A bijection between binary trees and some dissections of the hexagon

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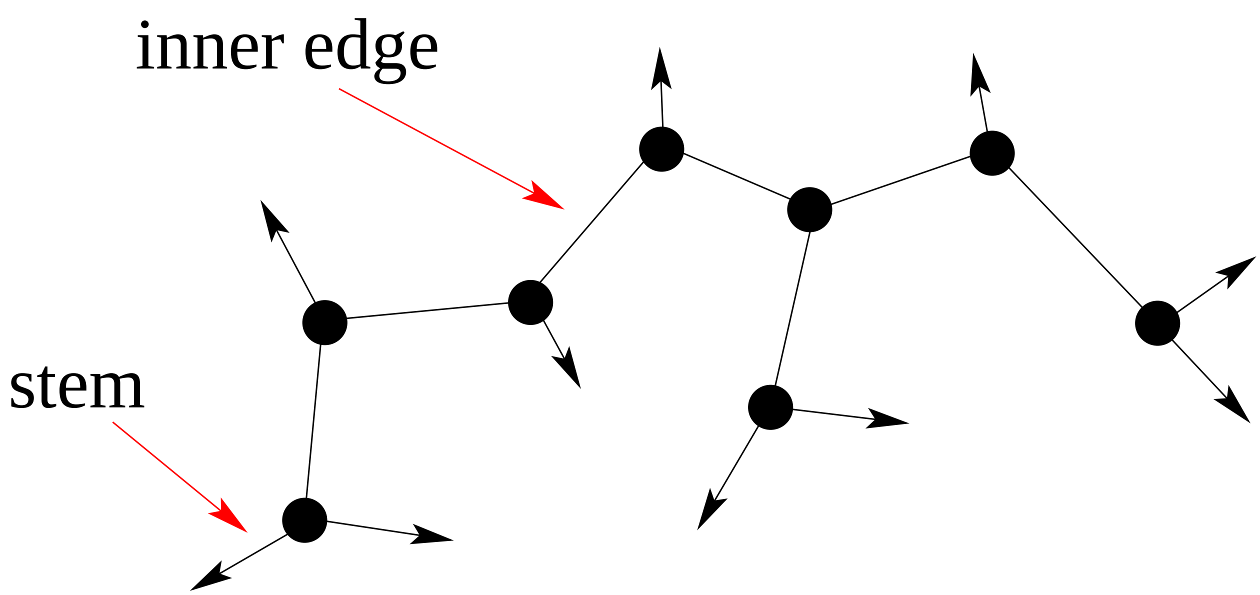
joint work with Dominique Poulalhon and Gilles Schaeffer

