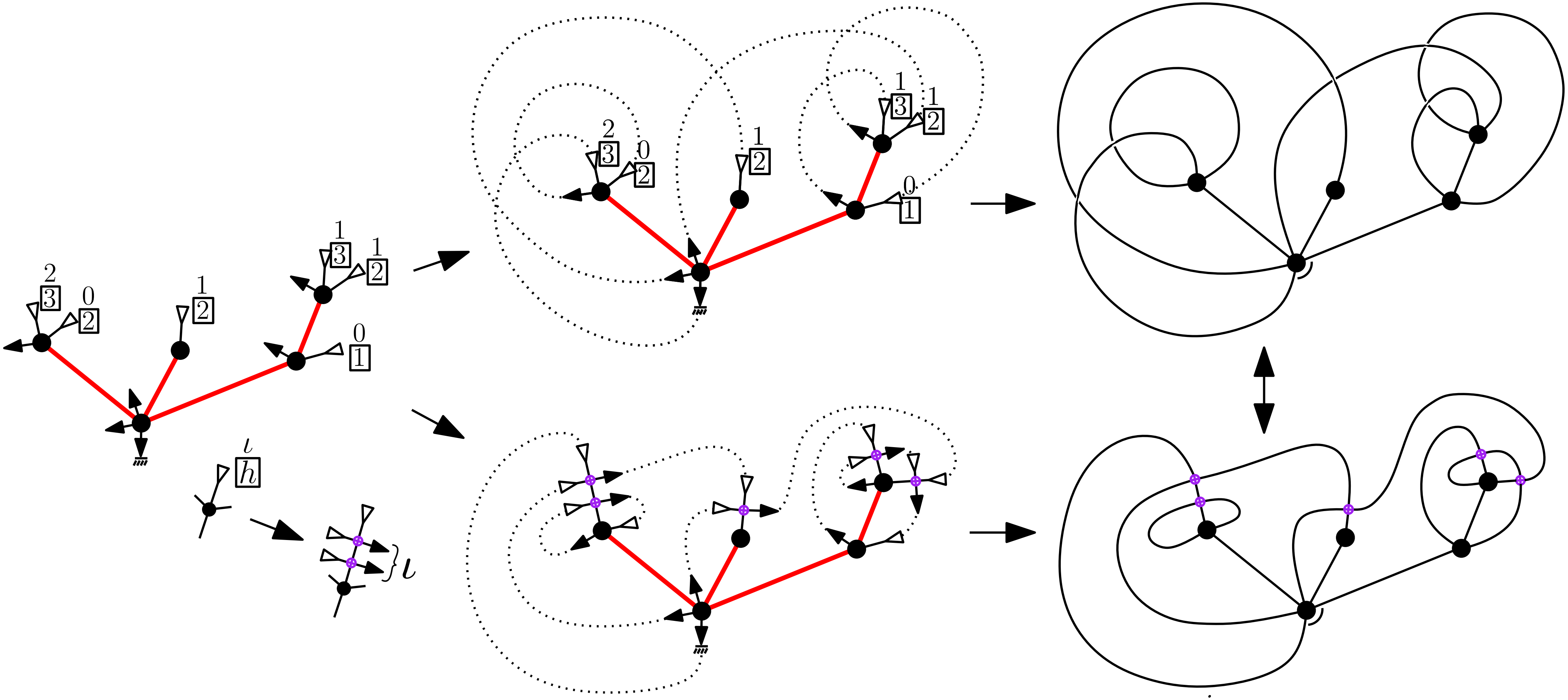

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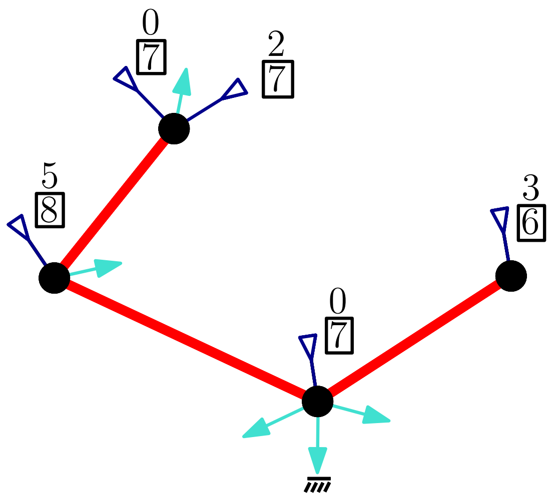
Rk: other extension of Schaeffer's bijection with control on the genus [Lepoutre'19]

