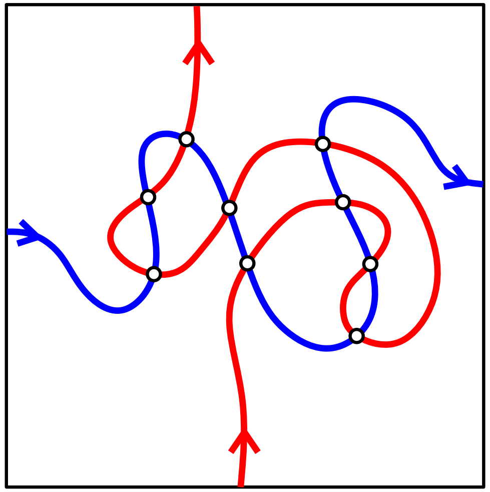
Baxter permutations and meanders

Éric Fusy (LIX, École Polytechnique)

Journées Viennot, 28-29 juin 2012, Labri

Meanders on two lines

- A 2-line meander



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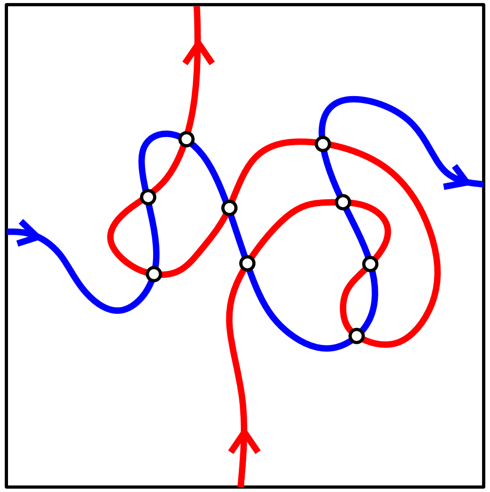
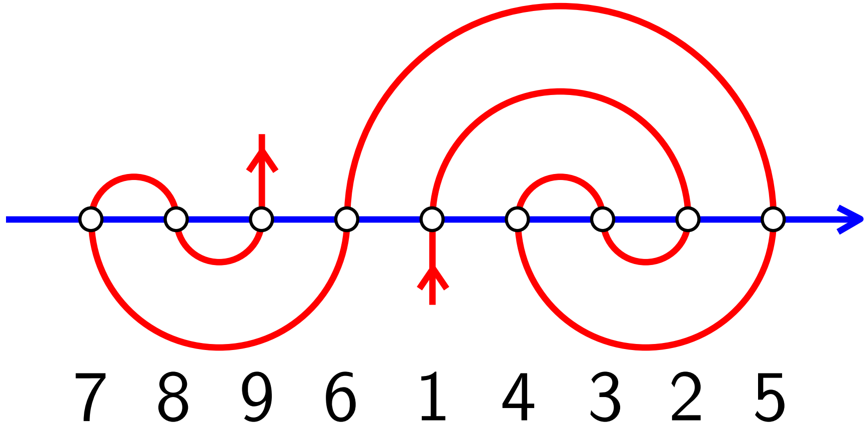


encoded by a permutation



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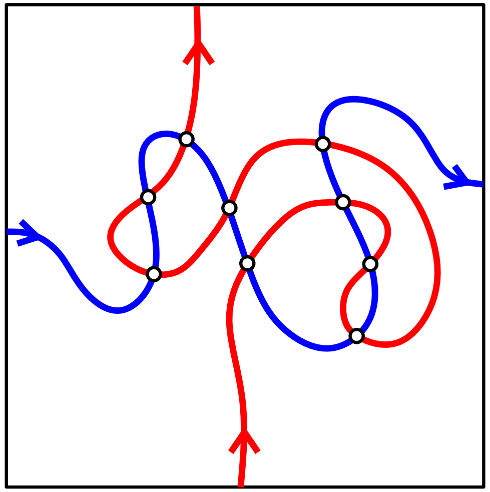

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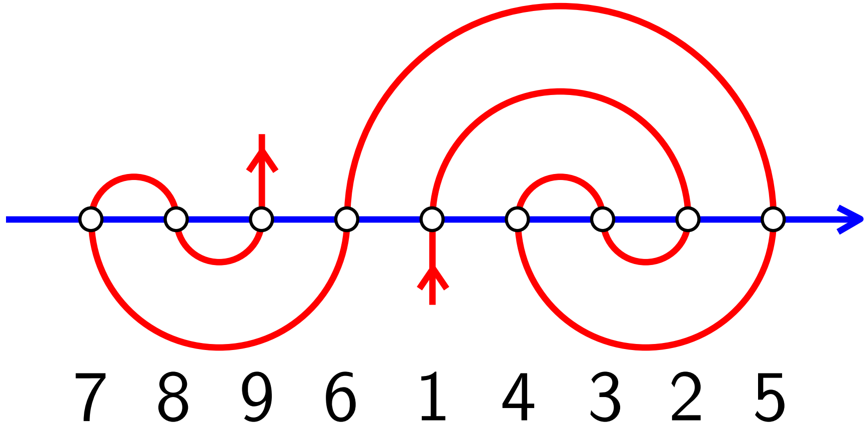
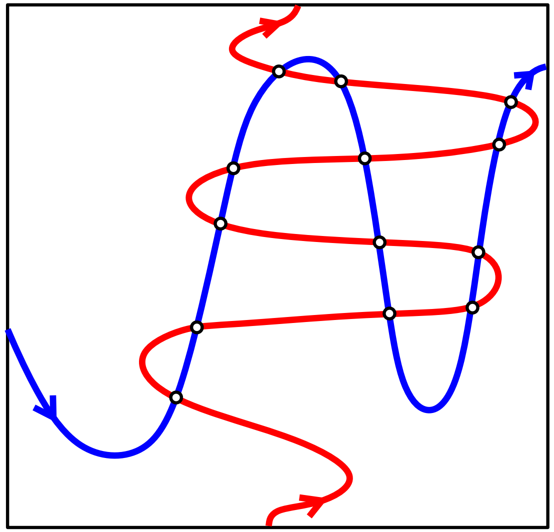


- Monotone 2-line meander:

can be obtained from two monotone lines (one in x, the other in y)



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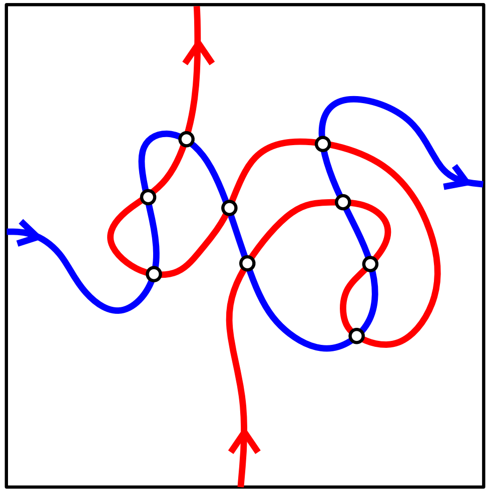

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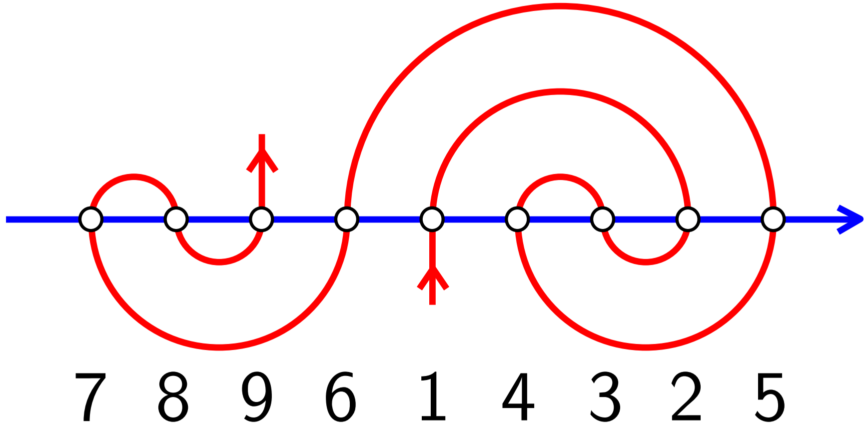

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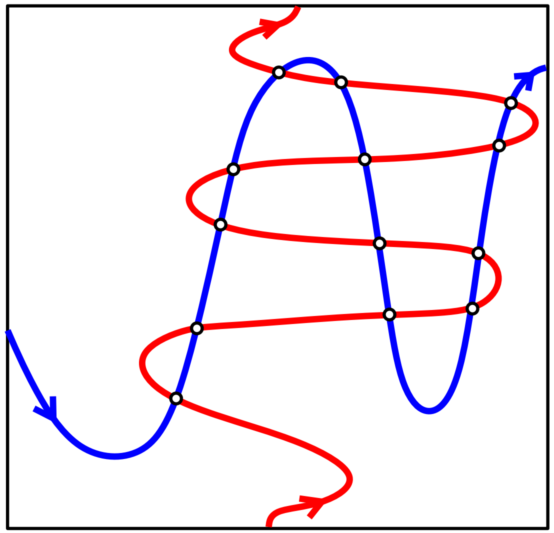

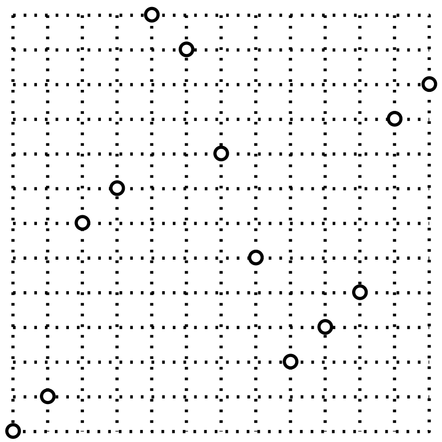

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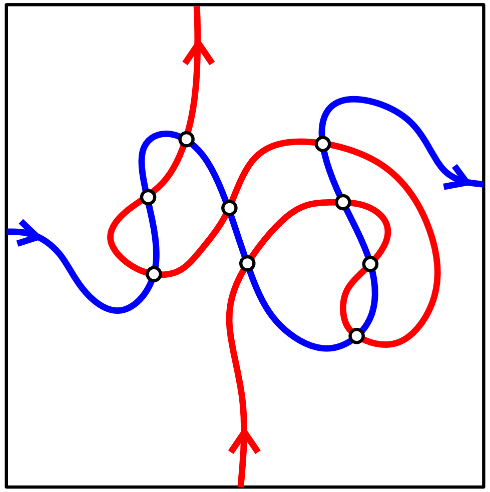

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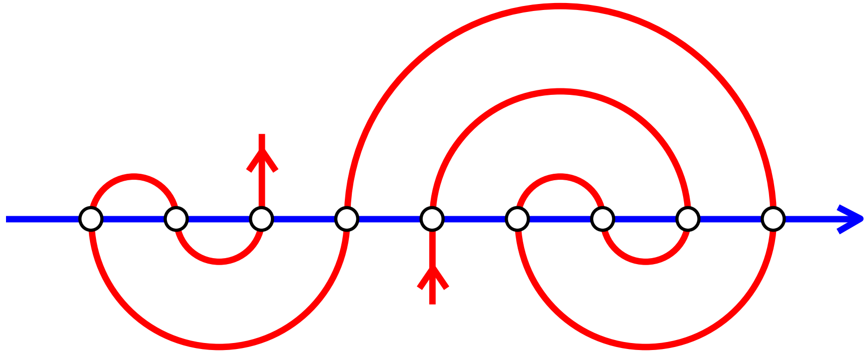

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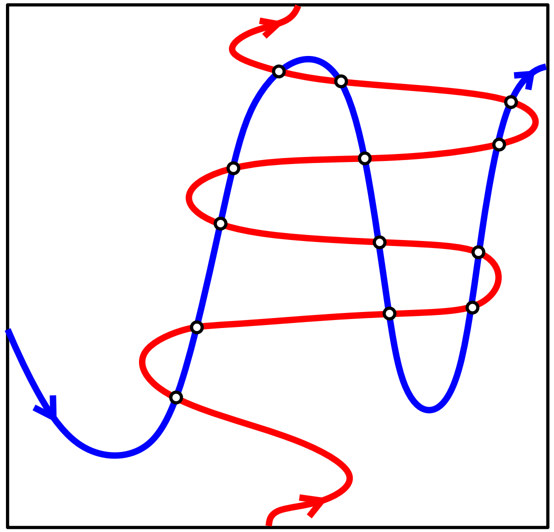

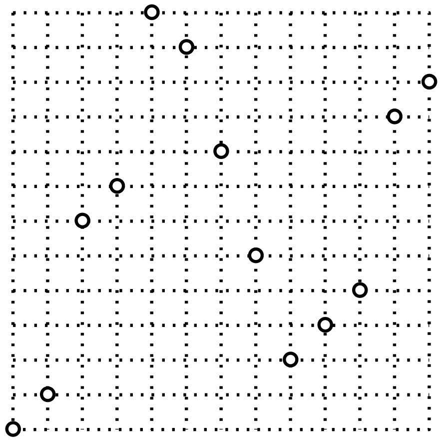

associated permutation

Which permutations can be obtained this way ?

Meanders on two lines

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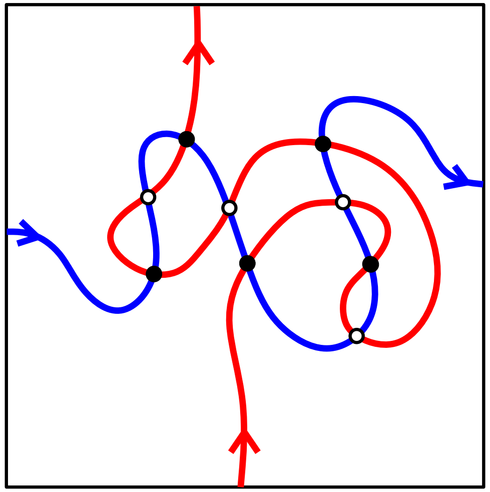

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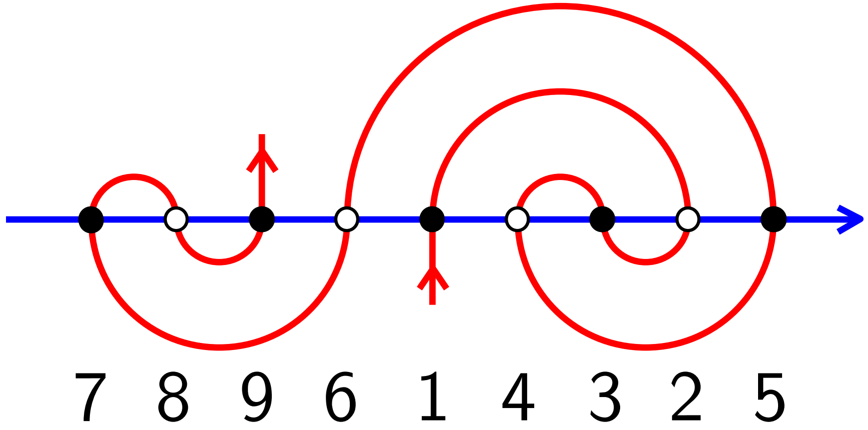

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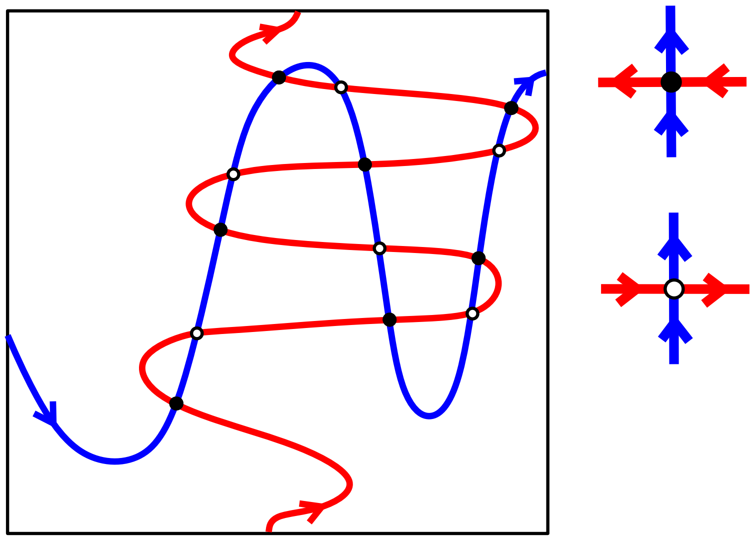

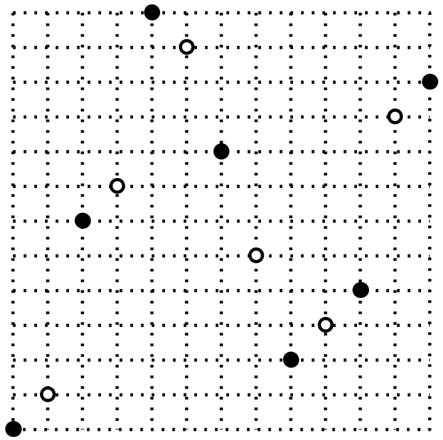

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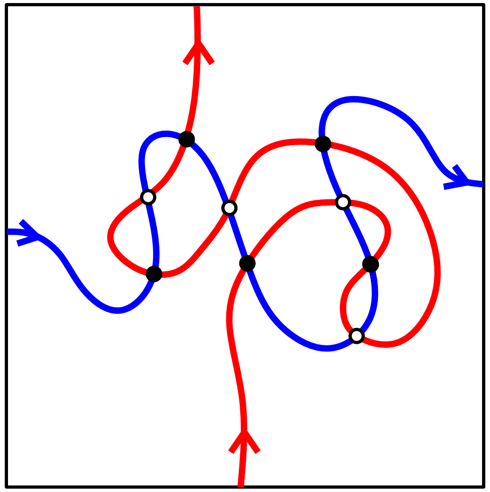

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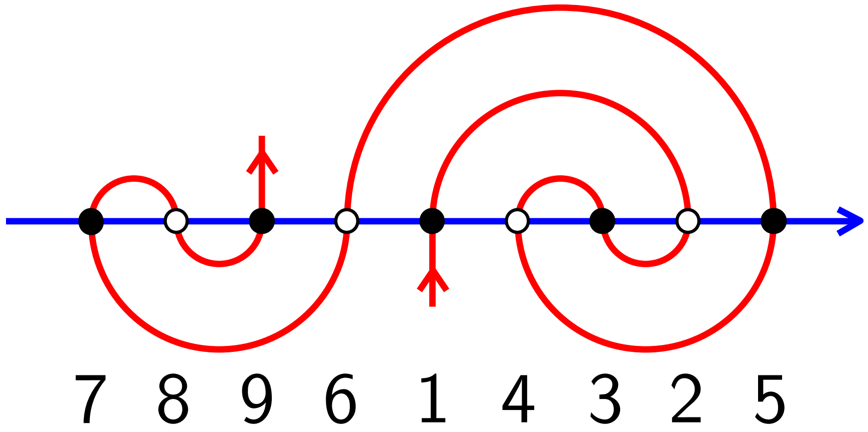

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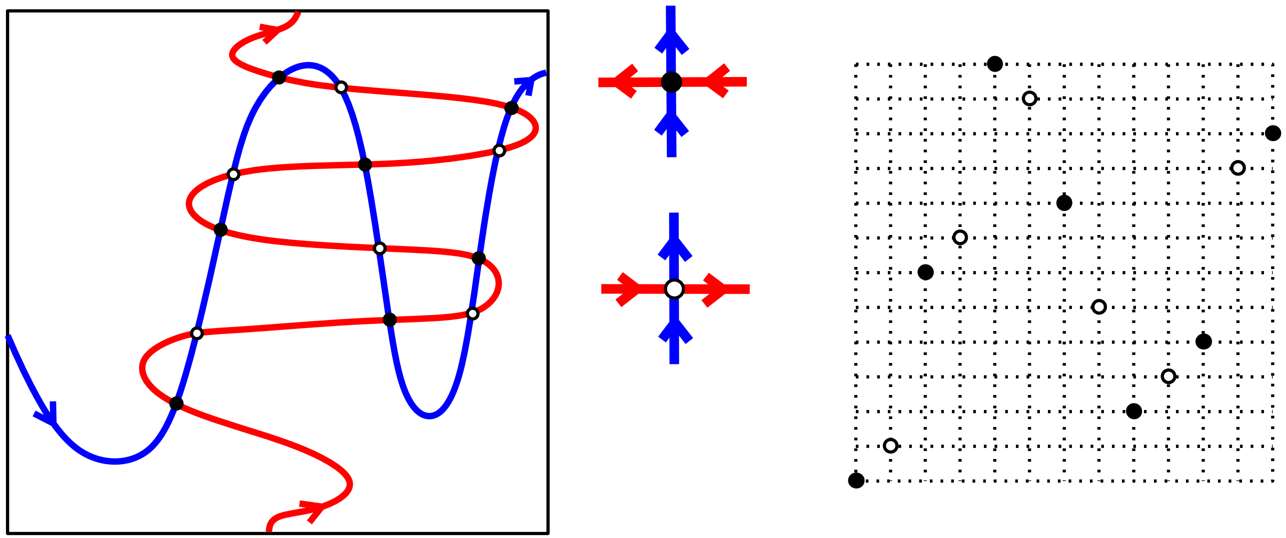


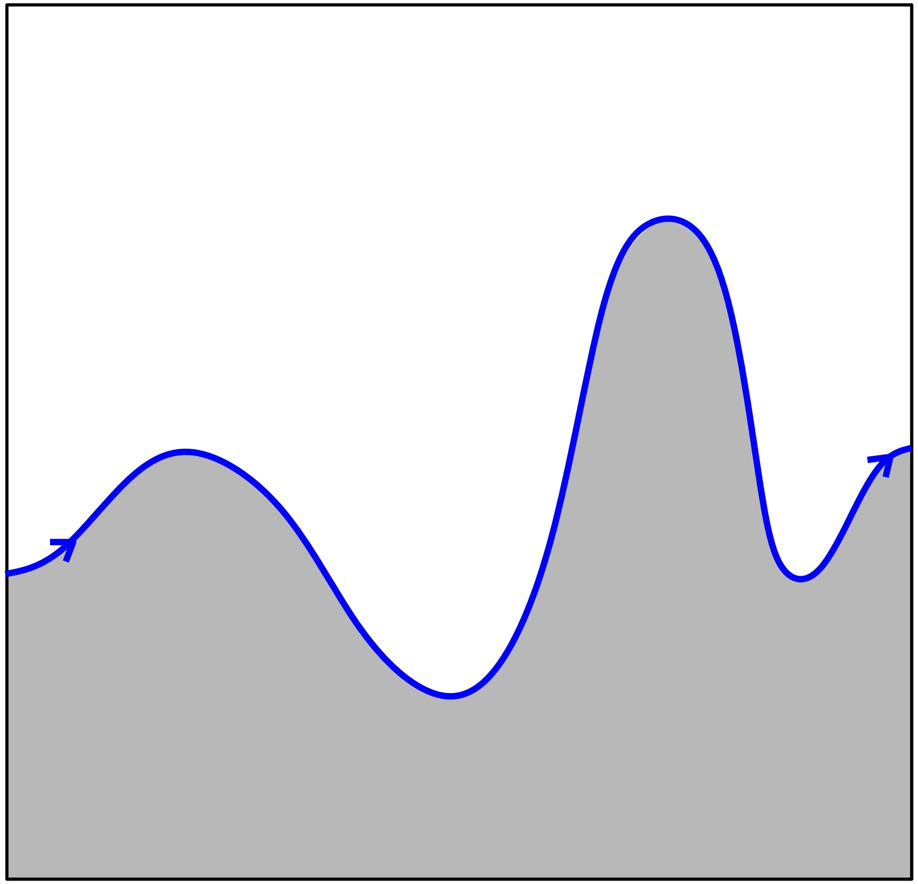
associated permutation

Which permutations can be obtained this way ?