Binary trees

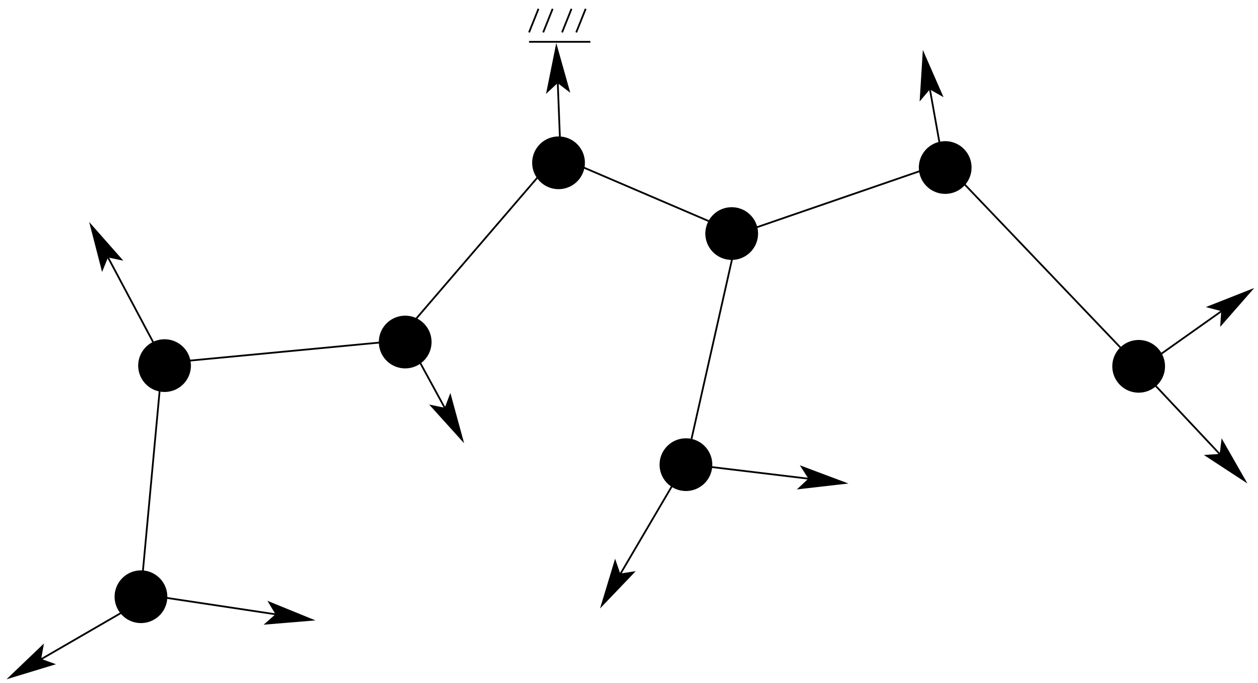
- A binary tree is a tree embedded on the plane with:
  - vertices of degree 3: inner nodes
  - vertices of degree 1: leaves
- An edge connecting two inner nodes is called an *inner edge*
- An edge incident to a leaf is called a *stem*



Rooted binary trees

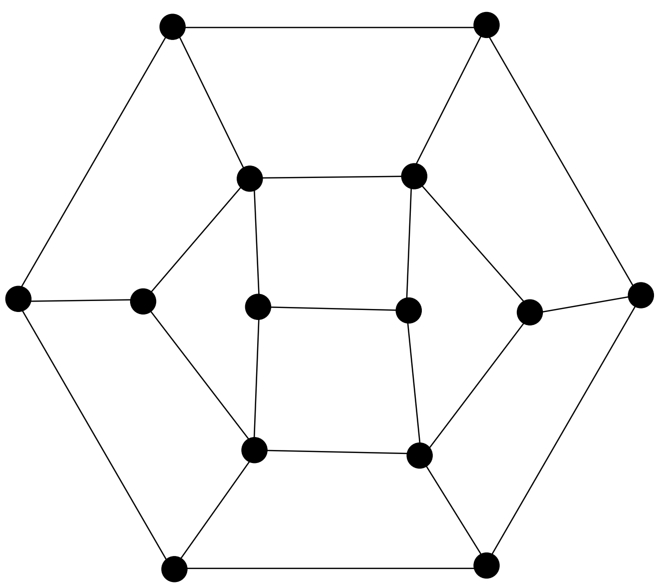
- A binary tree is rooted by marking one of its leaves
- Rooted binary trees are counted by the Catalan numbers:

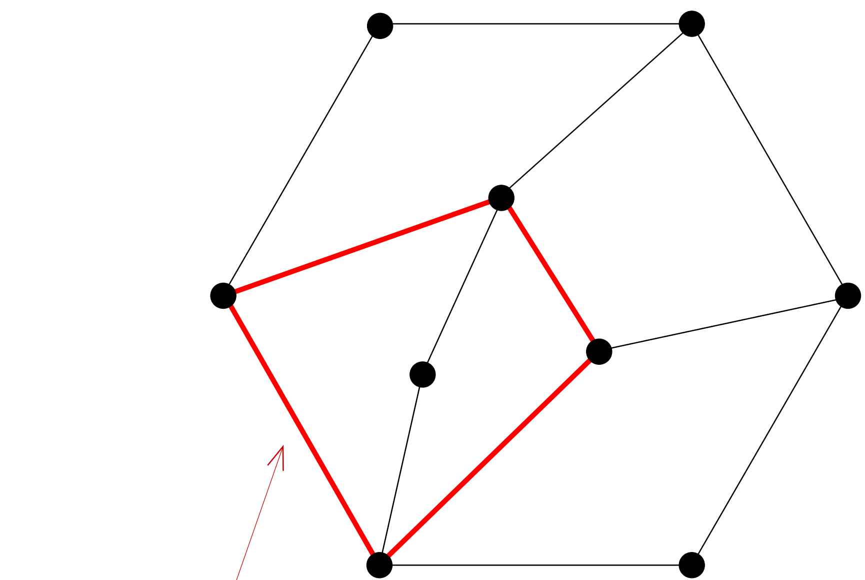
$$\frac{(2n)!}{(n+1)!n!}$$



Irreducible dissections

- Dissection of the hexagon in the plane with quadrangular inner faces
- Each 4-cycle delimits a face

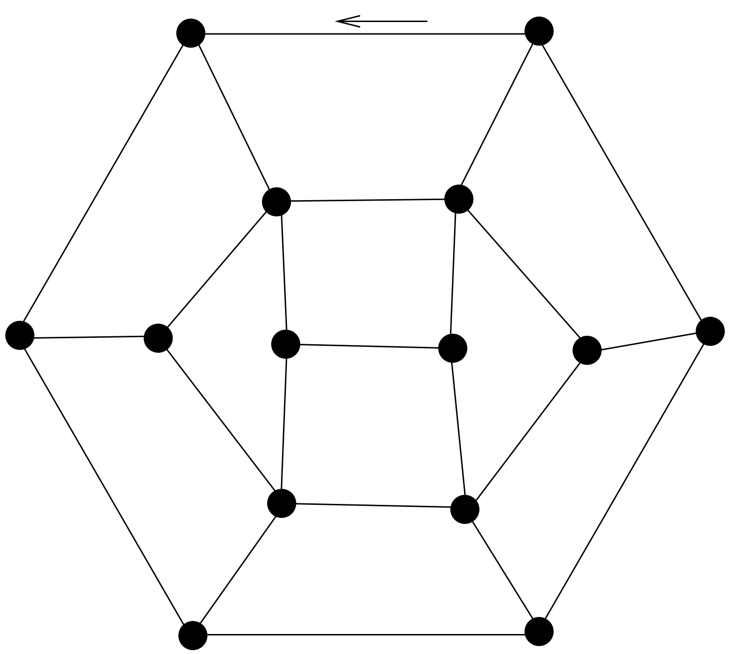

irreducible



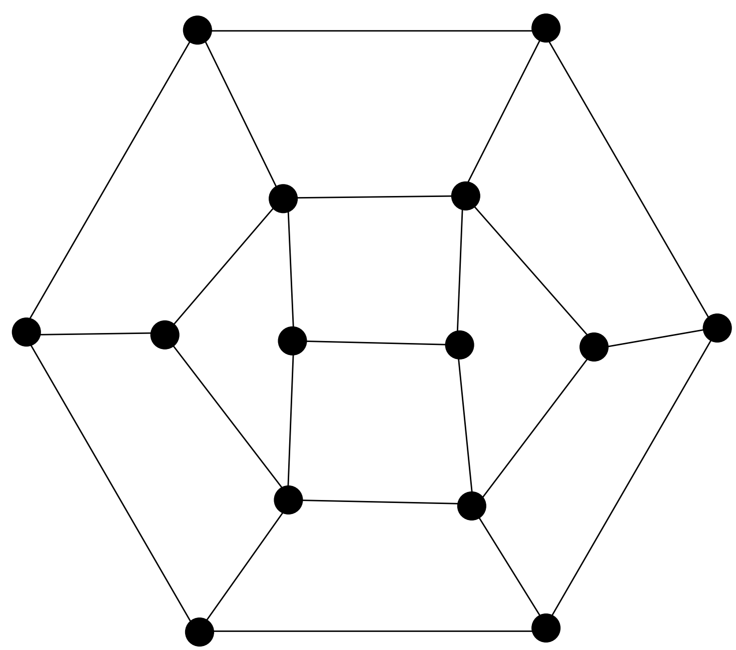
not irreducible

Rooted irreducible dissections

An irreducible dissection can be rooted by choosing and orienting an edge of the hexagon