Bijection for $r_i(t)$

$\boxed{i = 7}$



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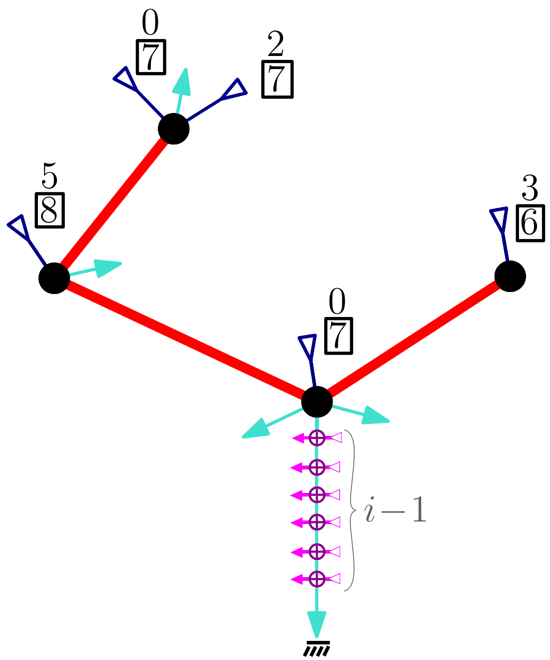

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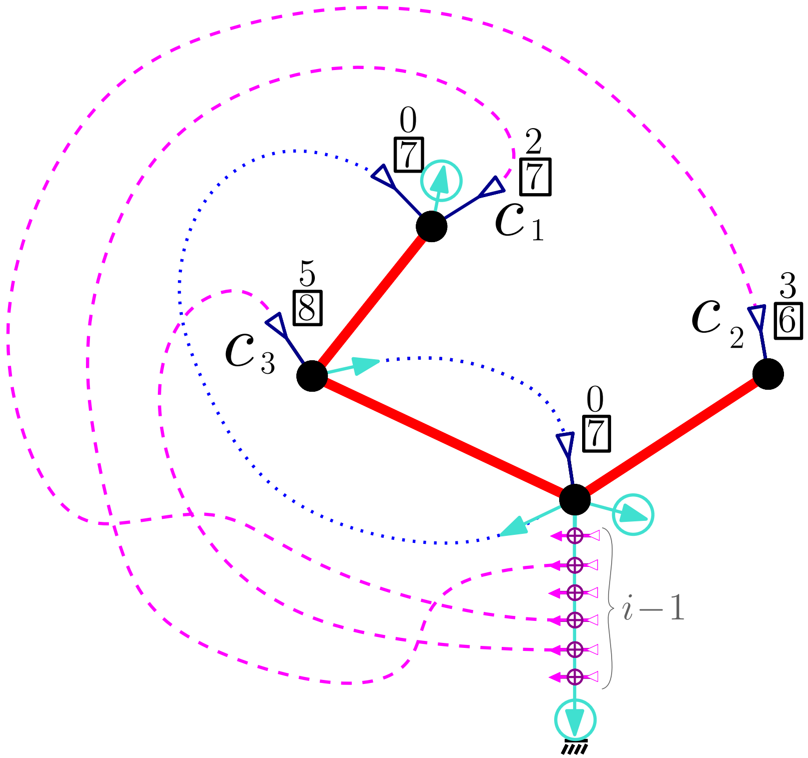

