Transversal structures on triangulations, with application to straight-line drawing

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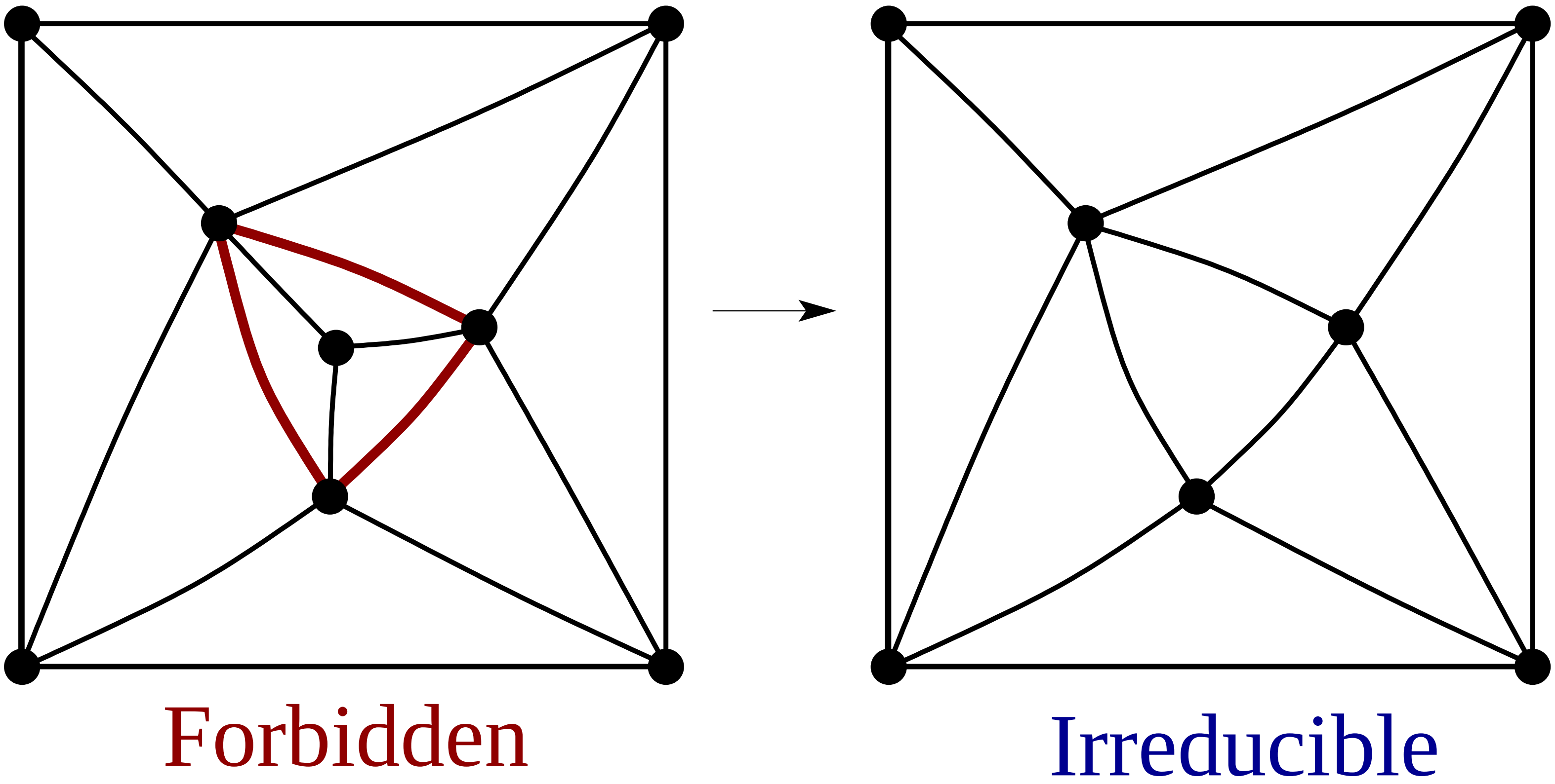
Overview

- Transversal structures on triangulations
  - Definition, cf Regular edge labelling: [Kant, He 97]
  - Algorithm computing a transversal structure
  - Combinatorial structure: distributive lattice
- Application: straight line drawing
  - n vertices $\Rightarrow$ grid of size $\frac{11}{27}n \times \frac{11}{27}n$ almost surely
  - Reflects the transversal structure
- Bijection triangulations $\Leftrightarrow$ ternary trees and applications

Definition and properties of transversal structures on triangulations

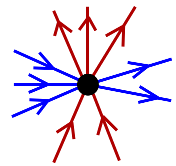
A particular family of triangulations

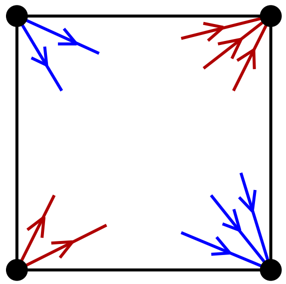
- We consider triangulations of the 4-gon (the outer face is a quadrangle)
- No separating triangle (irreducibility)

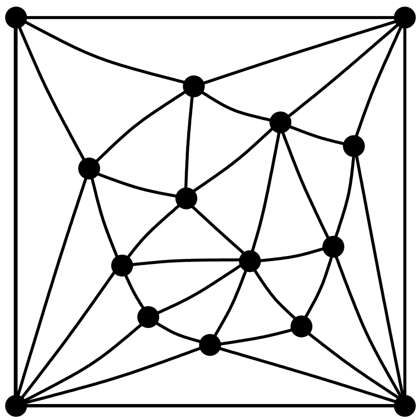
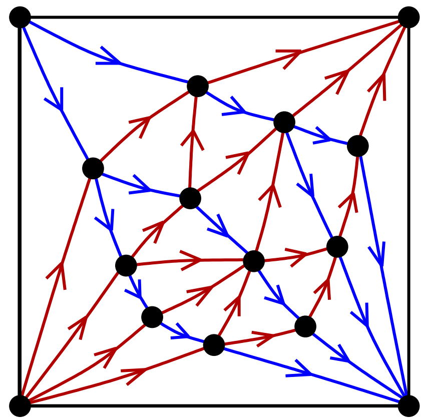


Transversal structures

A transversal structure is an orientation and bicoloration (in blue and red) of the inner edges such that:

Inner vertex ⇒ 

Border vertices ⇒ 

Example:  ⇒ 

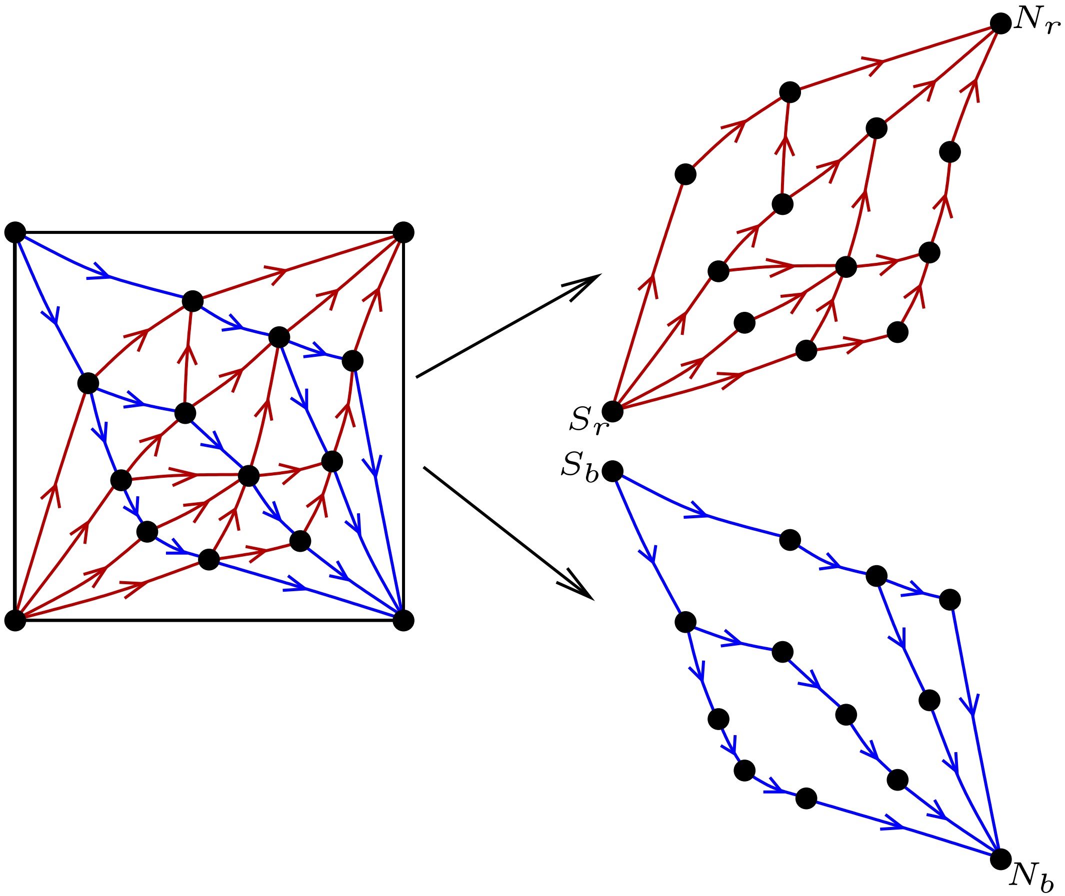
cf Regular edge labelling [Kant, He 1997]

Link with bipolar orientations

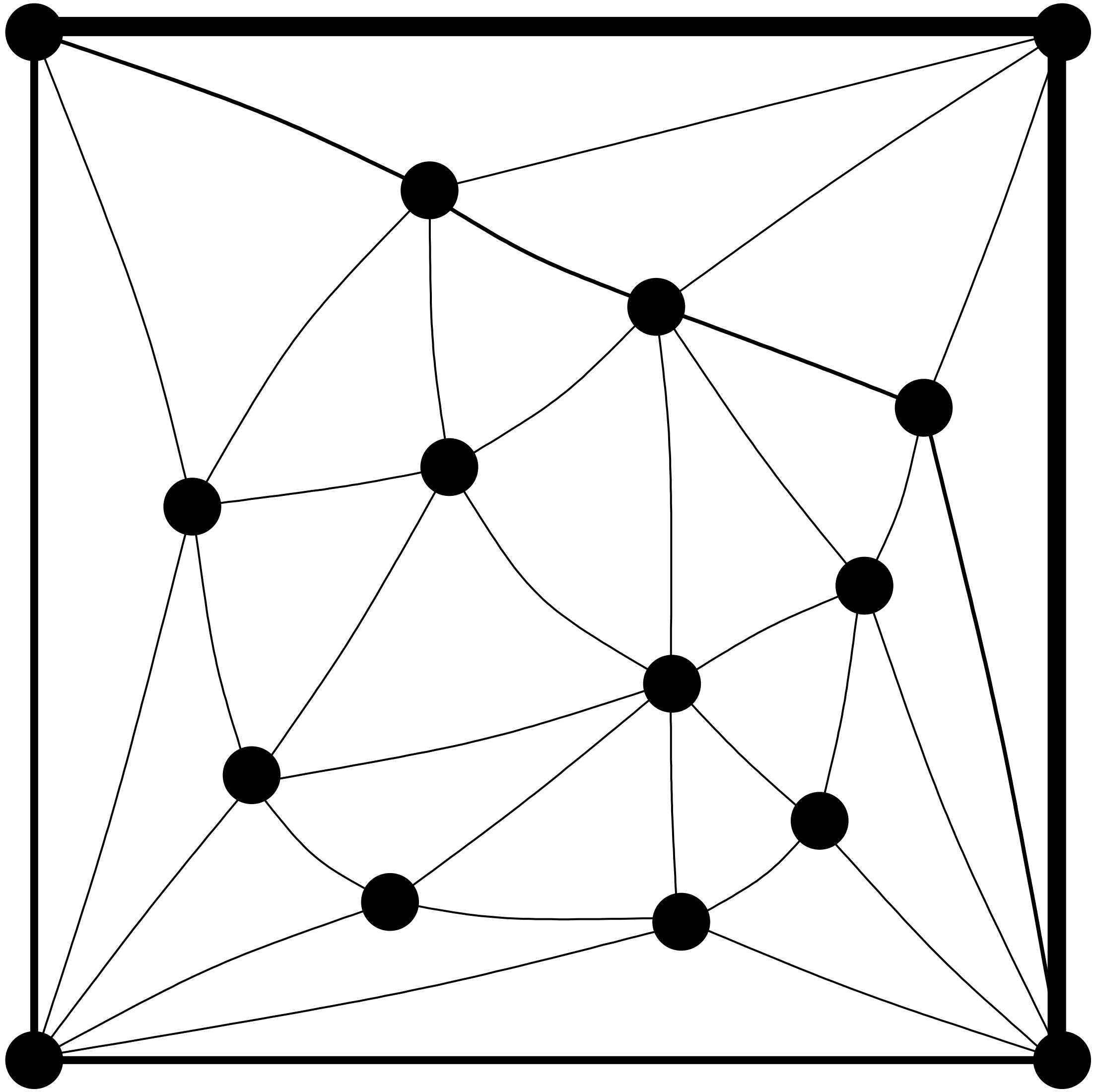
bipolar orientation = **acyclic** orientation with a unique minimum and a unique maximum

The blue (resp. red) edges give a bipolar orientation

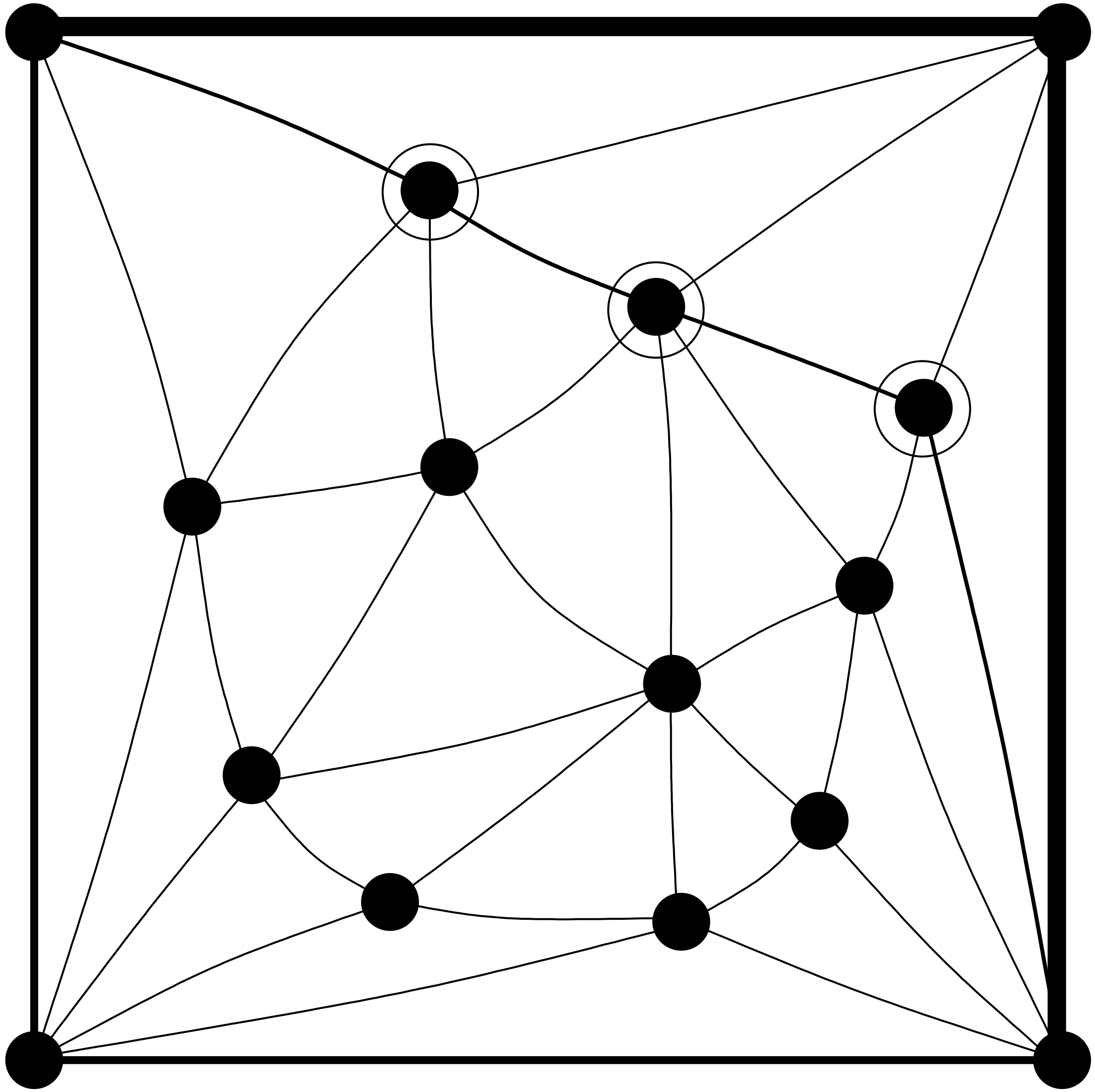
The two bipolar orientations are transversal