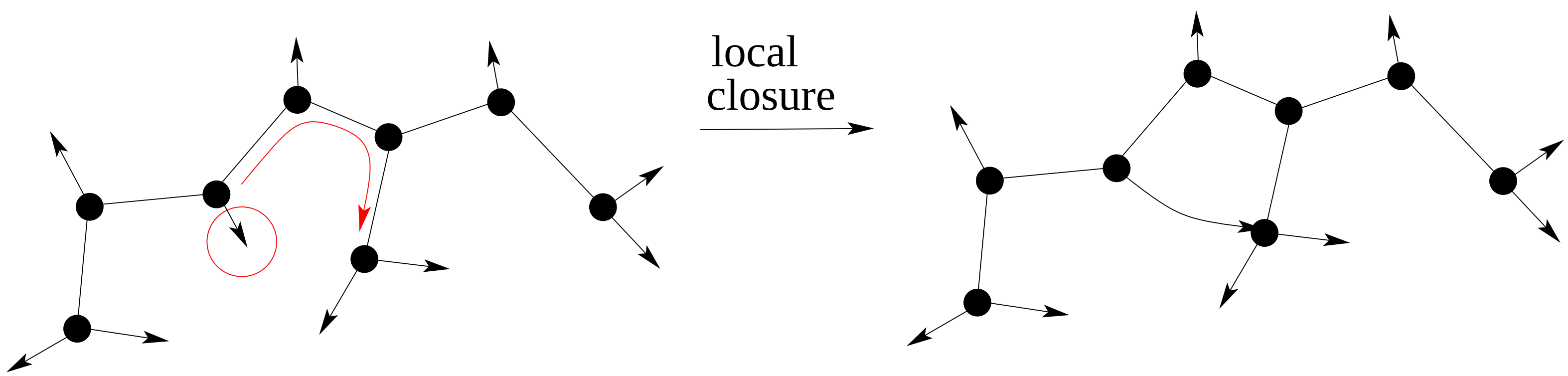
rooted



unrooted

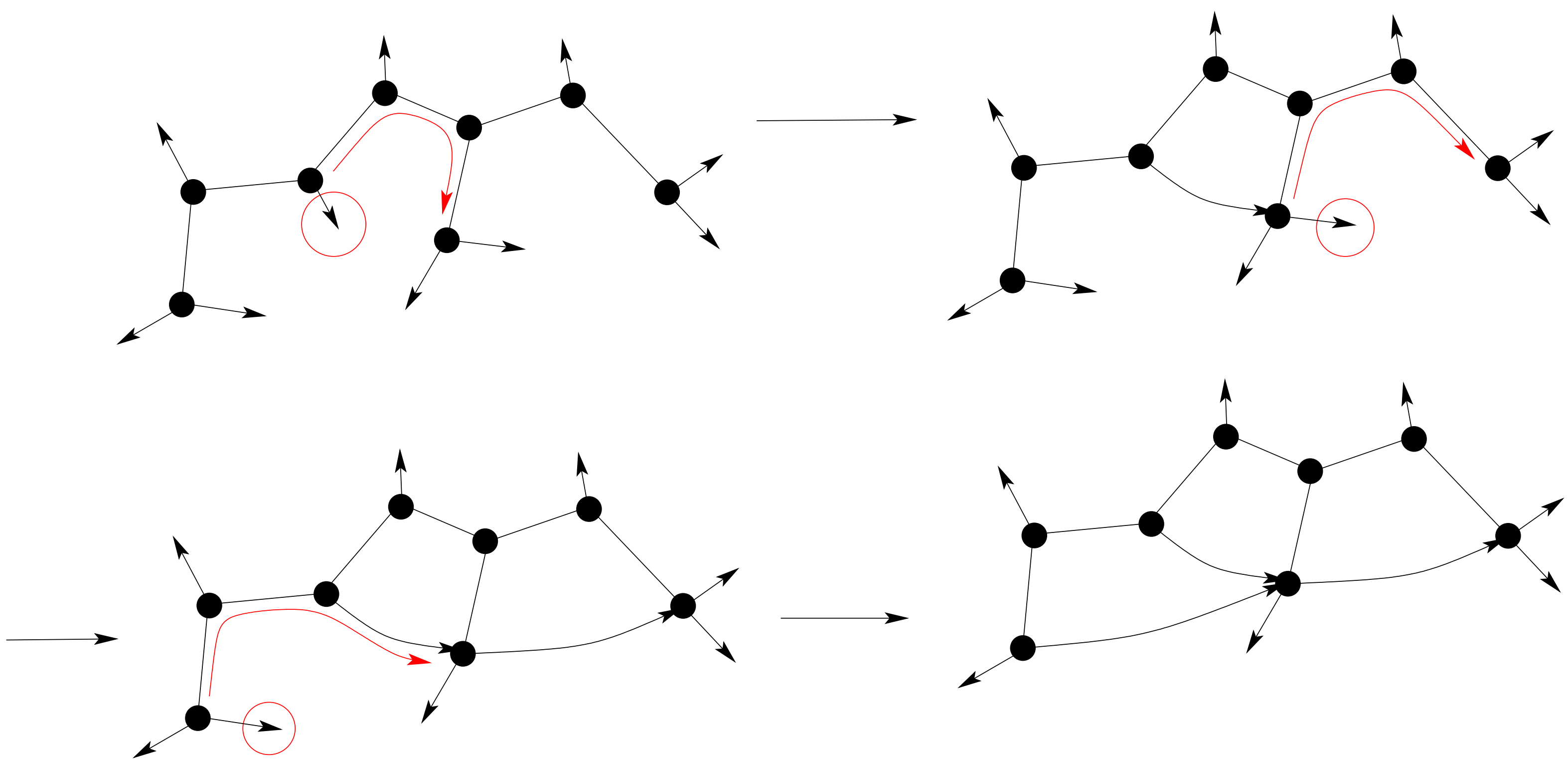
Local closure

- Choose a stem followed by three inner edges in a ccw traversal of the tree
- Merge the extremity of the stem with the extremity of the third edge