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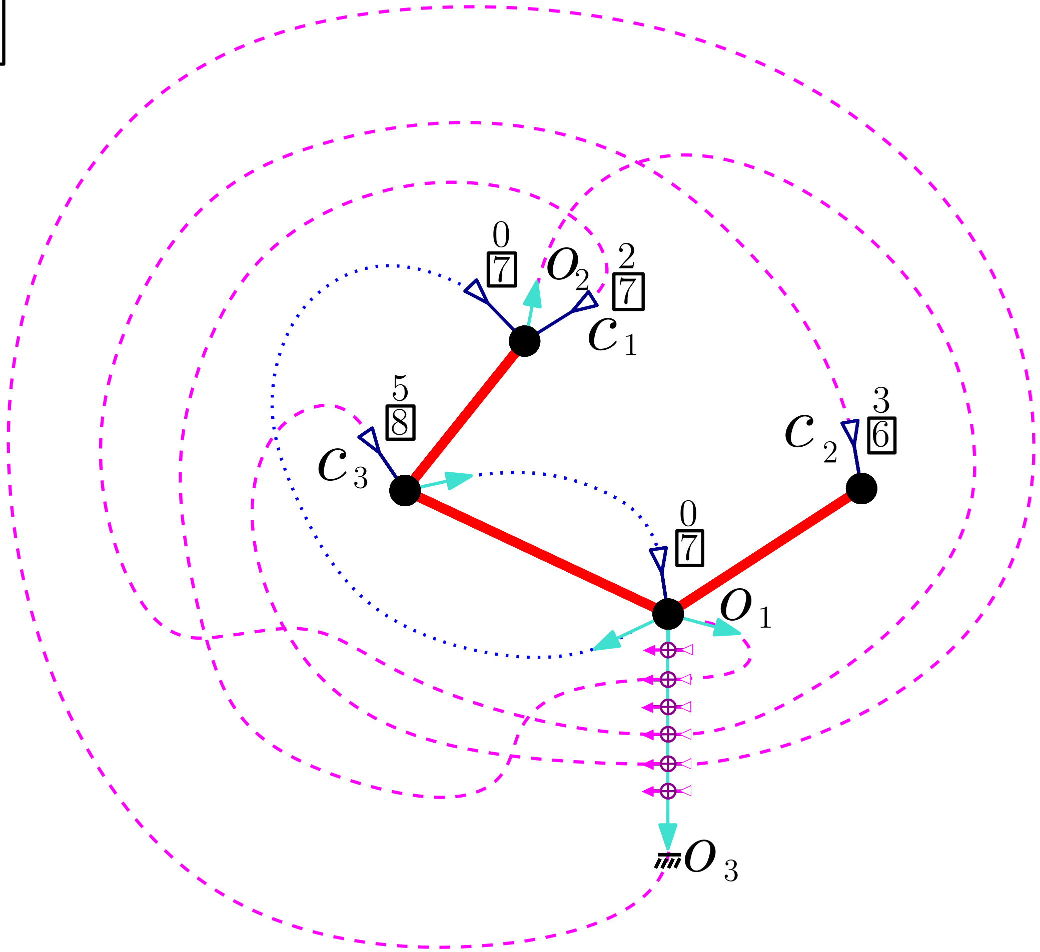

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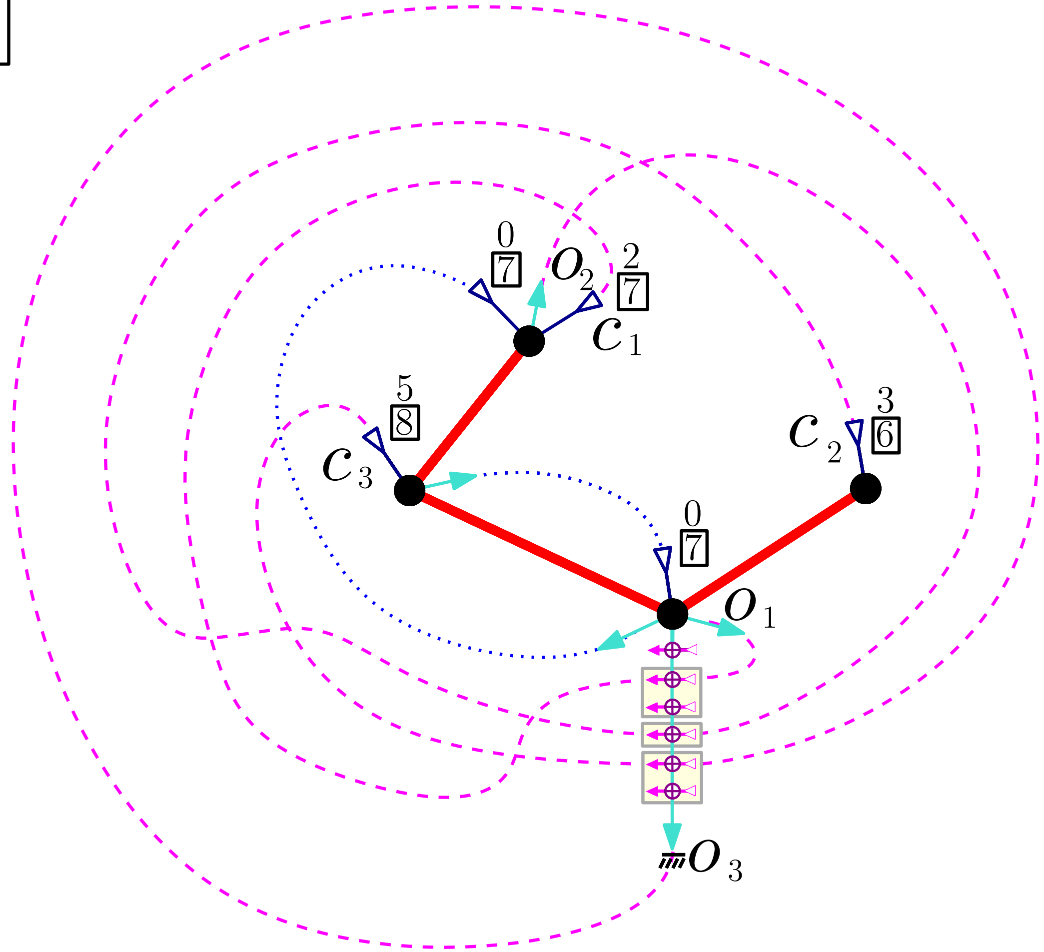

