Bijective counting of involutive Baxter permutations

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Part 1: Baxter families
Baxter permutations

- We adopt the diagram-representation of a permutation

\[ \pi = 5 \ 3 \ 4 \ 9 \ 7 \ 8 \ 10 \ 6 \ 1 \ 2 \]
Baxter permutations

- **Def:** Whenever there are 4 points in position

  or

  then the **dashed square is not empty.**

  (i.e., no pattern \(2 \, \overline{3} \, 1 \, 4\) nor \(4 \, \overline{1} \, \overline{3} \, 5 \, 2\) )

\[
\pi = 5 \, 3 \, 4 \, 9 \, 7 \, 8 \, 10 \, 6 \, 1 \, 2
\]
**Characterisation**

- **Inductive construction:** at each step, insert $n$ either
  - just before a left-to-right maximum (among $i$ of them)
  - just after a right-to-left maximum (among $j$ of them)

  **Insertion at left-to-right min:**
  - choose $k$ in $[1..i]$
  - update: $i:=k$, $j:=j+1$

  **Insertion at right-to-left min:**
  - choose $k$ in $[1..j]$
  - update: $j:=k$, $i:=i+1$
Baxter families

**Def:** Any combinatorial family with generating tree isomorphic to the generating tree $T$ with root $(1,1)$ and children rule

is called a Baxter family

- **Parallel with Catalan families:** one catalytic parameter

Dyck paths

Children rule is:
Other Baxter family: plane bipolar ori.

- Bipolar orientation = *acyclic* orientation with *unique* source and *unique* sink
- Planar map = graph embedded in the plane, no edge-crossing

Plane bipolar orientation =

- bipolar orientation on a *planar map*
- the source and the sink are incident to the *outer face*
Other Baxter family: plane bipolar ori.

- Two possibilities for inserting the topleft edge:

\[
\begin{align*}
  j &= \text{length(left boundary)} \\
  j &= 3 \text{ here}
\end{align*}
\]

\[
\begin{align*}
  i &= \text{indegree(sink)} \\
  i &= 4 \text{ here}
\end{align*}
\]

choose \( k \) in \([0..j-1]\)

\[
\begin{align*}
  i &= i + 1 \\
  j &= k + 1
\end{align*}
\]

\[
\begin{align*}
  k &= 1
\end{align*}
\]

choose \( k \) in \([0..i-1]\)

\[
\begin{align*}
  k &= 2
\end{align*}
\]

\[
\begin{align*}
  i &= k + 1 \\
  j &= j + 1
\end{align*}
\]
A countable Baxter family: triples of paths
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- **Counting** (by Gessel-Viennot’s lemma):

\[
q_n = \frac{1}{\binom{n+1}{1} \binom{n+1}{2}} \sum_{r=0}^{n-1} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2}
\]
Bijection links and bibliography

Triples of paths

[Baxter permutations]

[Baxter permutatiosn]

[Chung et al’78]
[Chung et al’78]

[Viennot’81]
[Viennot’81]

[Dulucq, Guibert’96]
[Dulucq, Guibert’96]

[Baxter permutations]

[Plane bipolar orientations]

[F, Poulalhon, Schaeffer’07]
[F, Poulalhon, Schaeffer’07]

[Felsner et al’08]
[Felsner et al’08]

[Ackerman et al’06]
[Ackerman et al’06]

[Bonichon et al’08]
[Bonichon et al’08]

[R. Baxter’01]
[R. Baxter’01]
Part 2: Baxter permutations and plane bipolar orientations
Baxter permutation $\rightarrow$ plane bipolar orientation
(hint: #ascents is distributed like #vertices)
Baxter permutation $\rightarrow$ plane bipolar orientation
(hint: #ascents is distributed like #vertices)

- Ascents of $\pi$ are in 1-to-1 correspondence with ascents of $\pi^{-1}$

- Place a white vertex at the intersection
Baxter permutation $\rightarrow$ plane bipolar orientation
(hint: \#ascents is distributed like \#vertices)