Maps odd numbers to odd numbers, even numbers to even numbers

Permutations for monotone 2-line meanders

[Baxter'64, Boyce'67&'81]



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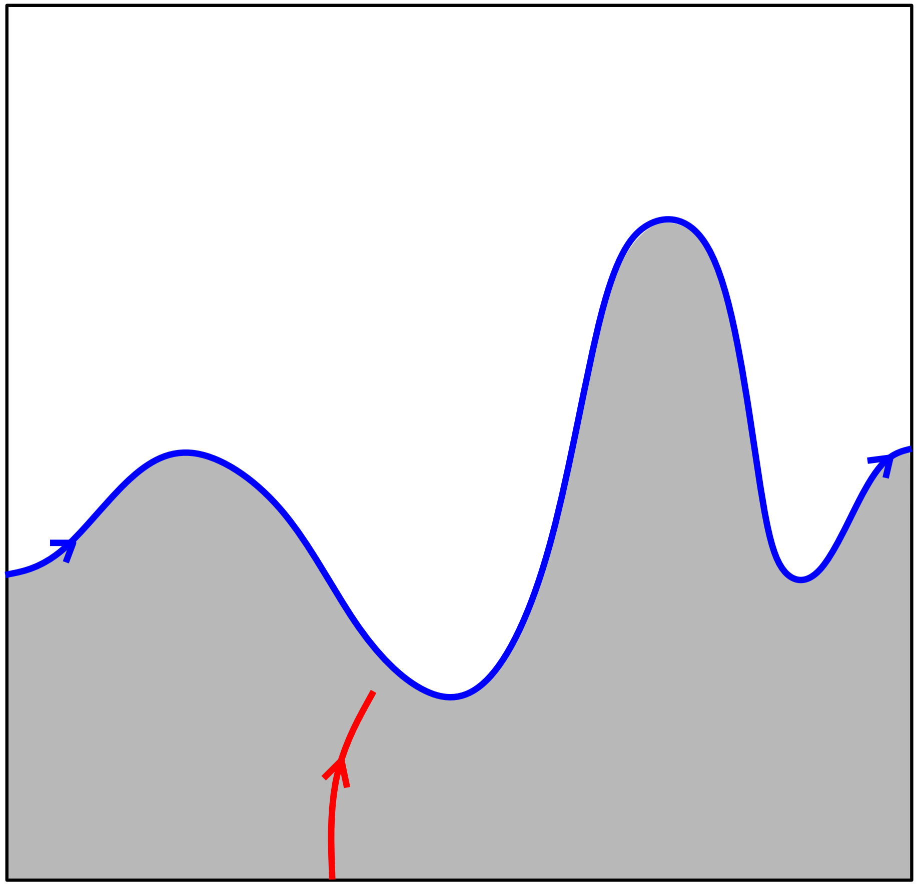

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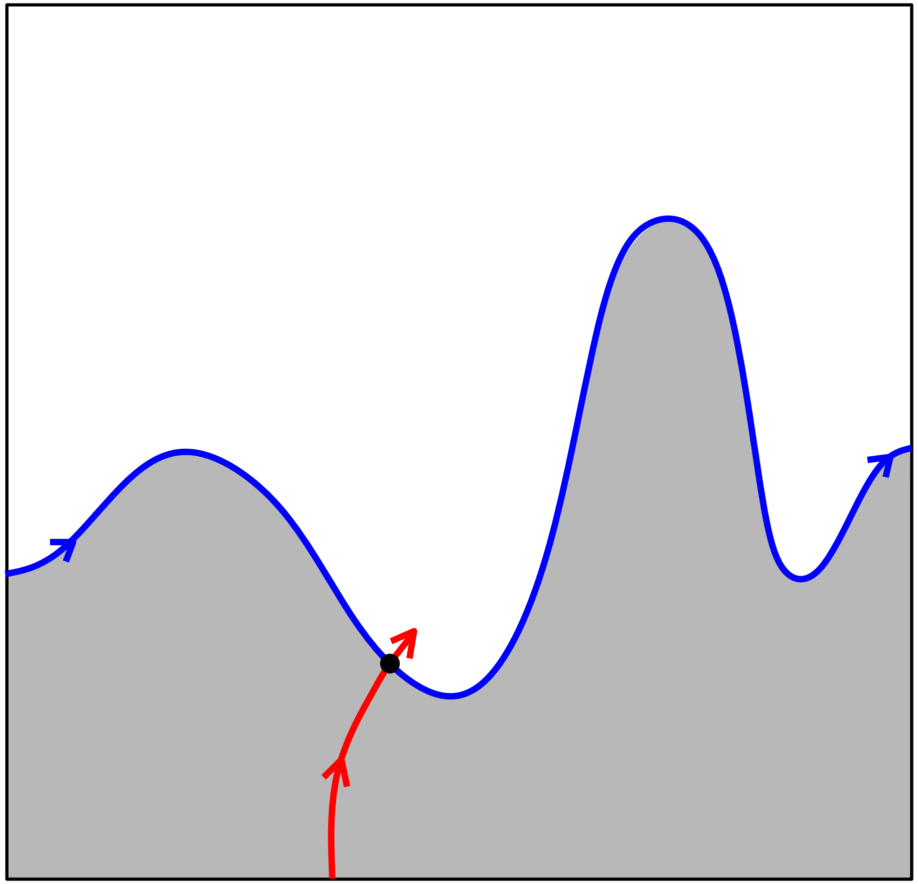

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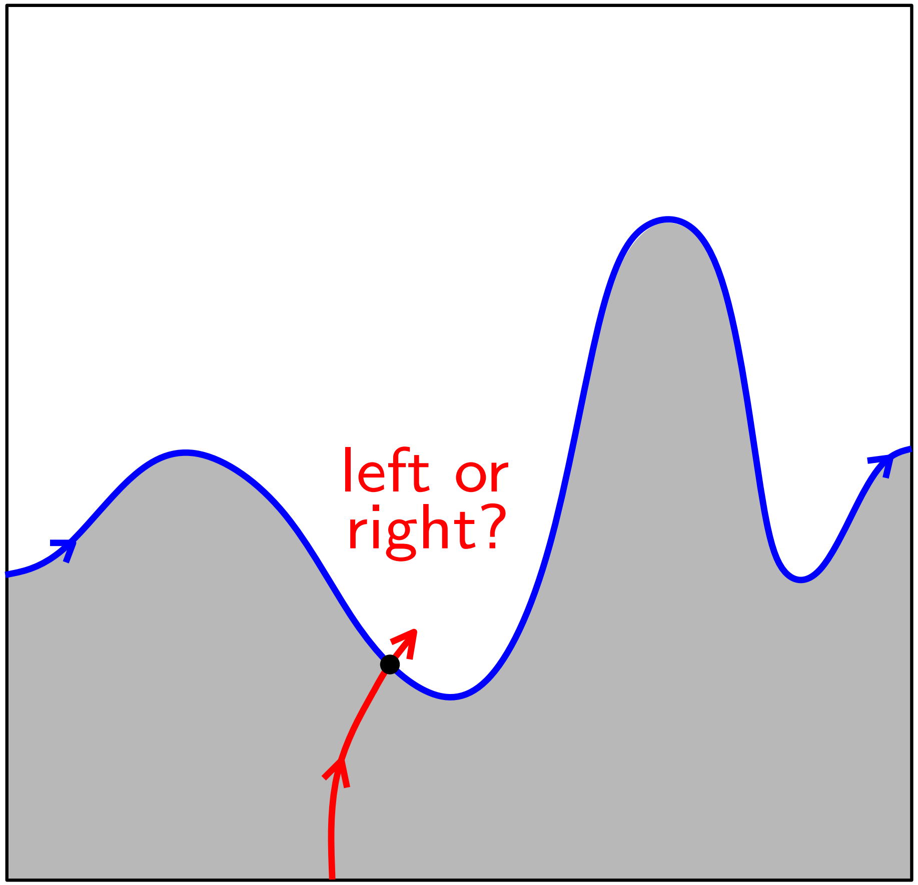

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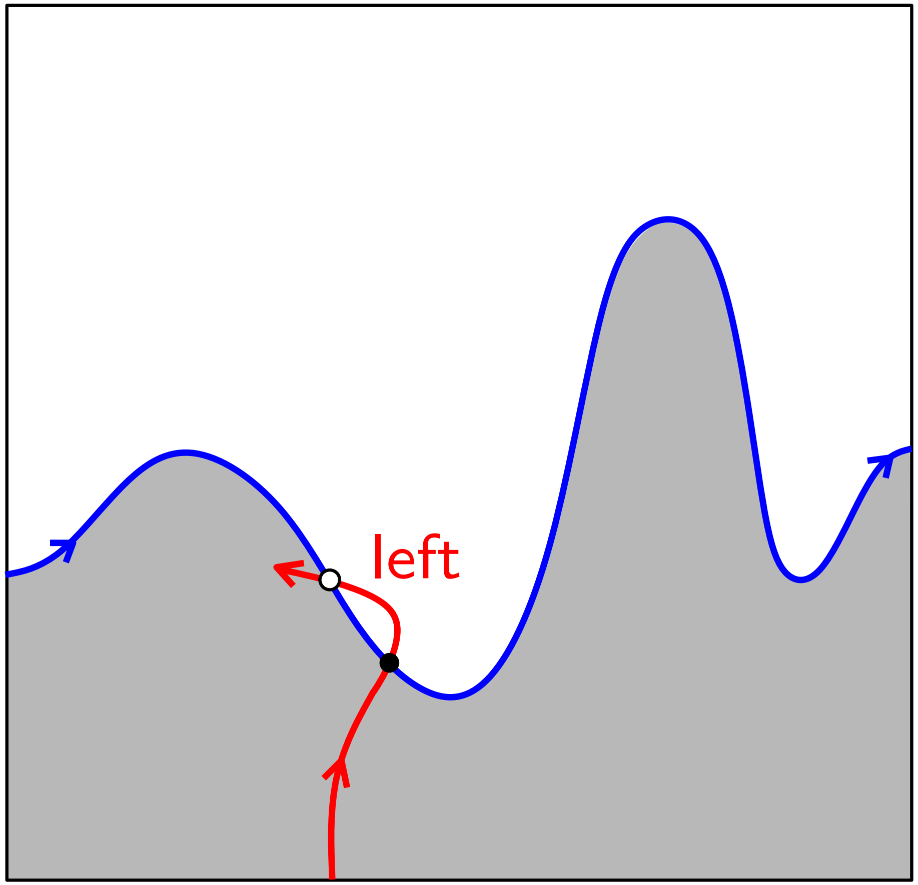

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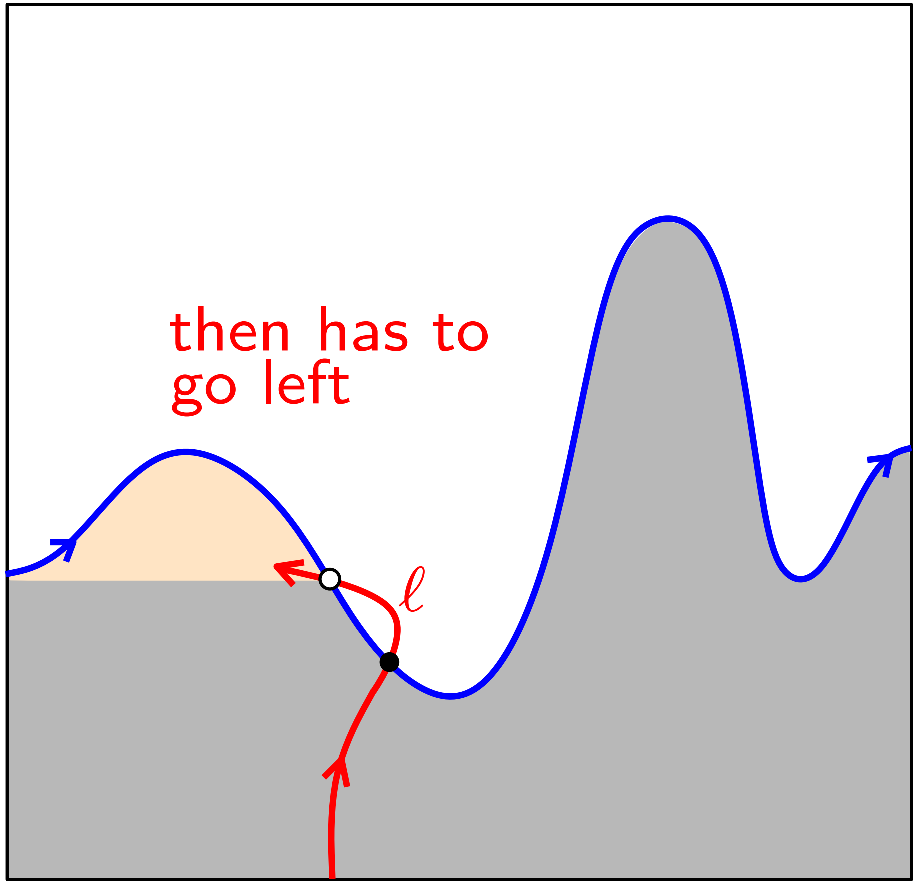

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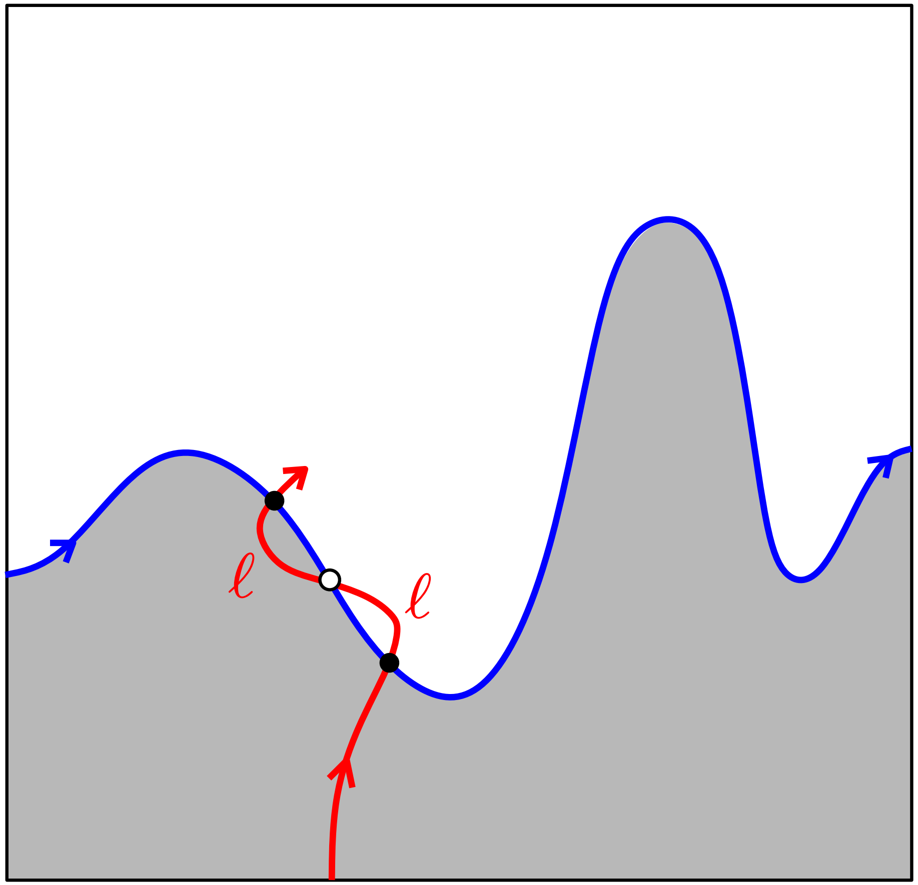

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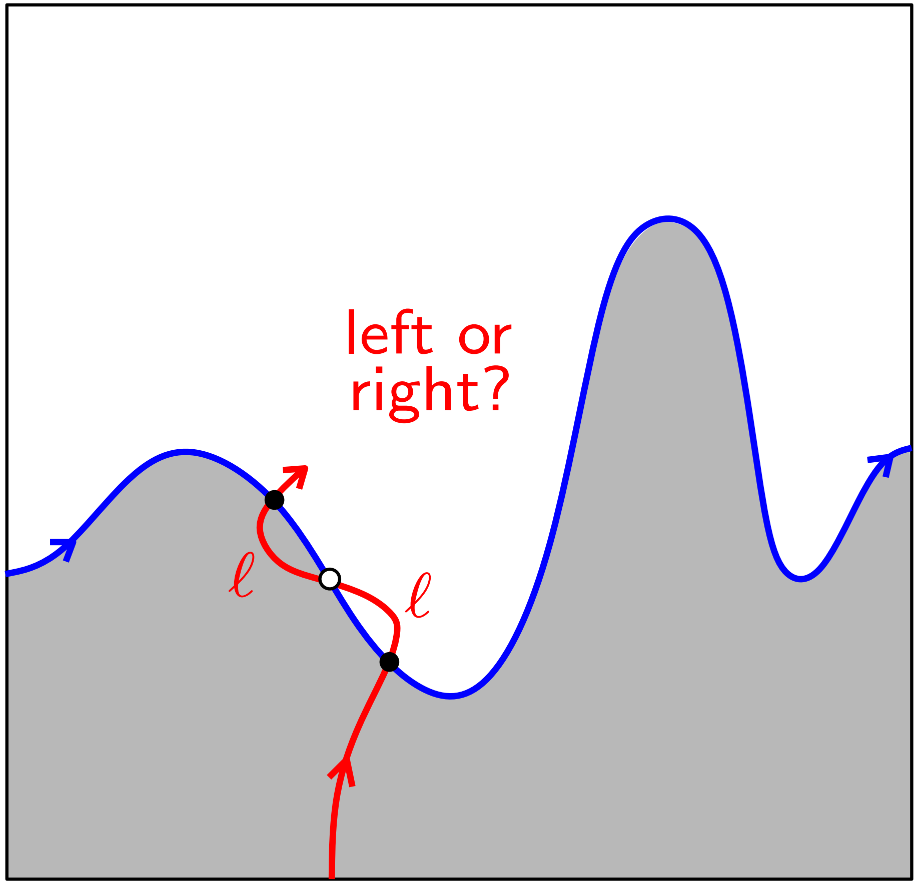

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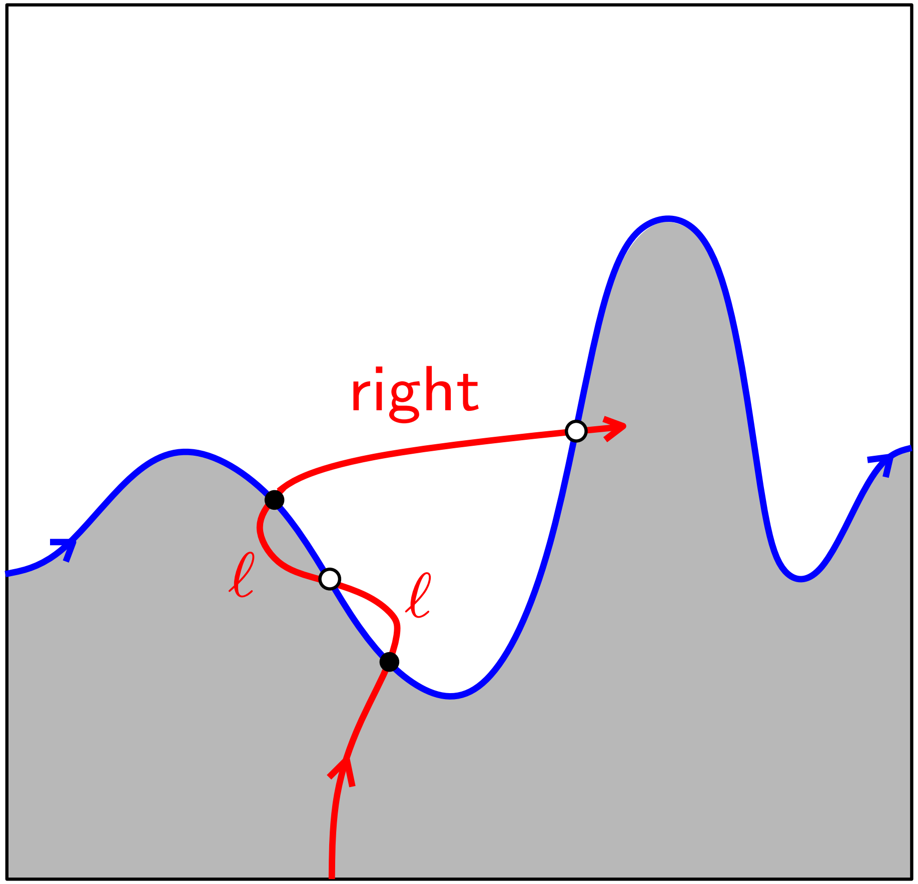

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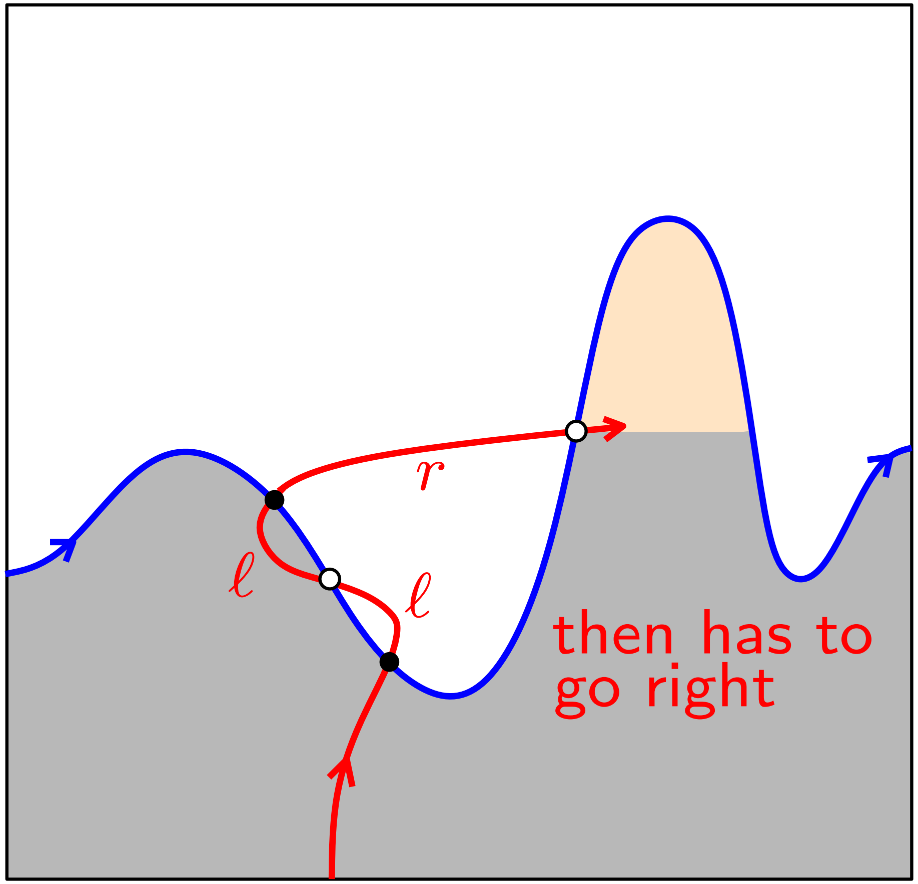

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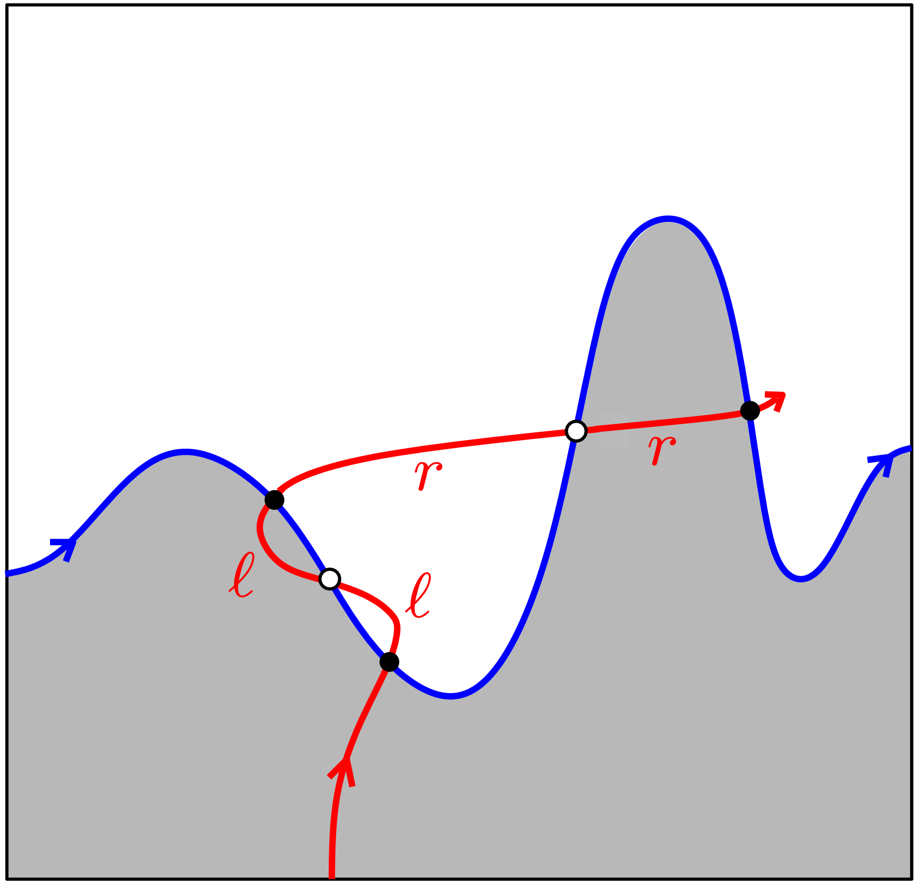

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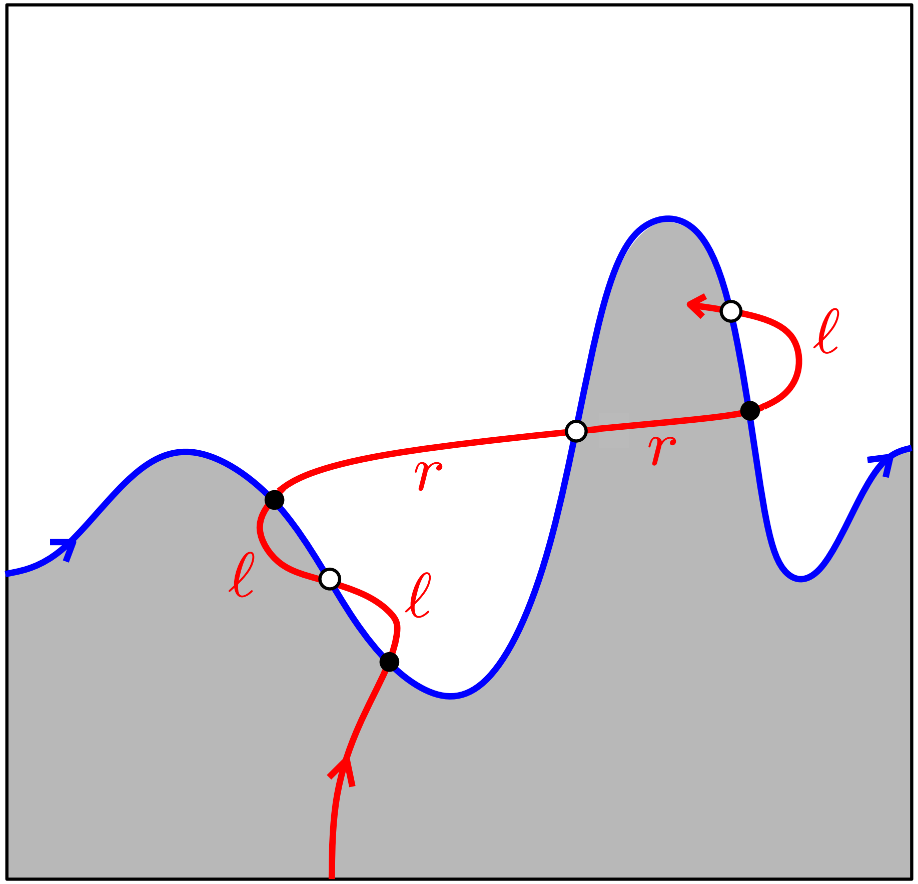

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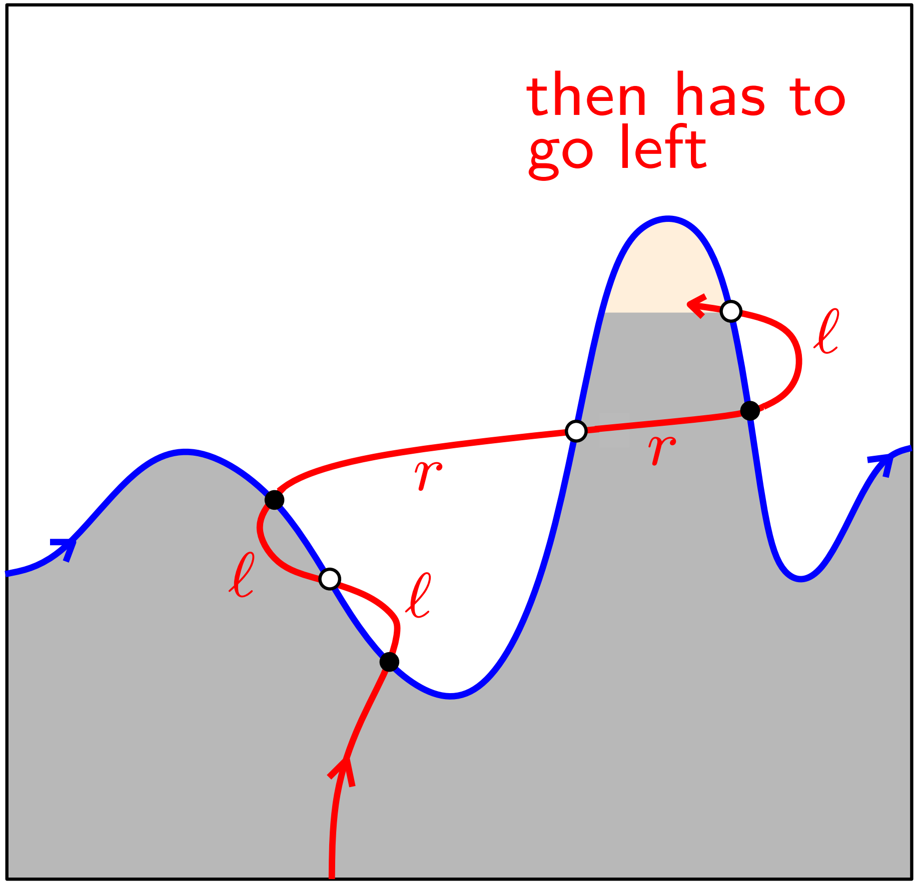