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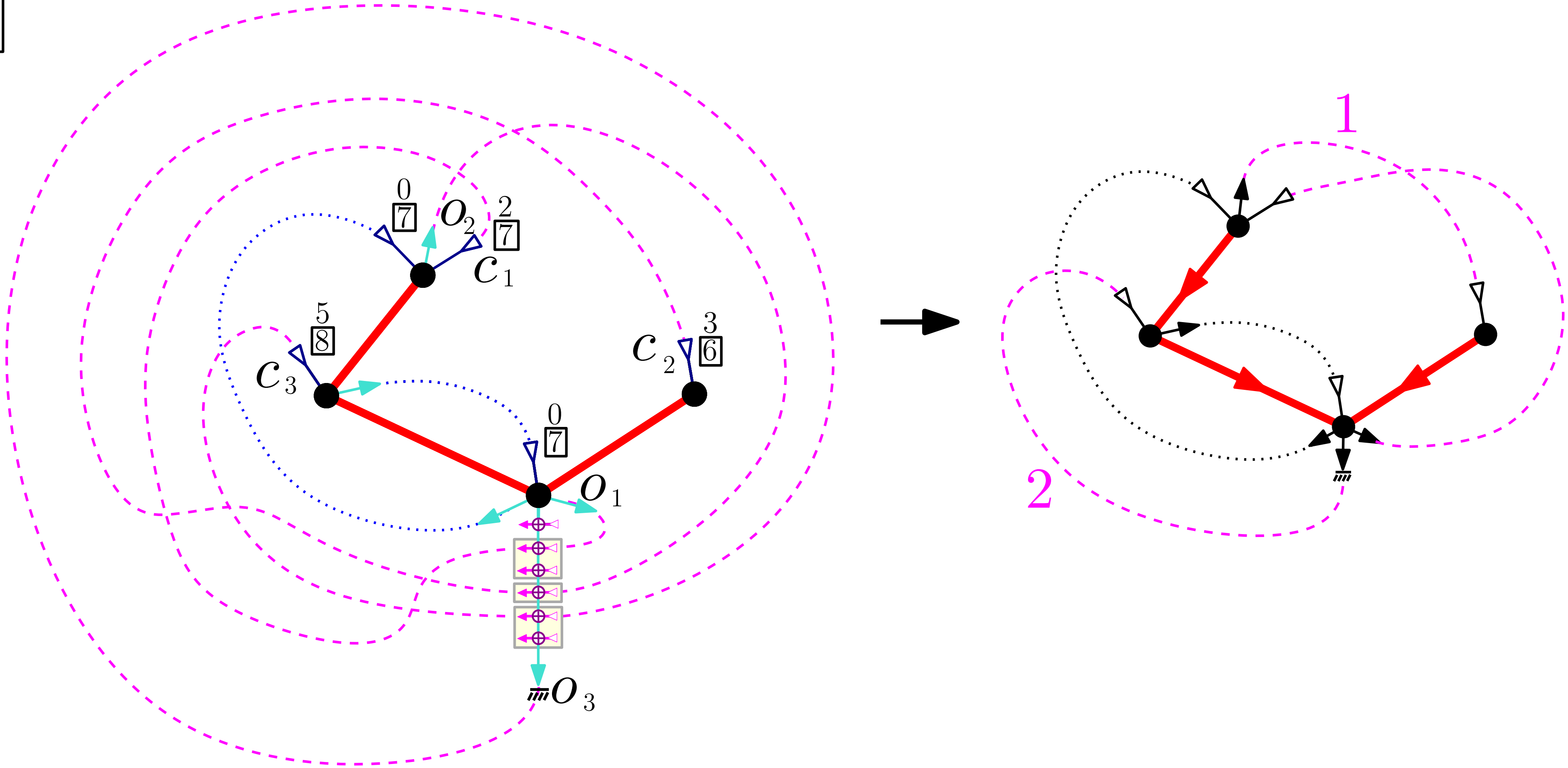