Finding a transversal structure



Finding a transversal structure



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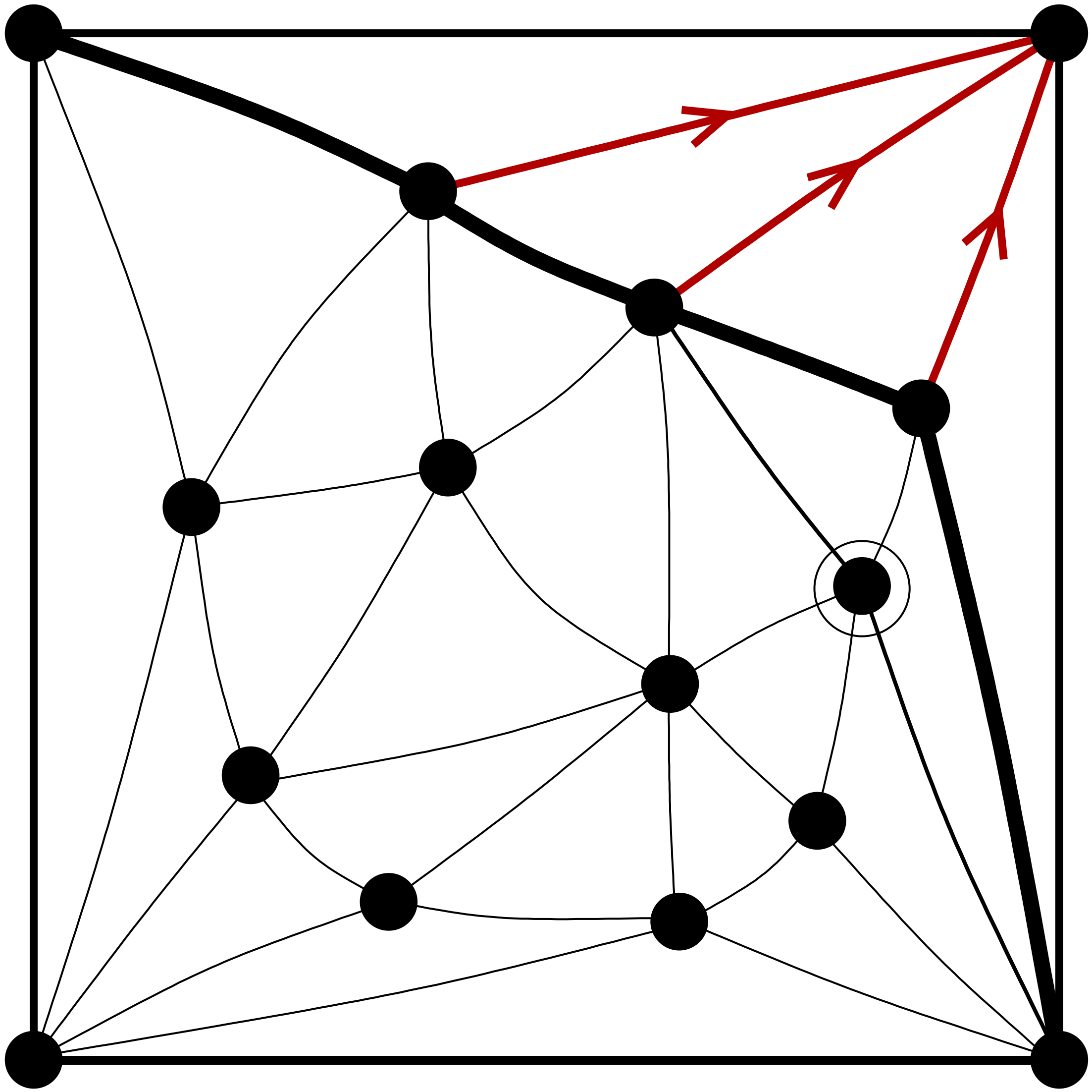
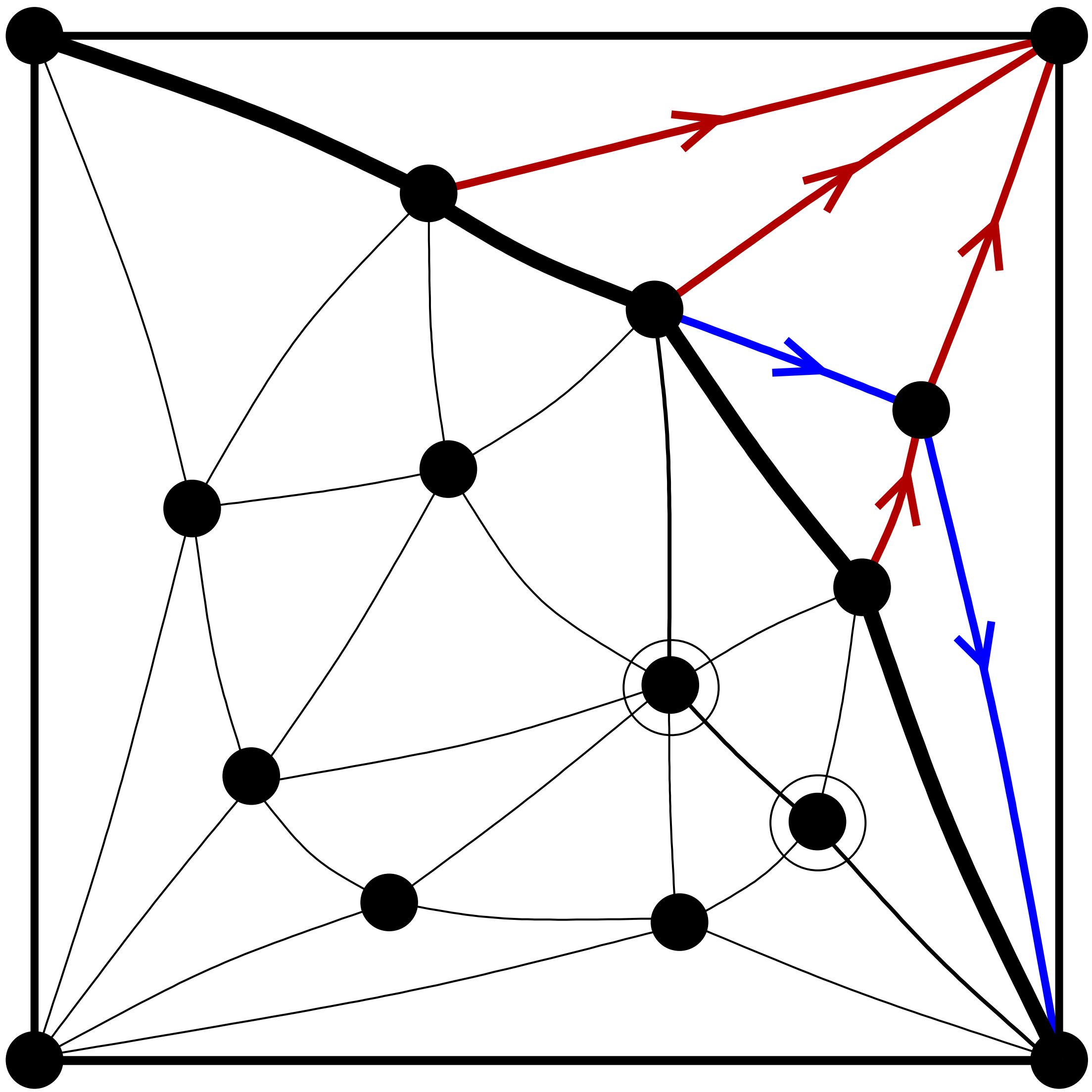
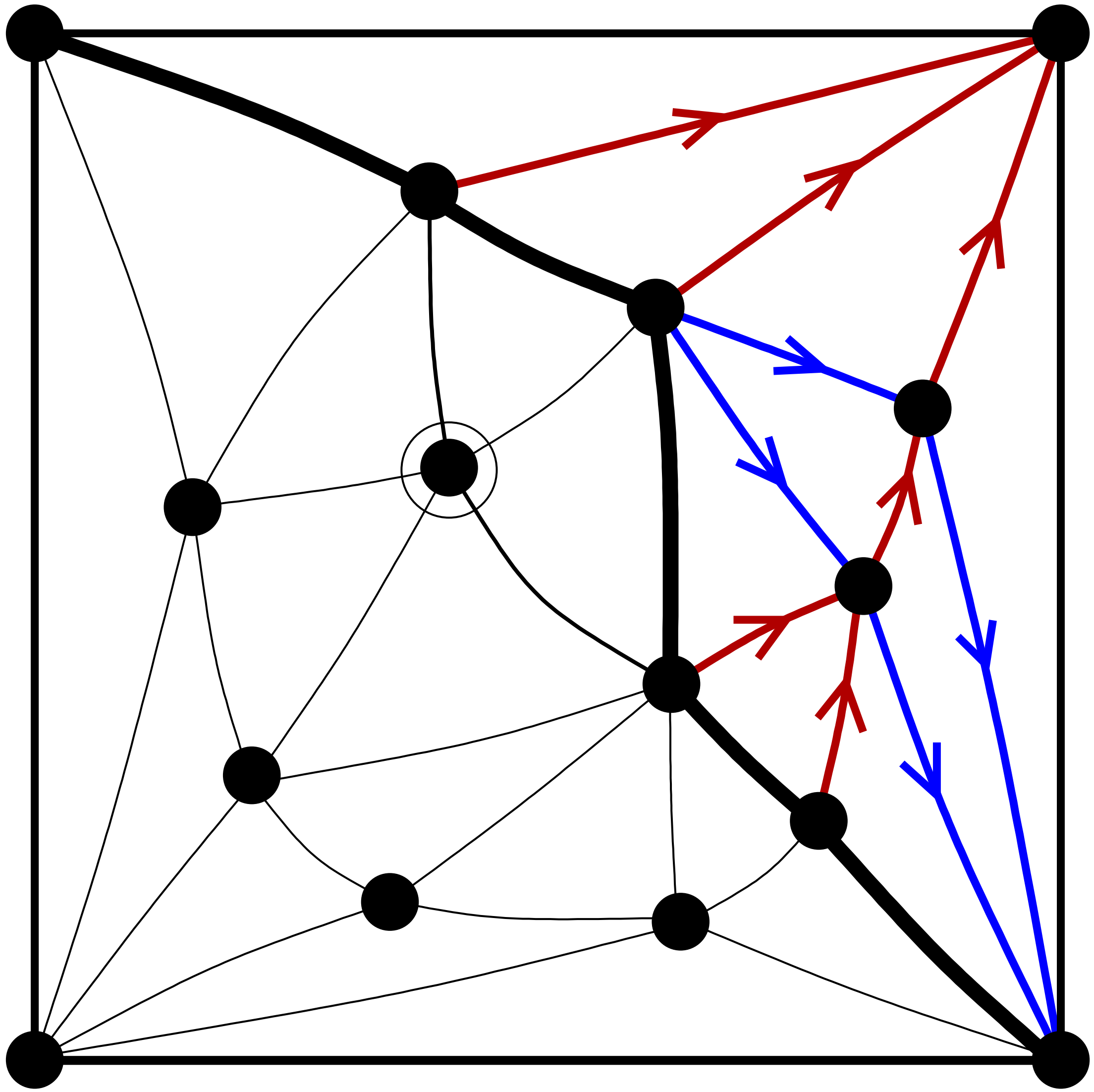


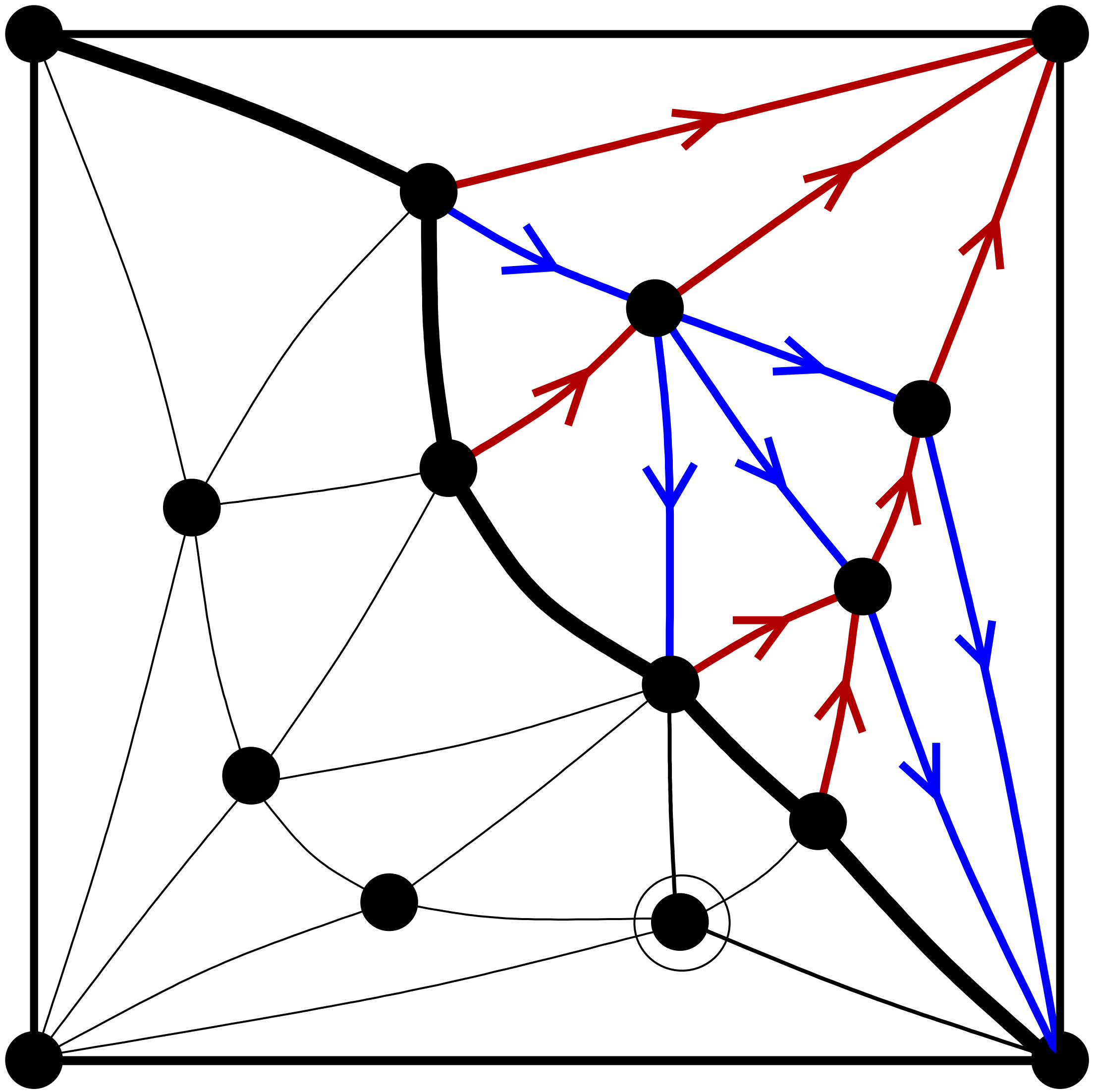
Finding a transversal structure



Finding a transversal structure



Finding a transversal structure



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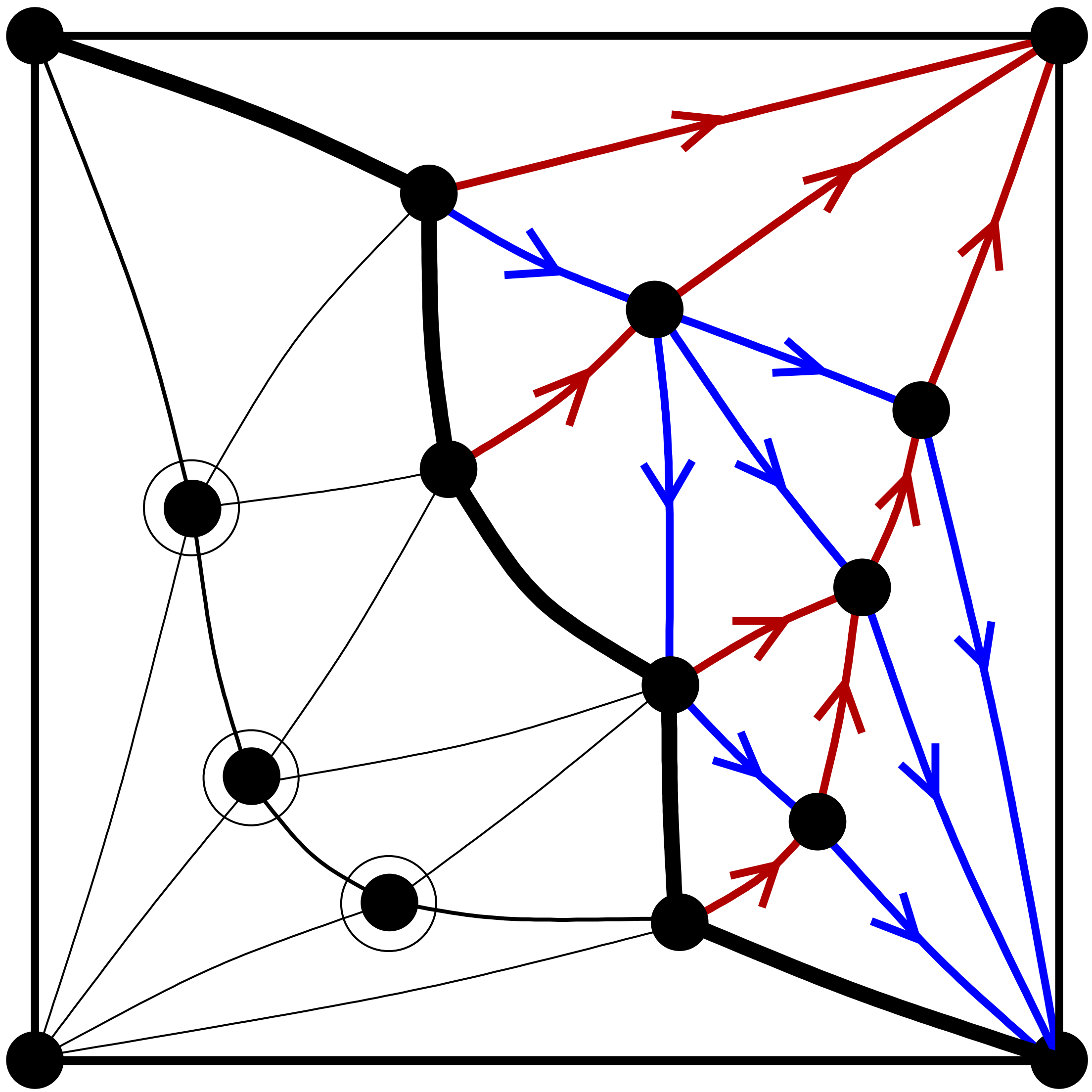