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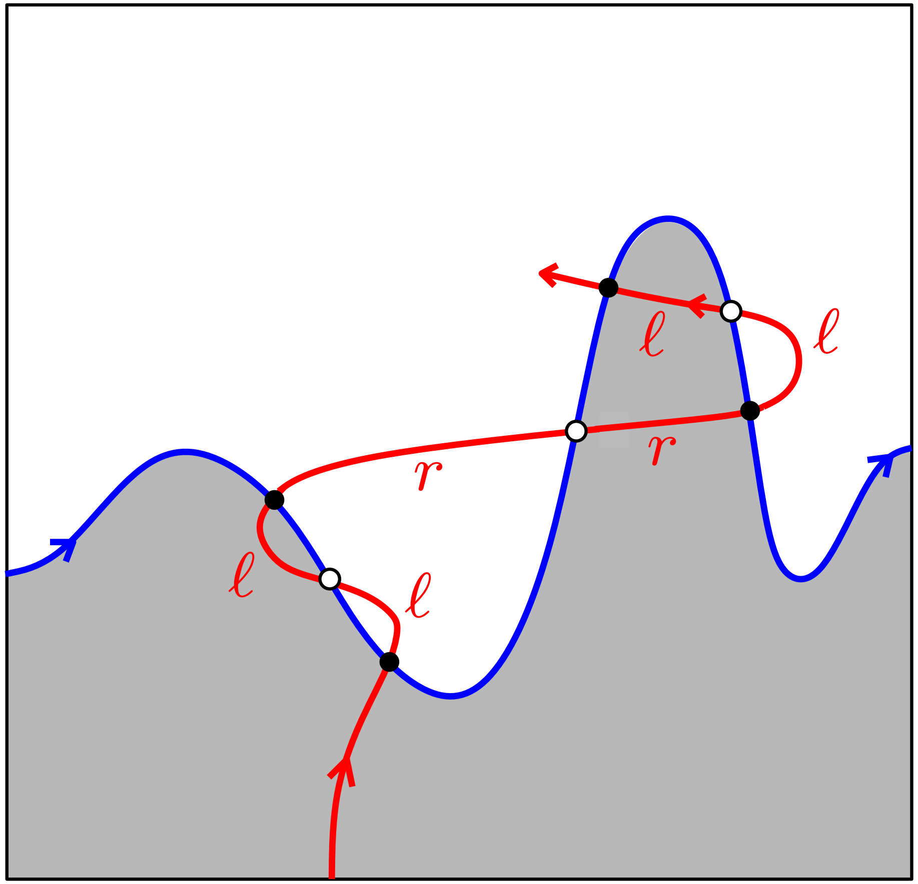

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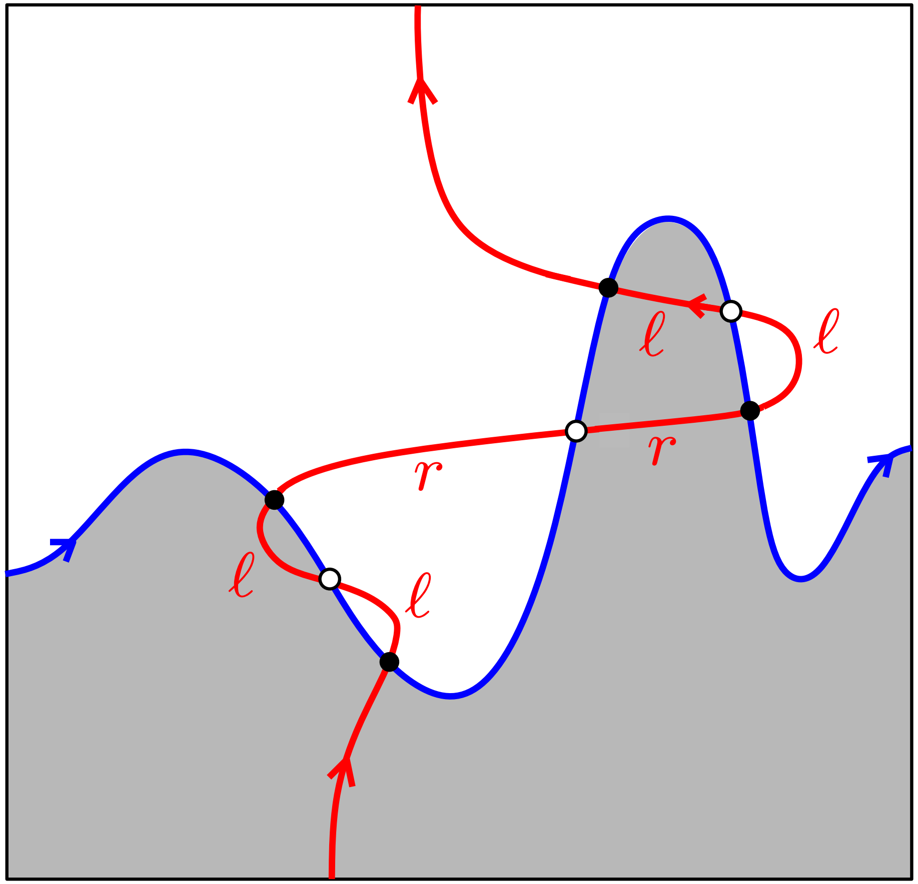

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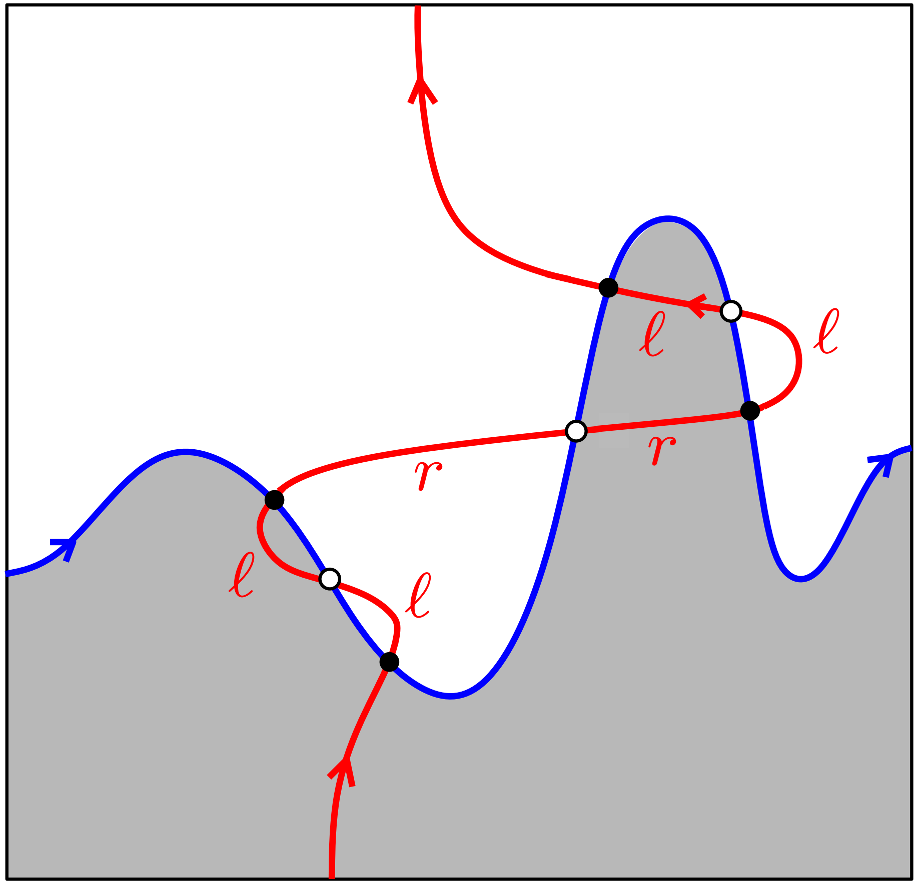

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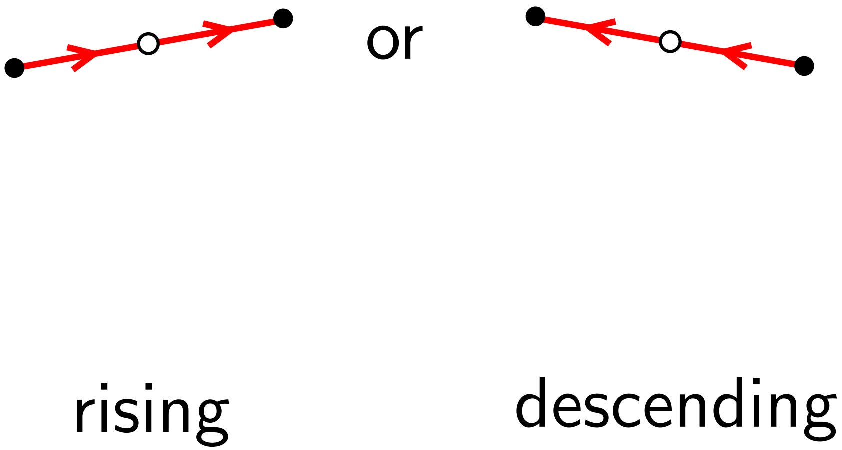

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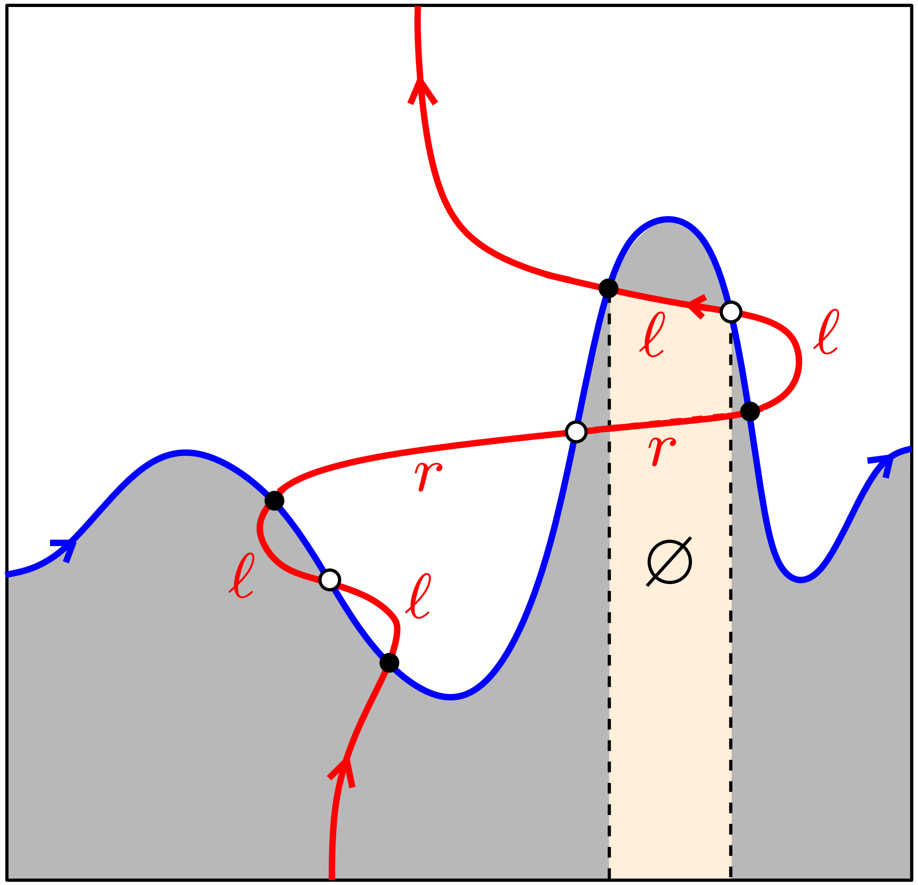

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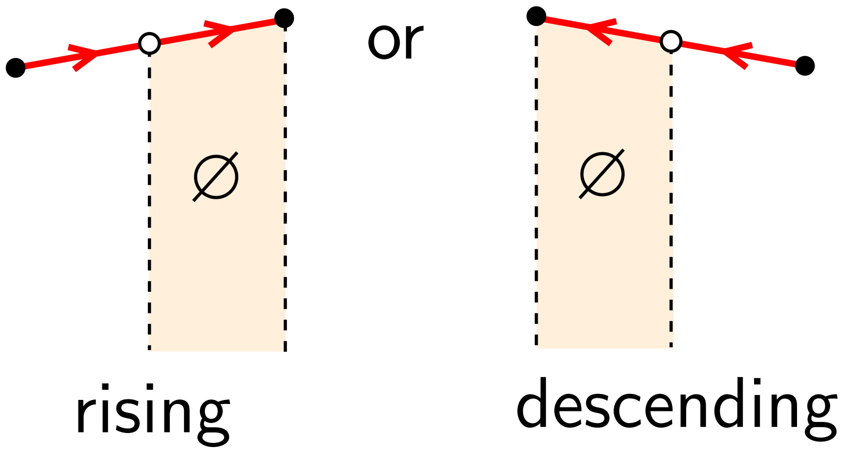

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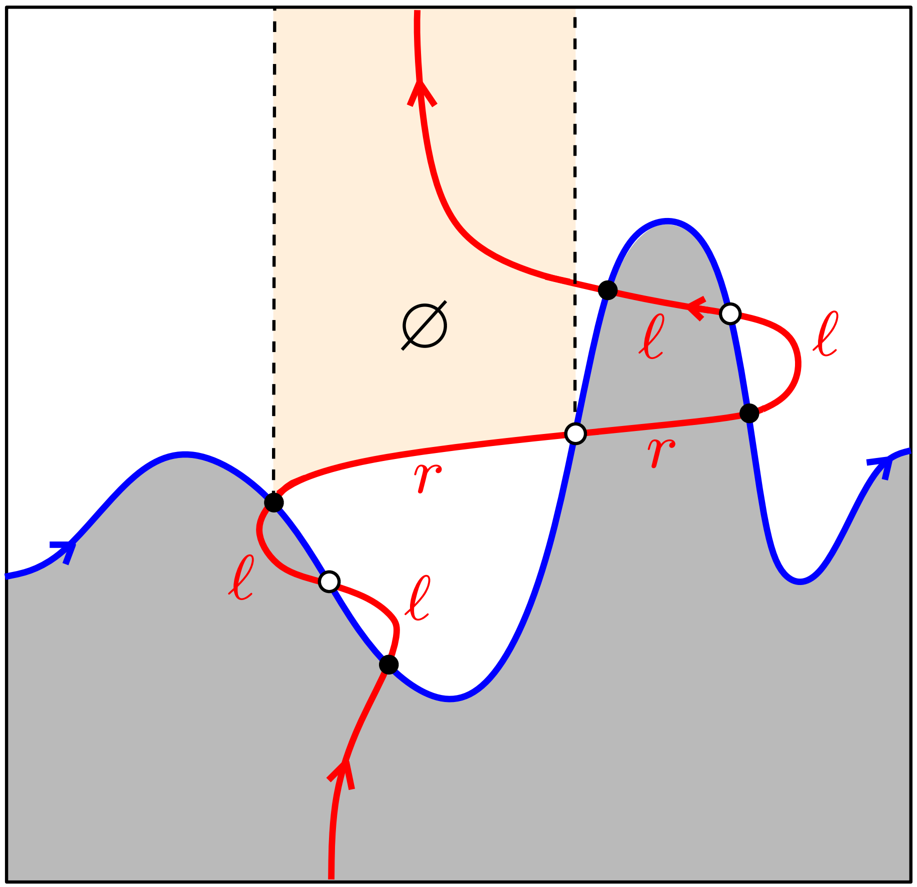

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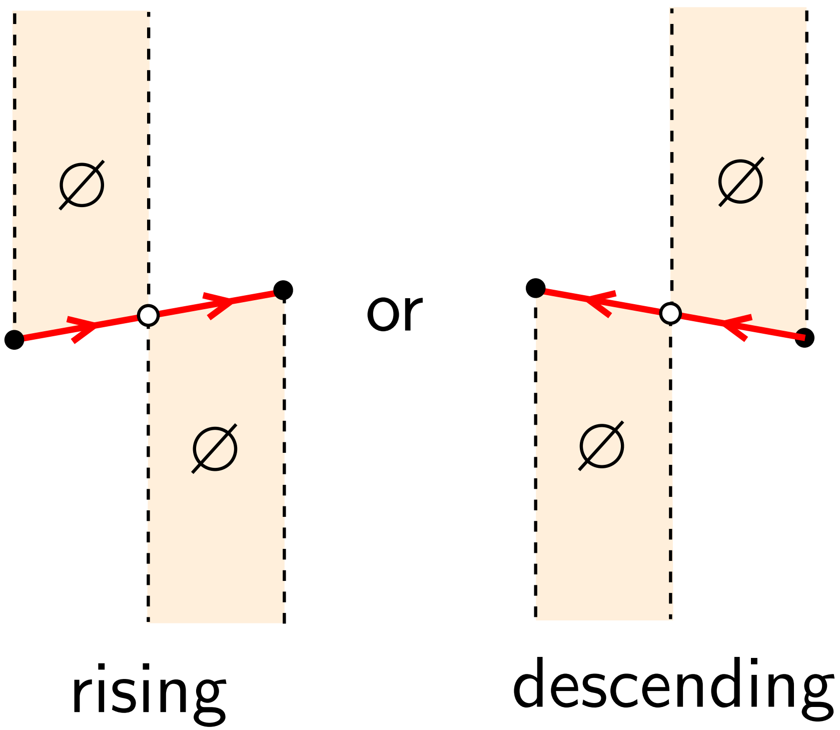

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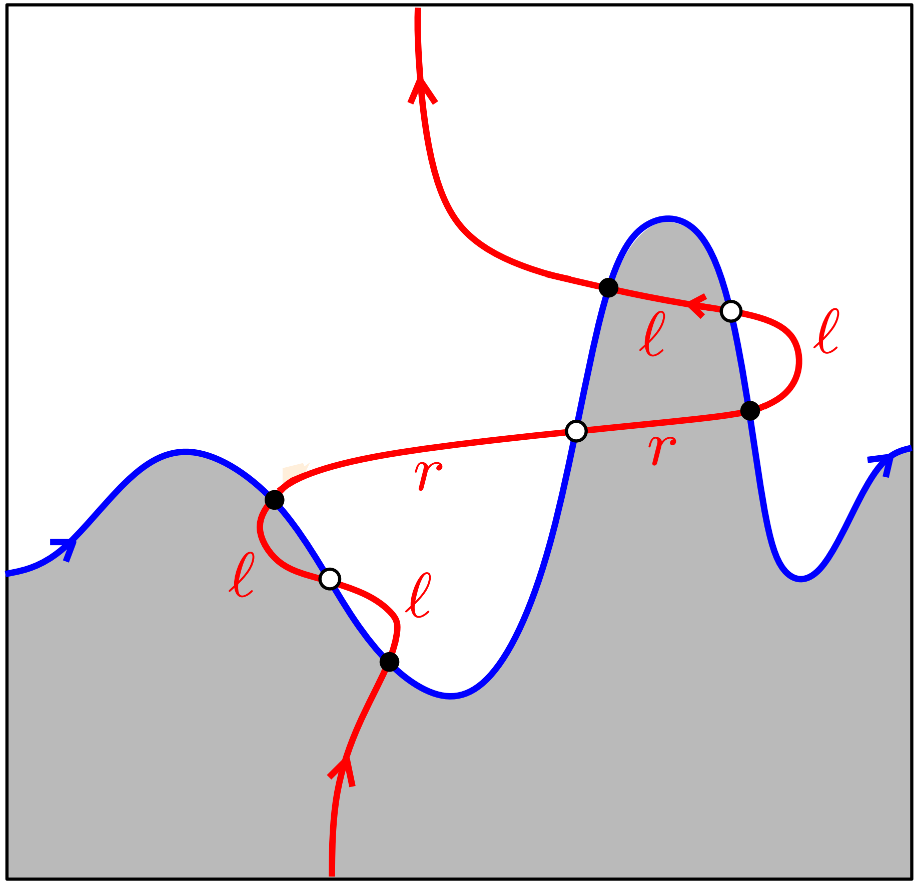

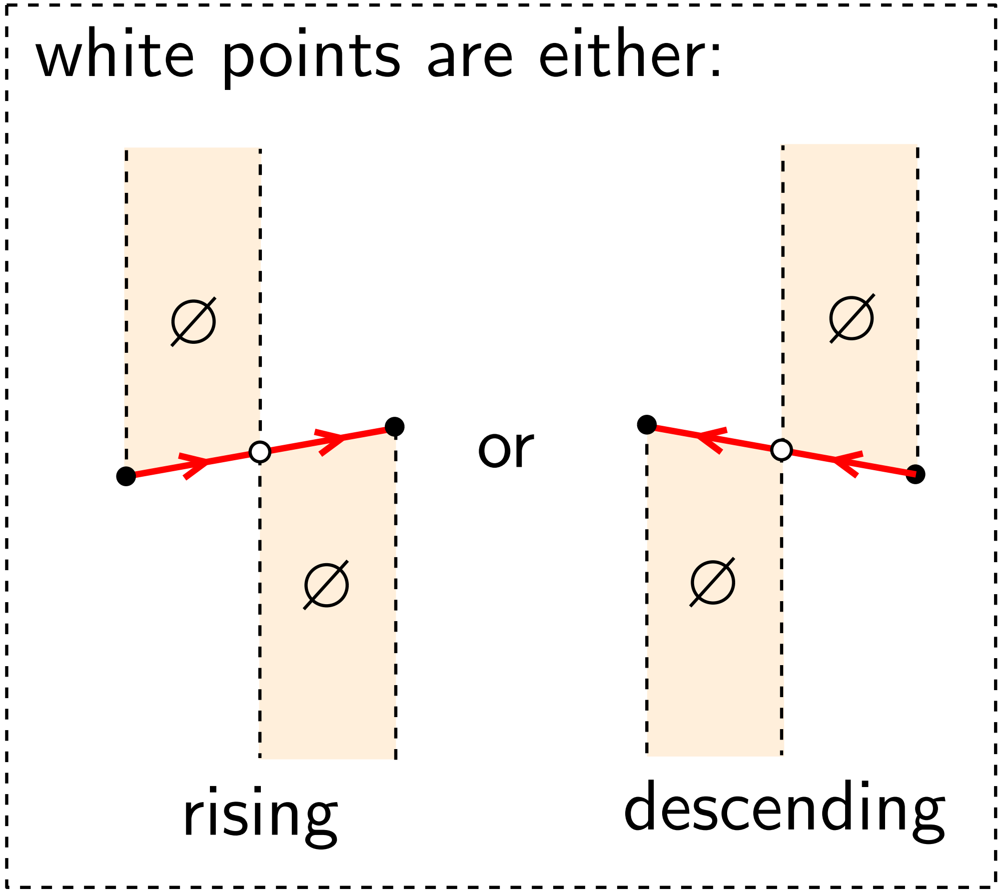

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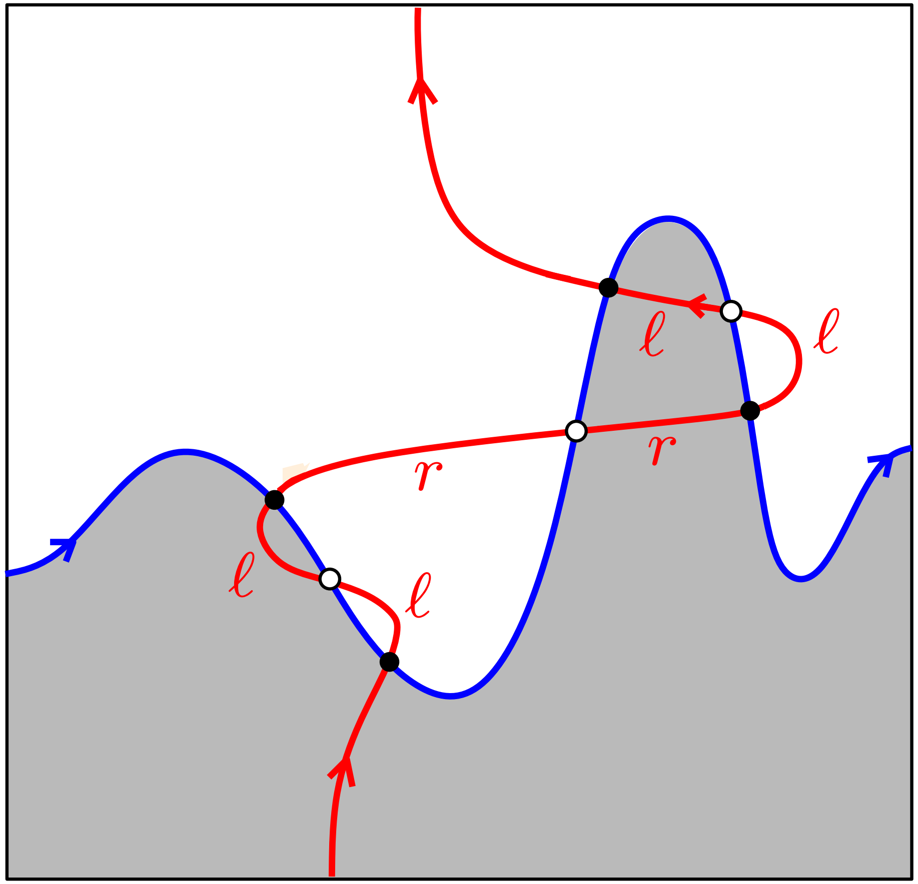

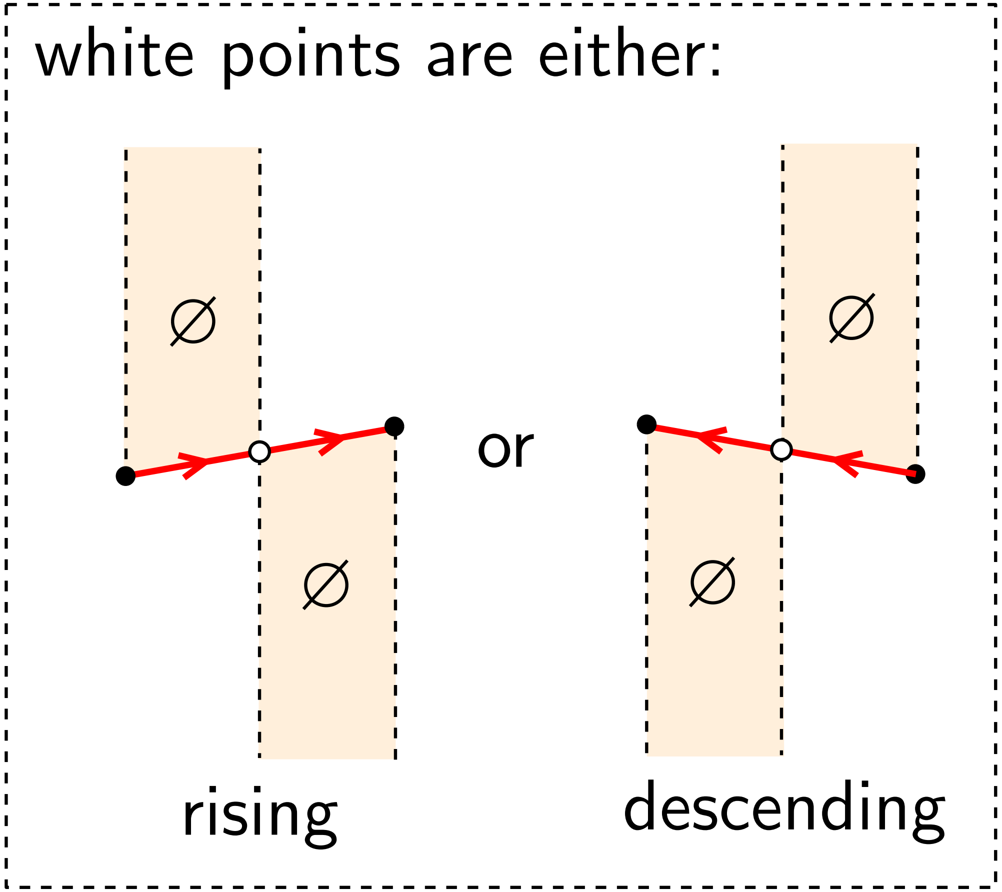

Permutations mapping even to even, odd to odd, and satisfying condition shown on the right are called complete Baxter permutations