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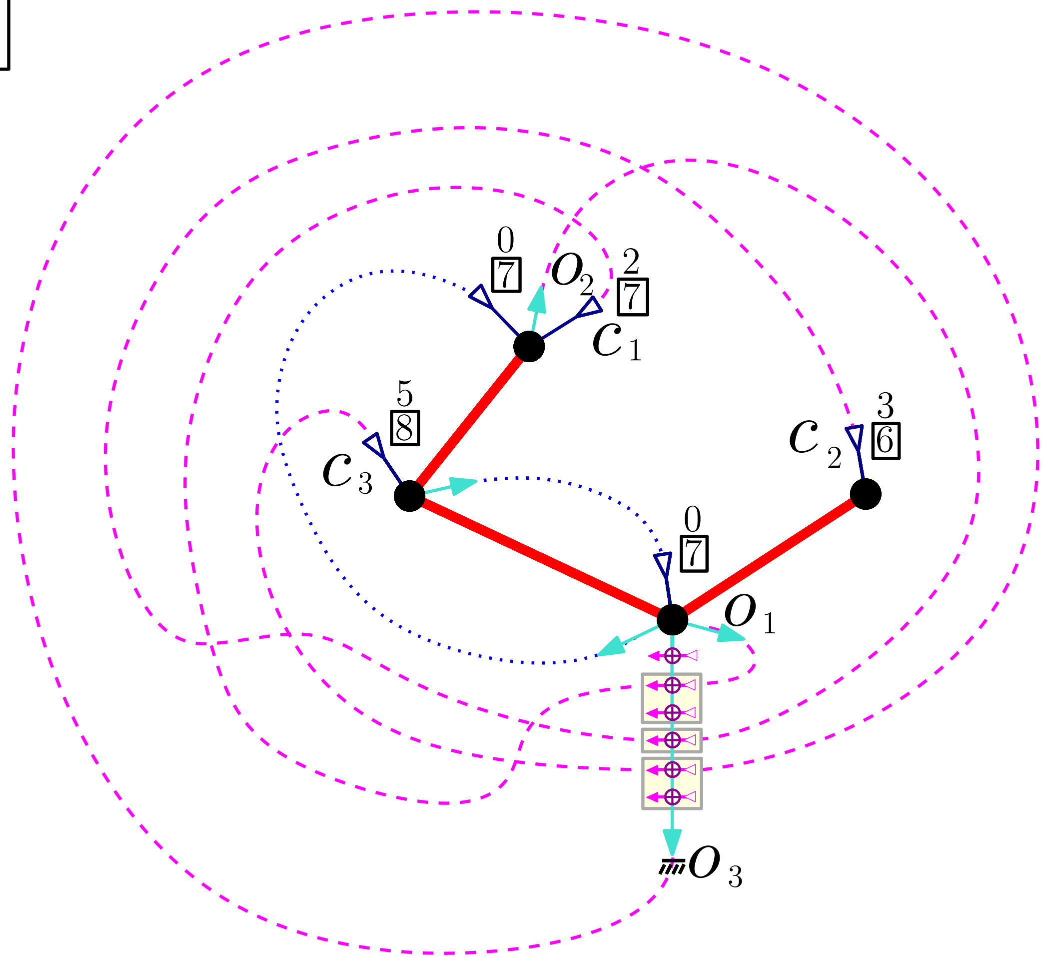
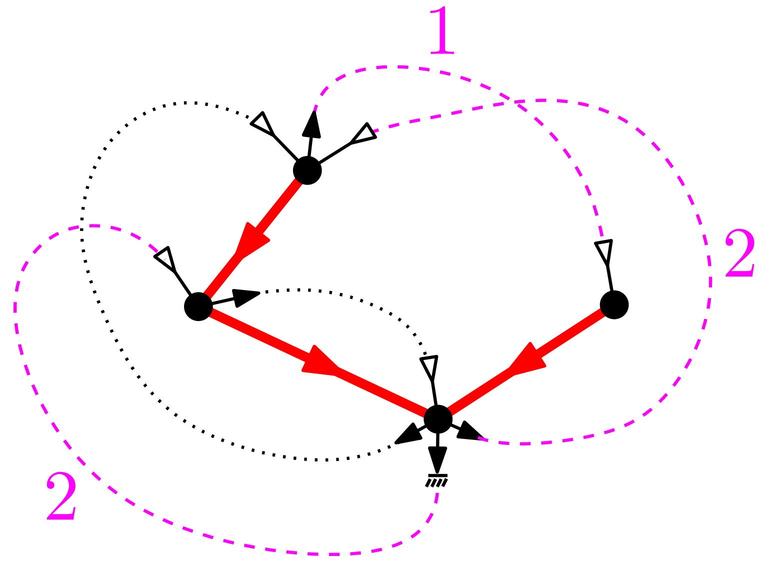
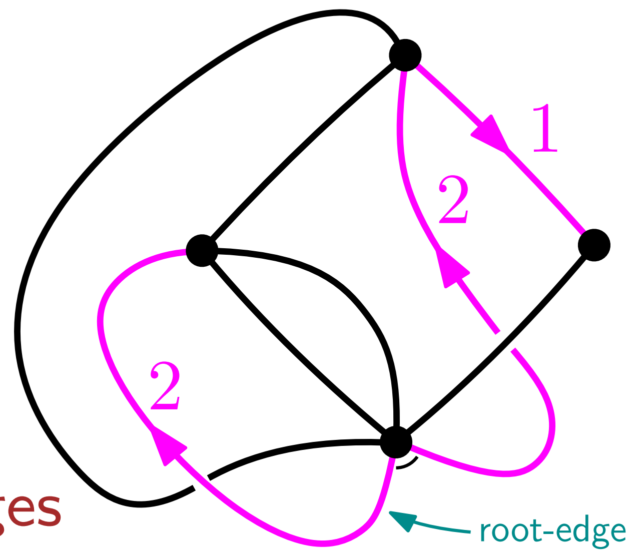