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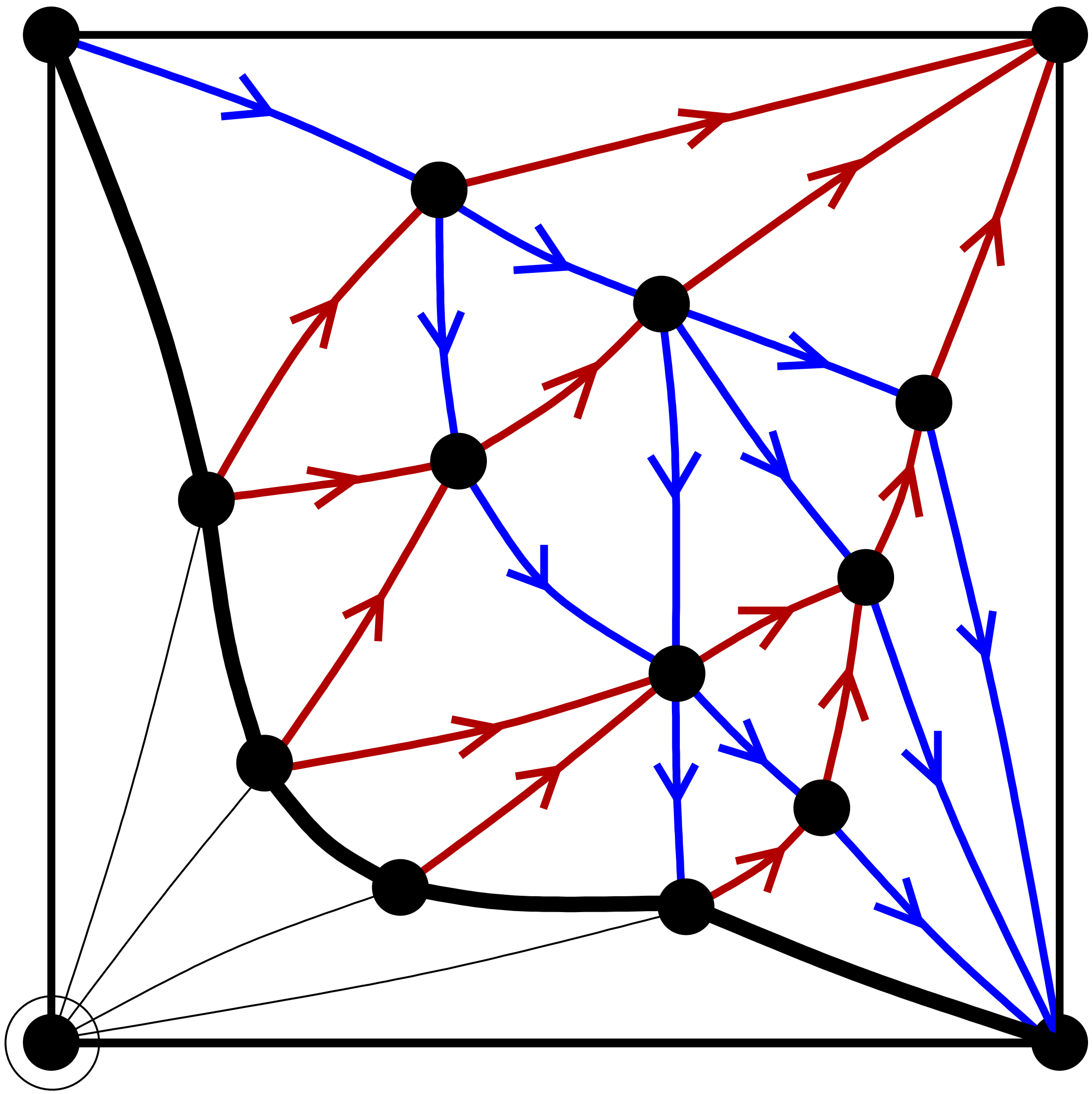
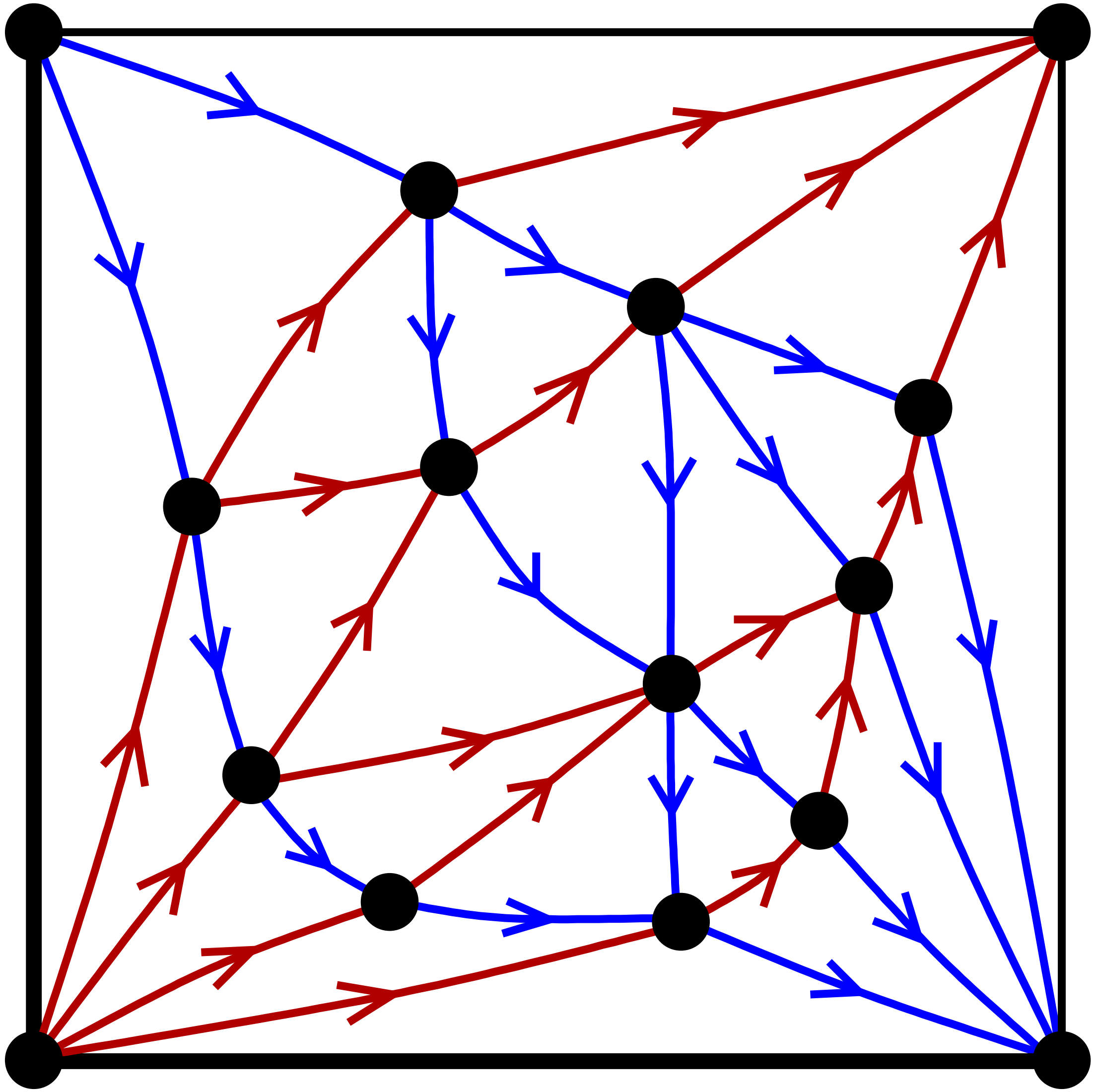
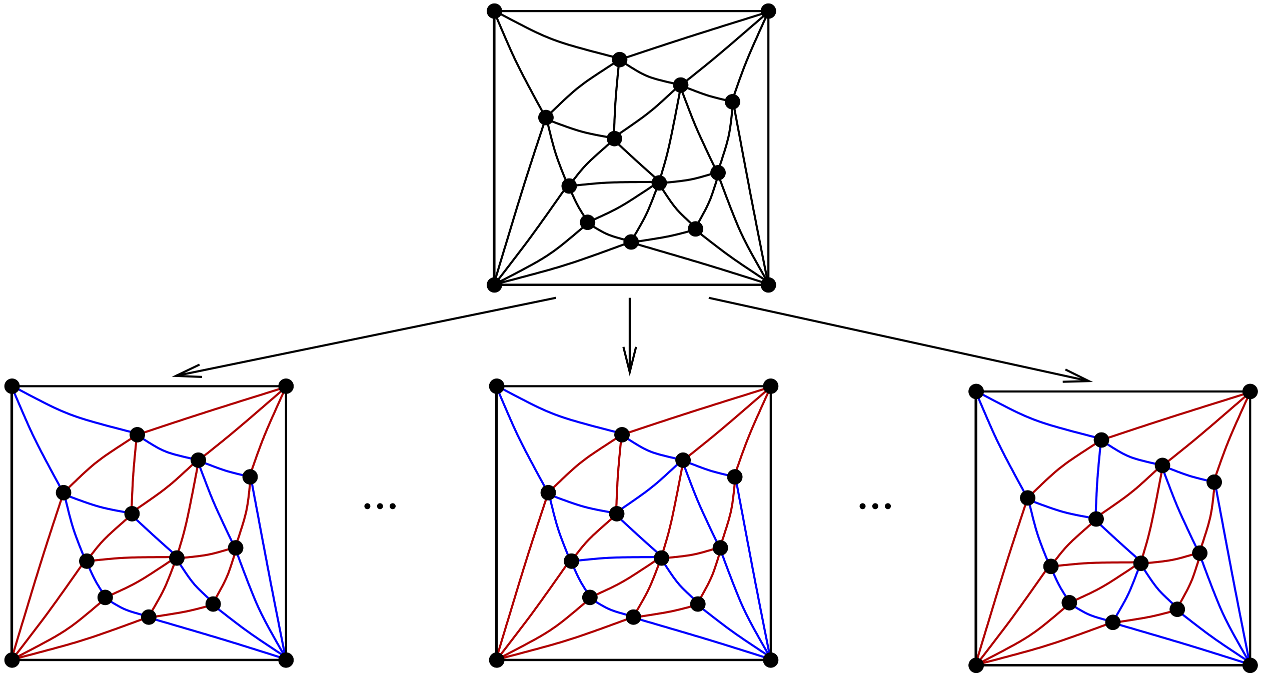


Finding a transversal structure



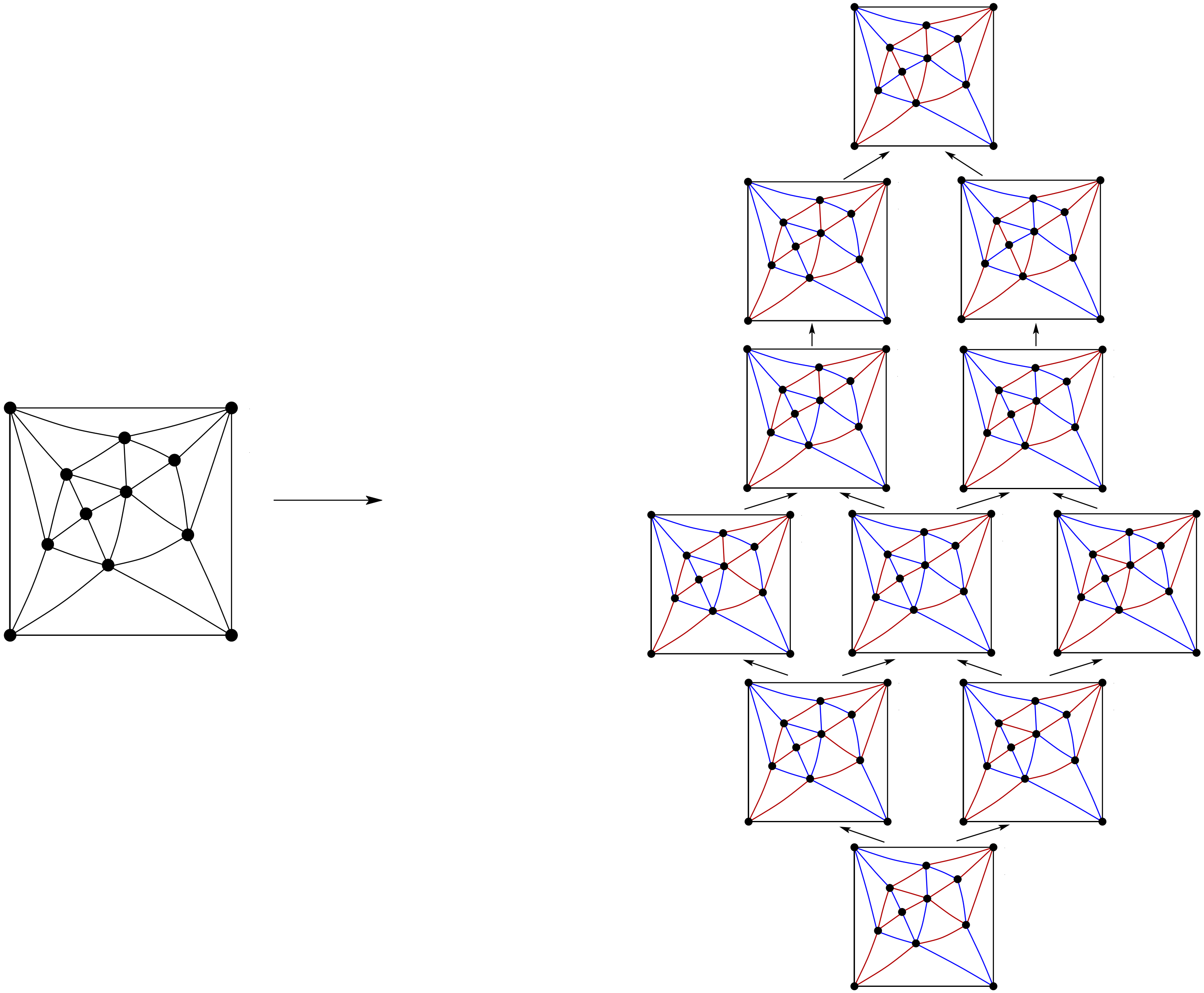
The set of transversal structures ?

- For each triangulation T, such transversal structures are not unique
- Let X_T be the set of transversal bicolorations of T
- What is the structure of X_T ?



The set X_T is a distributive lattice

T=



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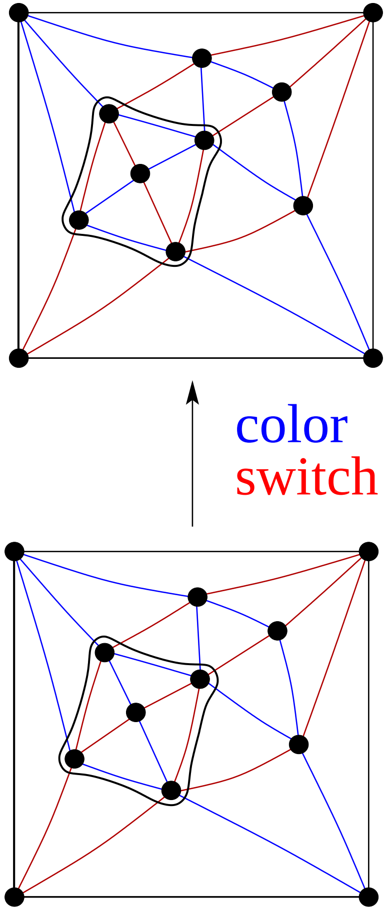
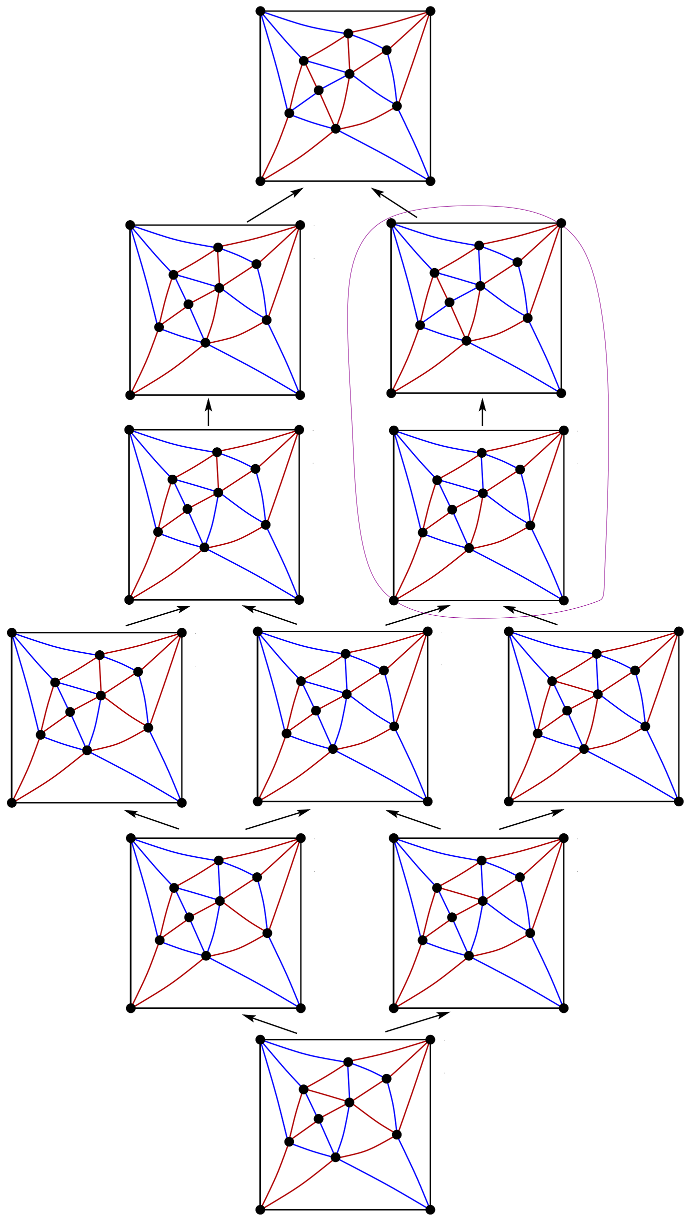
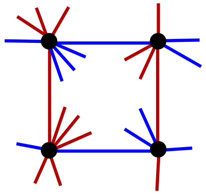