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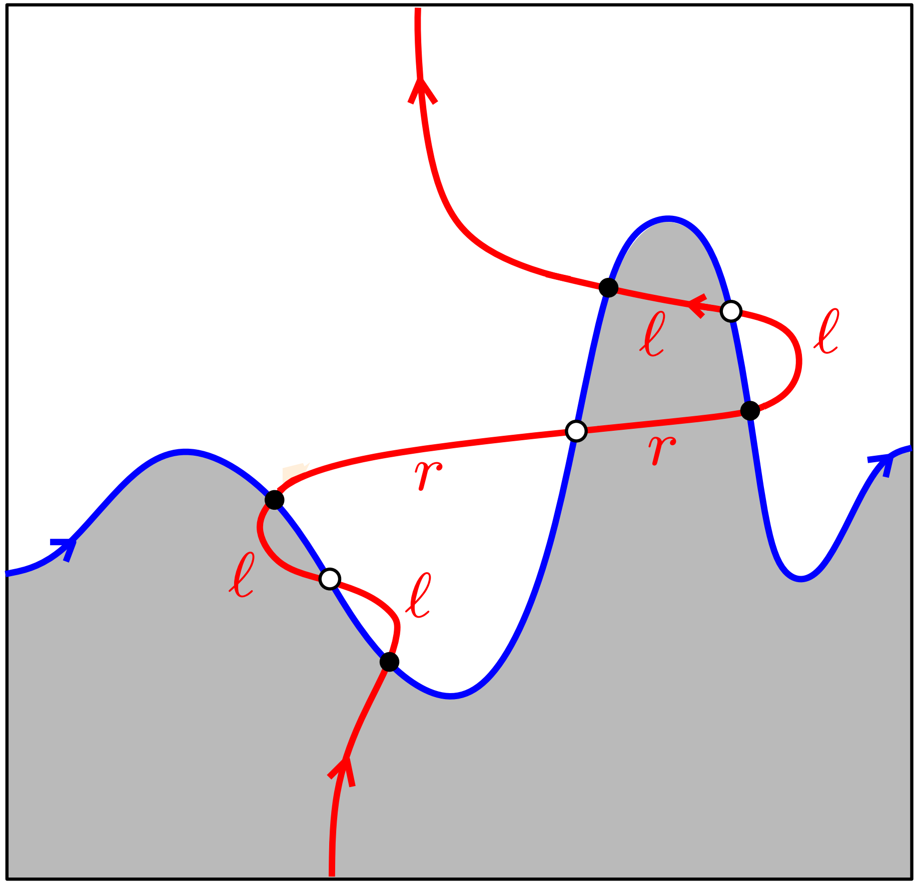

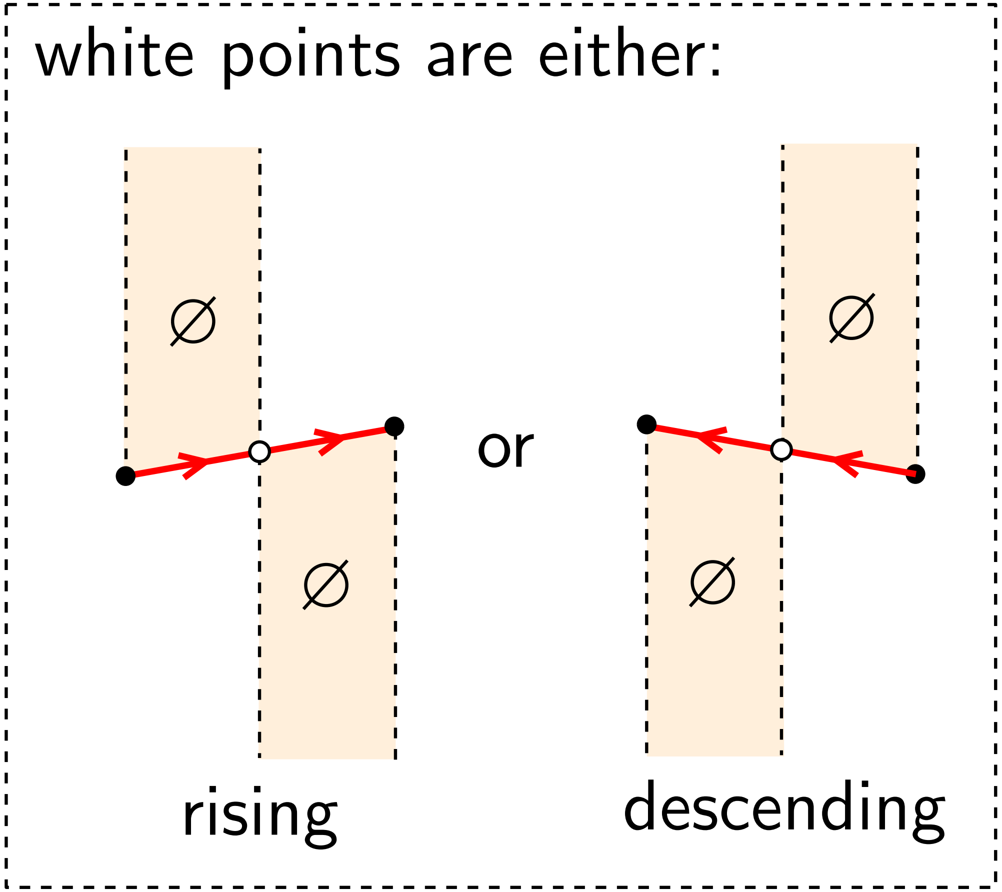

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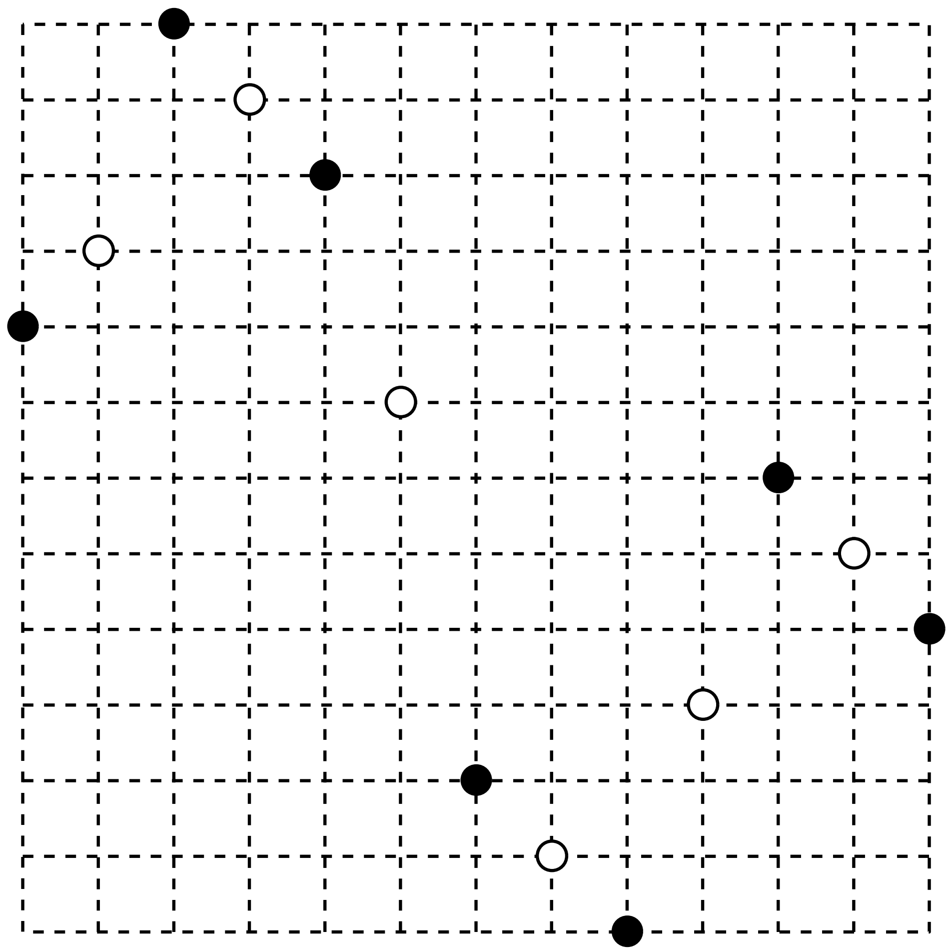
Theorem ([Boyce'81] reformulated bijectively):
Monotone 2-line meanders with $2n-1$ crossings are in bijection with complete Baxter permutations on $2n-1$ elements

Inverse construction

From a complete Baxter permutation to a monotone 2-line meander



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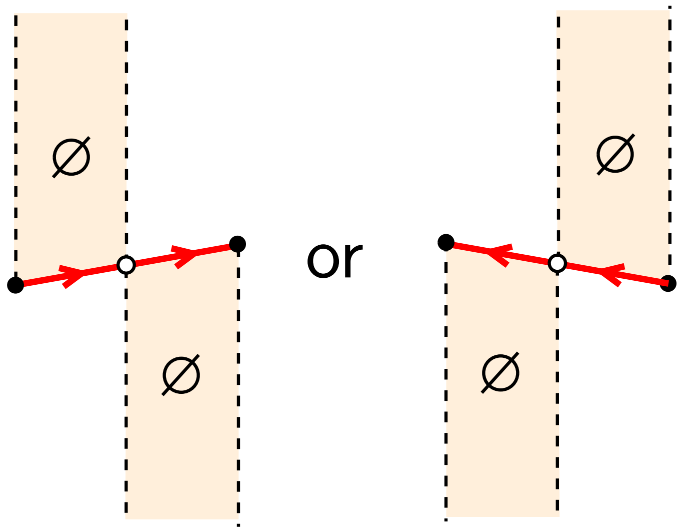

Inverse construction

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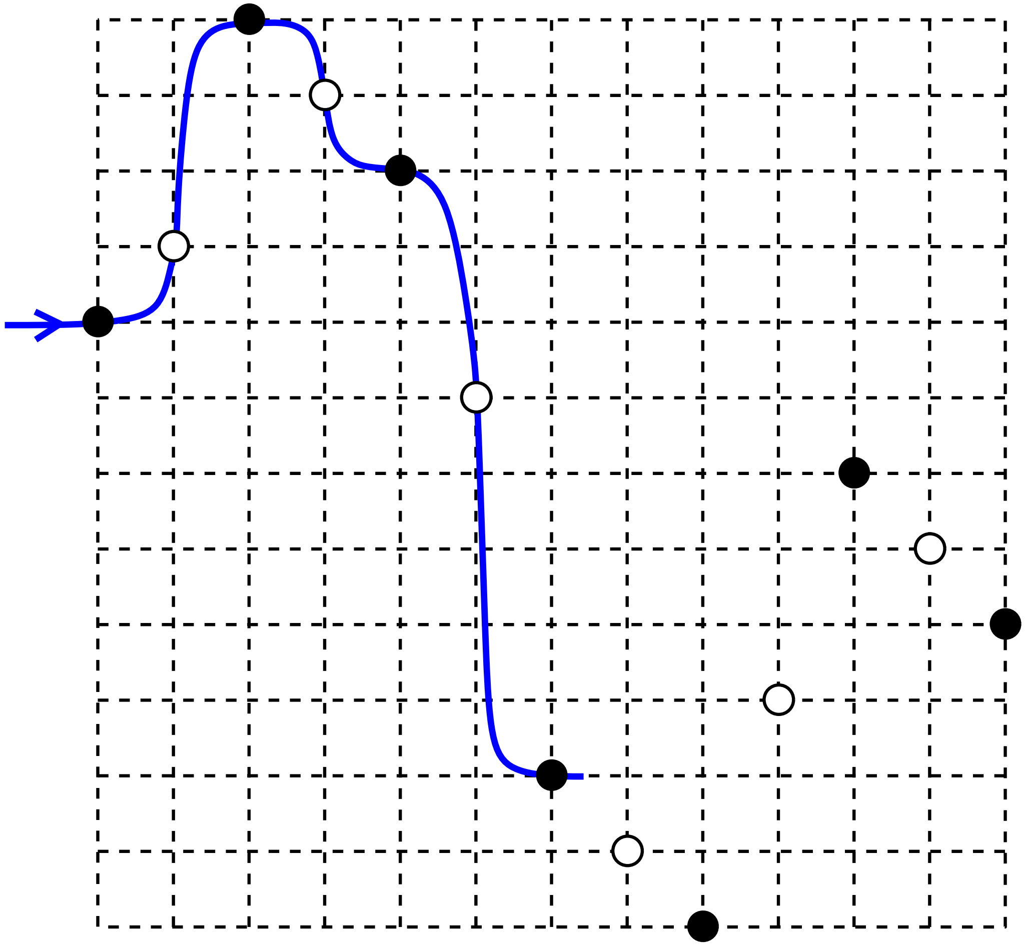

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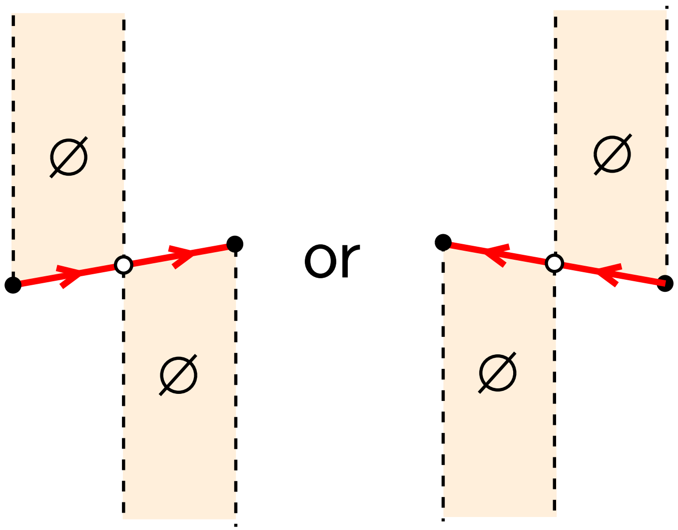

1) Draw the blue curve

Inverse construction

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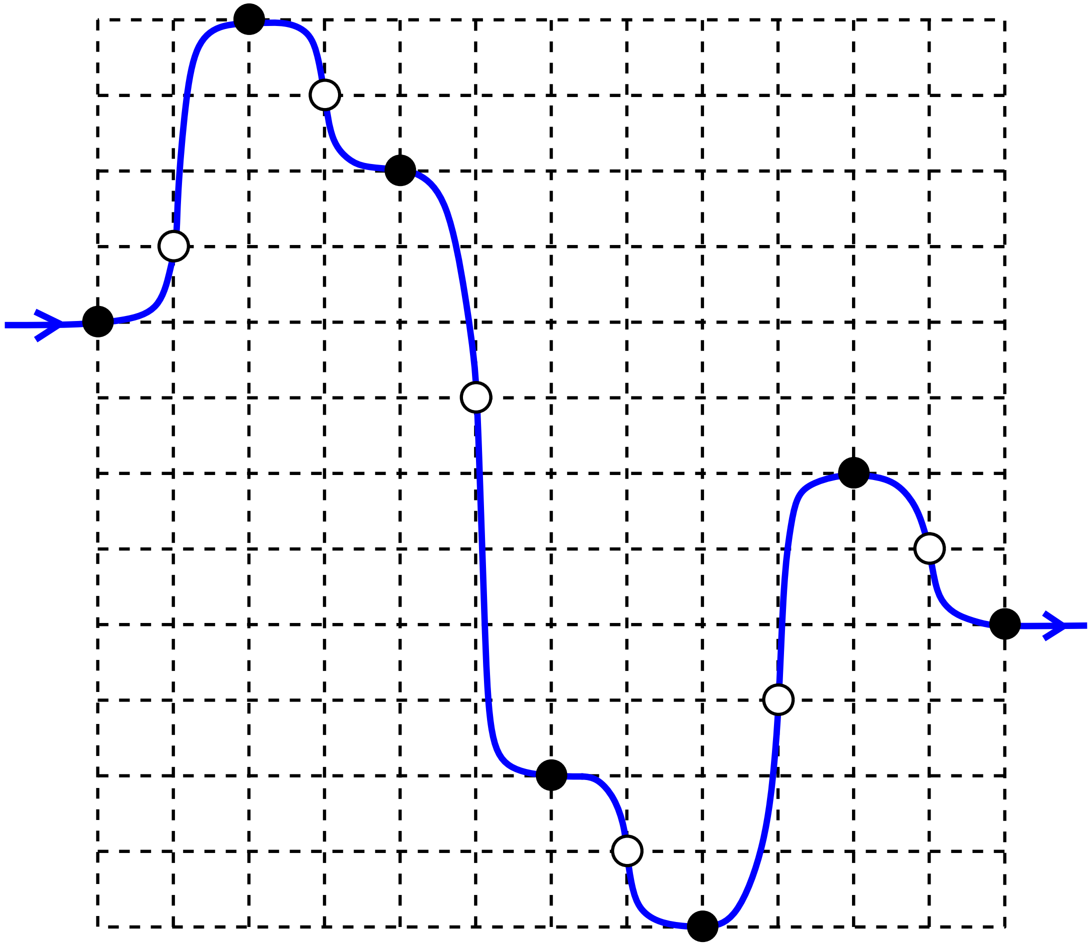

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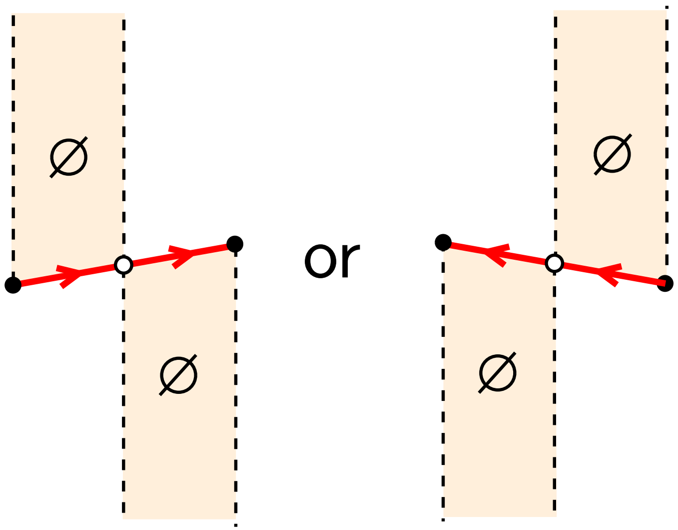

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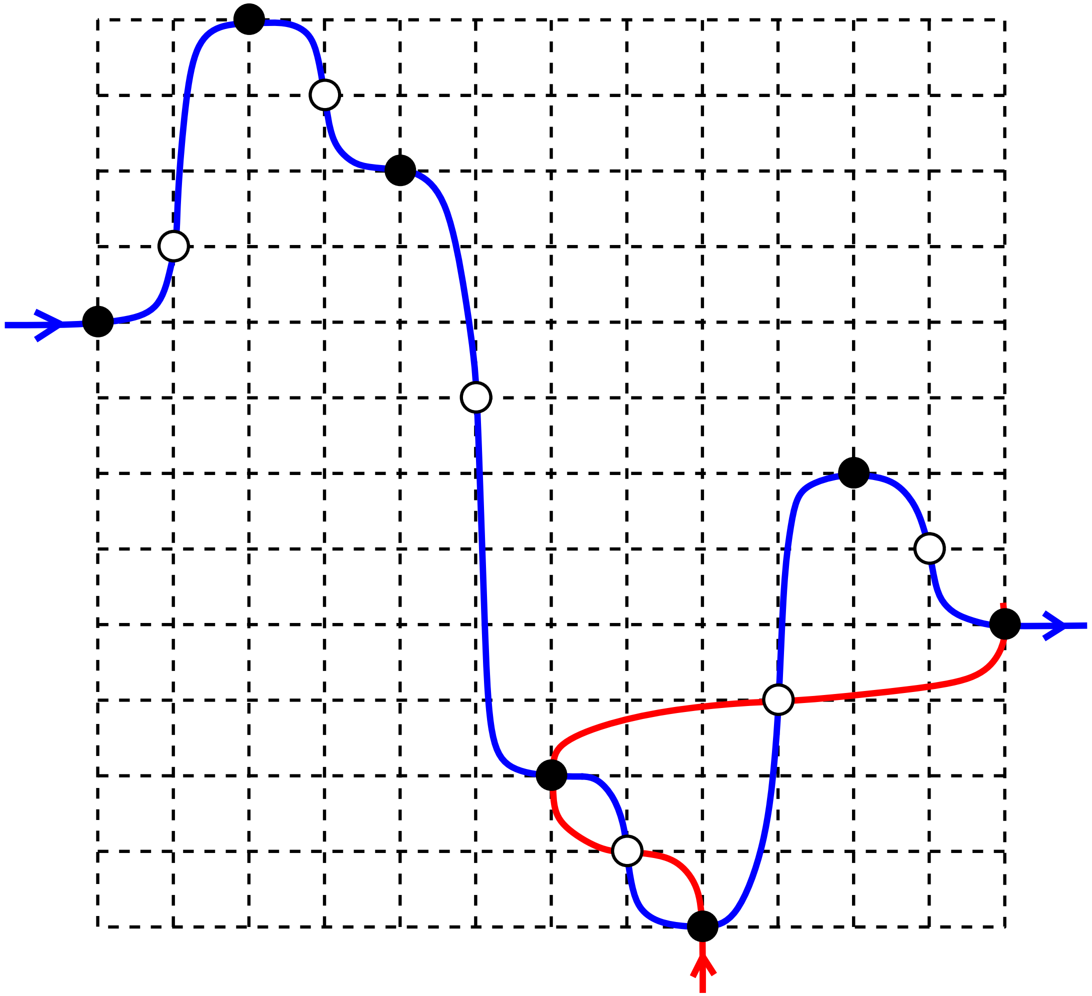

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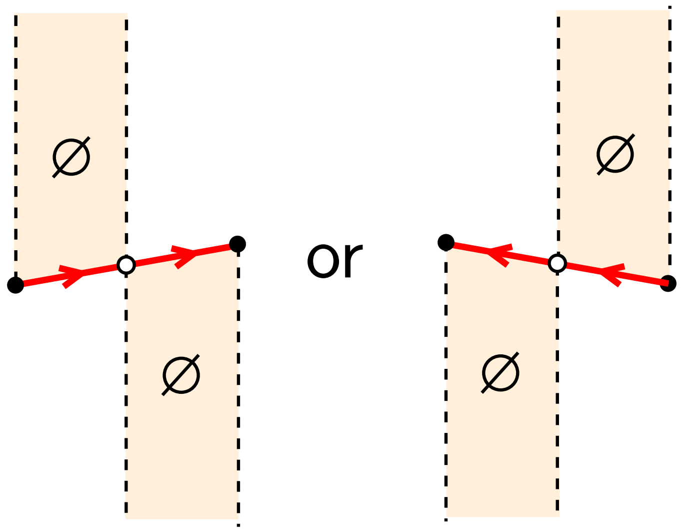

1) Draw the blue curve

2) Draw the red curve

Inverse construction

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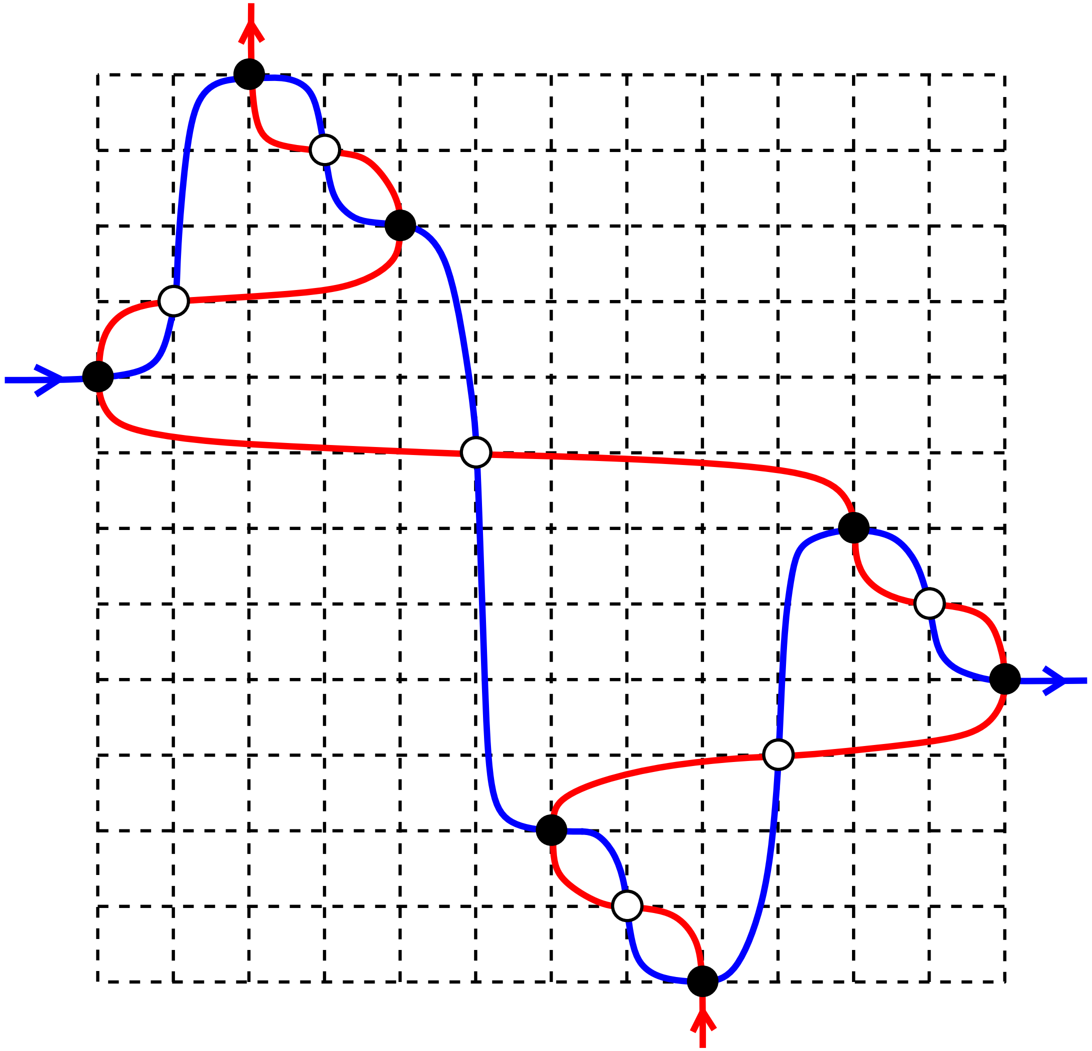

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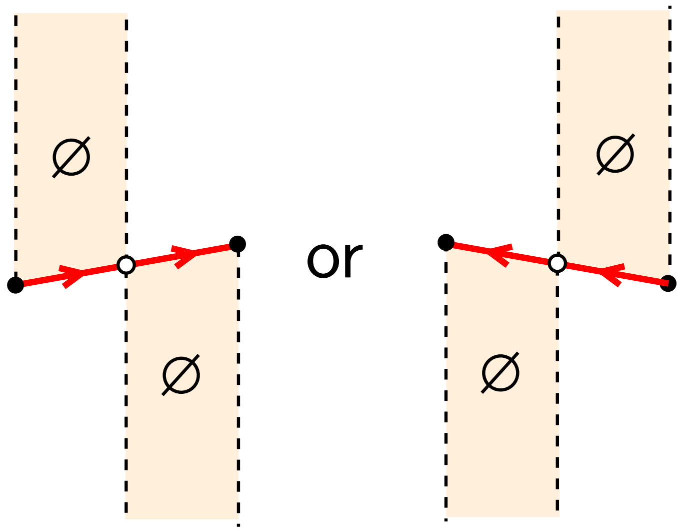