- Ascents of $\pi$ are in 1-to-1 correspondence with ascents of $\pi^{-1}$
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\[\pi = 5 \, 3 \, 4 \, 9 \, 7 \, 8 \, 10 \, 6 \, 1 \, 2\]
Baxter permutation $\rightarrow$ plane bipolar orientation

Dominance drawing:

- Draw segment $(x,y) \rightarrow (x',y')$
- Whenever $x < x'$ and $y < y'$
Baxter permutation  ->  plane bipolar orientation

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Baxter permutation $\rightarrow$ plane bipolar orientation

Erase the black vertices
(all have degree 2)

$\pi = 5 3 4 9 7 8 10 6 1 2$
Baxter permutation $\rightarrow$ plane bipolar orientation

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Baxter permutation $\rightarrow$ plane bipolar orientation

**Theorem** [Bonichon, Bousquet-Mélou, F’08]: The mapping is the canonical bijection (implements the isomorphism between generating trees)
Symmetry properties of the bijection

- The bijection \textit{``commutes''} with transformations in the dihedral group $D_4$.
Part 3: bijective counting of involutive Baxter permutations
Results

- **Univariate formula** (bijective proof of formula by M. Bousquet-Mélou): The number of involutive Baxter perm. with no fixed point and with 2n elements is

\[
\frac{3 \cdot 2^{n-1}}{(n + 1)(n + 2)} \binom{2n}{n}
\]

- **Multivariate formula**: number of involutive Baxter perm. with

<table>
<thead>
<tr>
<th>2n non-fixed points</th>
<th>2k descents not crossing the diagonal</th>
</tr>
</thead>
<tbody>
<tr>
<td>p fixed points</td>
<td>r descents crossing the diagonals</td>
</tr>
</tbody>
</table>

\[
\frac{(p+r) \binom{n+p-1}{k}^2 \binom{n}{t}}{nq^2(q+1)(k+1)(t+1)} \cdot \begin{vmatrix}
q(q+1) & q(q-1) & s(s-1) \\
k(q+1) & (k+1)q & s(t+1) \\
k(k-1) & k(k+1) & t(t+1)
\end{vmatrix}
\]

where \( q := n + p - k \), \( s := n - k - r \), \( t := k + r \)
Baxter invol. -> monosource ori.

Keep the part of the picture below the axis x=y

This yields a monosource orientation (acyclic, single source, possibly many sinks all in the outer face)
Encoding monosource orientations

merge sinks
Encoding monosource orientations

$\Rightarrow$ merge sinks

$i$ corners

Length $i$
Encoding monosource orientations

merge sinks

encode decorated corners

Length $i$
Generic picture

Baxter permutations $n$ elements

Baxter involutions $n$ two-cycles
no fixed point

$n-1$ steps

\[ \frac{1}{(n+1)(n+1)} \sum_{r=0}^{n-1} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2} \]

$n-1$ steps

\[ \frac{3 \cdot 2^{n-1}}{(n+1)(n+2)} \binom{2n}{n} \]
Counting

Want to count: \[ n-1 \] steps
Counting

Want to count:

$n-1$ steps

Useful lemma:
2 non-cross. paths
[J. Levine’59]

\[ a_{n,k} = (2k+1) \frac{(2n+2k)!}{n!(n+2k+1)!} \]
Counting

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\[
a_{n,k} = (2k+1) \frac{(2n+2k)!}{n!(n+2k+1)!}
\]

= free -
Counting

Want to count: 

Useful lemma: 2 non-cross. paths

[J. Levine’59]

\[ a_{n,k} = \frac{(2k+1)(2n+2k)!}{n!(n+2k+1)!} \]

\[ a_{n-1,0} \cdot 2^{n-1} \]
Counting

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\[ - \frac{1}{2} a_{n-1,1} \cdot 2^{n-1} \]
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Useful lemma:
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\[
a_{n,k} = \frac{(2k+1)(2n+2k)!}{n!(n+2k+1)!}
\]

\[
a_{n-1,0} \cdot 2^{n-1} - \frac{1}{2} a_{n-1,1} \cdot 2^{n-1} = \frac{3 \cdot 2^{n-1}}{(n+1)(n+2)} \binom{2n}{n}
\]