Marked Eulerian map: Eulerian map with marked oriented edges bearing ≥ 1 multiplicities

Admissible: possible to extend to Eulerian orientation where the root-edge is outgoing, and root-accessible without using marked edges

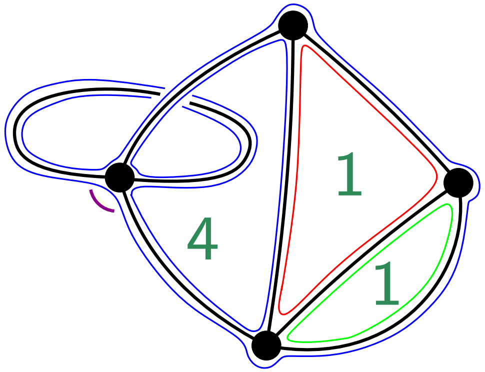
$r_i(t)$ counts admissible marked Eulerian maps with total multiplicity $\leq i - 1$

Face-colored maps

- recall on matrix integrals + orthogonal poly. for genus expansion
- interpretation of counting formula in terms of marked maps

Face-colored maps and relation to genus-expansion

N-face-colored map $=$ map where each face receives a color in $[1..N]$



4-face-coloring

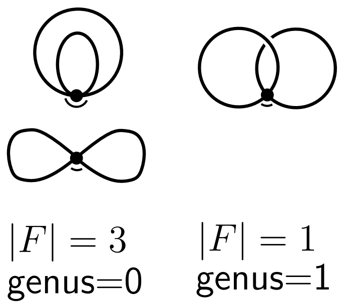
$\overline{M}(t,N) :=$ GF N-face-colored Eulerian maps

$\overline{U}(t,N) :=$ EGF unrooted N-face-colored Eulerian maps
(half-edge-labeled, not necessarily connected)

$$\overline{M}(t,N) = 2t\frac{\mathrm{d}}{\mathrm{d}t}\log(\overline{U}(t,N))$$

4-regular case ($g_k = \delta_{k=2}$):

$$\overline{M}(t,N) = (2N^3 + N)t^2 + O(t^4)$$



$$\text{genus} = \frac{|V| - |F|}{2} + 1$$

Matrix integral method (+ orthogonal poly.)