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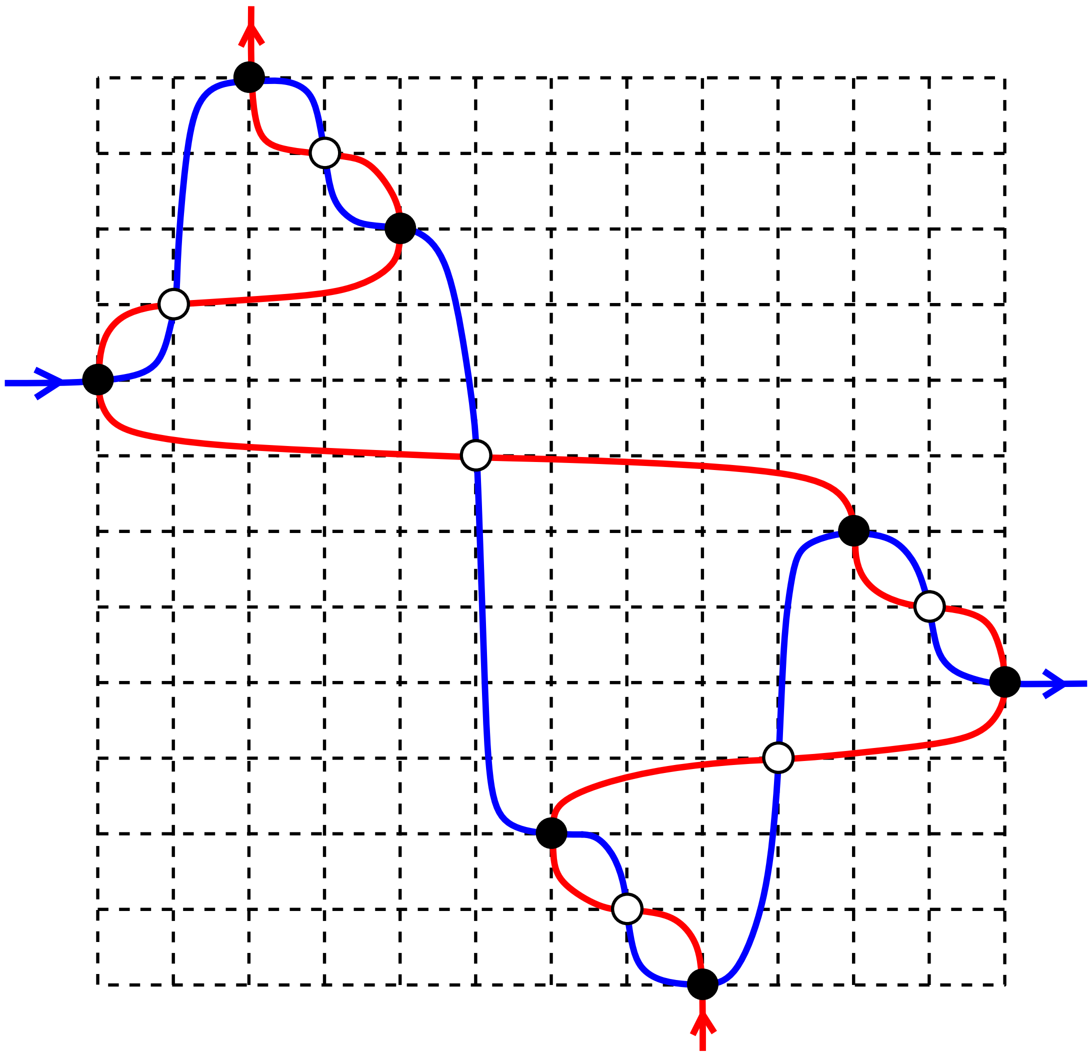

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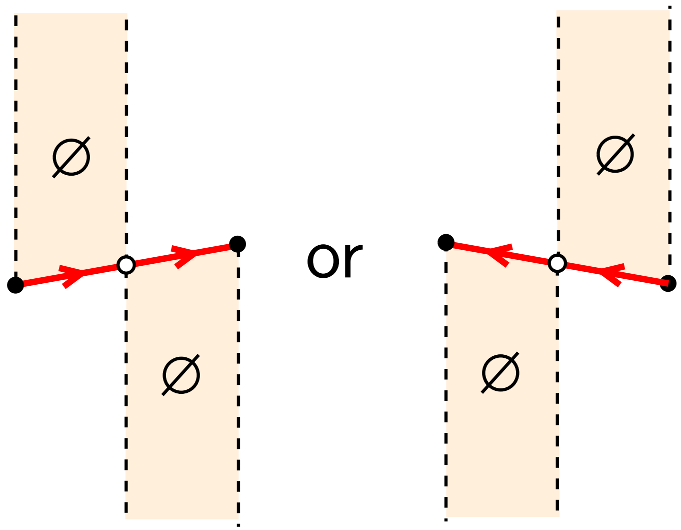

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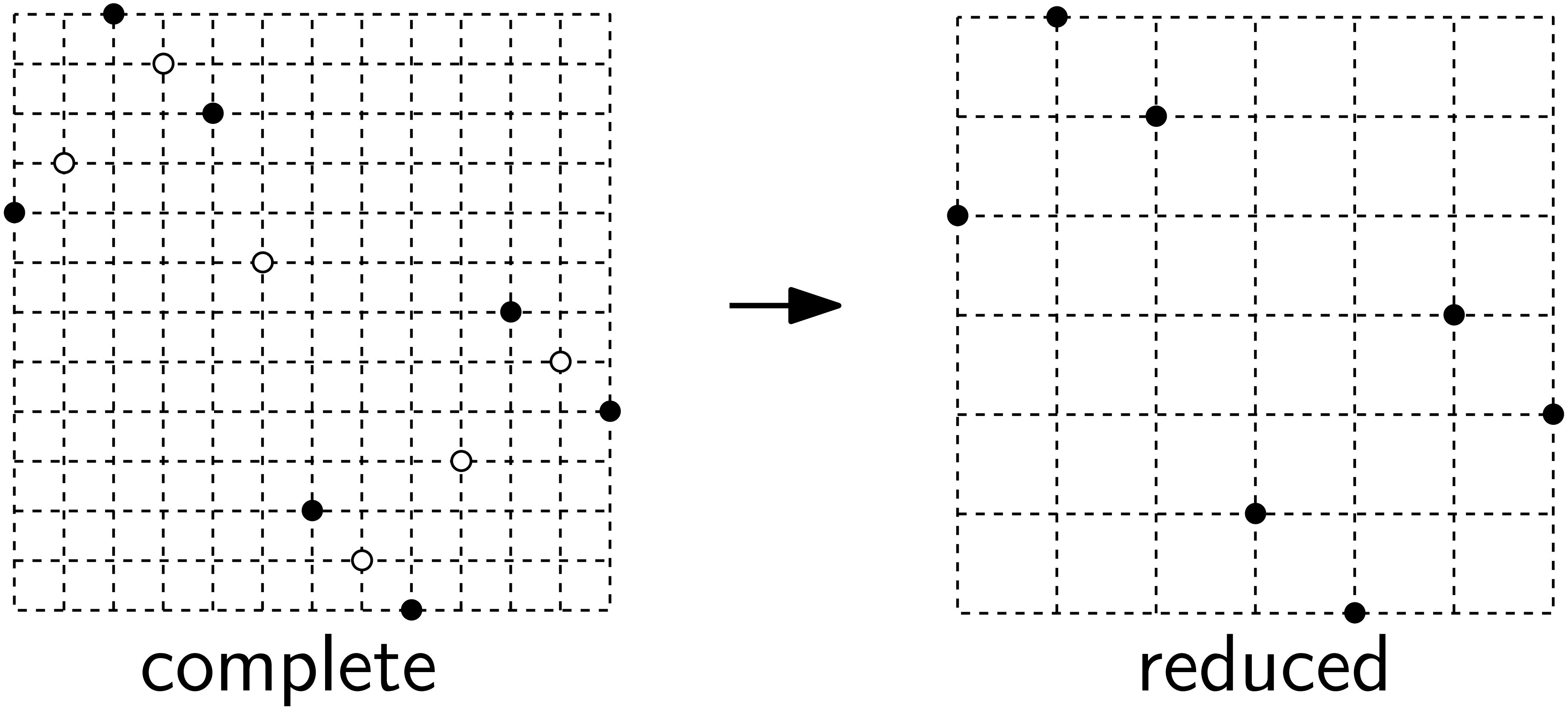
The two curves meet only at the permutation points
(because of the empty area-property at white points)

Complete and reduced Baxter permutations



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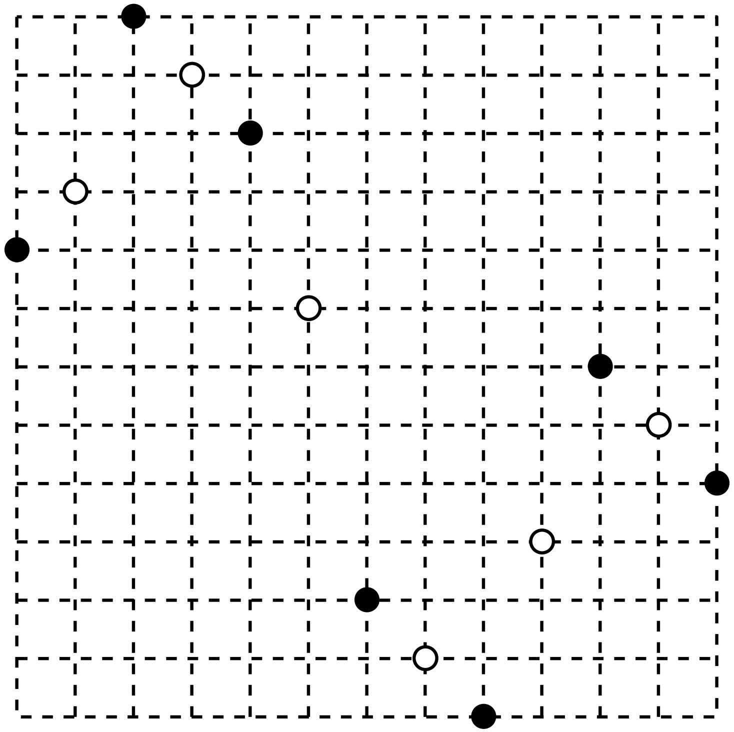
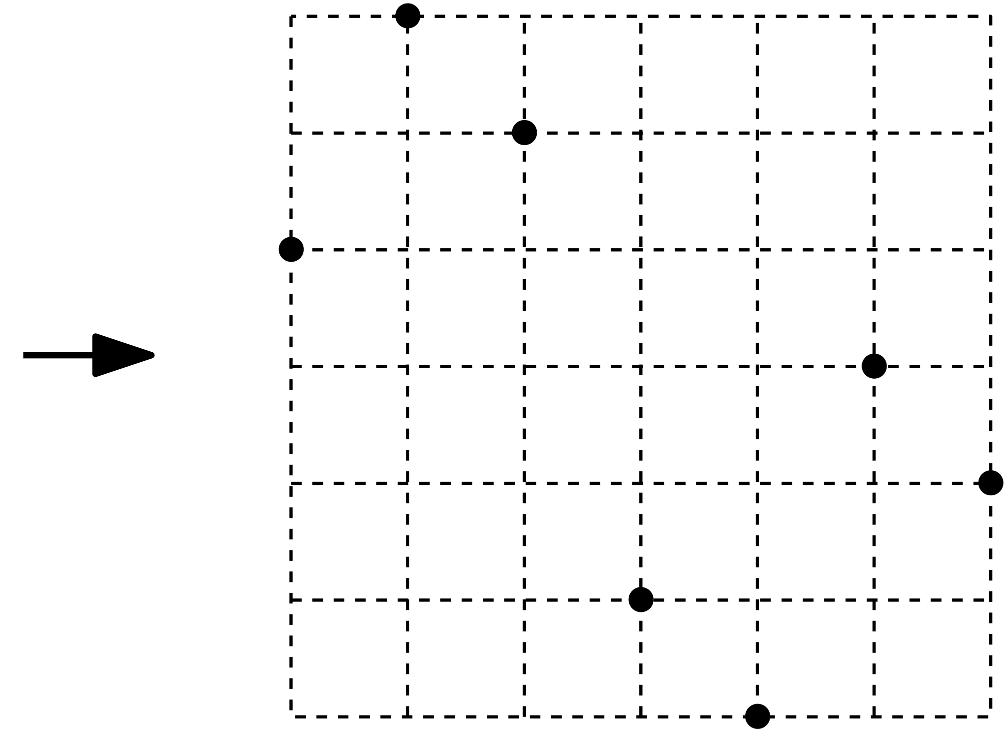


complete



reduced

- complete one can be recovered from reduced one

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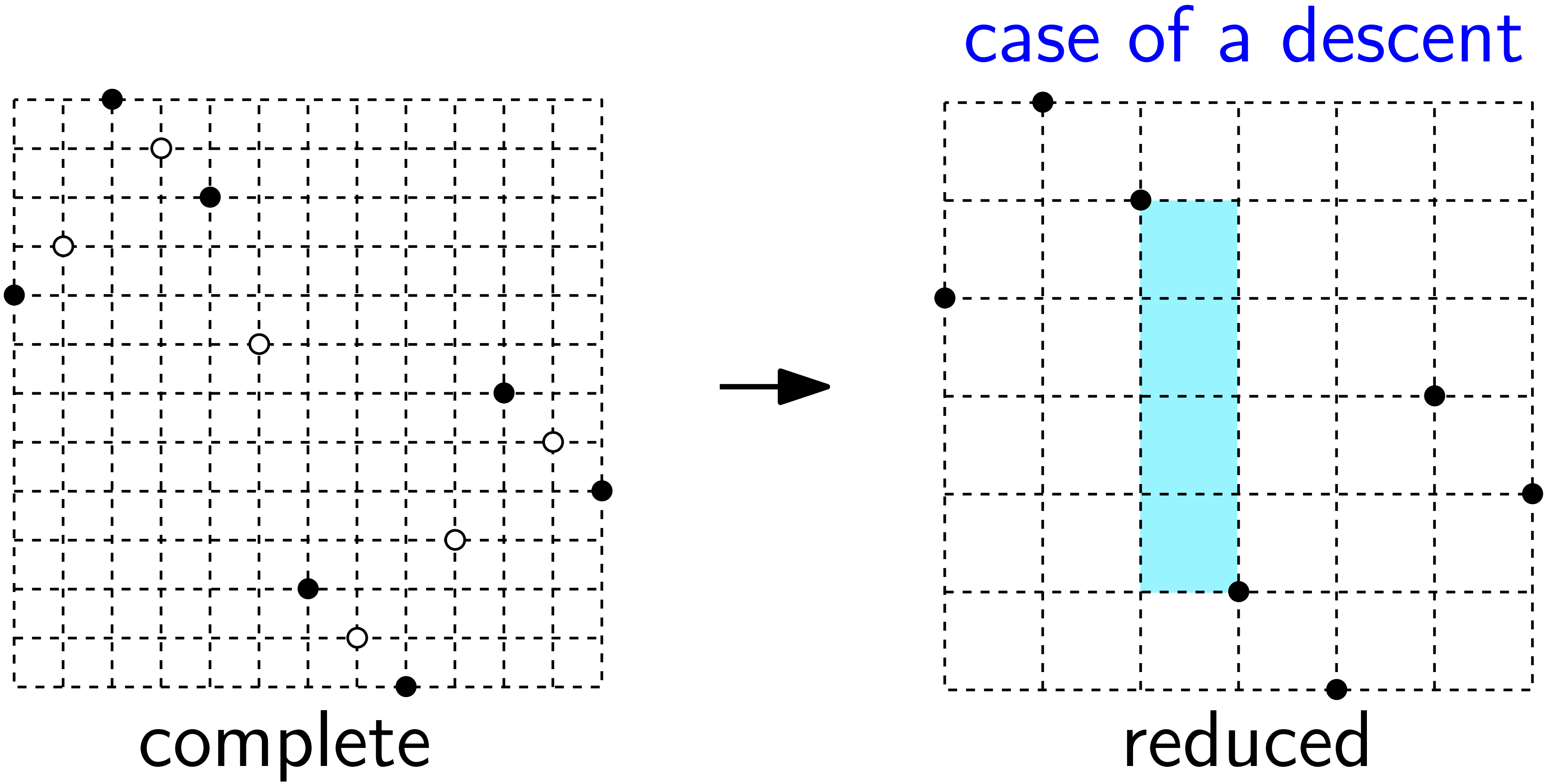

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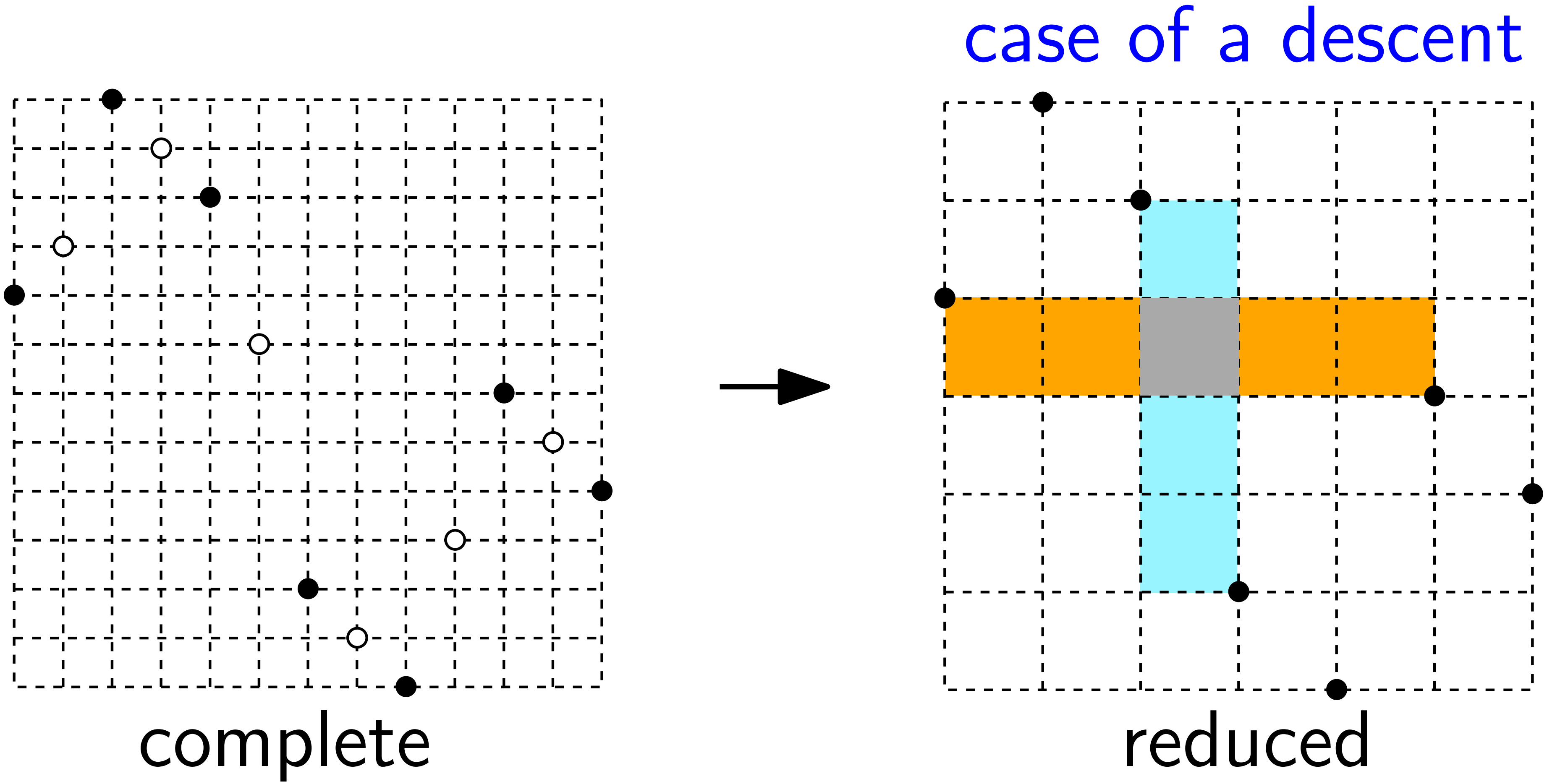
Complete and reduced Baxter permutations



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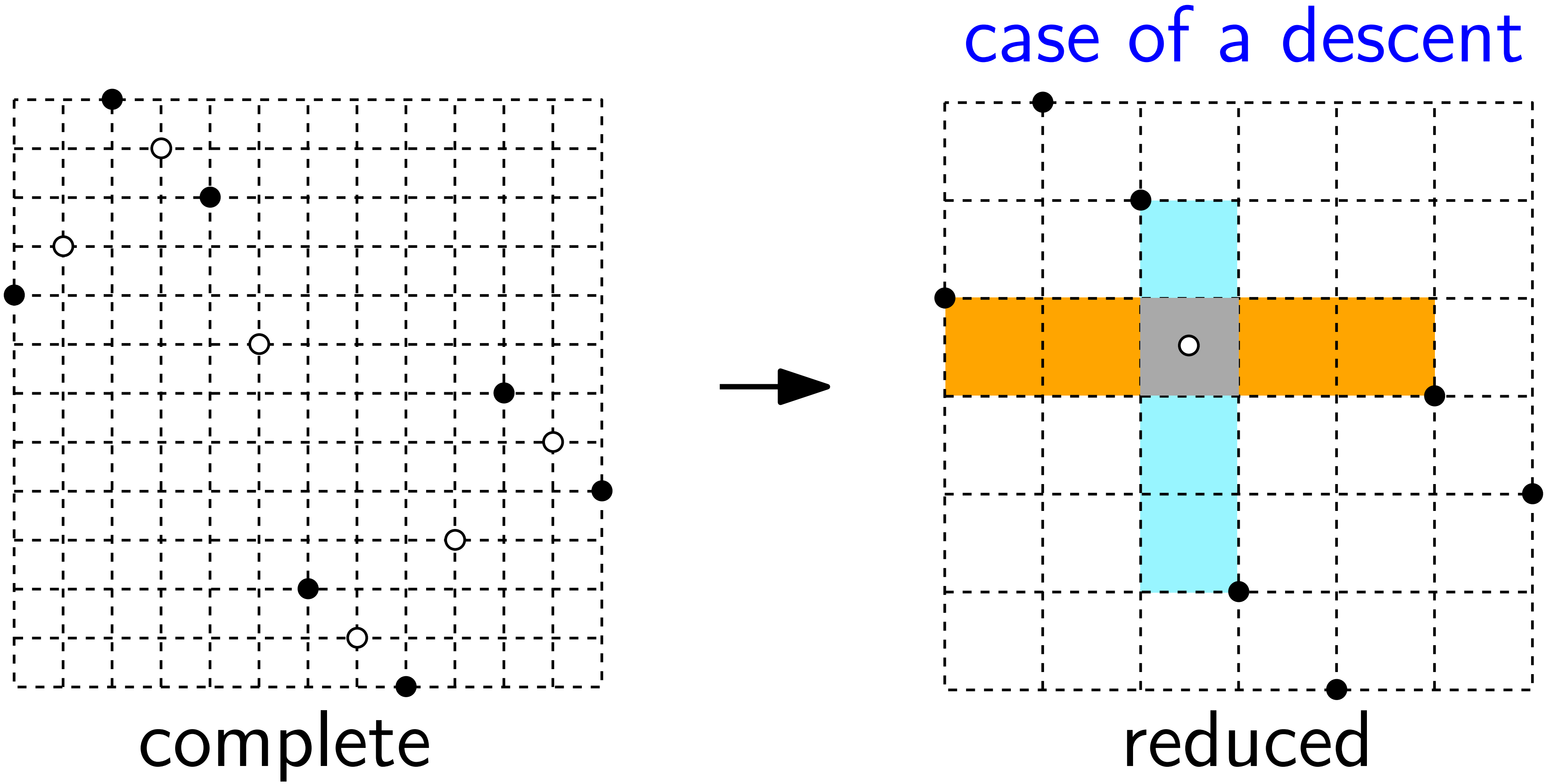

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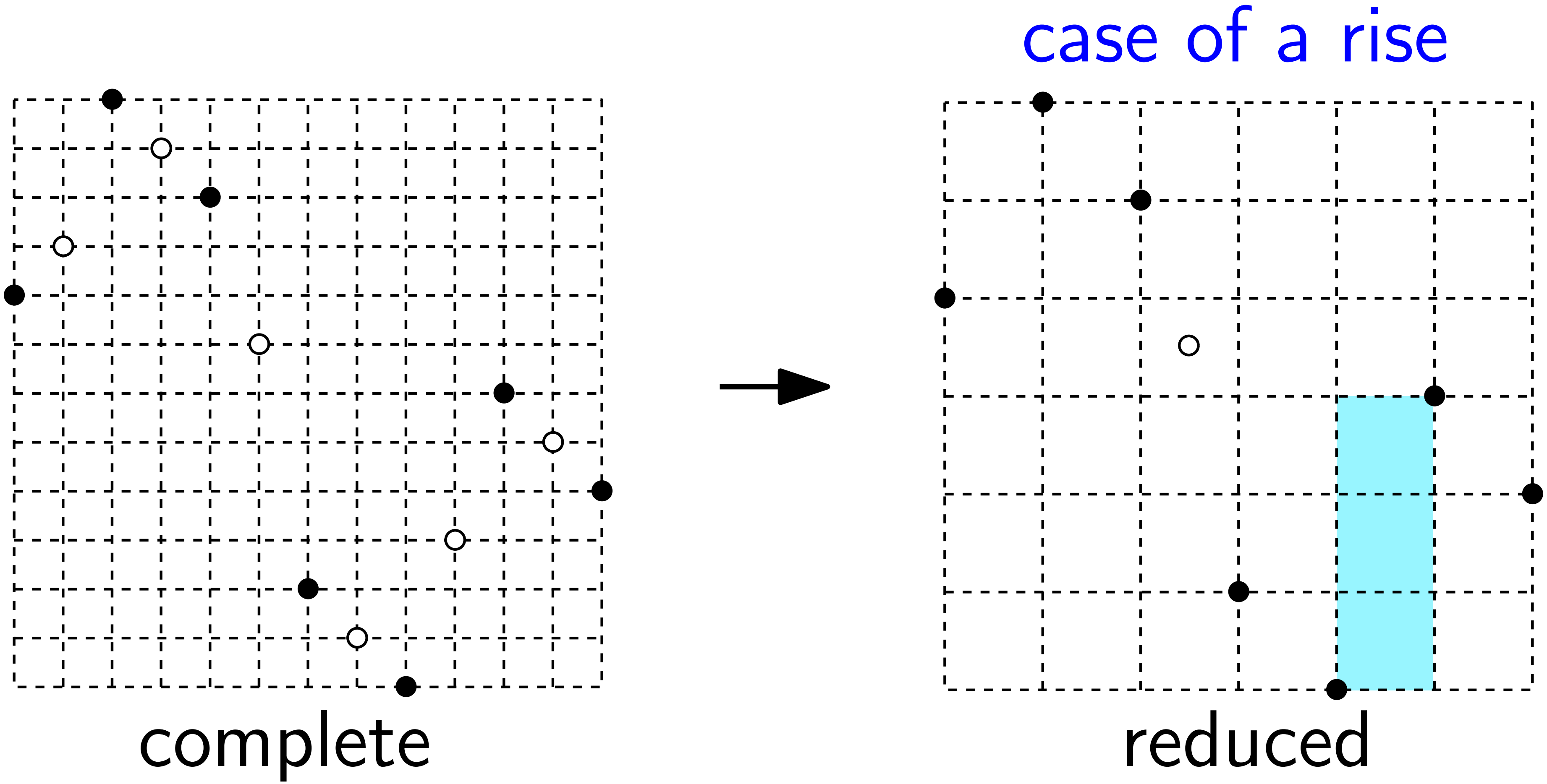