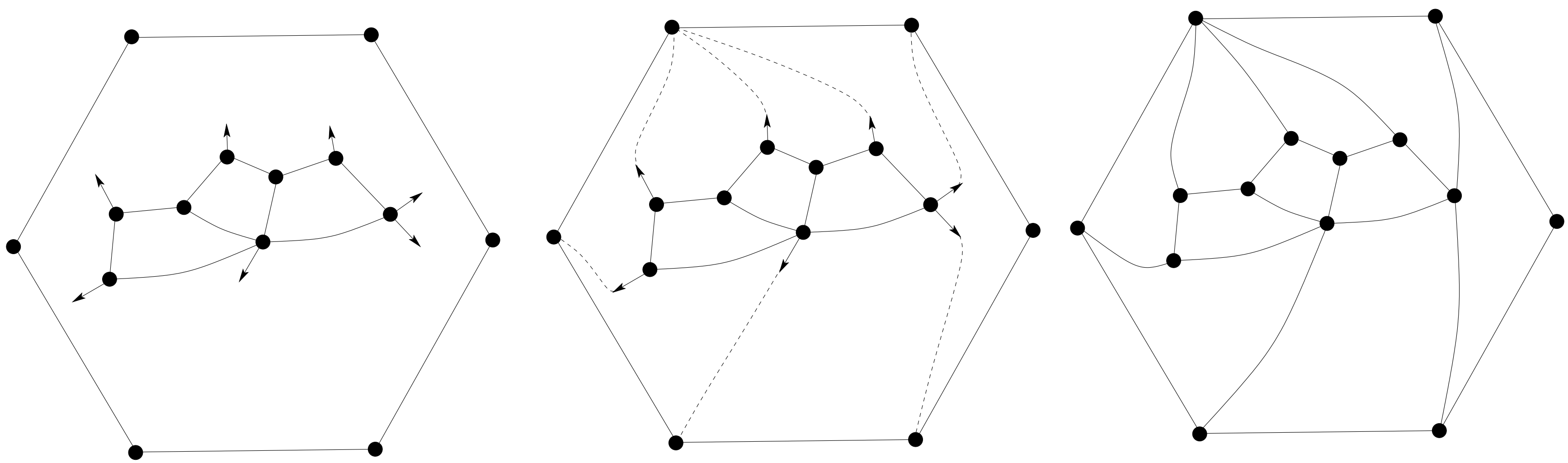
Partial closure

Perform all local closures greedily until no local closure is possible



Complete closure

- Draw an hexagon outside of the figure
- Merge all remaining stems with vertices of the hexagon so as to create quadrangular faces

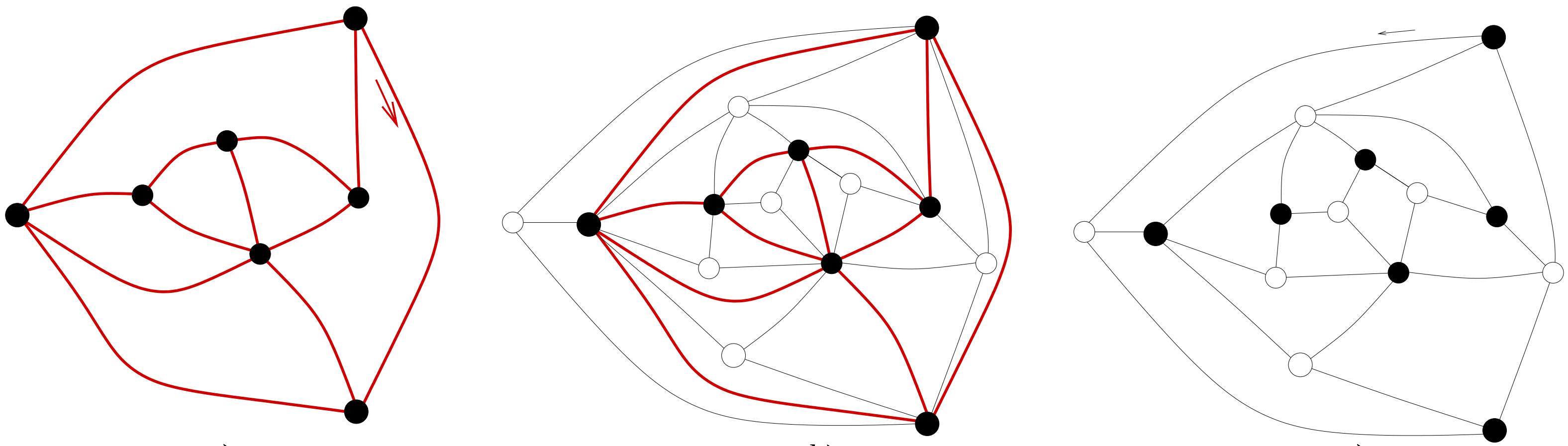


Results

- The closure application is a bijection between unrooted binary trees and unrooted irreducible dissections
- The closure application is a 6-to-$(n+2)$ application between rooted binary trees and rooted irreducible dissections
- $|\mathcal{D}'_n| = \frac{6}{n+2}|\mathcal{B}'_n| = \frac{6(2n)!}{n!(n+2)!}$ already found by Mullin and Schellenberg