['t Hooft'74], [Brézin-Itzykson-Parisi-Zuber'78], [Bessis-Itzykson-Zuber'80]

$$\overline{U}(t,N) \xrightarrow{\text{configuration-model + Wick's formula}} \frac{2^{N(N-1)/2}}{(2\pi)^{N^2/2}} \int_{\mathcal{H}_N} dH \ \mathrm{e}^{\mathrm{Tr}\left(-H^2/2+V(t,H)\right)}$$

$$\boxed{V(t,X) = \sum_{k\geq 1} \frac{1}{2k} g_k t^k X^{2k}}$$

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$$\Big\downarrow$$

change of variable $H = \mathfrak{U}\Lambda\mathfrak{U}^\dagger$, with $\Lambda = \mathrm{Diag}(\lambda_0, \ldots, \lambda_{N-1})$

Jacobian $\propto \prod_{i\neq j}(\lambda_i - \lambda_j) = \Delta(\Lambda)^2$

$$c_N \int_{\mathbb{R}^N} d\Lambda \ \Delta(\Lambda)^2 \ e^{\mathrm{Tr}\left(-\Lambda^2/2+V(t,\Lambda)\right)}$$

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$$\Big\downarrow \; \Delta(\Lambda) = \det(p_i(t,\lambda_j))_{0\leq i,j\leq N-1}$$

$$(2\pi)^{N/2} c_N \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}(\sigma) \sum_{\tau\in\mathfrak{S}_N} \mathrm{sgn}(\tau) \prod_{i=0}^{N-1} < p_{\sigma(i)}, p_{\tau(i)} >$$

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$$(2\pi)^{N/2} c_N \; N! \; h_0 h_1 \cdots h_{N-1} \longrightarrow \tilde{c}_N \; h_0^N r_1^{N-1} r_2^{N-2} \cdots r_{N-1}$$

Expressions for $\overline{M}(t, N)$

$$\overline{U}(t, N) = \tilde{c}_N \; h_0^N r_1^{N-1} r_2^{N-2} \cdots r_{N-1}$$

$$\Downarrow$$

$$\overline{M}(t, N) = 2t \frac{\mathrm{d}}{\mathrm{d}t} \log \overline{U}(t, N) = \; 2tN \frac{h_0'(t)}{h_0(t)} \; + \; 2t \sum_{i=1}^{N-1} (N-i) \frac{r_i'(t)}{r_i(t)} \quad \textbf{(Exp1)}$$

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Simpler expression: [course notes Di Francesco'14]

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Then $\overline{M}(t,N) = \frac{< \mathrm{Tr}(H^2) >}{\overline{U}(t,N)} - N^2 = -N^2 + \sum_{i=0}^{N-1} \left(r_i(t) + r_{i+1}(t)\right)$ **(Exp2)**

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Rk: No bijective proofs of **(Exp1)** or **(Exp2)** for $N \geq 2$, but they are linked by the differential identity $r_i'(t) = r_i(t)\big(r_{i+1}(t) - r_{i-1}(t) - 2\big)$ for which we have a bijective proof (using marked maps)

Genus expansion

For 4-regular maps $r_i(t)$ satisfy the recursive system

$$\boxed{r_i(t) = i + t^2 r_i(t)(r_{i-1}(t) + r_i(t) + r_{i+1}(t))}$$

$$\Rightarrow r_i(t) = i + 3i^2t^2 + (18i^3 + 6i)t^4 + (135i^4 + 162i^2)t^6 + \cdots$$

(by induction on $k \geq 0$, $P_k(i) := [t^{2k}]r_i(t)$ is a polynomial in i)

$$P_0(i) = i \quad \text{and for } k \geq 1, \; P_k(i) = \sum_{\ell=0}^{k-1} P_{k-\ell-1}(i)\big(P_\ell(i-1) + P_\ell(i) + P_\ell(i+1)\big)$$