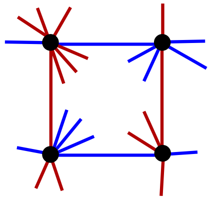
The set X_T is a distributive lattice

We distinguish:

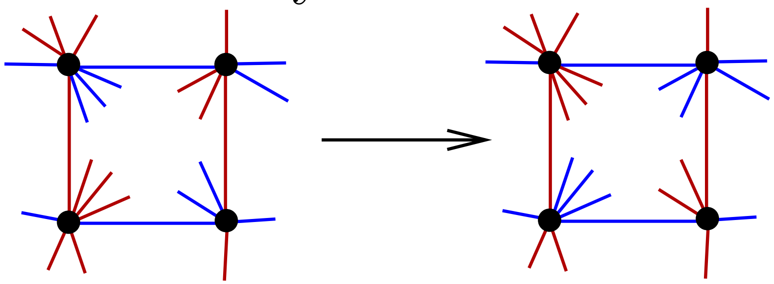
left alternating 4-cycles



right alternating 4-cycles



Flip operation: switch colors inside a right alternating 4-cycle

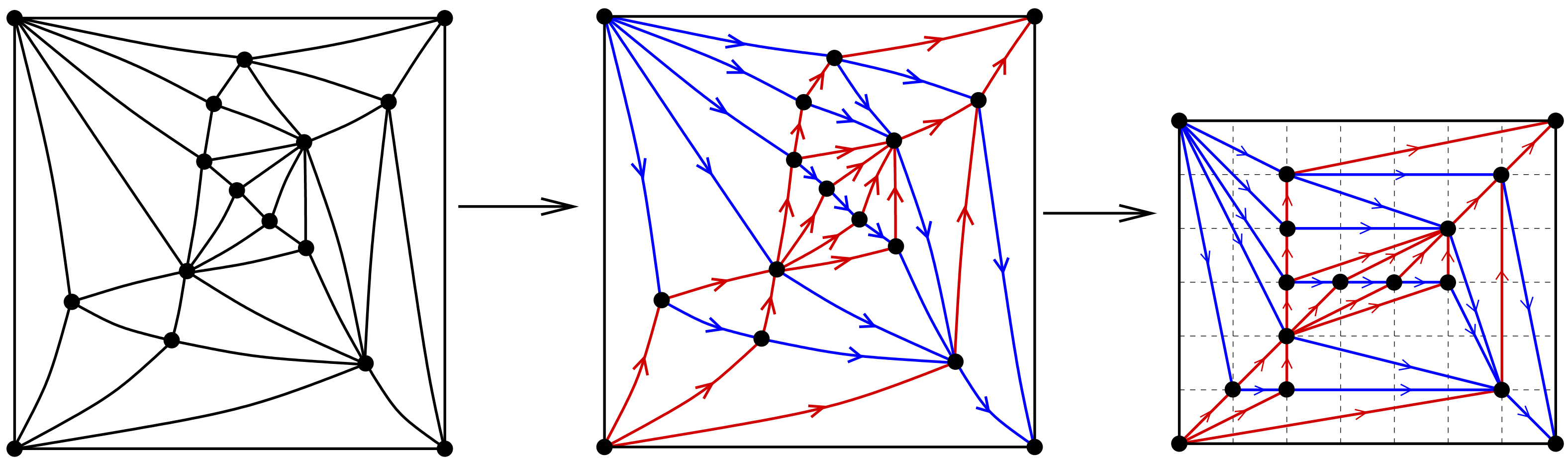


The unique transversal bicoloration of T without right alternating 4-cycle is said minimal

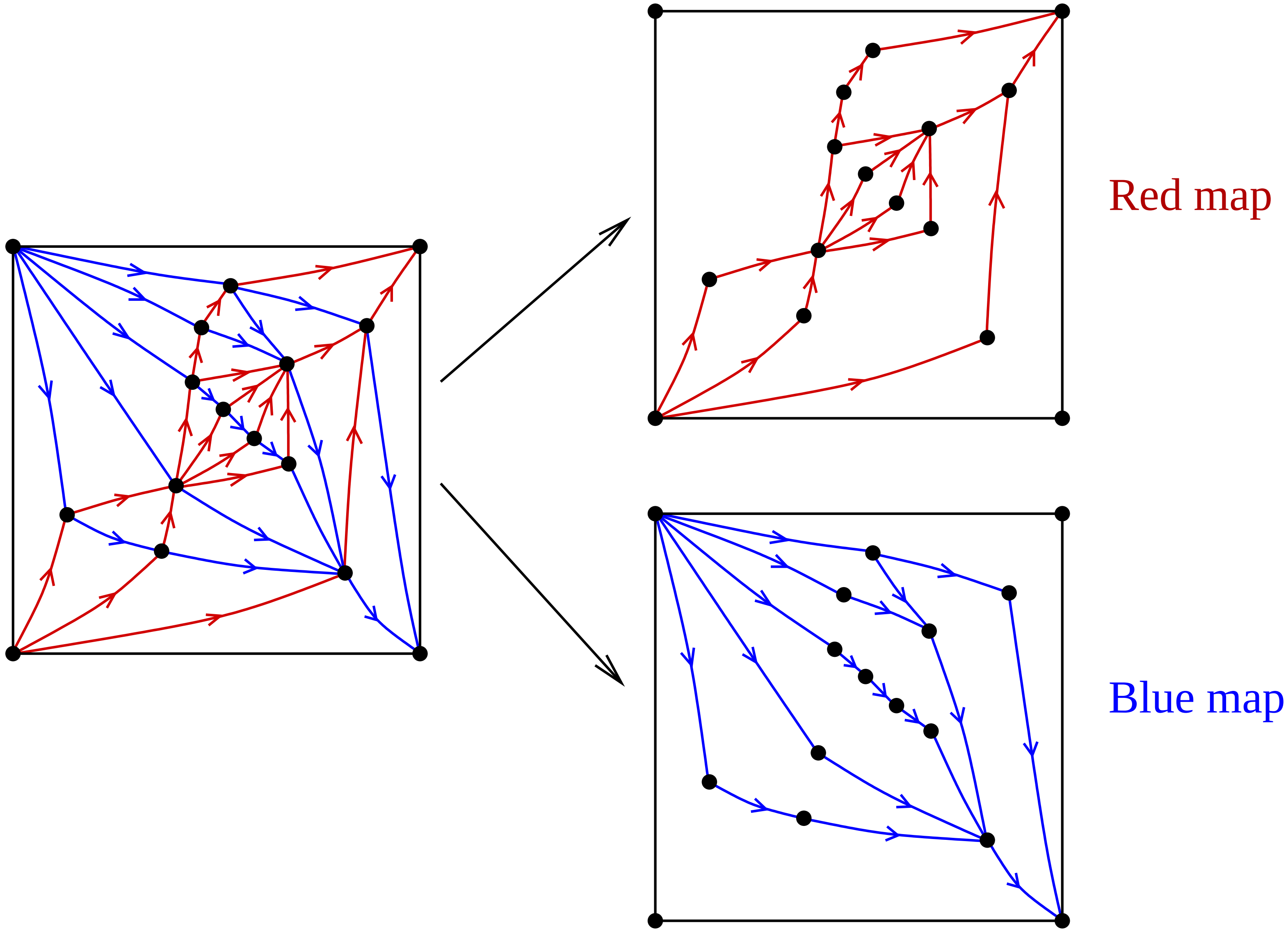
Straight-line drawing algorithm from the transversal structures

Application to graph drawing

The transversal structure can be used to produce a planar drawing on a regular grid



The red map and the blue map of T

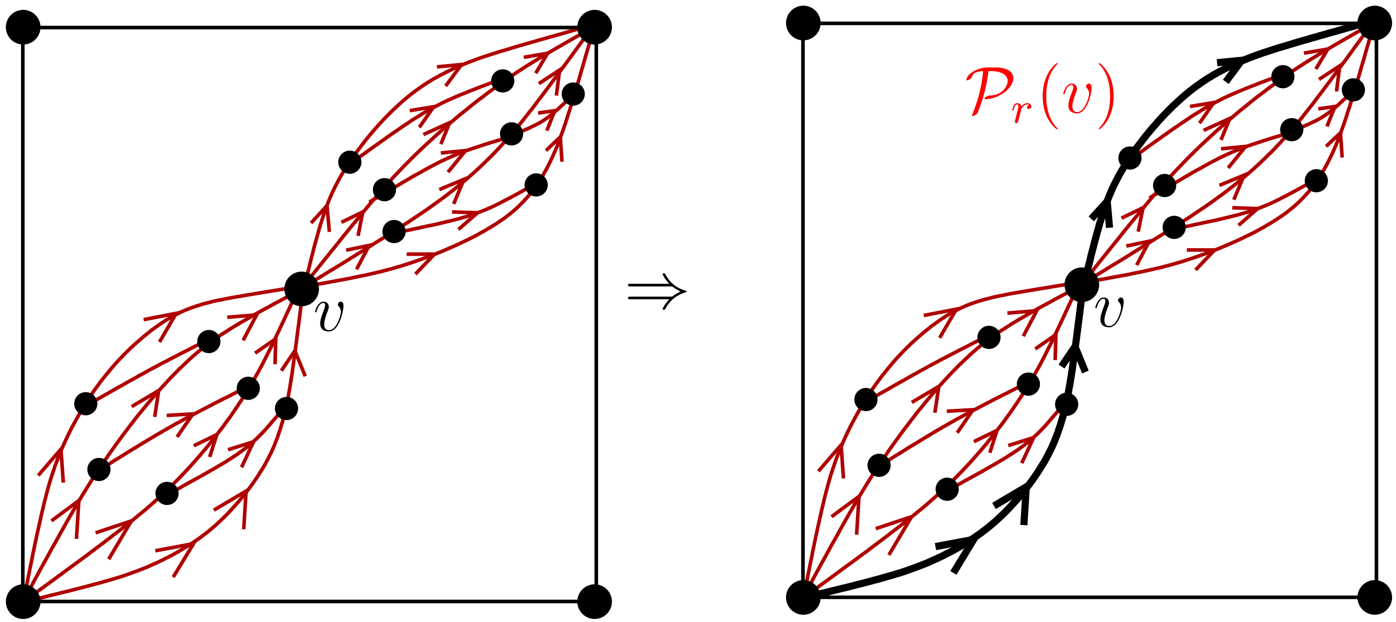


The red map gives abscissas (1)

Let v be an inner vertex of T

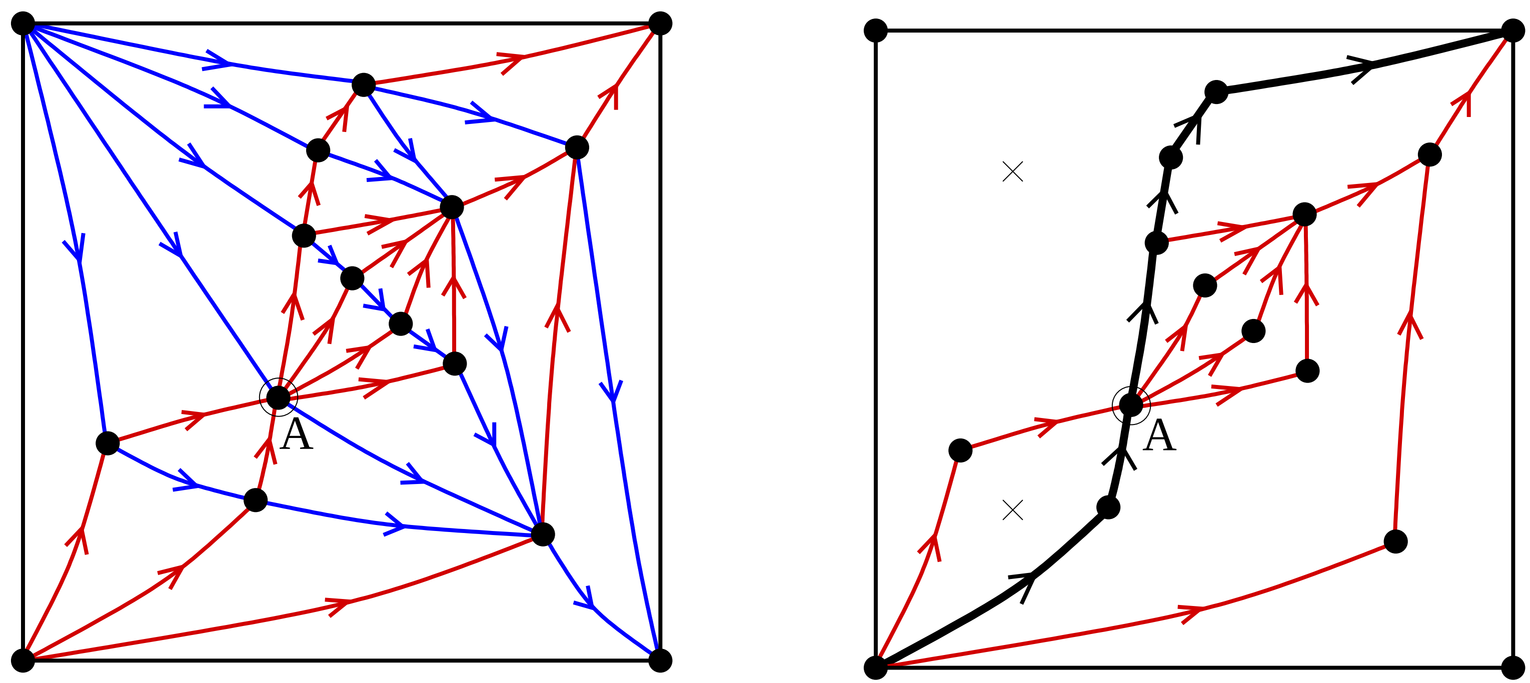
Let $\mathcal{P}_r(v)$ be the unique path passing by v which is:

- the rightmost one before arriving at v
- the leftmost one after leaving v



The red map gives abscissas (2)

The abscciss of v is the number of faces of the red map on the left of $\mathcal{P}_r(v)$

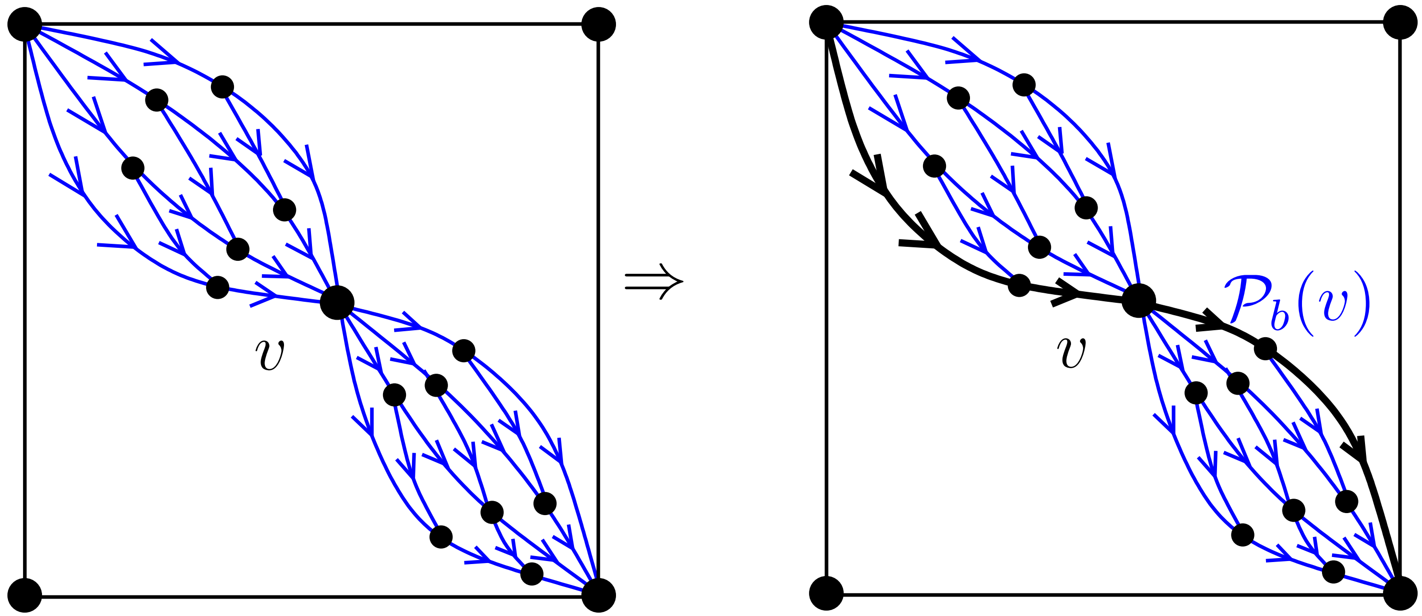


$\Rightarrow$ A has absciss 2

The blue map gives ordinates (1)

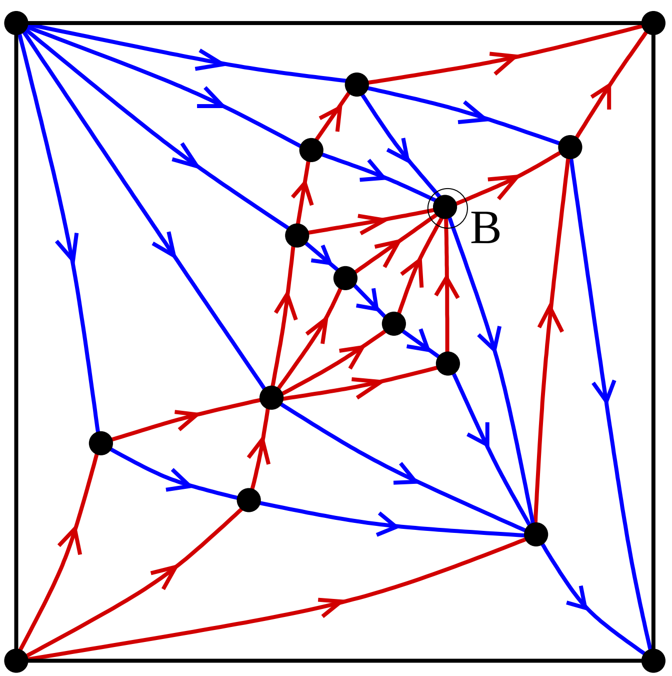
Similarly we define $\mathcal{P}_b(v)$ the unique blue path which is:

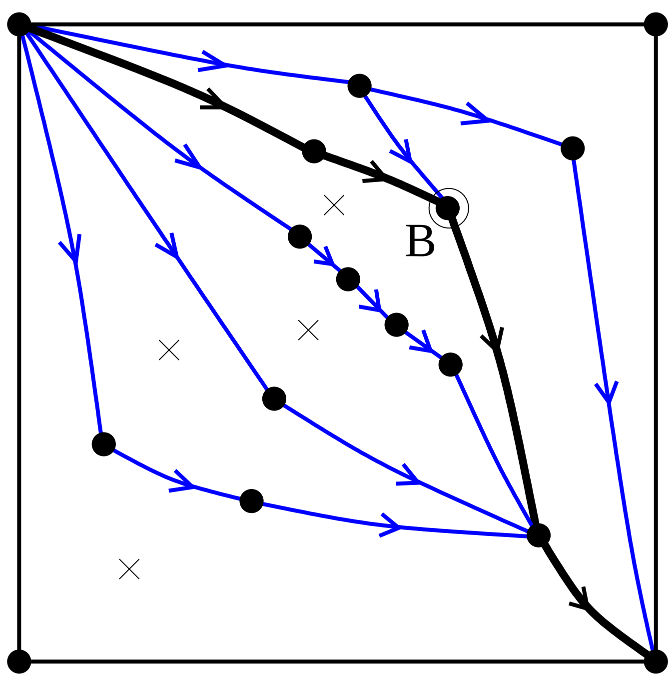
- the rightmost one before arriving at v
- the leftmost one after leaving v



The blue map gives ordinates (2)

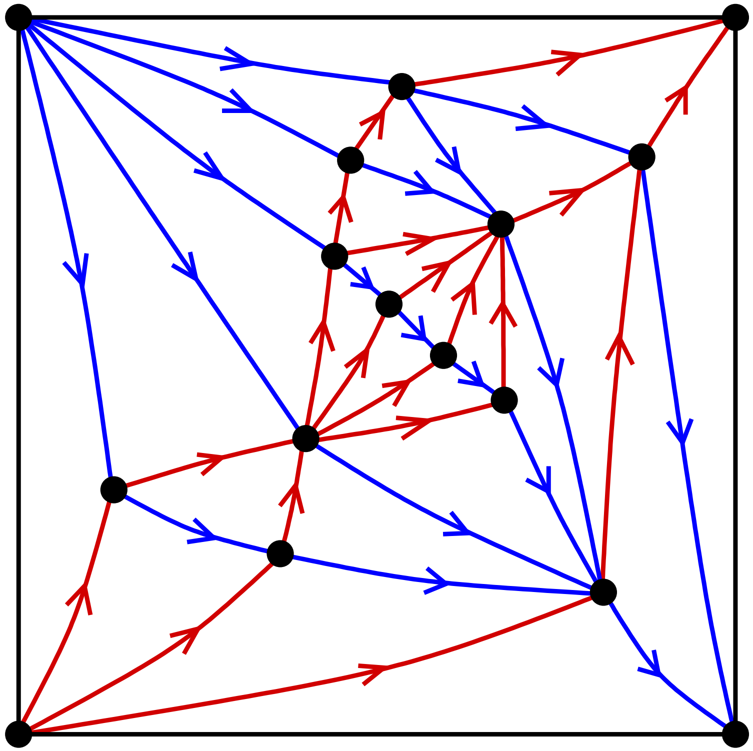
The ordinate of v is the number of faces of the blue map below $\mathcal{P}_b(v)$





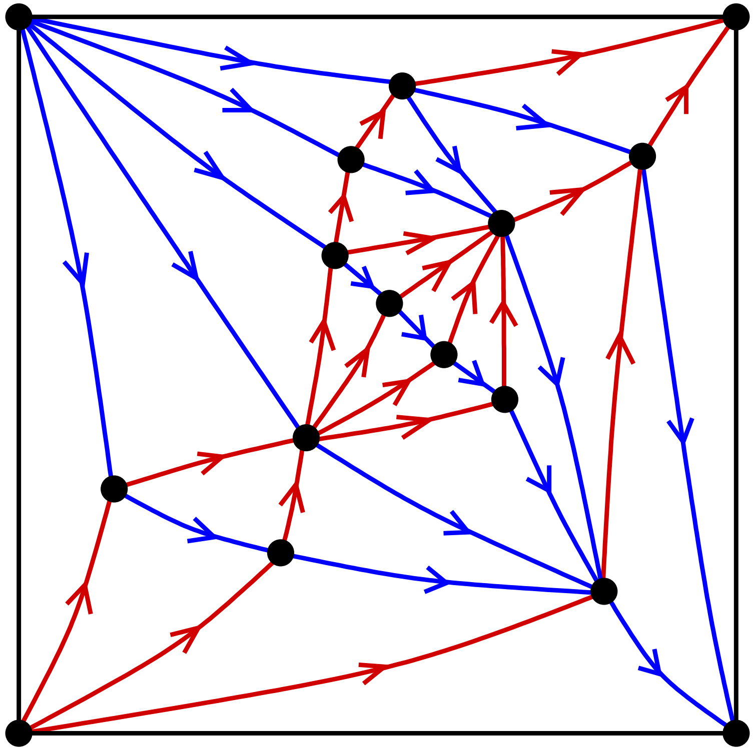
$\Rightarrow B$ has ordinate 4

Execution of the algorithm

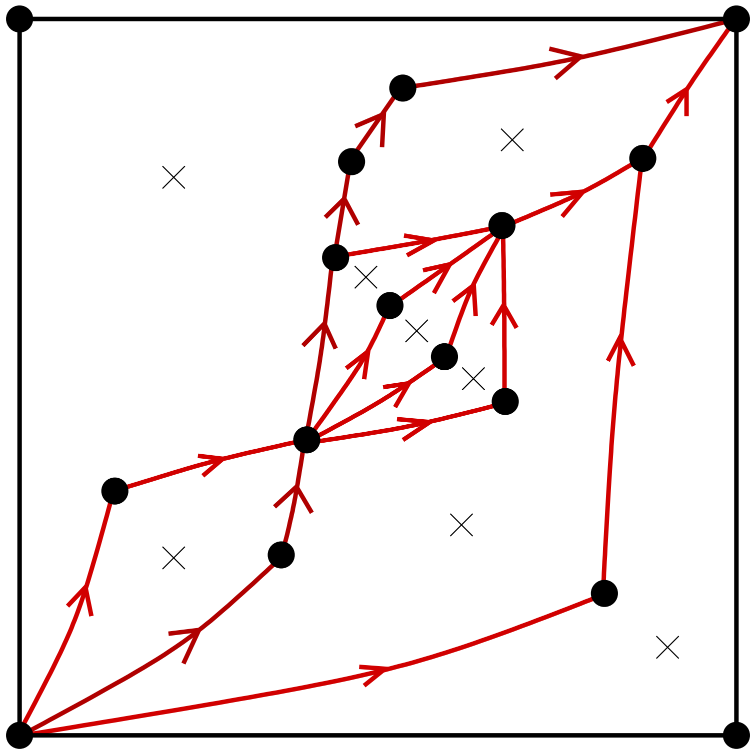


Execution of the algorithm

Let f_r be the number of faces of the red map

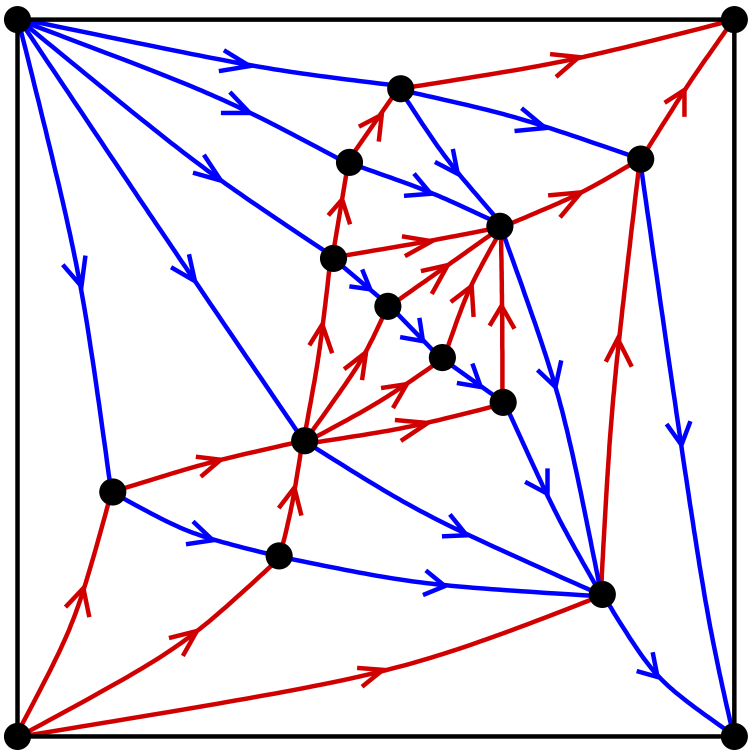


$f_r = 8$

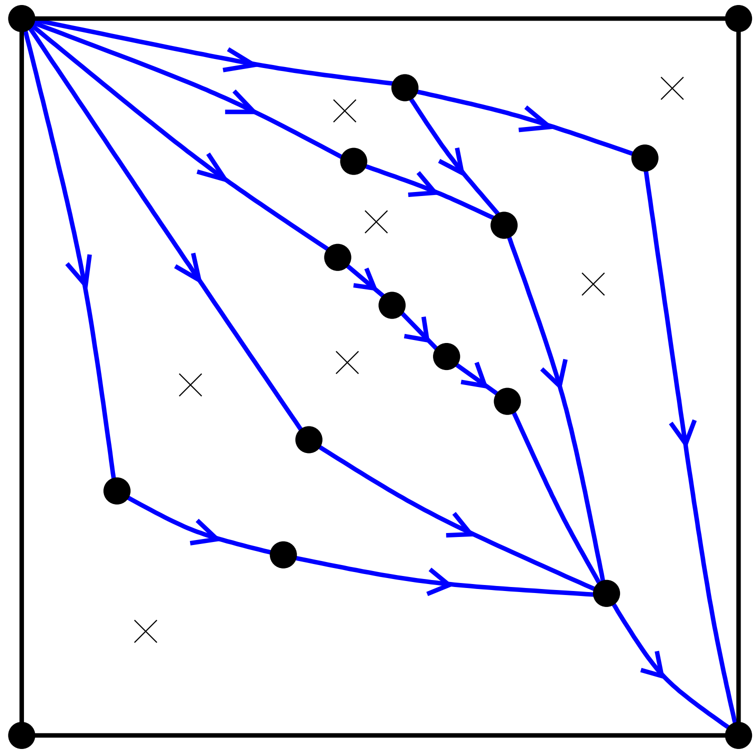


Execution of the algorithm

Let f_b be the number of faces of the blue map

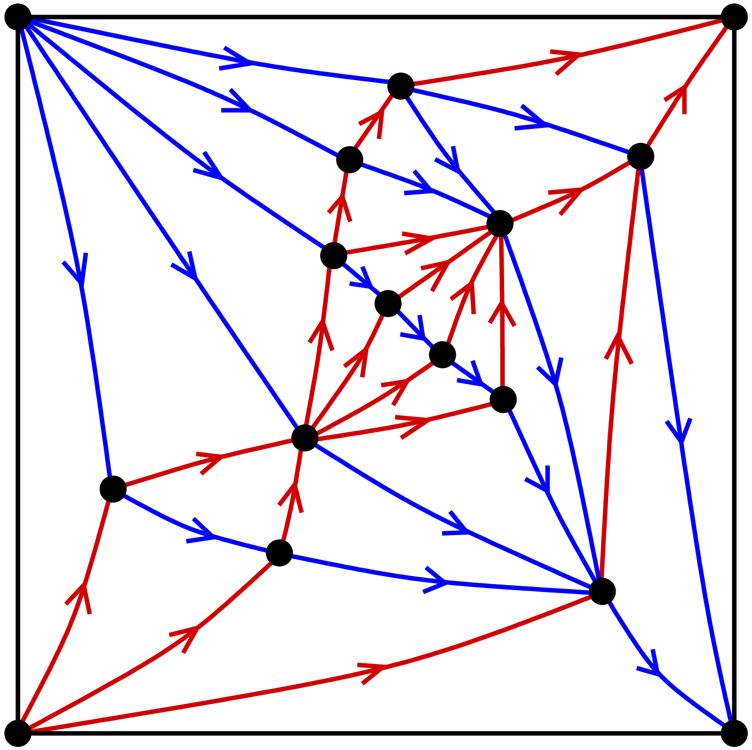


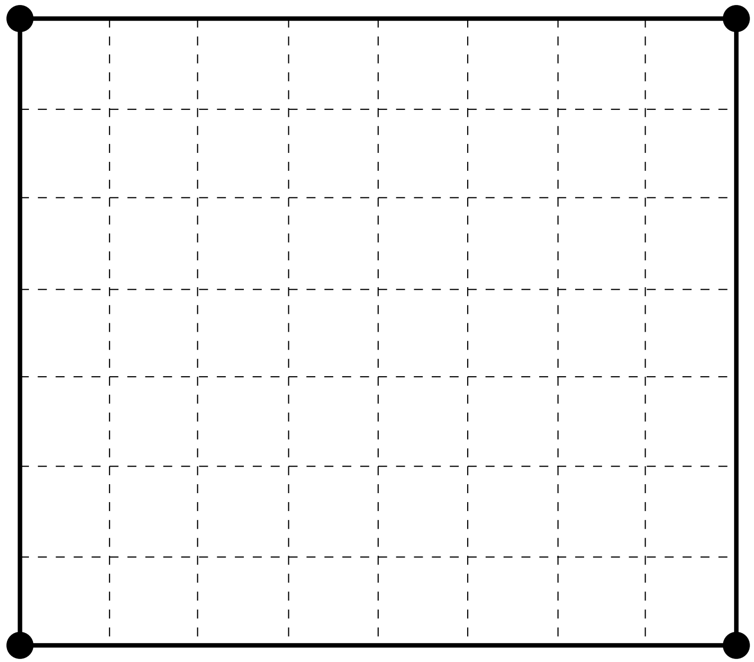
$f_r = 8$
$f_b = 7$



Execution of the algorithm

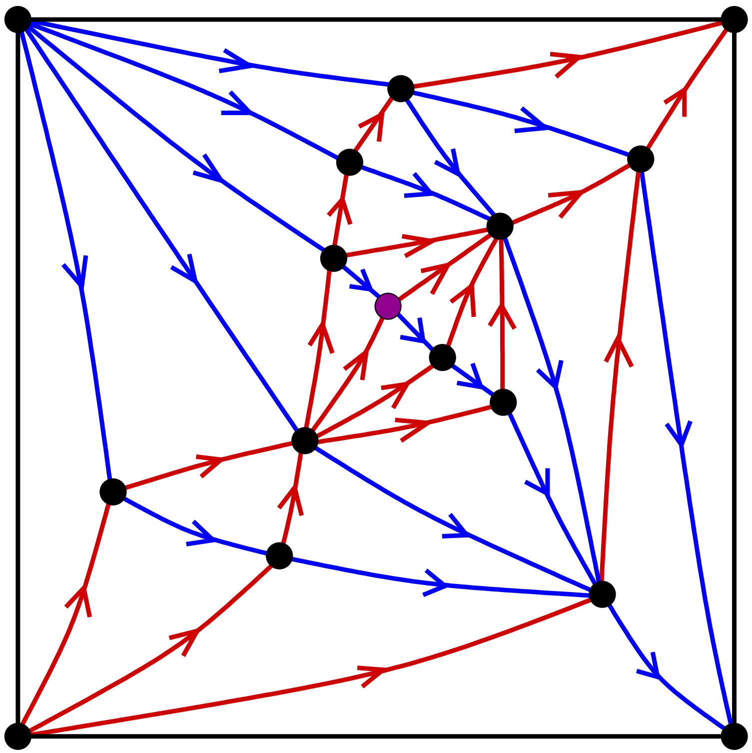
Take a regular grid of width f_r and height f_b and place the 4 border vertices of T at the 4 corners of the grid

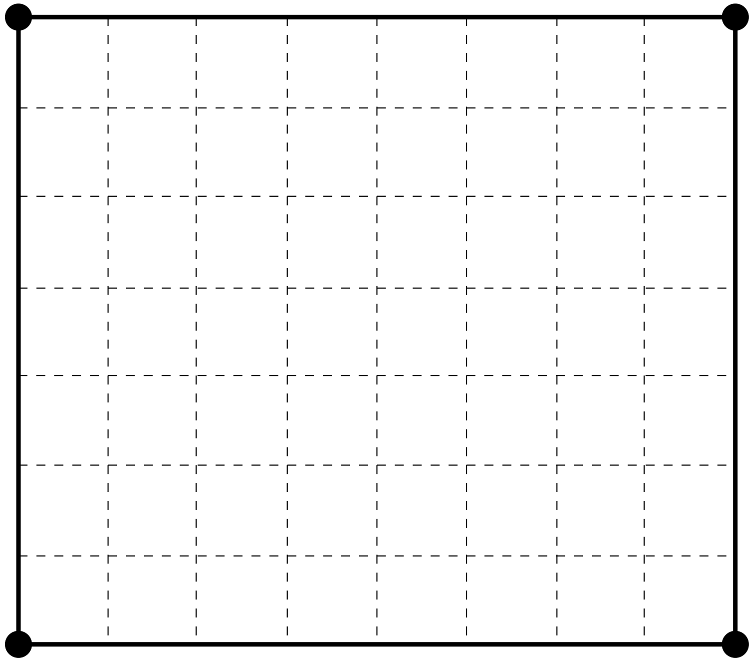




Execution of the algorithm

Place all other points using the red path for absciss and the blue path for ordinate