Links with 3-connected maps

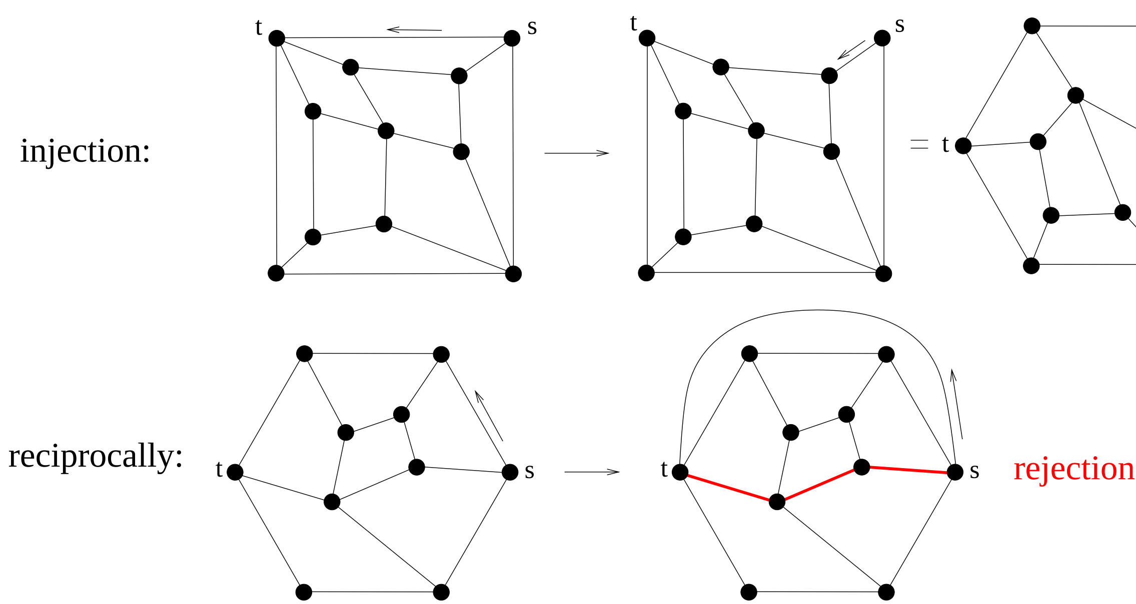
- A 3-connected map is a map that can not be disconnected by the removal of two of its vertices.
- An irreducible quadrangulation is a quadrangulation such that each 4-cycle delimits a face
- 3-connected rooted planar maps with n edges are in bijection with irreducible rooted quadrangulations with n faces