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&= (2N^3 + N)t^2 + (9N^4 + 15N^2)t^4 + (54N^5 + 198N^3 + 45N)t^6 + \cdots
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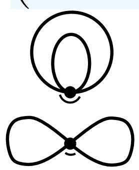 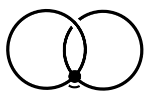

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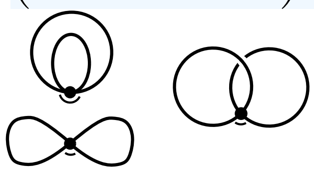
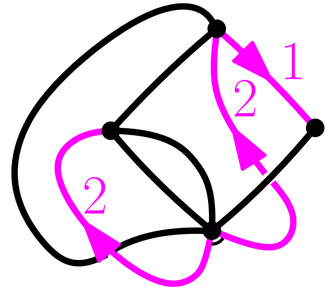


Rk: Expansion with $N^{2-2\text{genus}}$ instead of N^F and letting $N \to \infty$

$$\Rightarrow \quad M_{\text{planar}}(t) = 2\int_0^1 R(x,t)\mathrm{d}x \quad \text{with } R(x,t) = x + \sum_{k\geq 1} t^k g_k \binom{2k-1}{k-1} R(x,t)^k$$

Interpretation in terms of marked maps



$$\overline{M}(t,N) = -N^2 + \sum_{i=0}^{N-1} \left(r_i(t) + r_{i+1}(t)\right)$$

$\Rightarrow \overline{M}(t,N) =$ GF of admissible marked Eulerian maps of total multiplicity $\mu \leq N-1$ where each such map is counted $2(N-1-\mu) + \delta_{\mu=N-1}$ times

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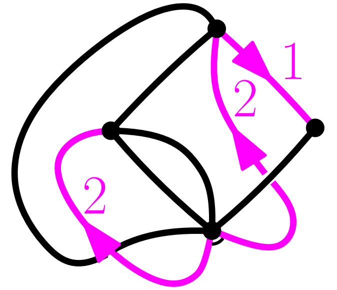

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(every color $\in [1..N]$ is used by at least one face)

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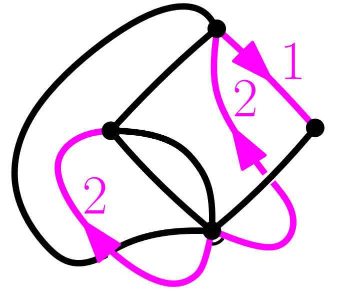

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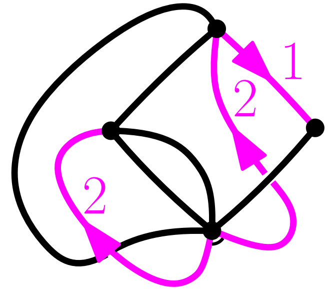

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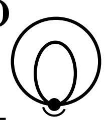
6

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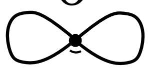
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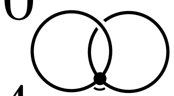
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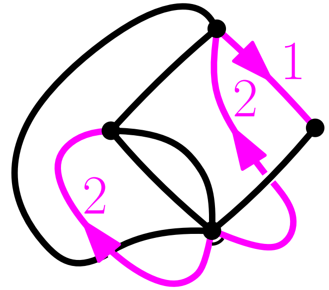
Interpretation in terms of marked maps



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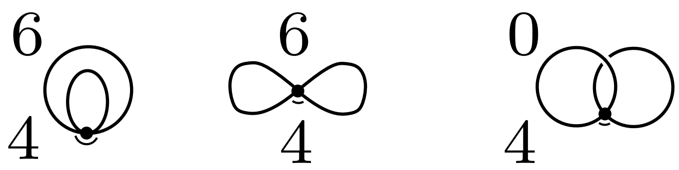

More generally $[t^n g_n]\widehat{M}(t,L) = [t^n g_n] u_L(t) = (2n-1)!! \binom{n}{L-1} 2^{L-1}$

Harer-Zagier summation formula

(bijective proofs: [Goulden-Nica'05, Bernardi'12, Chapuy-Feray-Fusy'13])