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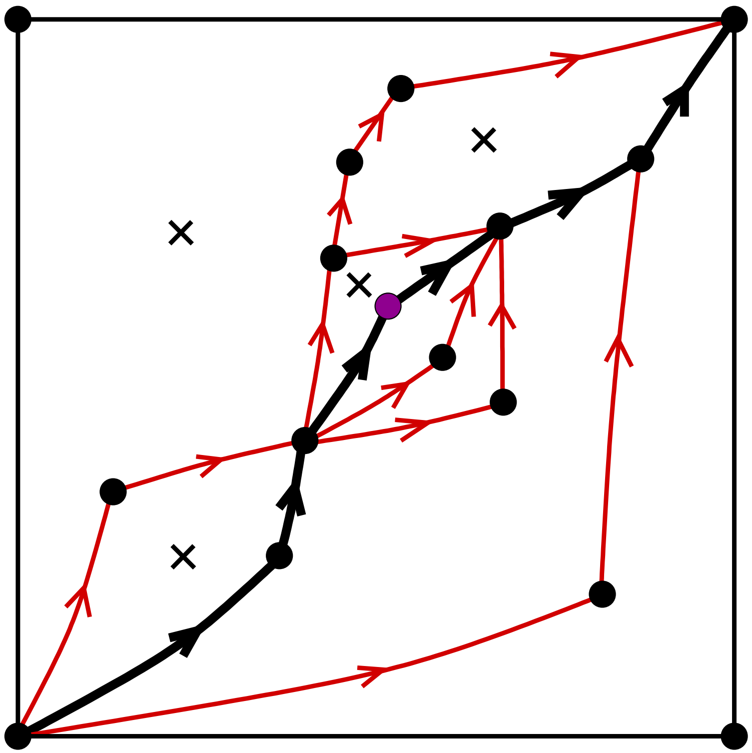
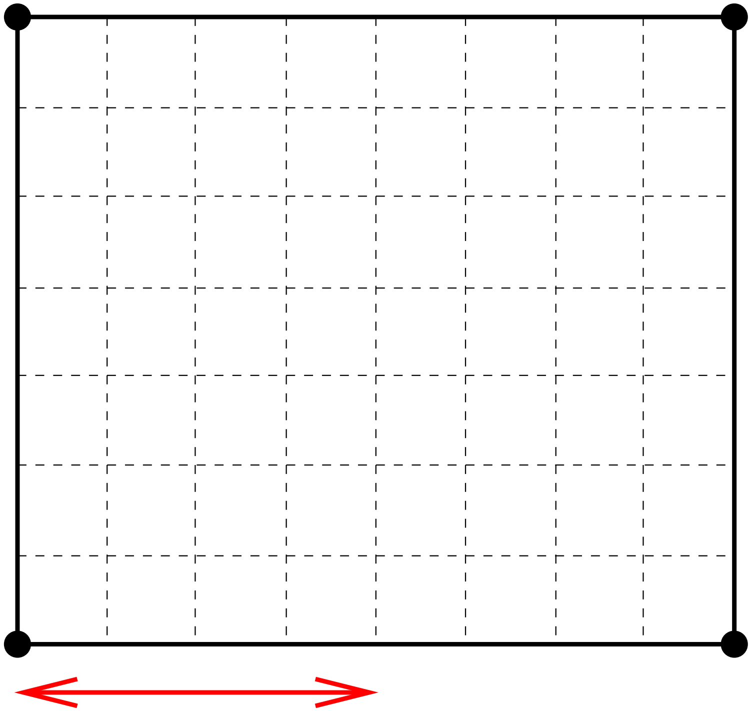
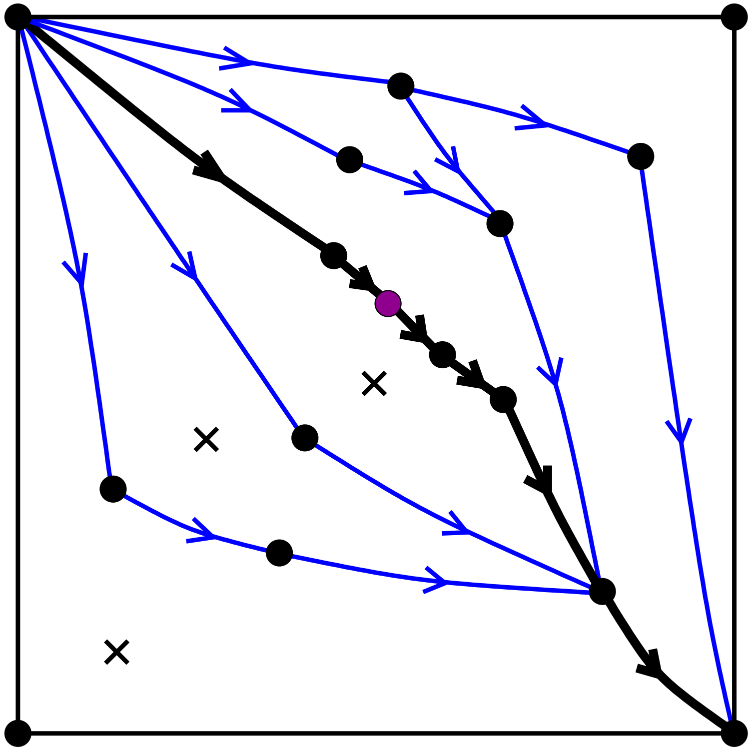


4 faces on the left

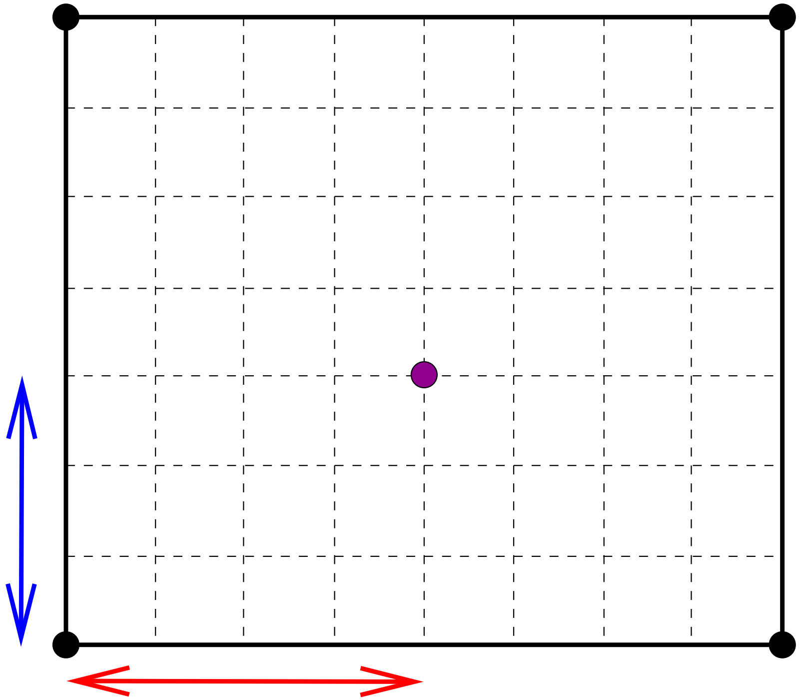


Execution of the algorithm

Place all other points using the red path for absciss and the blue path for ordinate

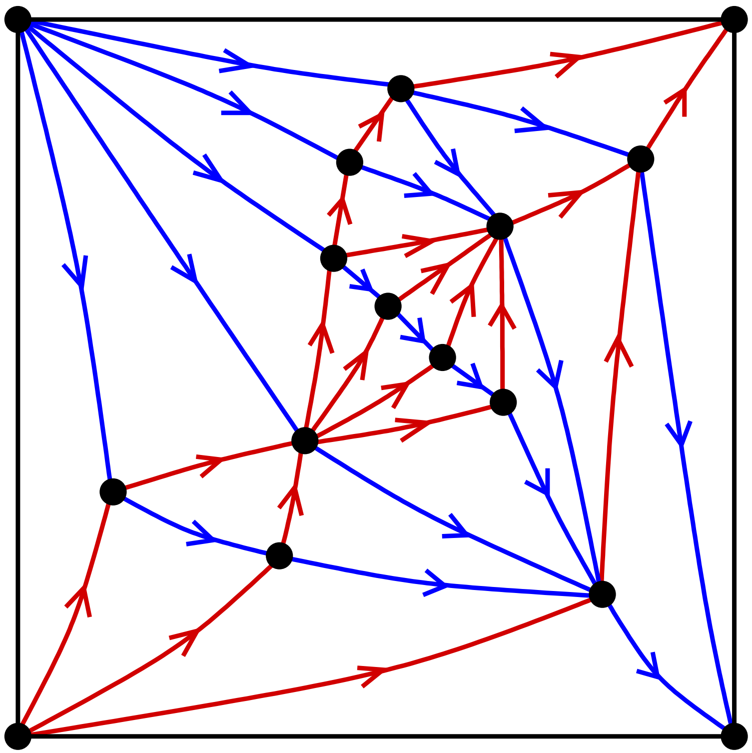


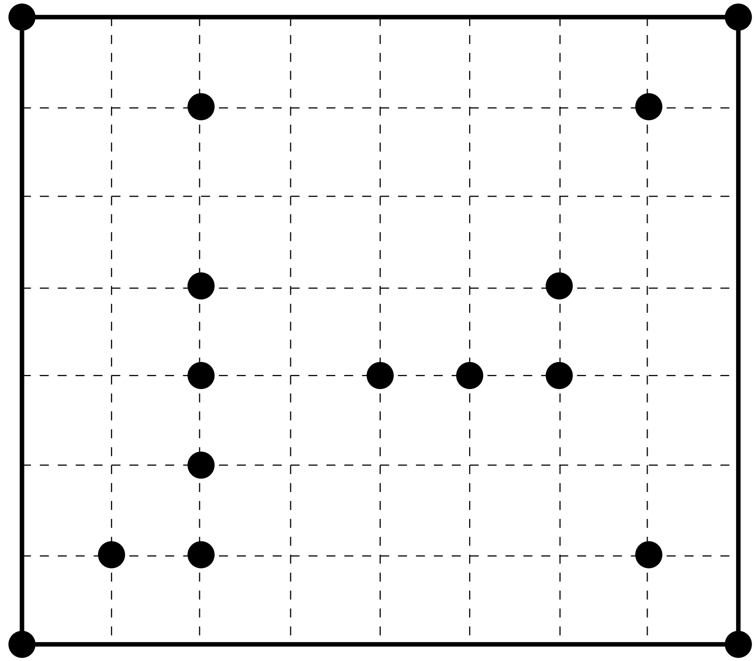
3 faces below



Execution of the algorithm

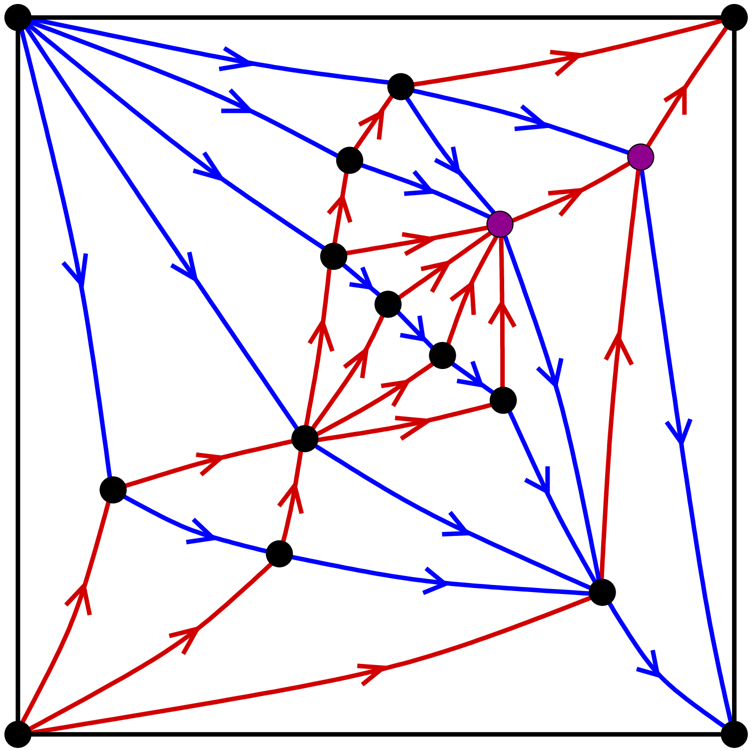
Place all other points using the red path for absciss and the blue path for ordinate

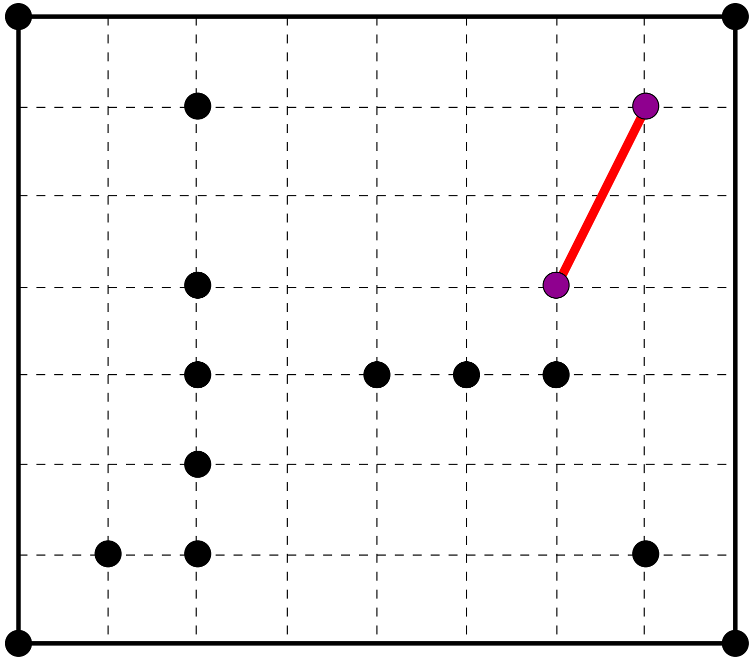




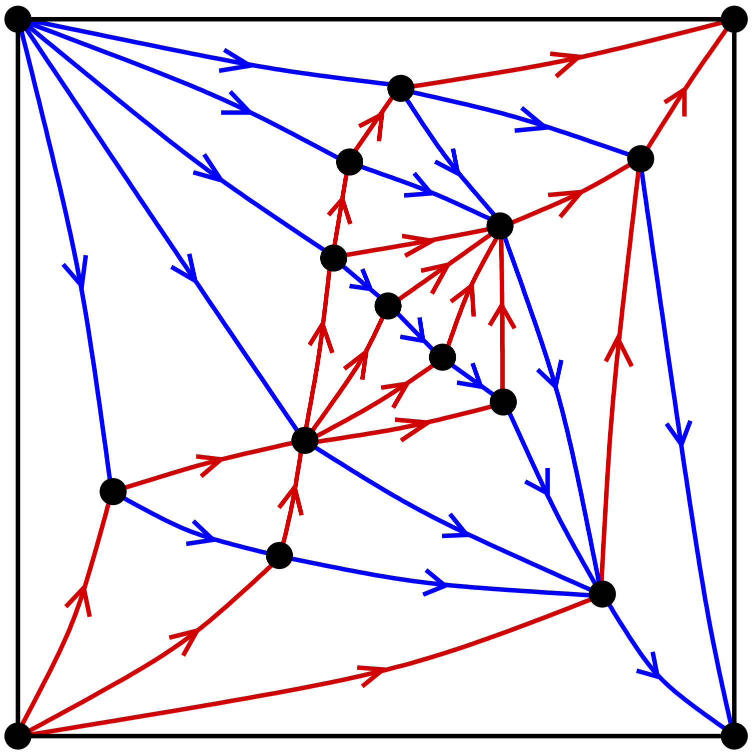
Execution of the algorithm

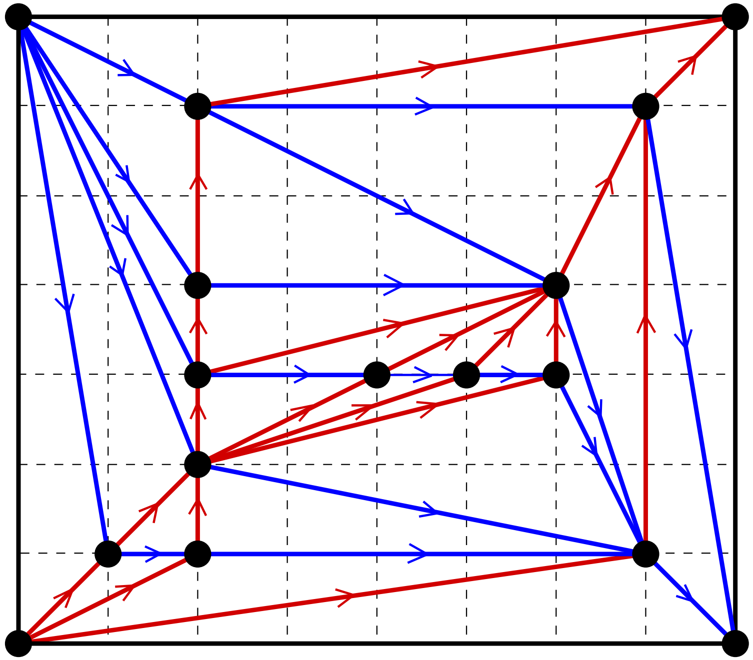
Link each pair of adjacent vertices by a segment