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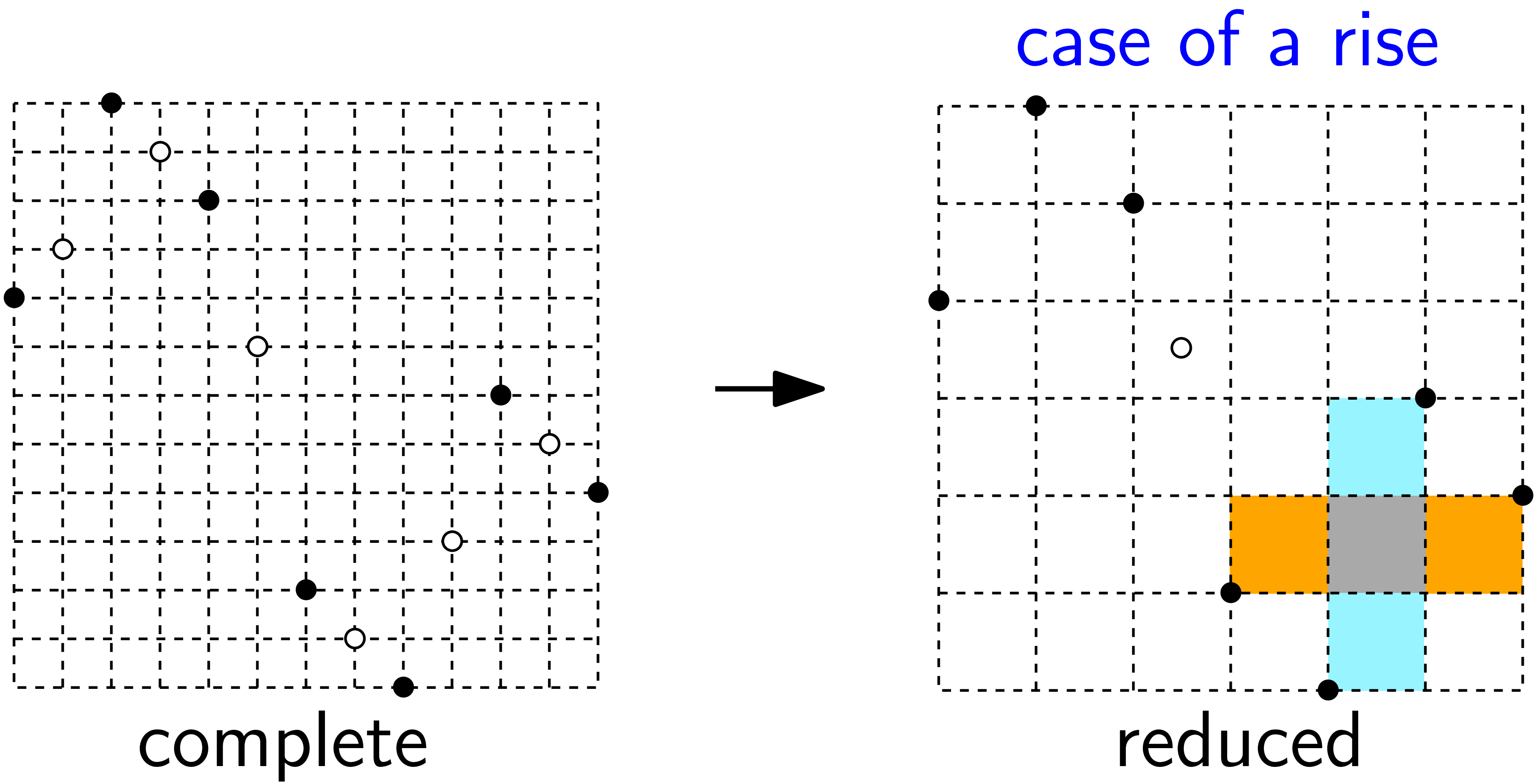

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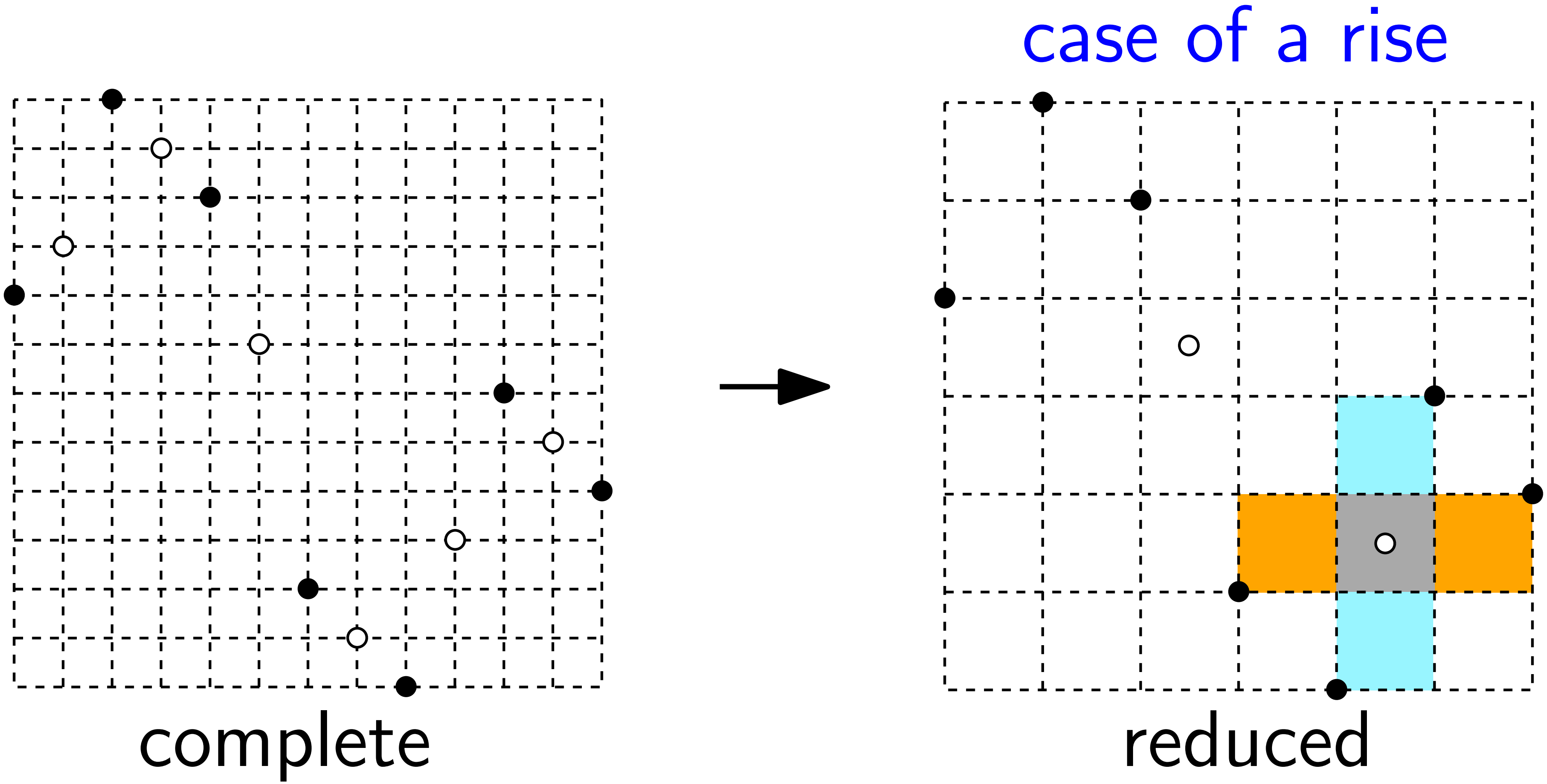

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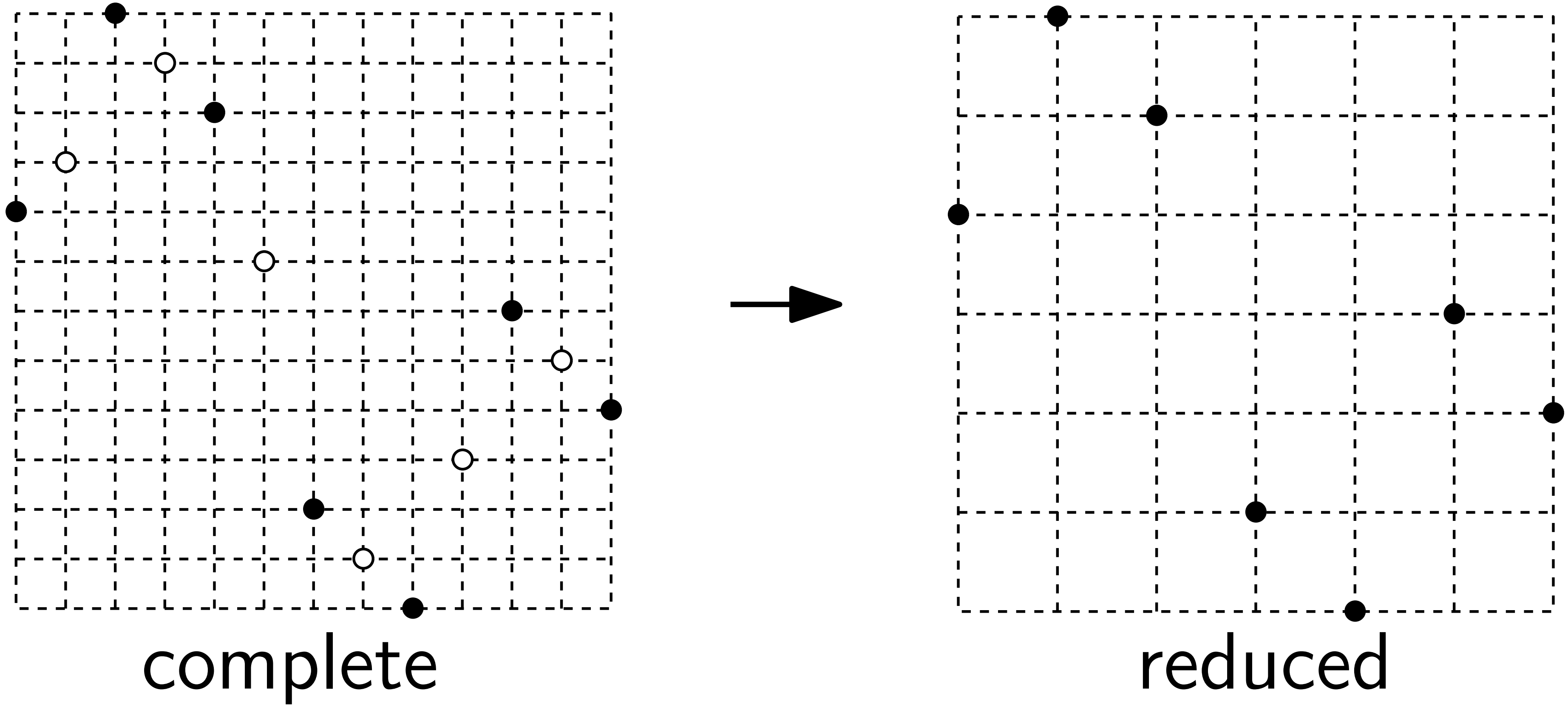

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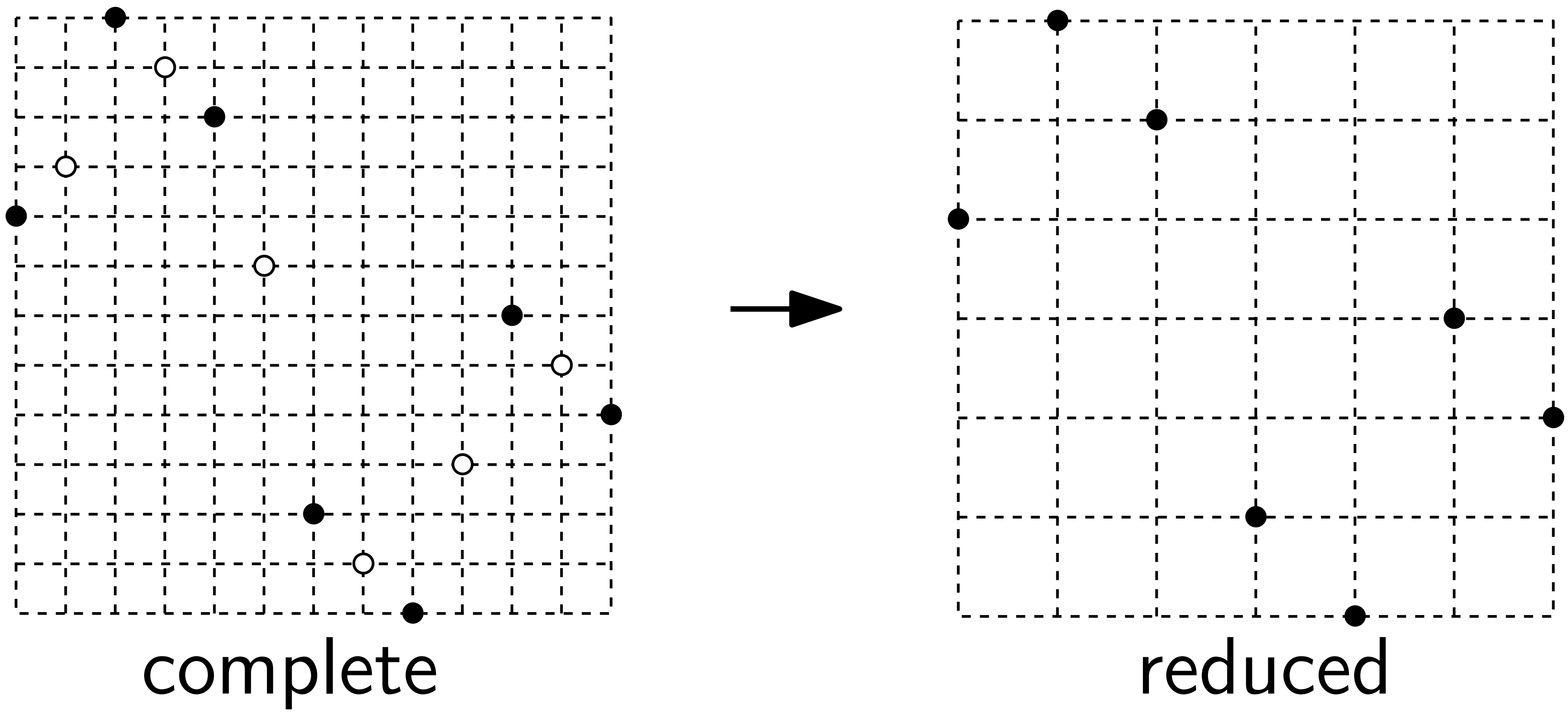
$$2-41-3 \text{ and } 3-14-2$$

Complete and reduced Baxter permutations



- complete one can be recovered from reduced one
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$$2-41-3 \text{ and } 3-14-2$$

- permutation on white points (called anti-Baxter) is characterized by forbidden patterns

$$2-14-3 \text{ and } 3-41-2$$

Counting results

- Baxter permutations
  - Number of reduced Baxter permutations with n elements

$$b_n = \sum_{r=0}^{n-1} \frac{2}{n(n+1)^2} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2}$$

[Chung et al'78] [Mallows'79]

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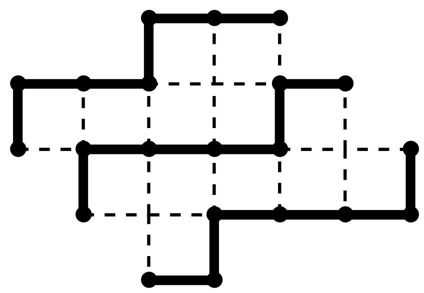
$5\ 7\ 6\ 2\ 1\ 4\ 3 \quad \Leftrightarrow$



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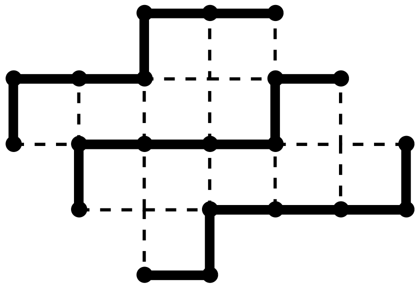
5 7 6 2 1 4 3 $\Leftrightarrow$ 

- Subfamilies
  - alternating **[Cori-Dulucq-Viennot'86], [Dulucq-Guibert'98]**

    $\mathrm{Cat}_k \mathrm{Cat}_k$ if $n = 2k$ $\mathrm{Cat}_k \mathrm{Cat}_{k+1}$ if $n = 2k+1$
  - doubly alternating **[Guibert-Linusson'00]**

    Cat_k where $k = \lfloor n/2 \rfloor$

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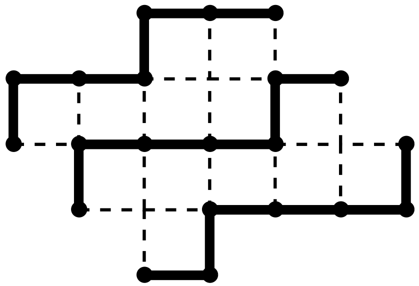

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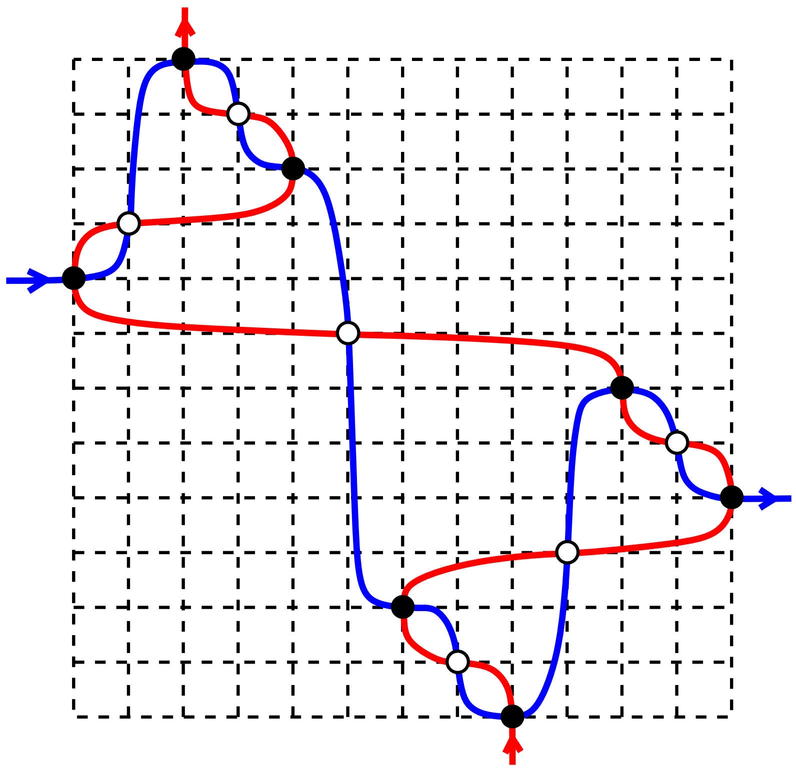
- anti-Baxter permutations

[Asinowski et al'10]

$$a_n = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} (-1)^i \binom{n+1-i}{i} b_{n+1-i}$$

Local conditions for monotone 2-line meanders



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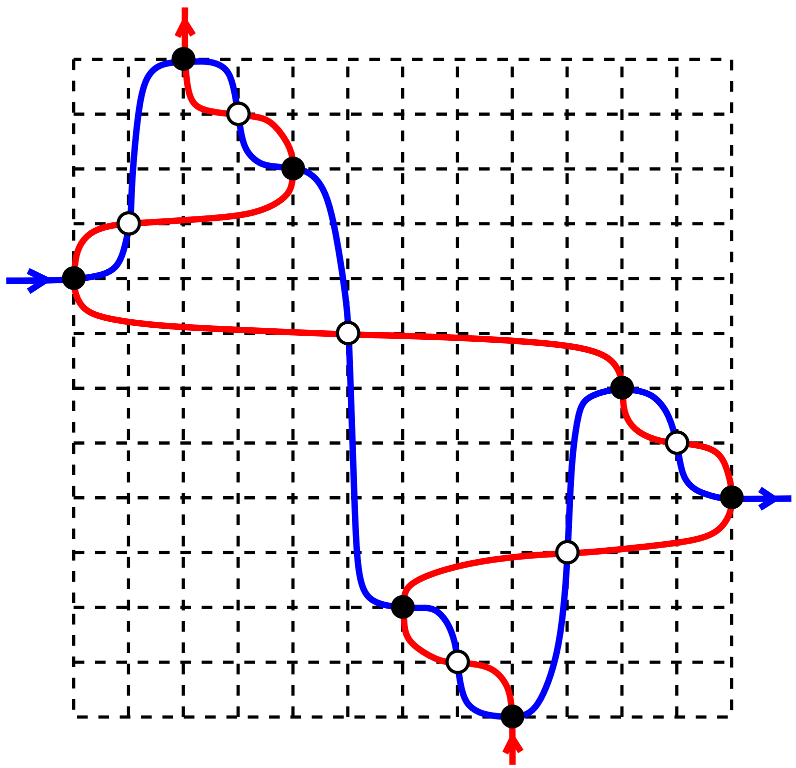
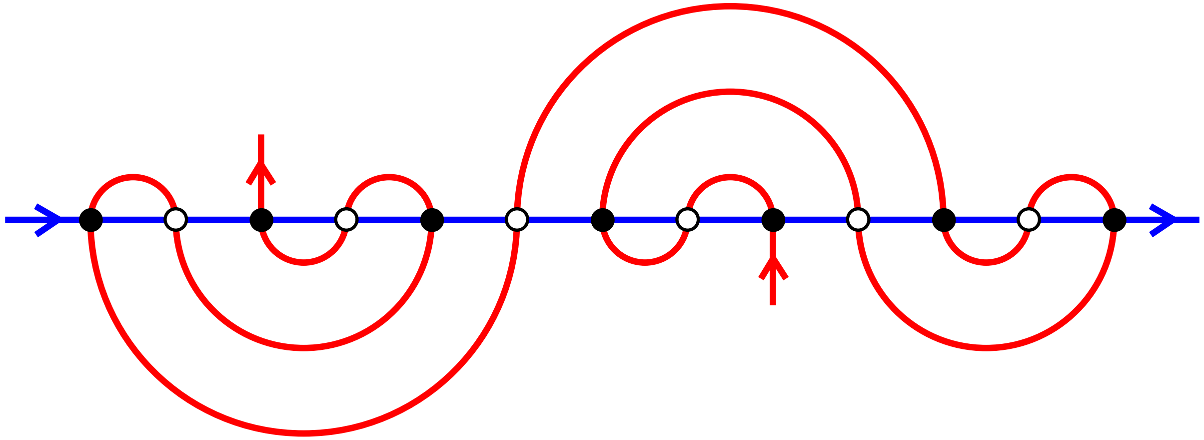


$\Longleftrightarrow$



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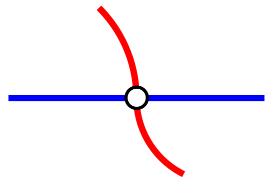
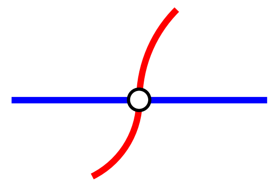
- white points are either  or 

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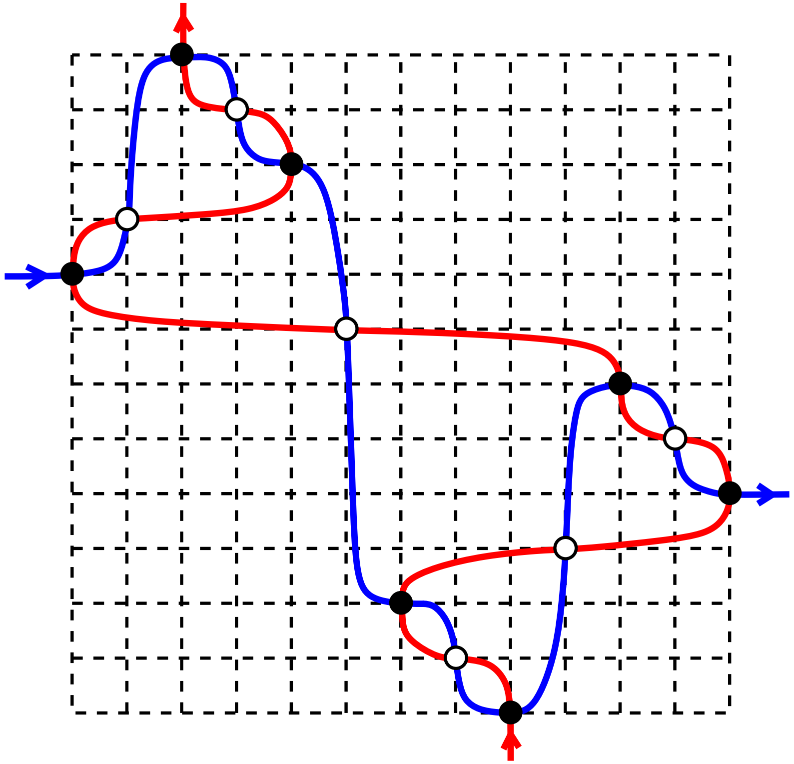

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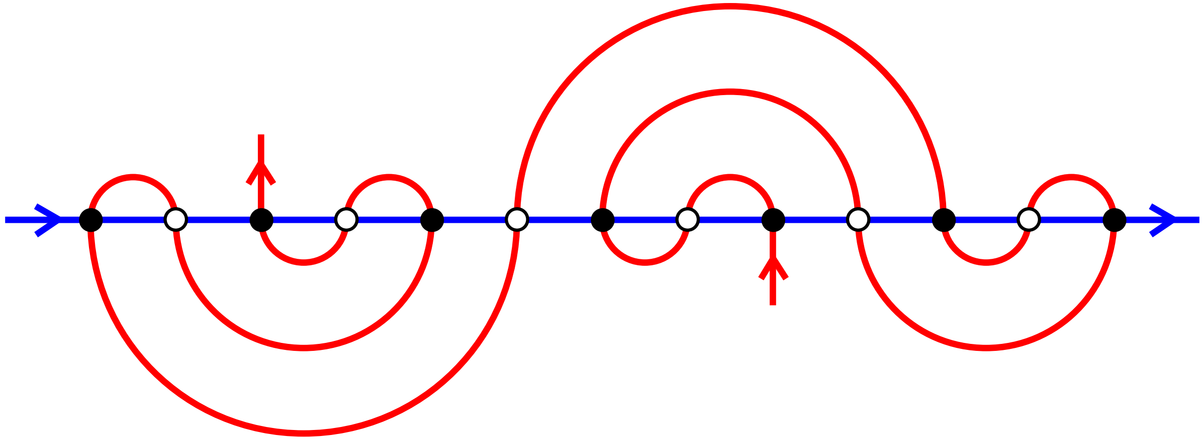

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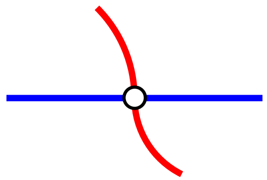
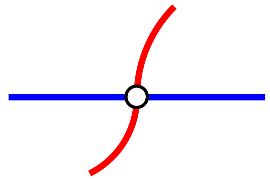
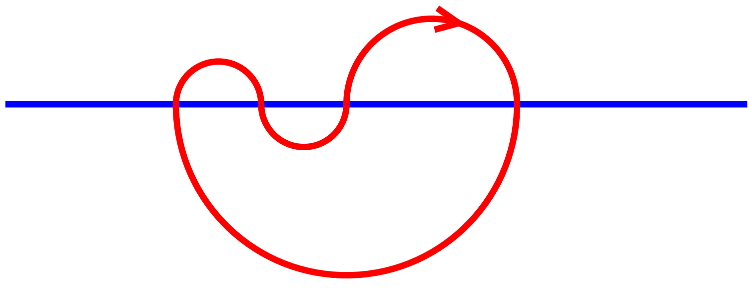
- white points are either  or 

rising descending

Proof of $\Leftarrow$

Assume there is a red loop (say, clockwise):



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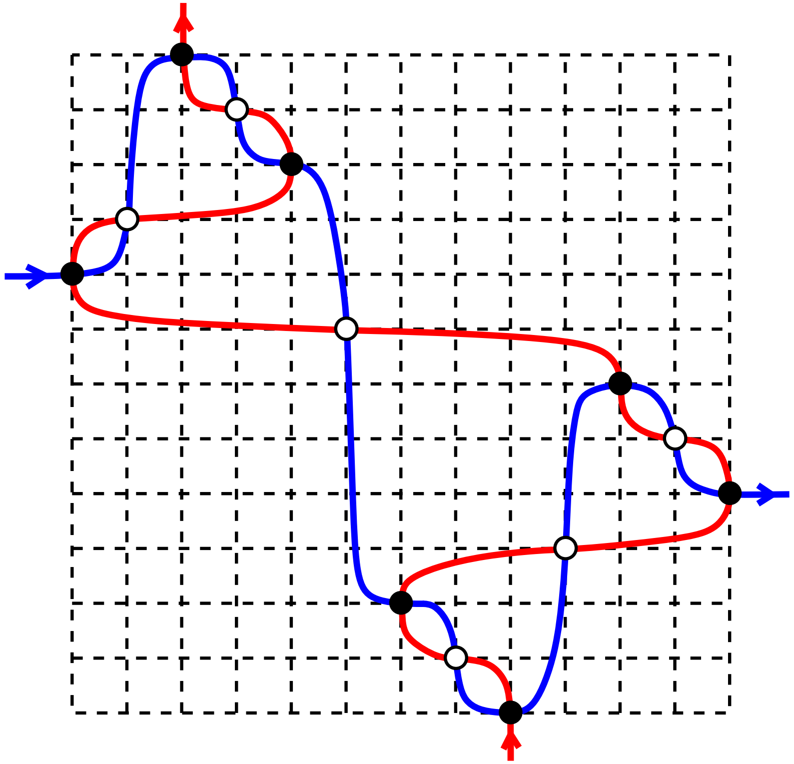