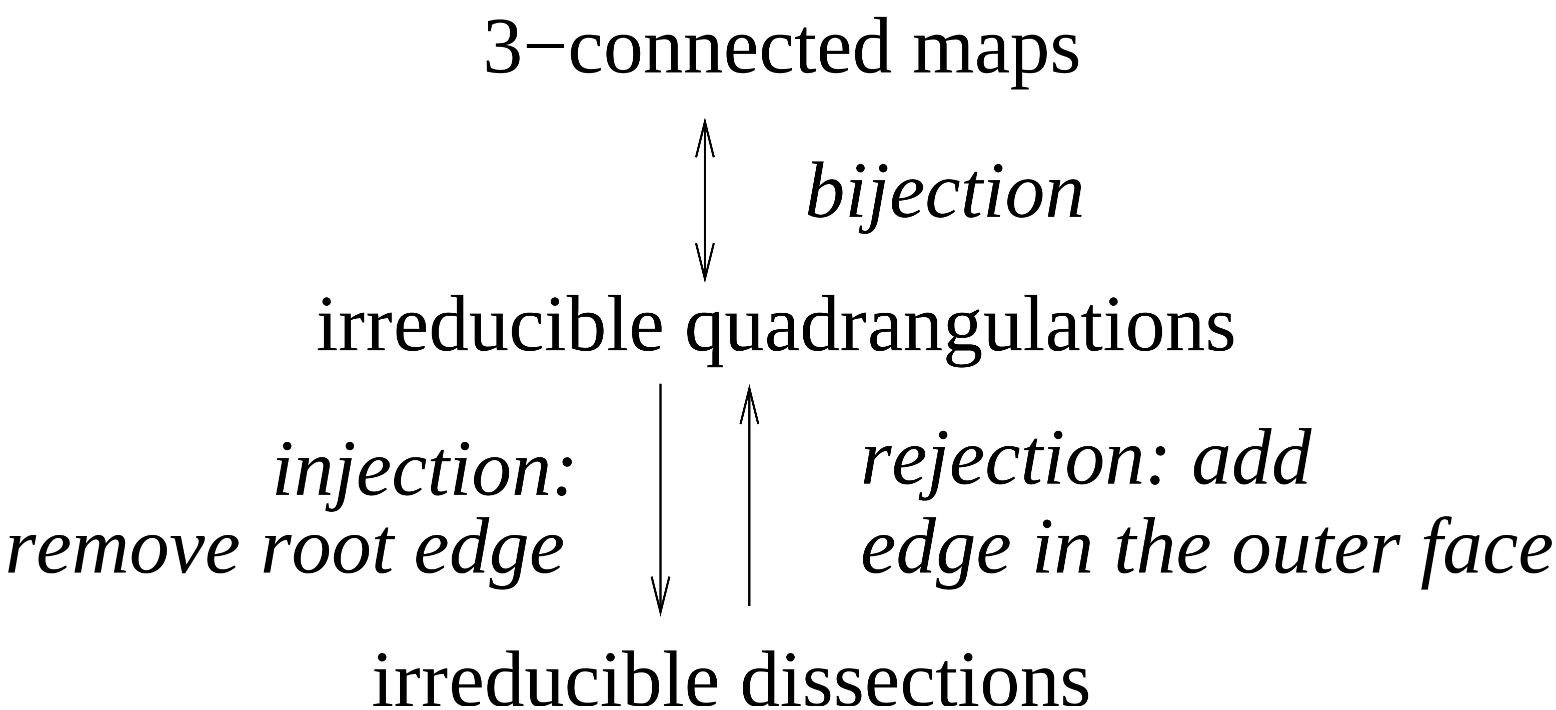
a) b) c)

Irreducible quadrangulations (all faces are squares) and irreducible dissections (outer face hexagon)

- Removing the root defines an injection between irreducible quadrangulations and irreducible dissections
- The reciprocal fails when there is an internal path of length 3 between s and t



Summary of the links



Application: sampling 3-connected maps

Algorithm:

1. Sample a rooted binary tree $T \in \mathcal{B}'_n$ using a parenthesis word
2. Close T to obtain an irreducible dissection $D \in \mathcal{D}'_n$
3. If no internal path of length 3 between s and t add an edge between s and t to obtain an irreducible quadrangulation Q. Otherwise reject and return to 1.
4. Return the 3-connected map with n edges whose quadrangulation is Q

Complexity:

- linear time for each try
- number of rejects: geometric law with parameter $p_n = \frac{|\mathcal{Q}'_n|}{|\mathcal{D}'_n|} \to_{n\to\infty} 2^6/3^5$. Mean number rejects: $1/p_n \to 3^5/2^6$

Application: counting 3-connected maps

$M(x)$, $Q(x)$ and $D(x)$ generating functions of 3-connected maps, irreducible quadrangulations, and irreducible quadrangulations.
$r(x) = x(1 + r(x))^2$ generating function of rooted binary trees

- Bijection: $M(x) = Q(x)$
- Rejection (decomposition along paths of length 3 between s and t): $D(x) = \left(\frac{2x}{1-x} + 1\right) \frac{1}{1-Q(x)\left(\frac{2x}{1-x}+1\right)} - 2x - 1$
- Closure application: $D(x) = r(x)(2 - r(x))$

Finally

$$M(x) = -\frac{1}{1 + 2x + x^2 r(x)(2 - r(x))} + \frac{1-x}{1+x}$$

Idea of the reciprocal application

- A binary tree can be oriented with outdegree 3 for each inner node
- The closure induces a certain orientation of edges of the irreducible dissection with outdegree 3 for each inner vertex
- Reciprocal: find this particular orientation and use it to open the dissection in a binary tree