Execution of the algorithm





Results

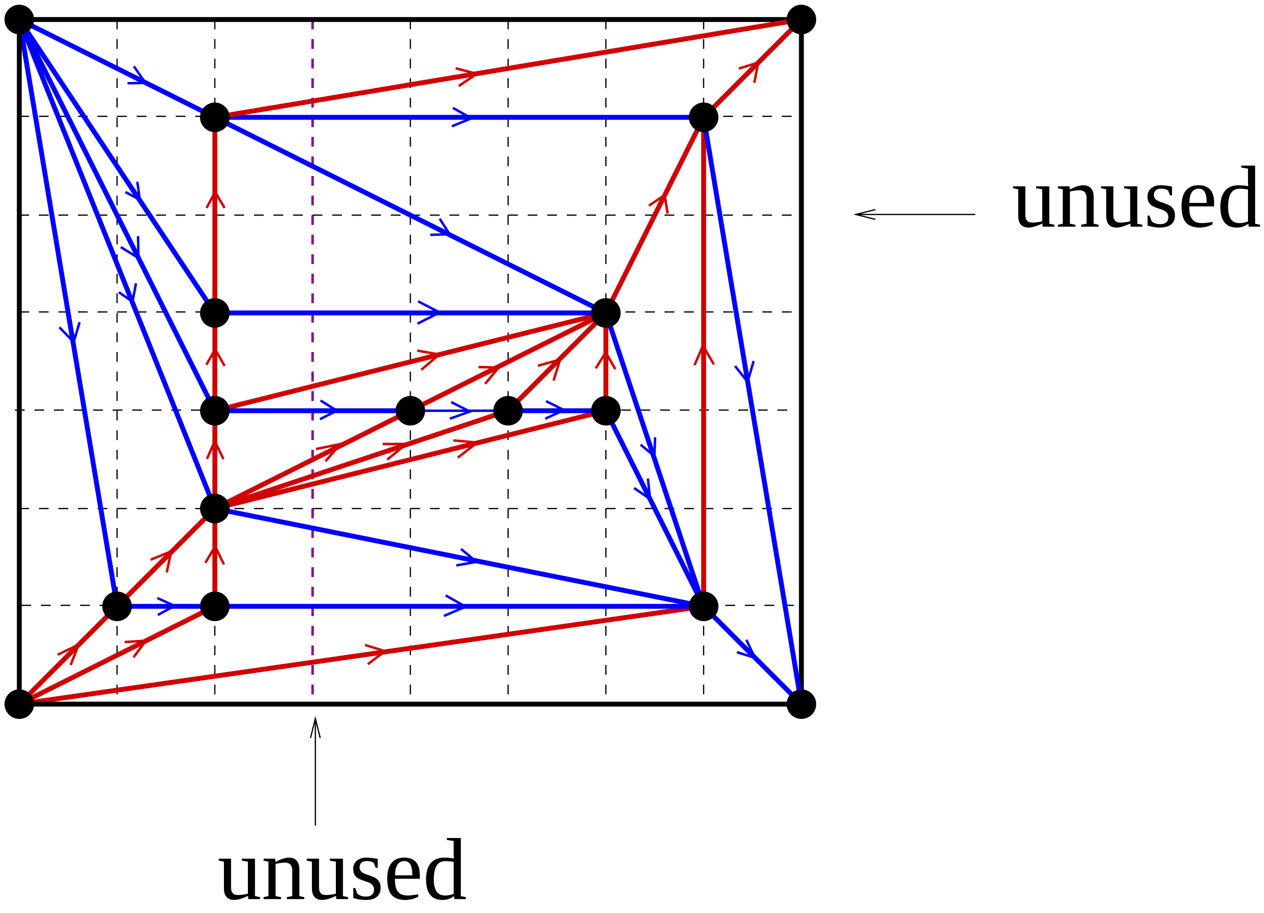
- The obtained drawing is a straight line embedding
- The drawing respects the transversal structure:
  - Red edges are oriented from bottom-left to top-right
  - Blue edges are oriented from top-left to bottom-right
- If T has n vertices, the width W and height H verify

  $$\boxed{W + H = n - 1}$$

  similar grid size as He (1996) and Miura et al (2001)

Compaction step

- Some abscissas and ordinates are not used
- The deletion of these unused coordinates keeps the drawing planar



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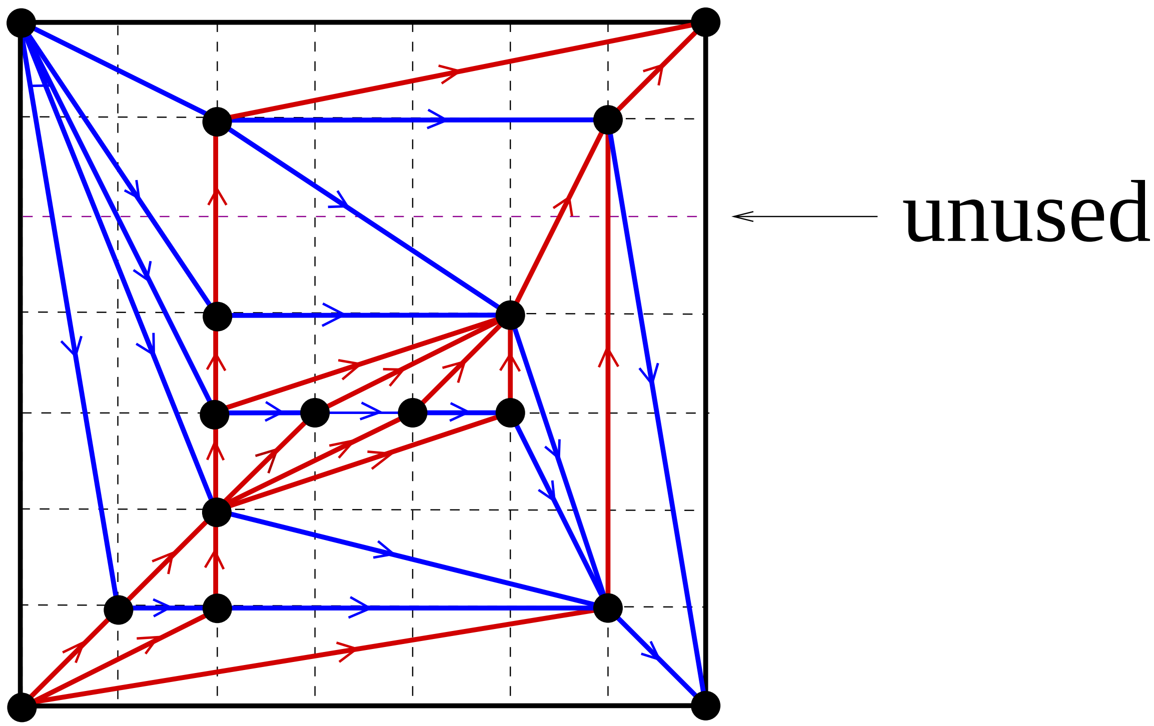

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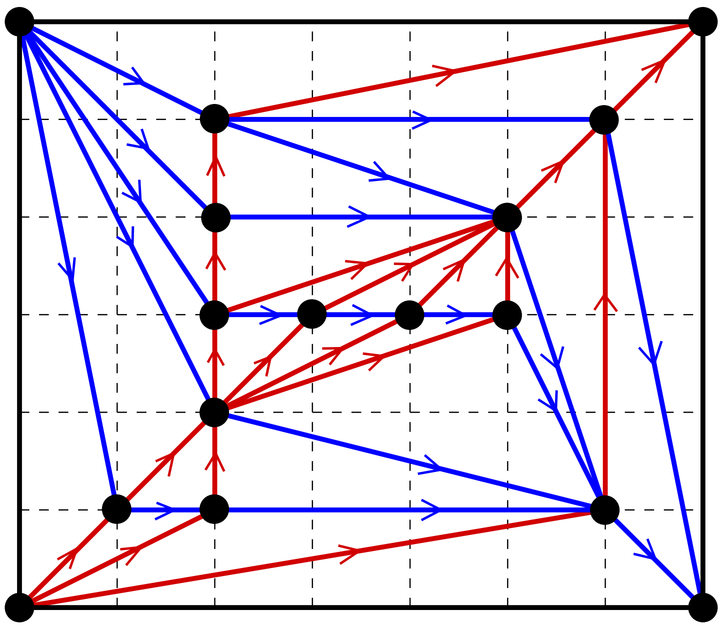
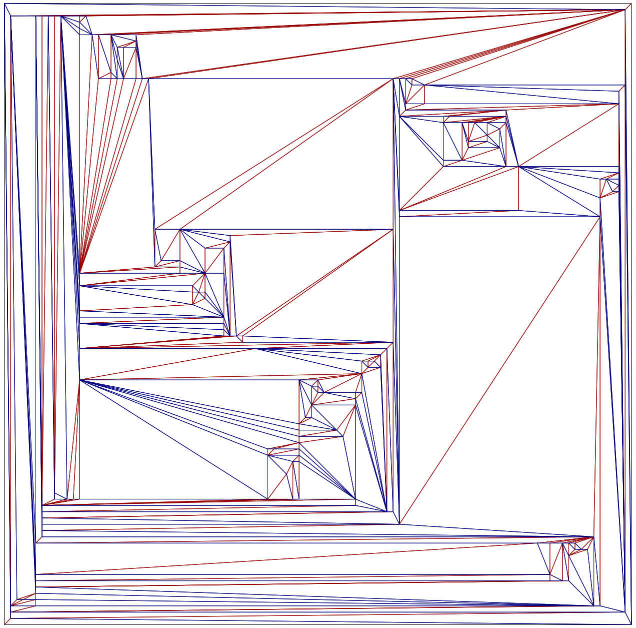
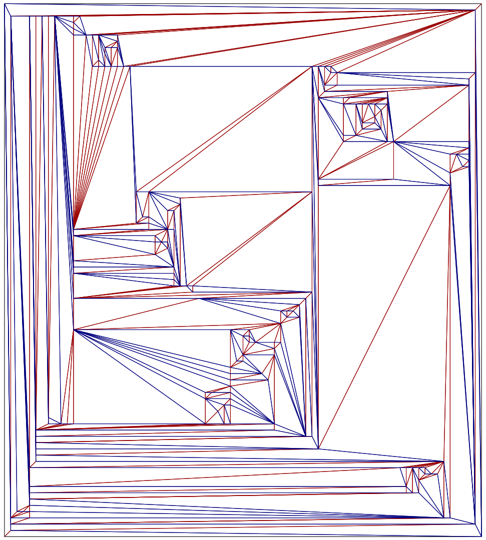


Size of the grid after deletion

- If the transversal structure is the minimal one, the number of deleted coordinates can be analyzed:
- After deletion, the grid has size $\frac{11}{27}n \times \frac{11}{27}n$ "almost surely"
- Reduction of $\frac{5}{27} \approx 18\%$ compared to He and Miura et al

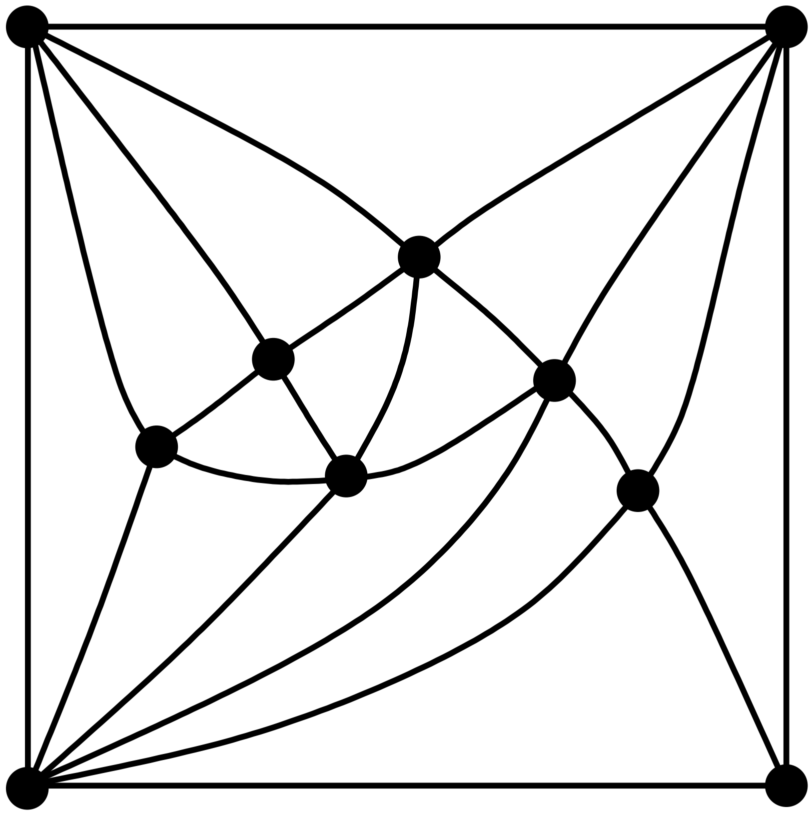




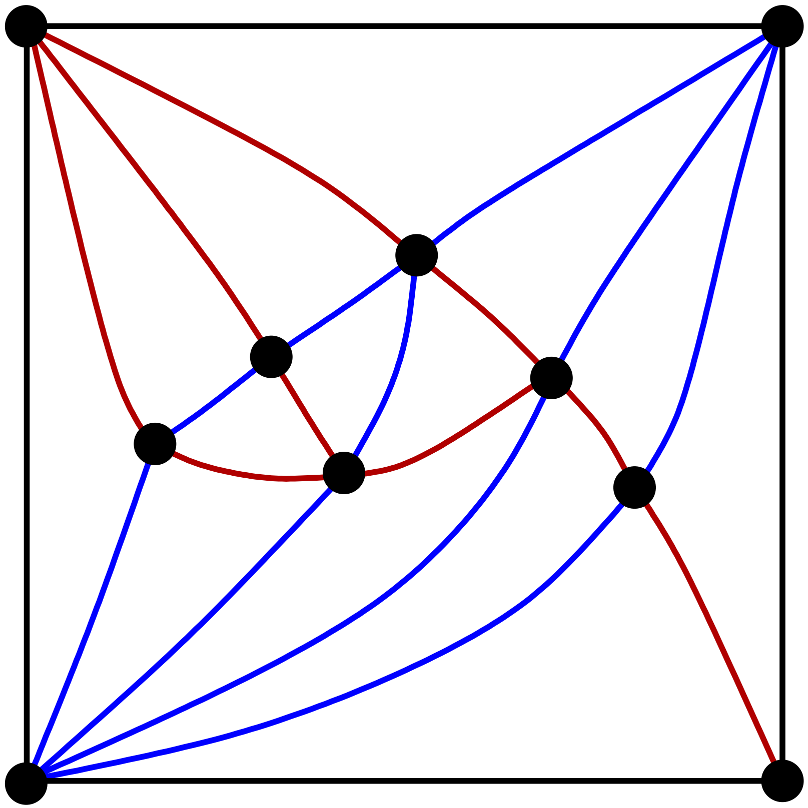
Bijection between triangulations and ternary trees