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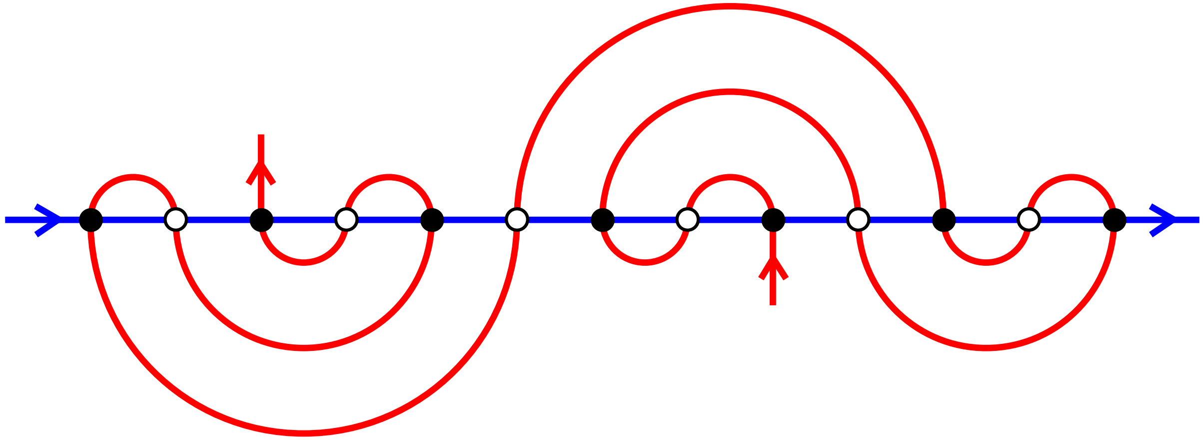

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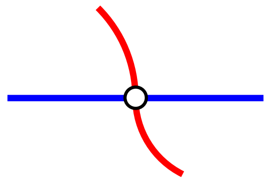
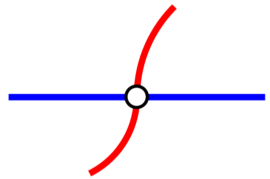
- white points are either  or 

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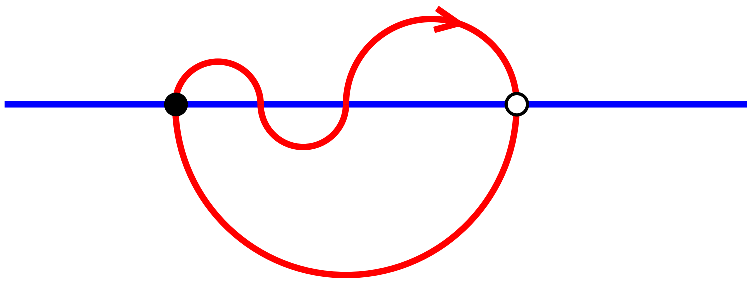

then the leftmost and the righmost point on the loop are of different colors

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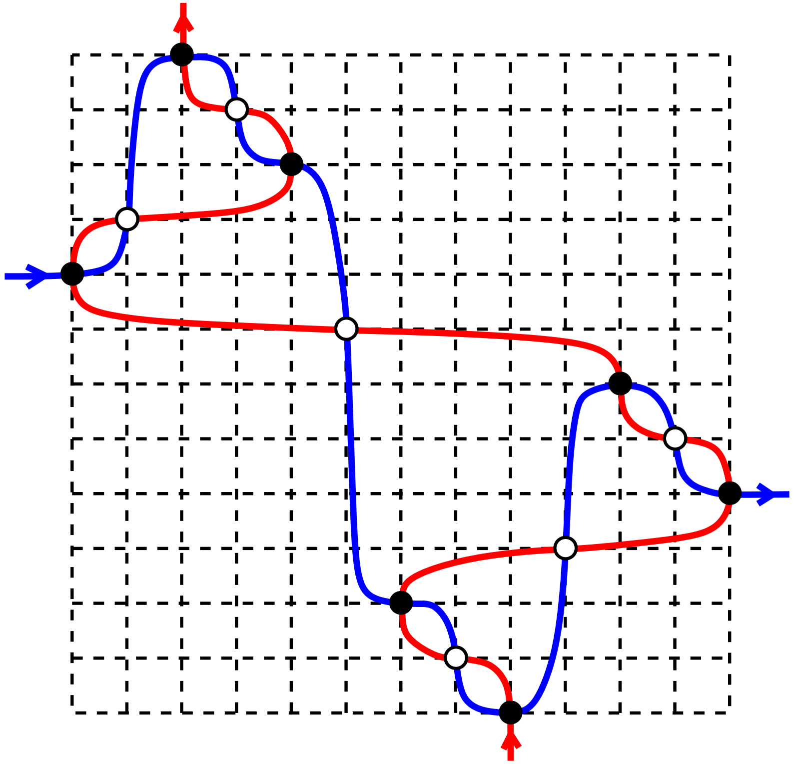

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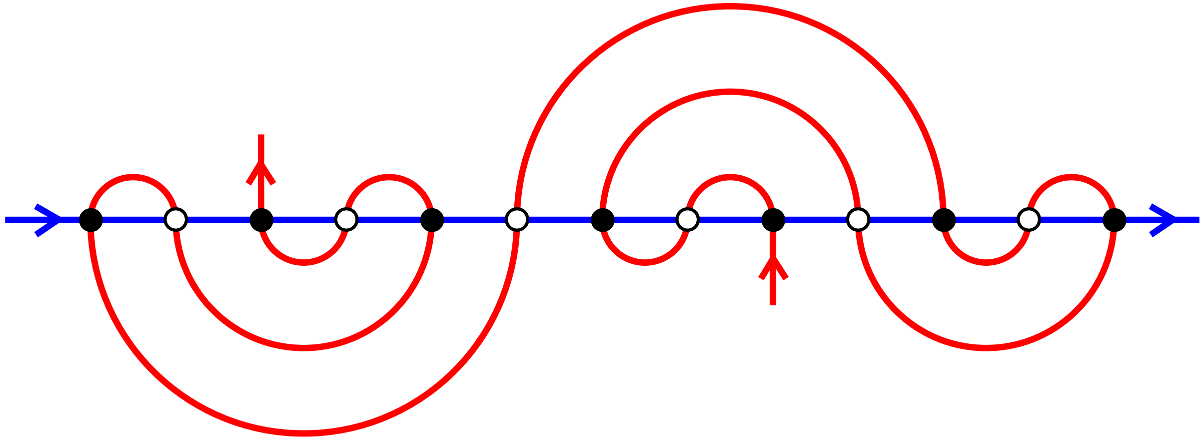

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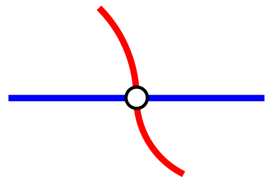

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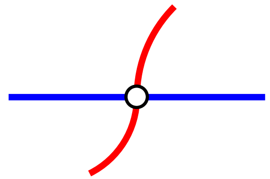

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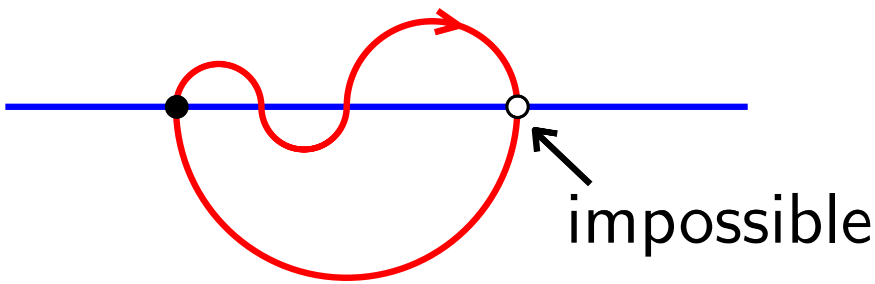

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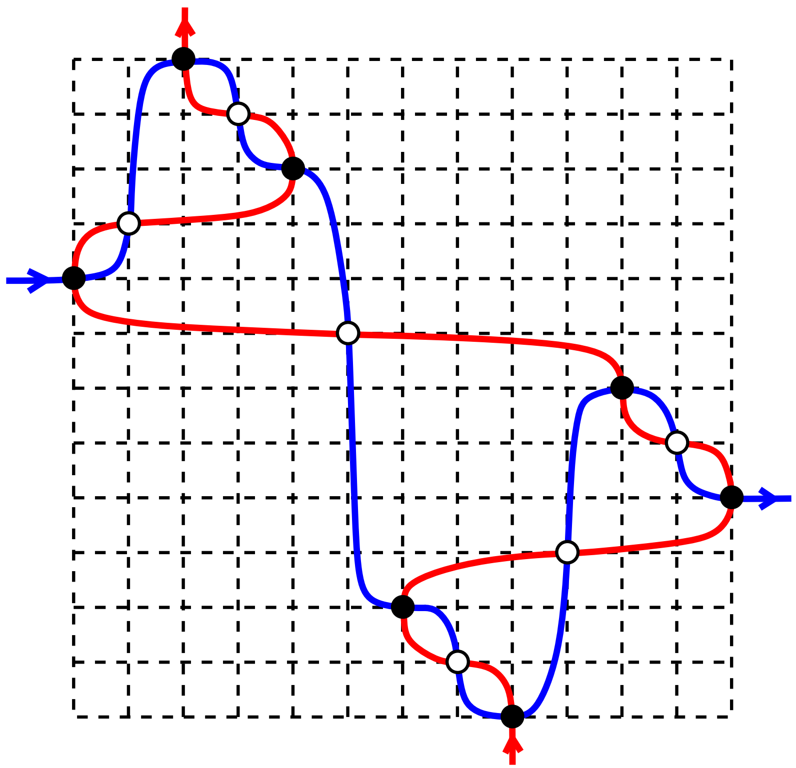

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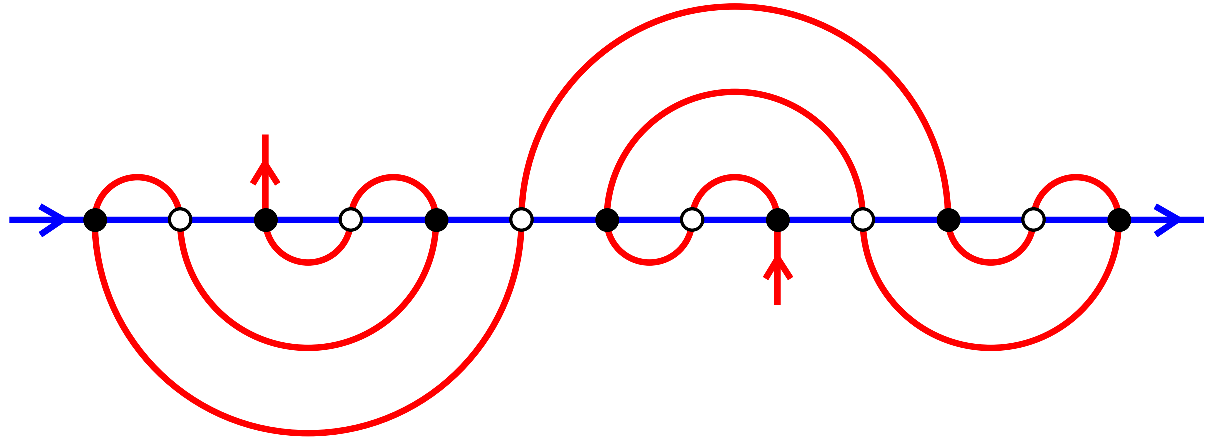

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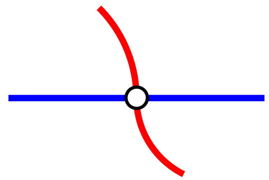

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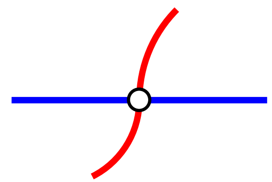

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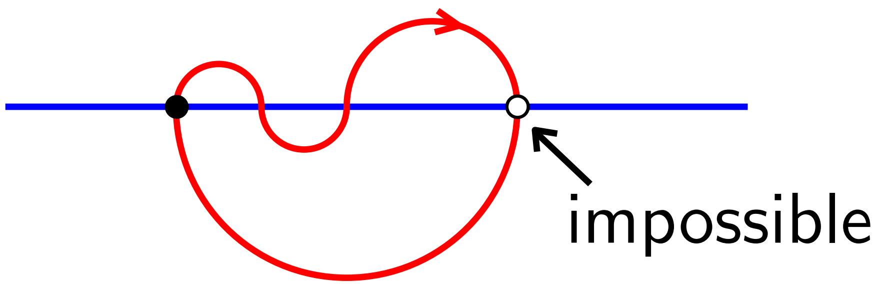

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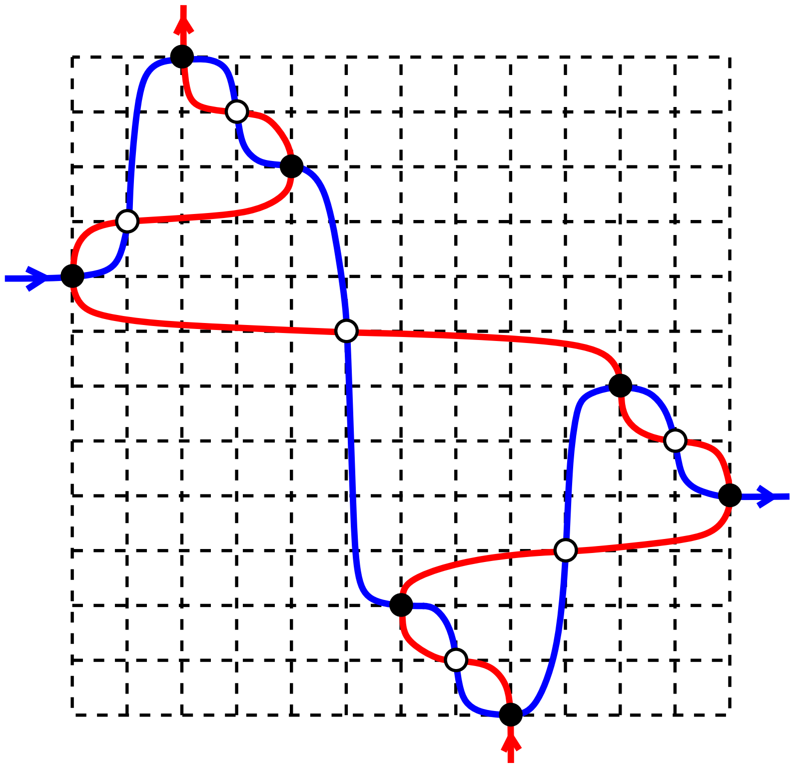
$\Rightarrow$ we have a 2-line meander

Local conditions for monotone 2-line meanders



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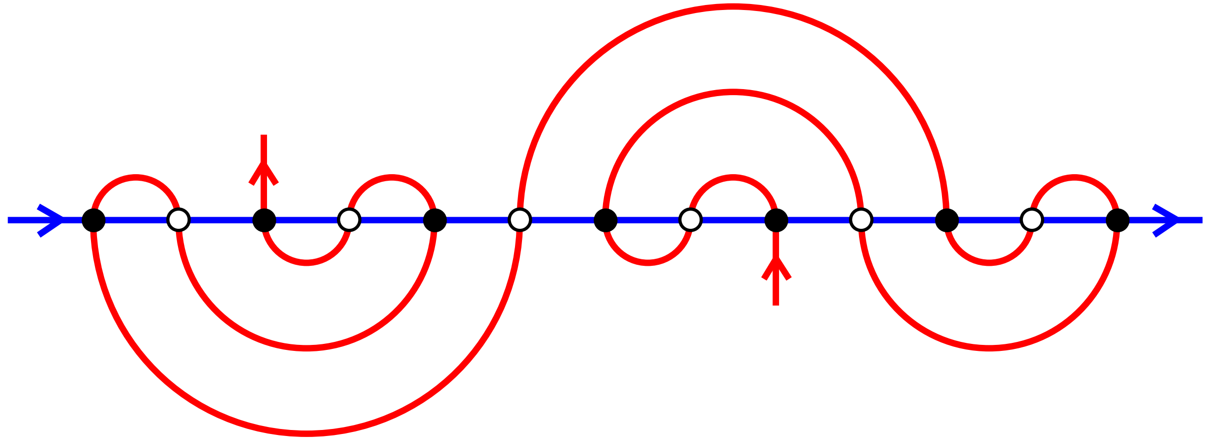

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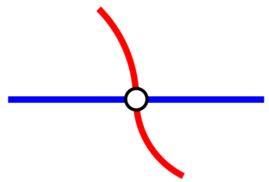
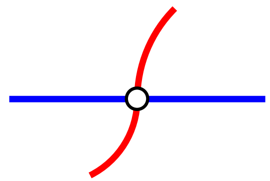
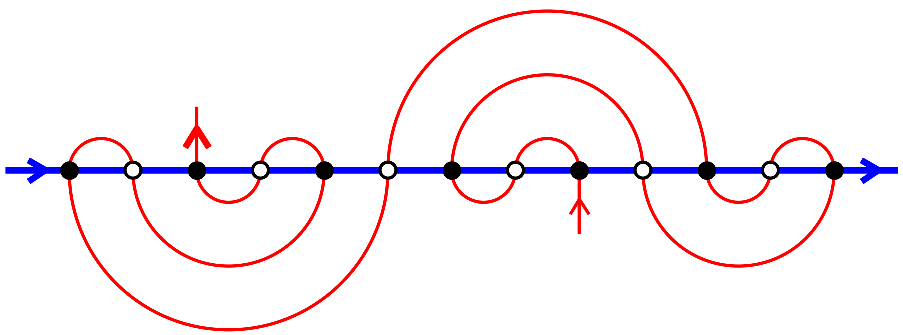
- white nodes are either  or 

Proof of $\Leftarrow$: construct permutation step by step



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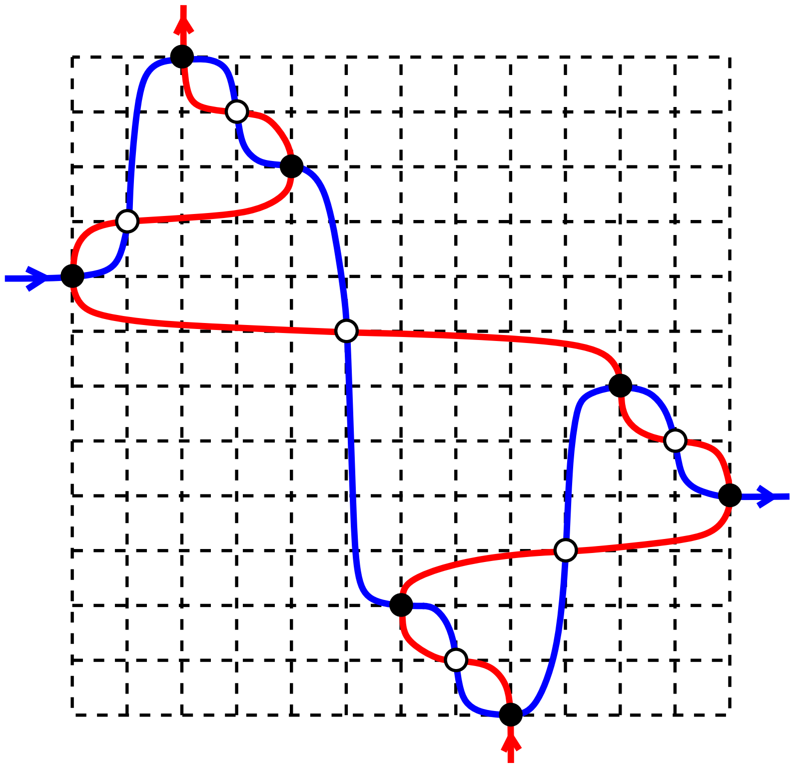

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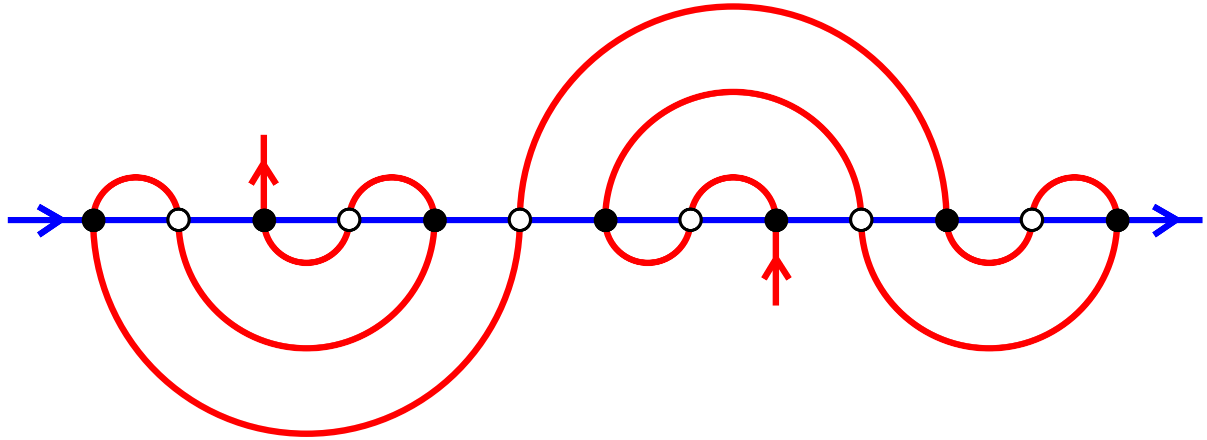

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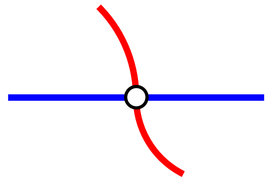
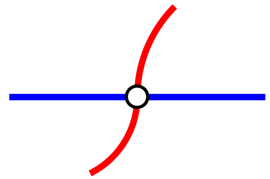
- white nodes are either  or 

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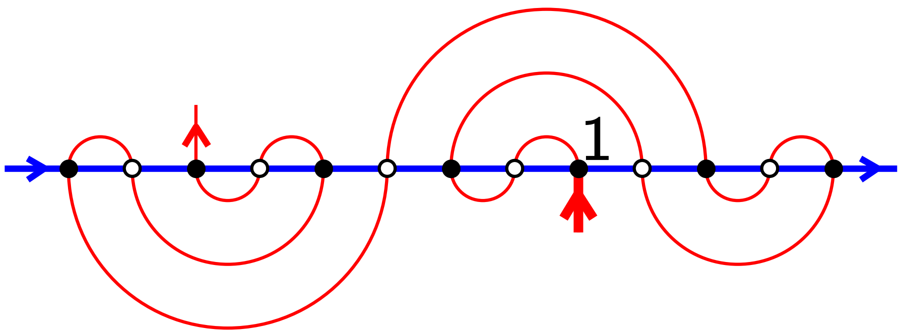

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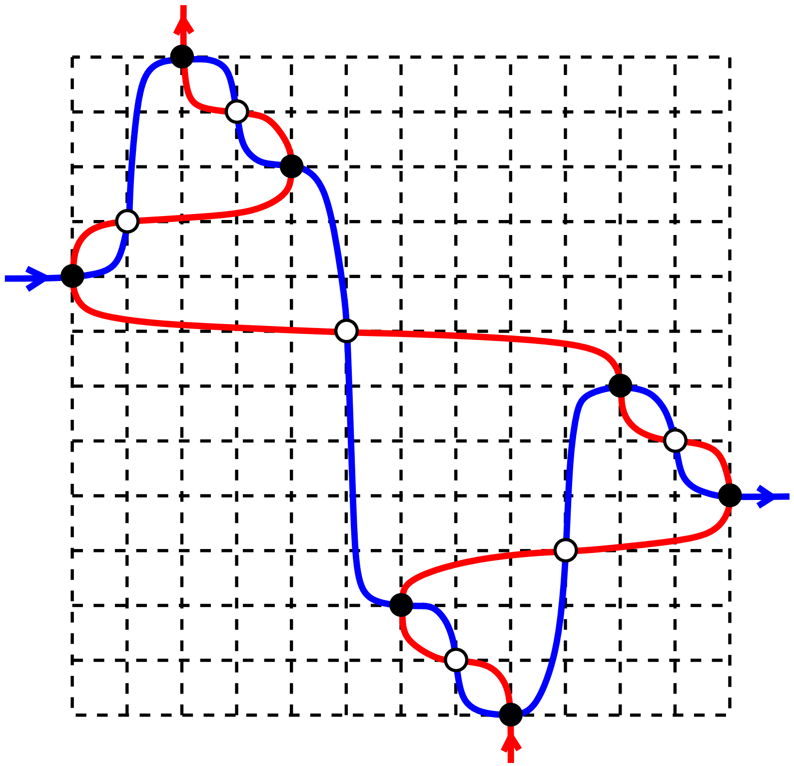

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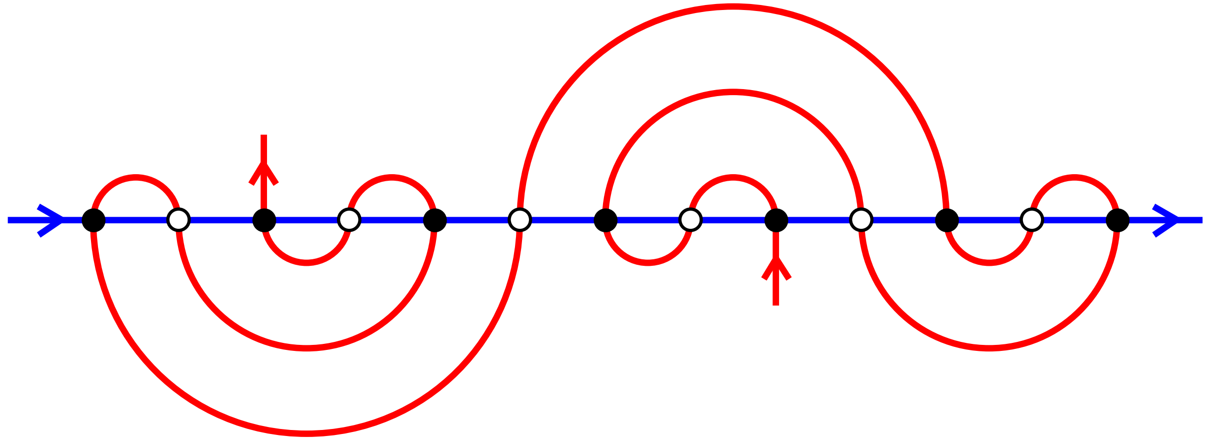

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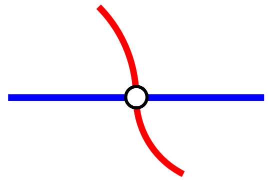
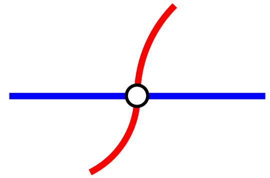
- white nodes are either  or 

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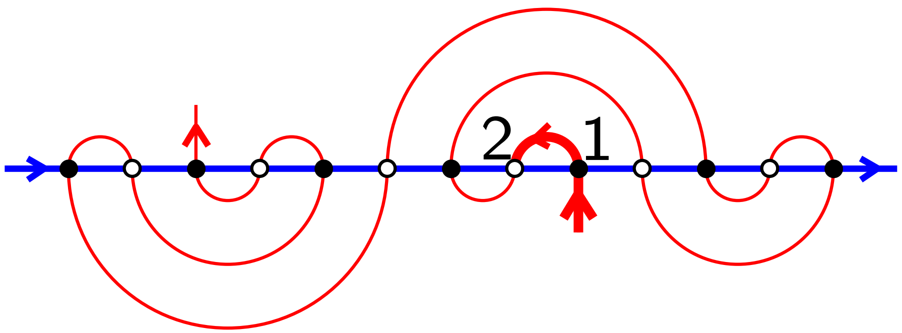