Opening: triangulation⇒ternary tree

Compute the minimal transversal structure

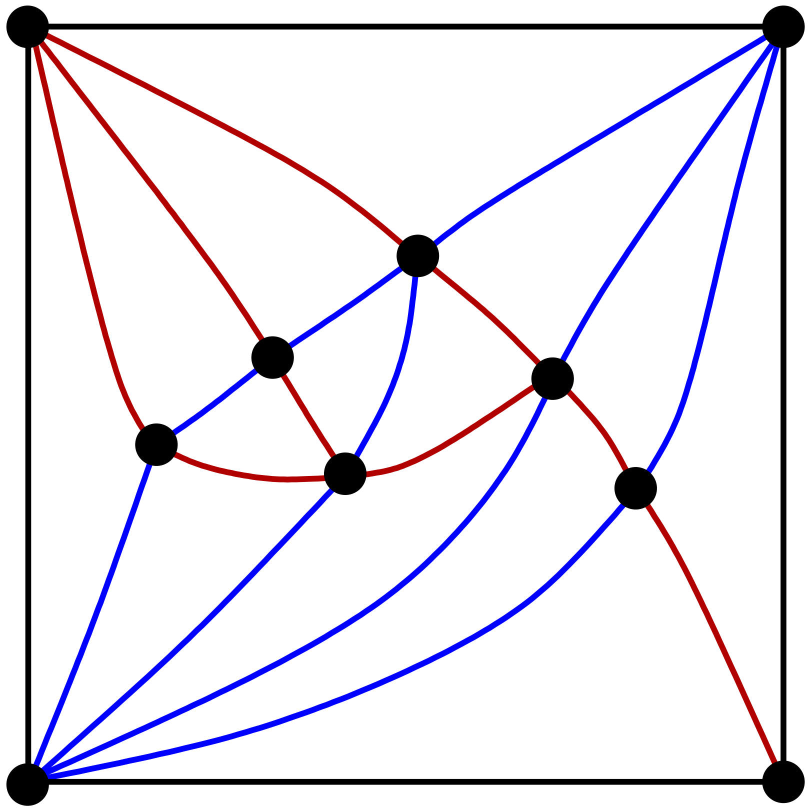


Opening: triangulation⇒ternary tree

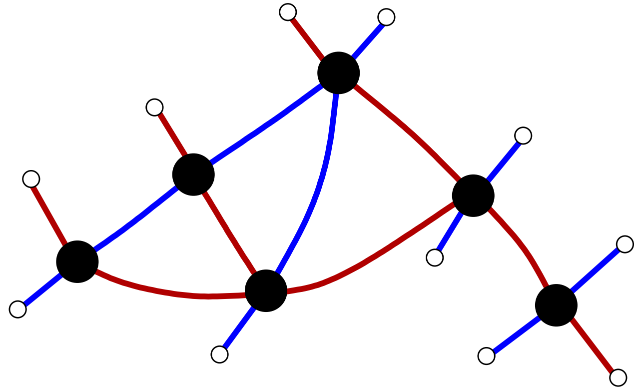


Opening: triangulation⇒ternary tree

Remove quadrangle

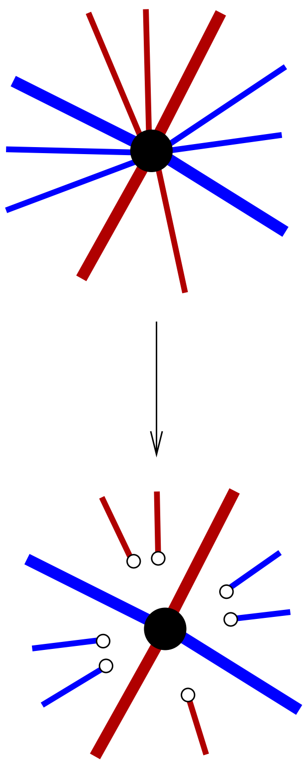


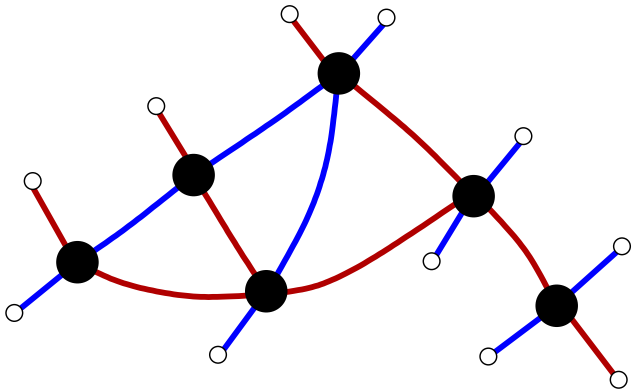
Opening: triangulation⇒ternary tree



Opening: triangulation⇒ternary tree

Keep the clockwisemost edge in each bunch





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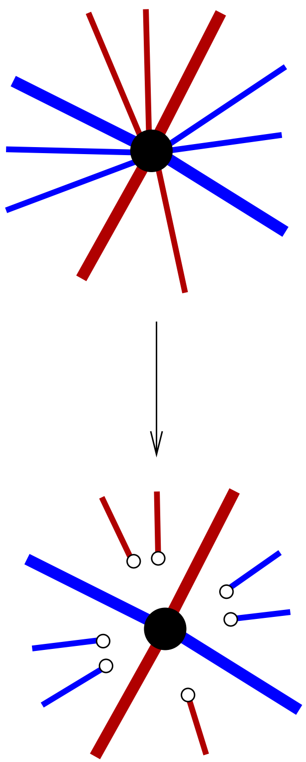
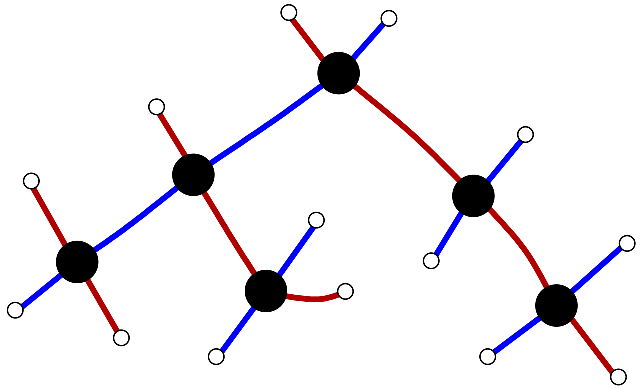
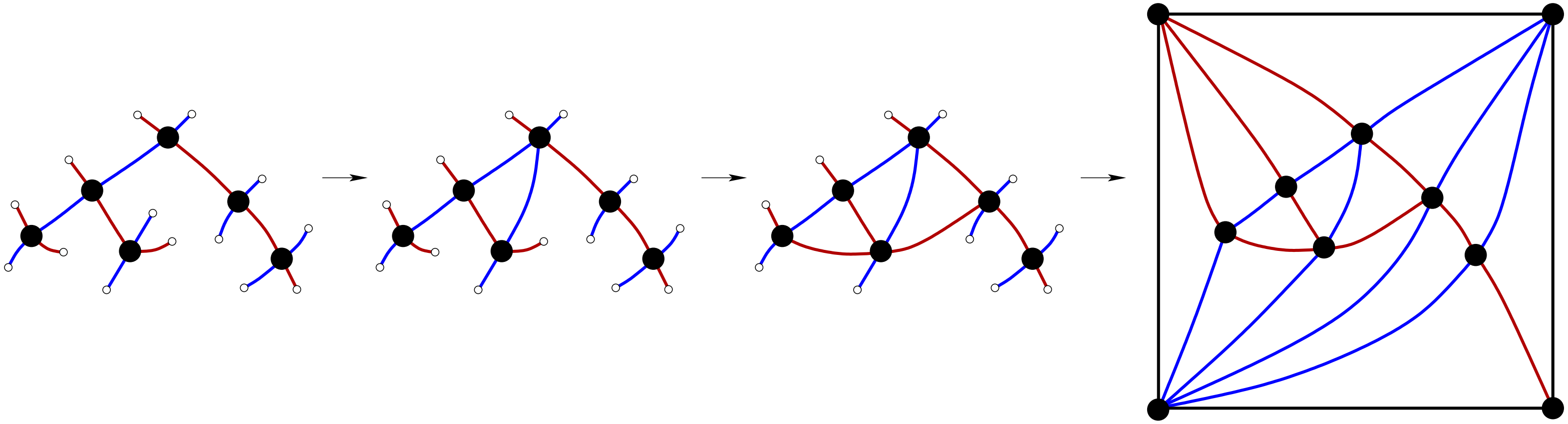




Result

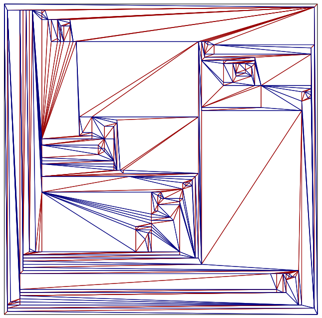
Theorem This mapping is a bijection between triangulations with n inner nodes and ternary trees with n inner nodes

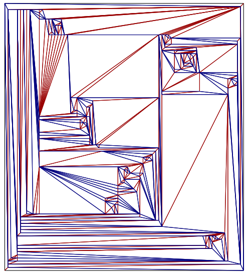
The inverse mapping: ternary trees→triangulations is also explicit:



Applications of the bijection

- Enumeration:$\Rightarrow$ $\boxed{T_n = \frac{4}{2n+2}\frac{(3n)!}{n!(2n+1)!}}$
- Random generation: linear-time uniform random sampler of triangulations with n vertices





- Analysis of the grid size: almost surely $5n/27$ deleted coordinates for a random triangulation with n vertices

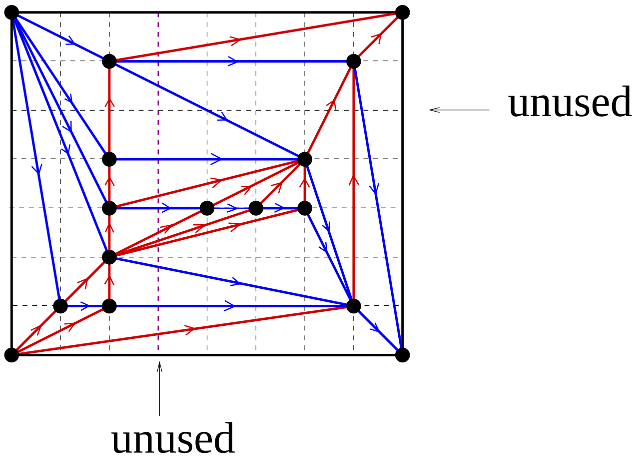
Analysis of the size of the grid using the bijection

Size of the compact drawing ?

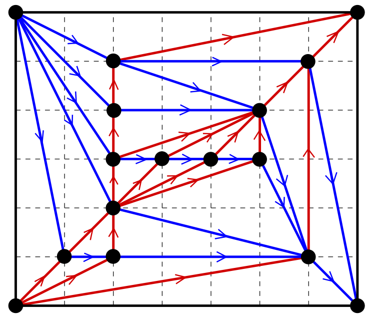
Let T be a triangulation with n vertices endowed with its minimal transversal structure

- Unoptimized drawing: $W + H = n - 1$
- Delete unused coordinates⇒Compact drawing:

$$W_c + H_c = n - 1 - \#(unused\ coord.)$$



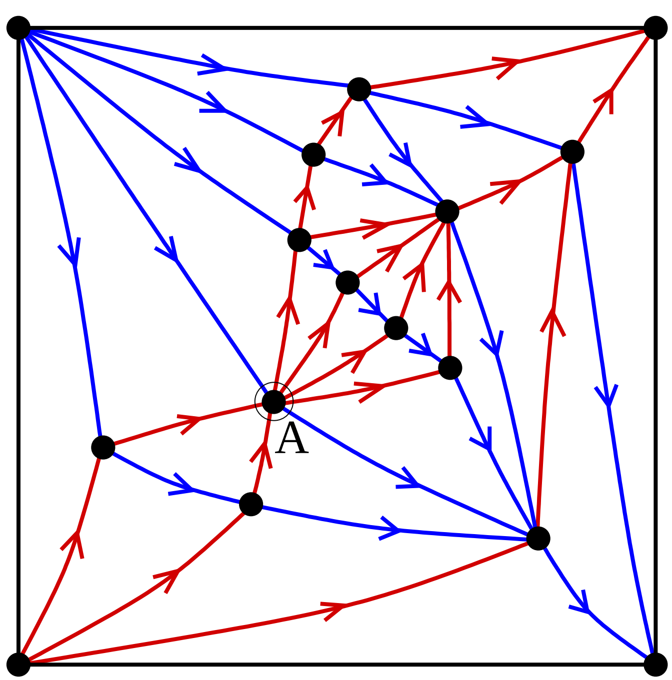
⇒

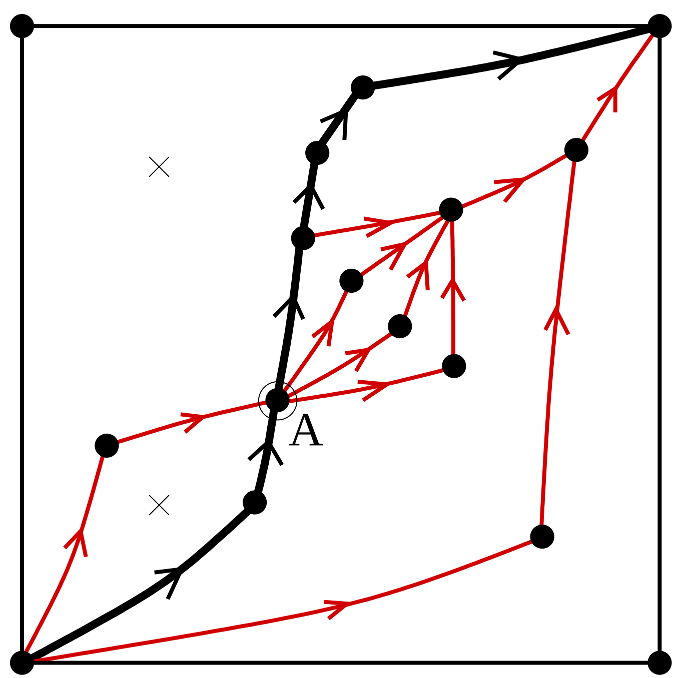


Theorem: $\#(unused\ coord.) \sim \frac{5n}{27}$ almost surely

Rule to give abscissa

The absciss of v is the number of faces of the red map on the left of $\mathcal{P}_r(v)$





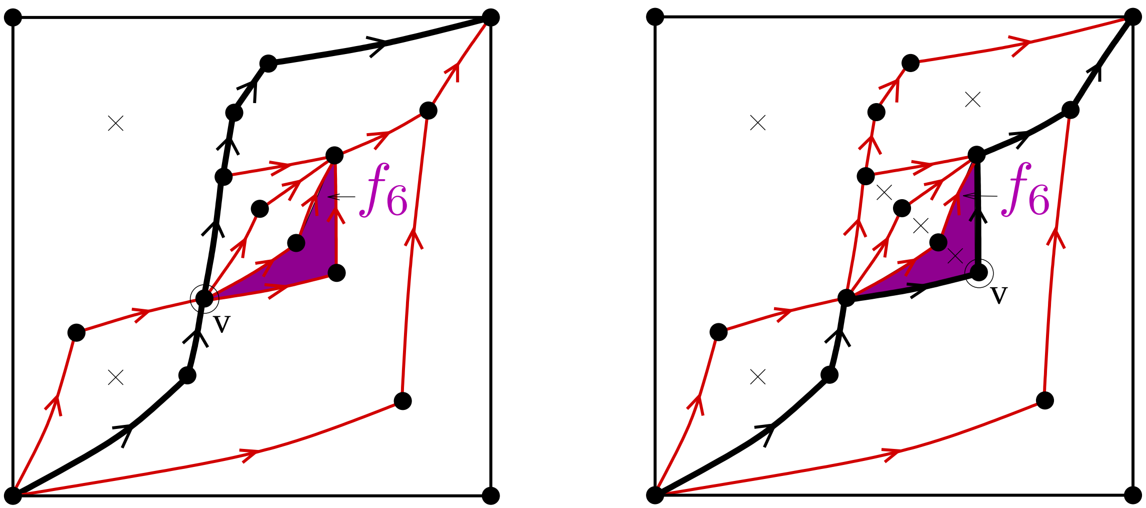
$\Rightarrow$ A has absciss 2

Absciss $\leftrightarrow$ face of the red-map

- Let f_r be the number of faces of the red-map
- Let $i \in [1, f_r]$ be an absciss-candidate
- There exists a face f_i of the red-map such that:

$$\boxed{Abs(v) \geq i \Leftrightarrow f_i \text{ is on the left of } \mathcal{P}_r(v)}$$

Example: $i = 6$



$Abs(v) = 2 < 6$ $\quad$ $Abs(v) = 6 \geq 6$

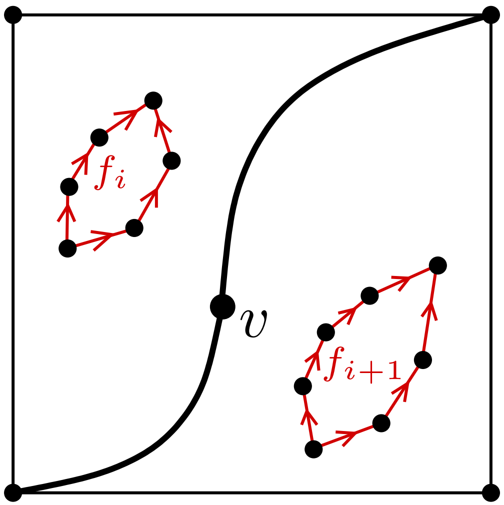
Unused abscissa

An absciss-candidate $i \in [1, f_r]$ is unused iff:

$$\boxed{Abs(v) \geq i \Rightarrow Abs(v) \geq i+1}$$

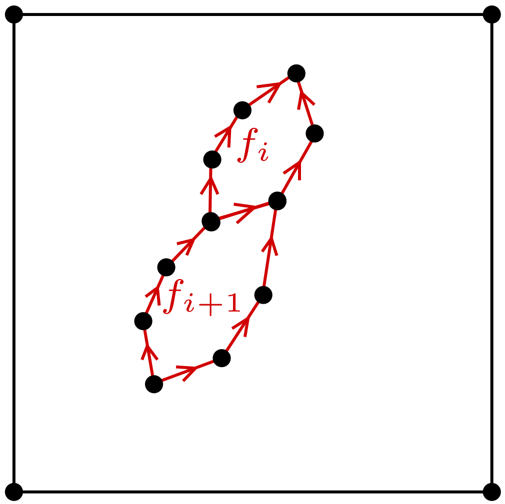
$\Rightarrow$ Faces f_i and f_{i+1} can not be separated by a path $\mathcal{P}_r(v)$

$\Rightarrow$ f_i and f_{i+1} are contiguous



CAN NOT HAPPEN

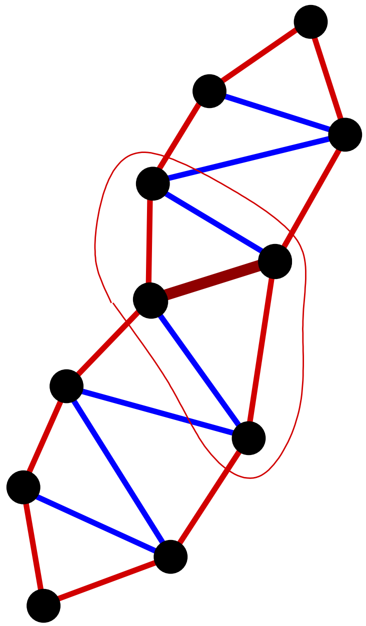
$\Rightarrow$

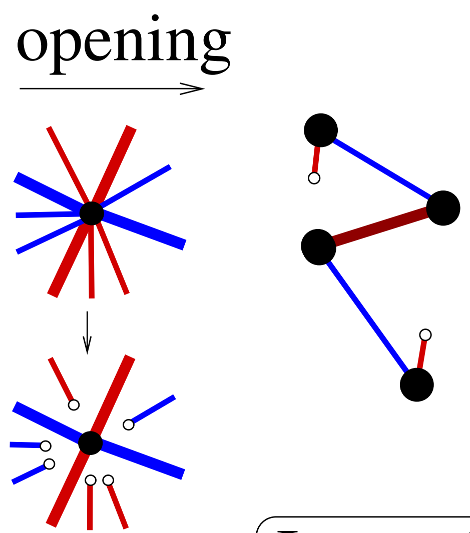


BottomRight(f_i)=TopLeft(f_{i+1})

Unused abscissa and opening

Triangulation ⟷ Ternary tree





Unused abscissa ⟷ Internal edge such that cw–consecutive edge at each extremity is an internal edge

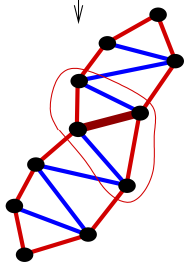
Reduction to a tree-problem

Width of the grid of the compact drawing ?

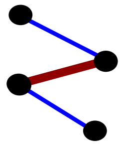
↓

How many unused abscissas in a random triangulation

↓

How many  in a random triangulation

↓

How many  in a random ternary tree

↓

$\Rightarrow \boxed{\sim \frac{1}{2}\frac{5n}{27}}$ (using generating functions)