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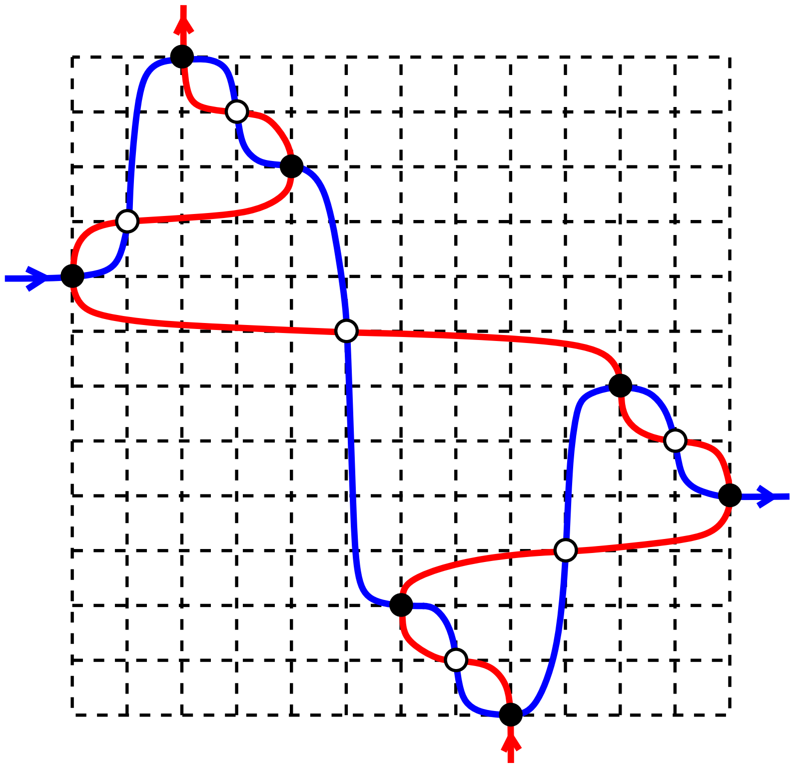

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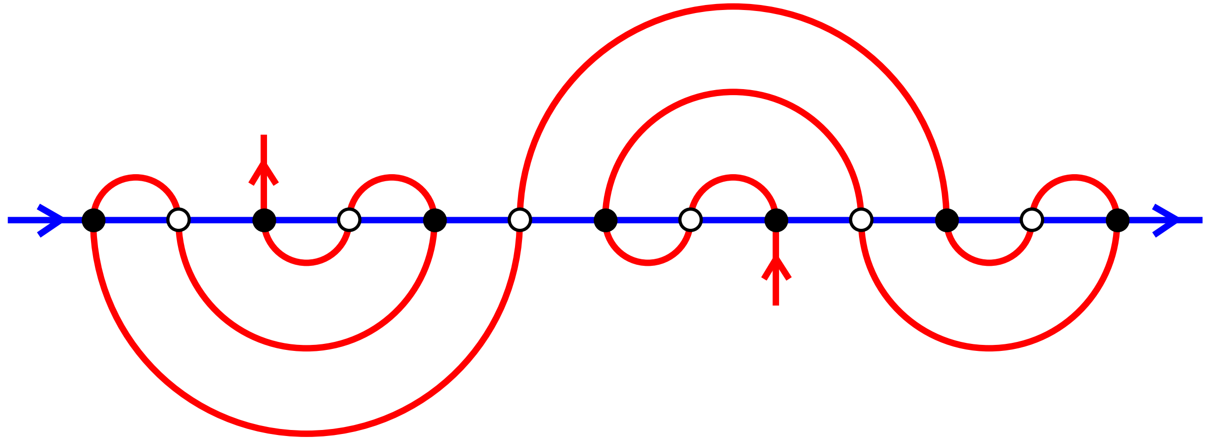

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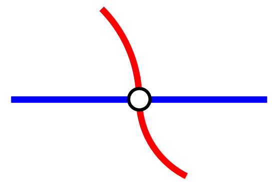
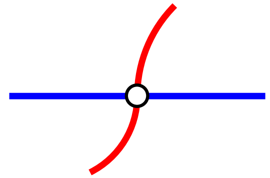
- white nodes are either  or 

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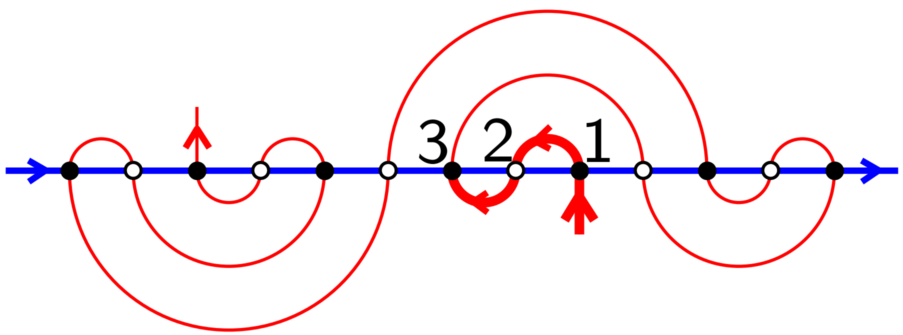

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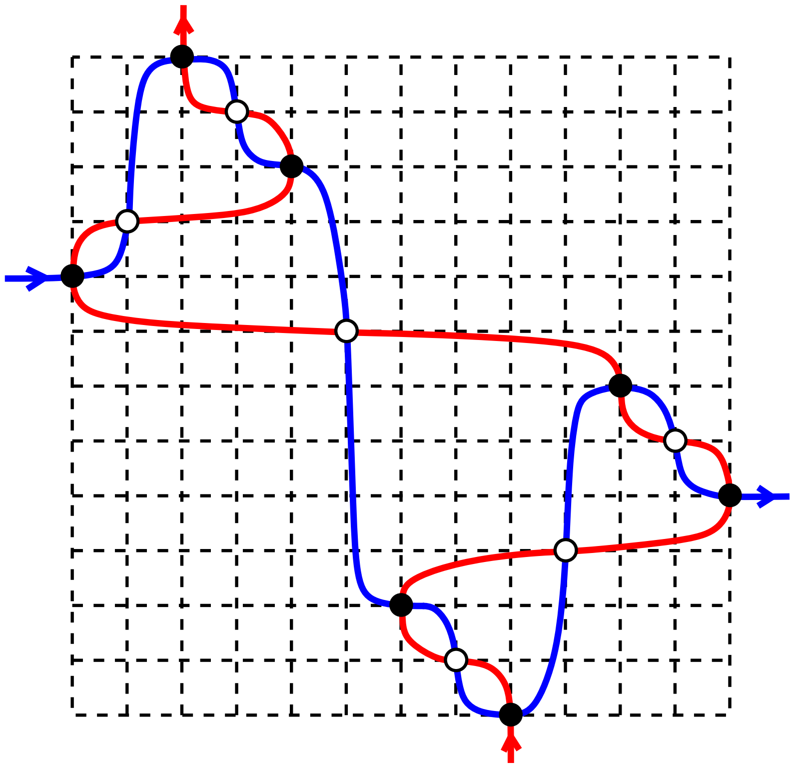

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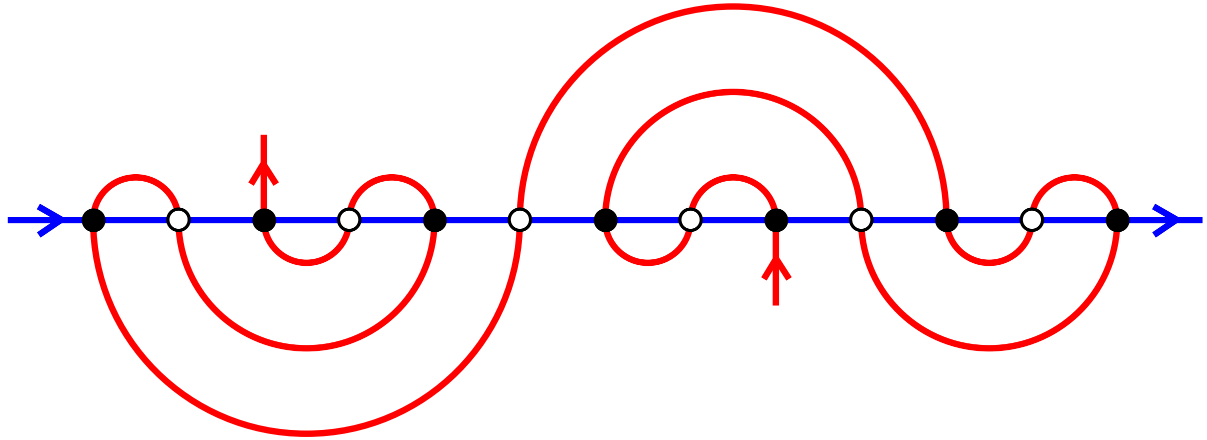

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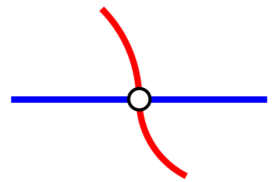
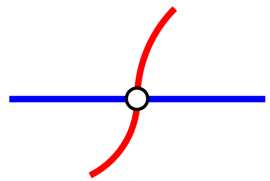
- white nodes are either  or 

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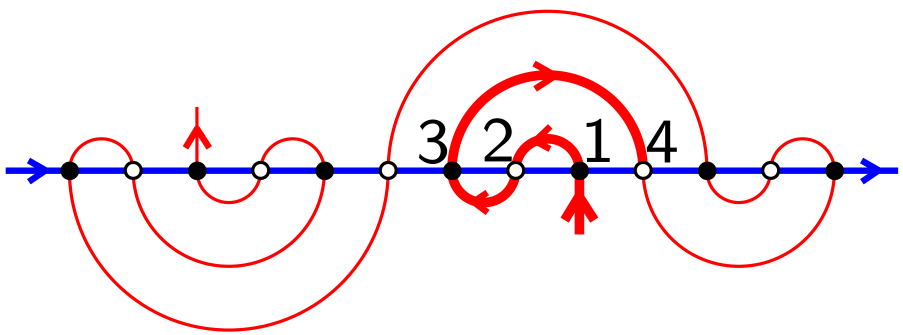
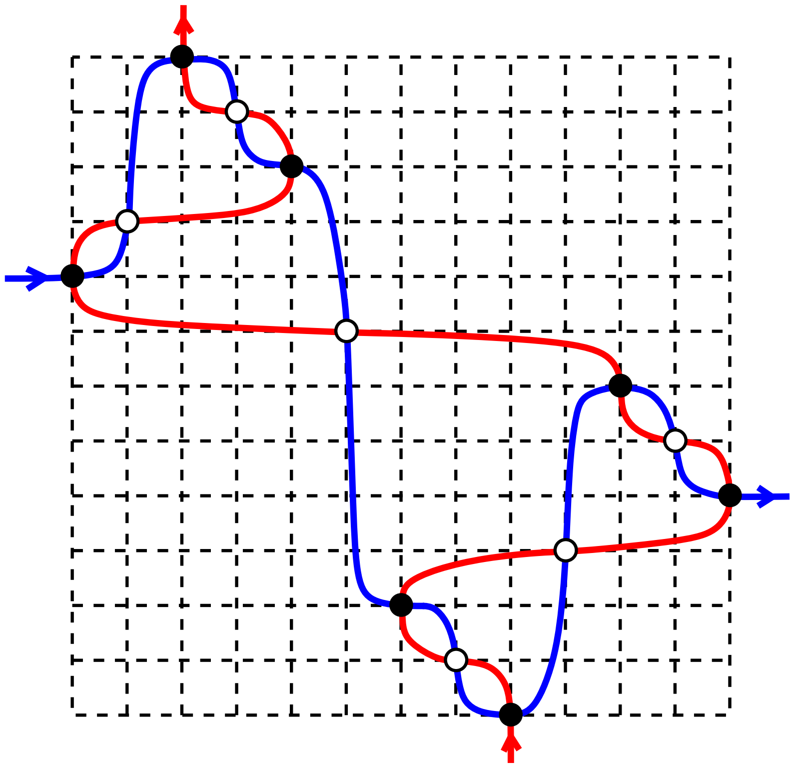


Local conditions for monotone 2-line meanders



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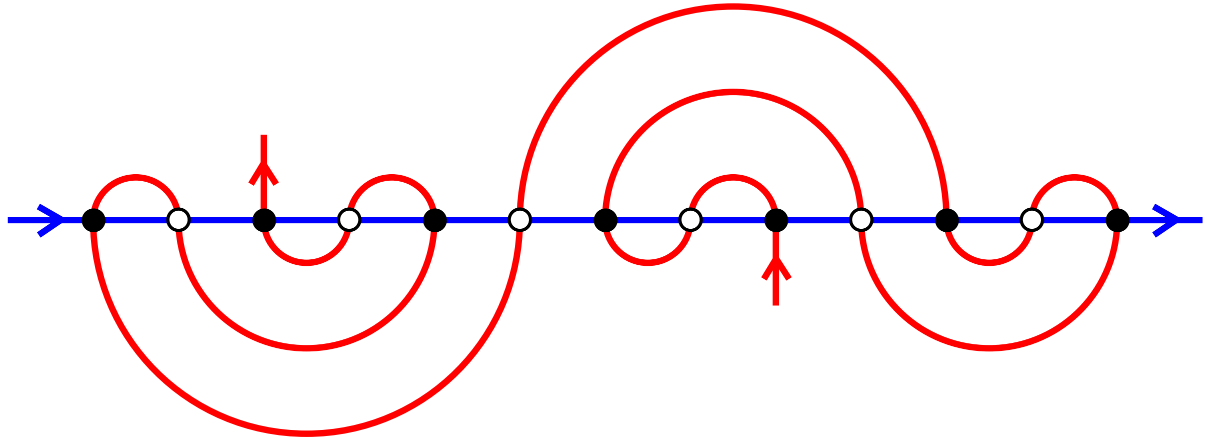

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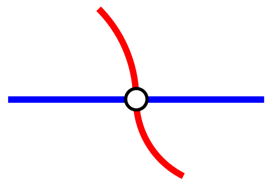
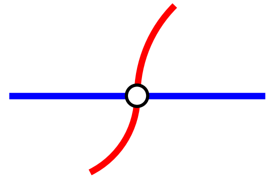
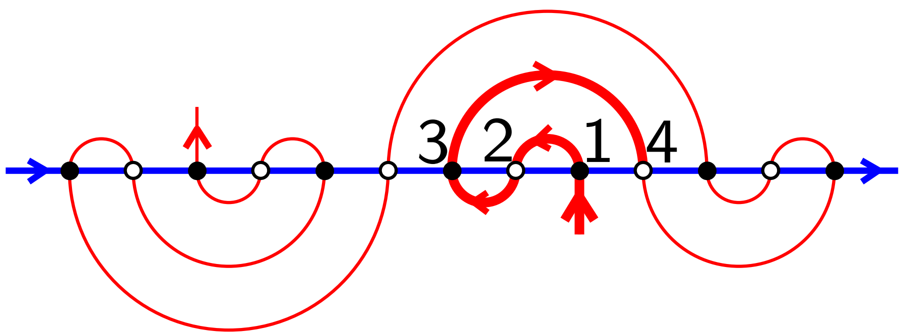
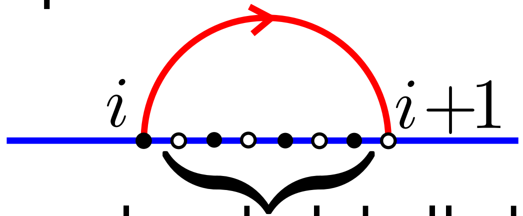
- white nodes are either  or 

Proof of $\Leftarrow$: construct permutation step by step

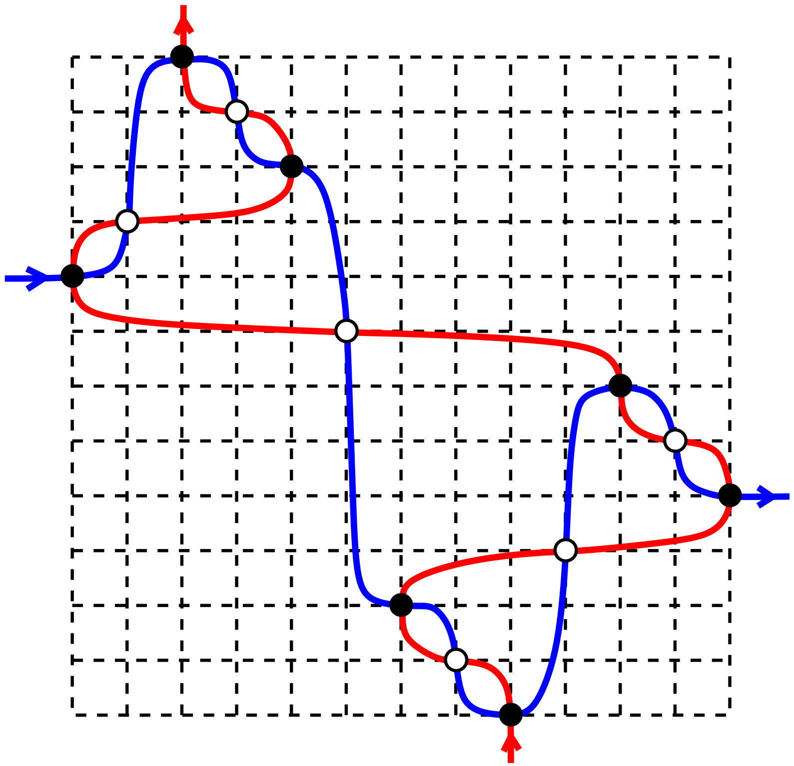


important observation:



By similar argument as to show there is no red loop

Local conditions for monotone 2-line meanders



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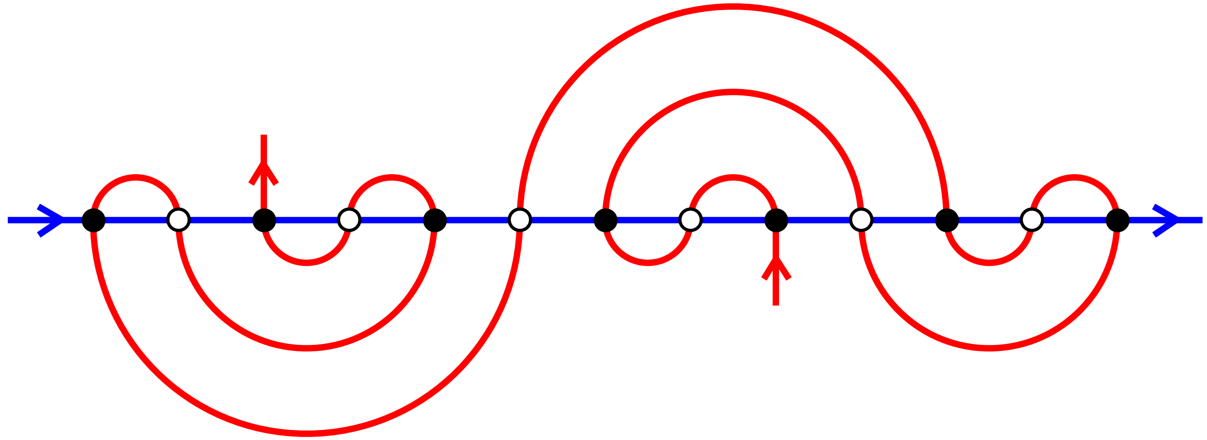

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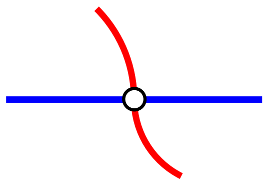
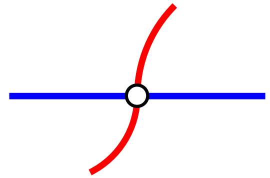
- white nodes are either  or 

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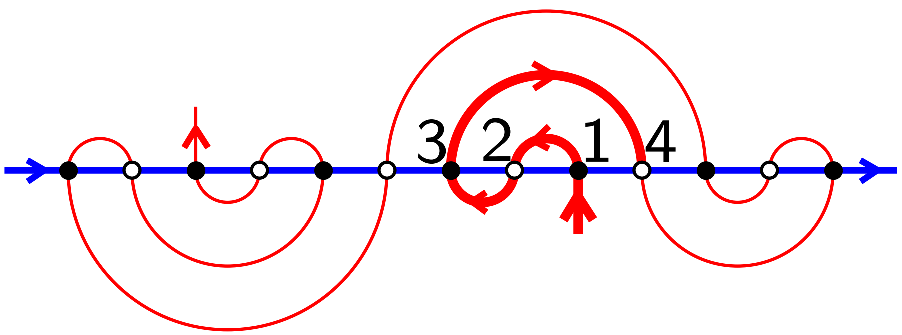

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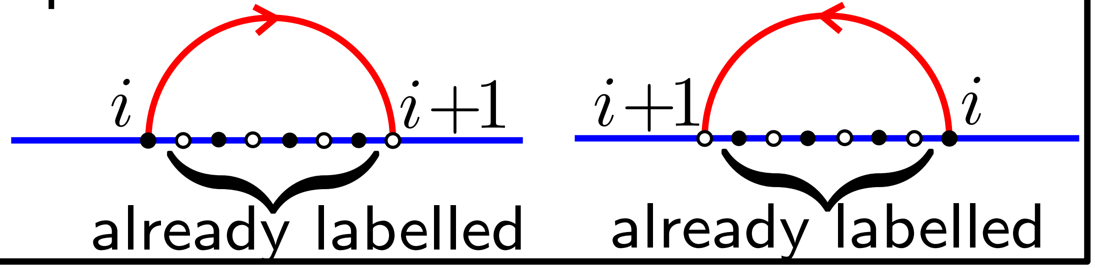

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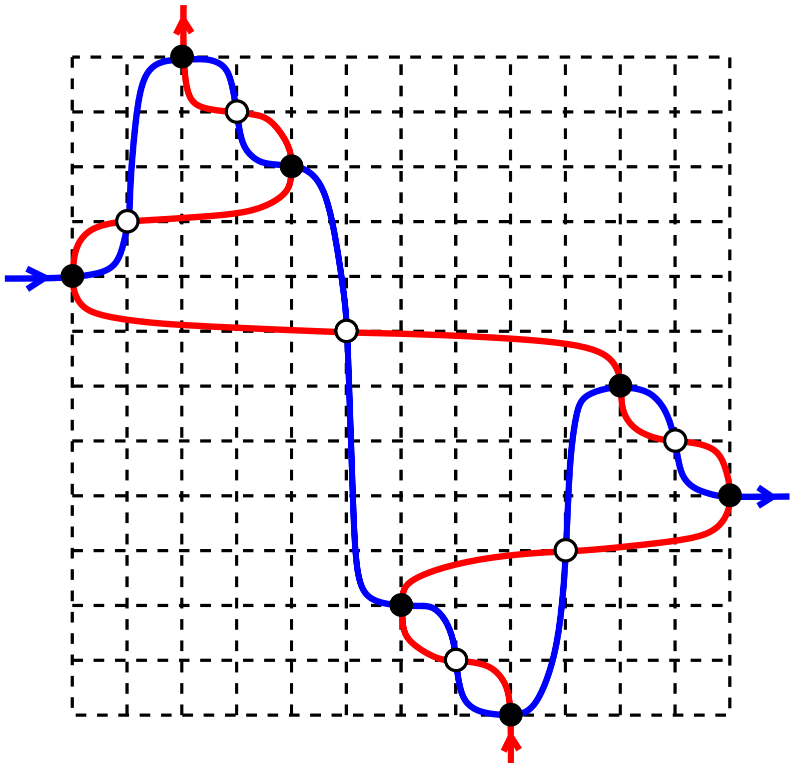

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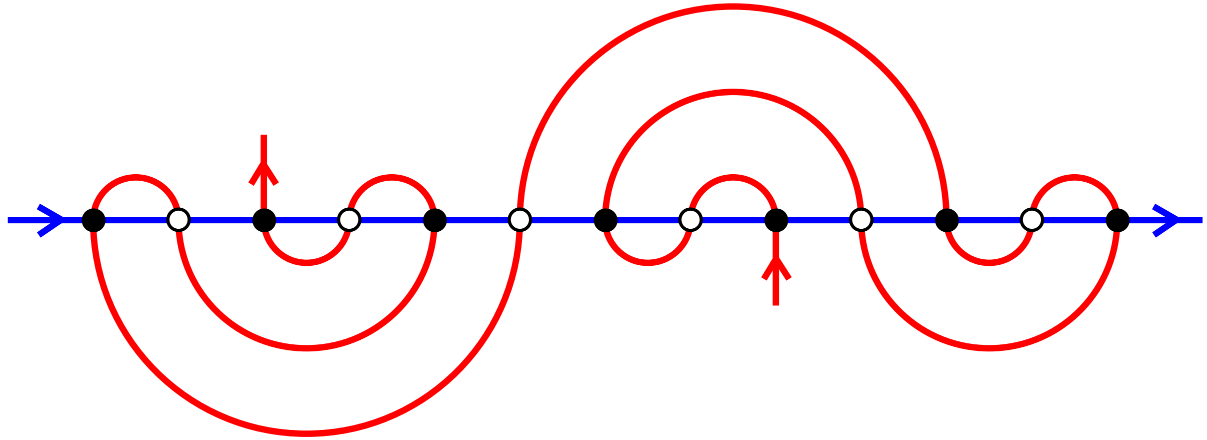

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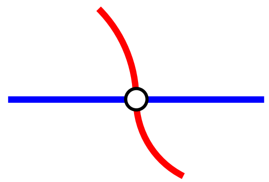

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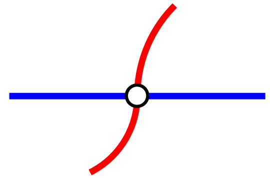

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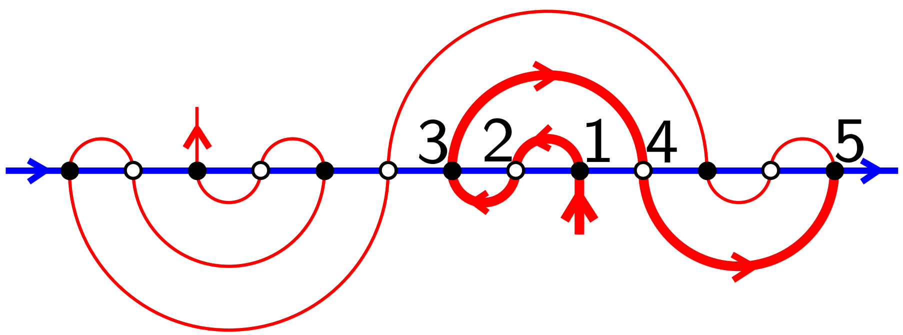

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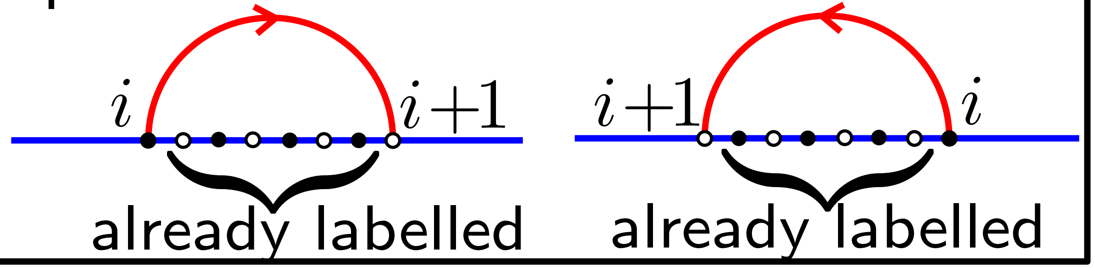
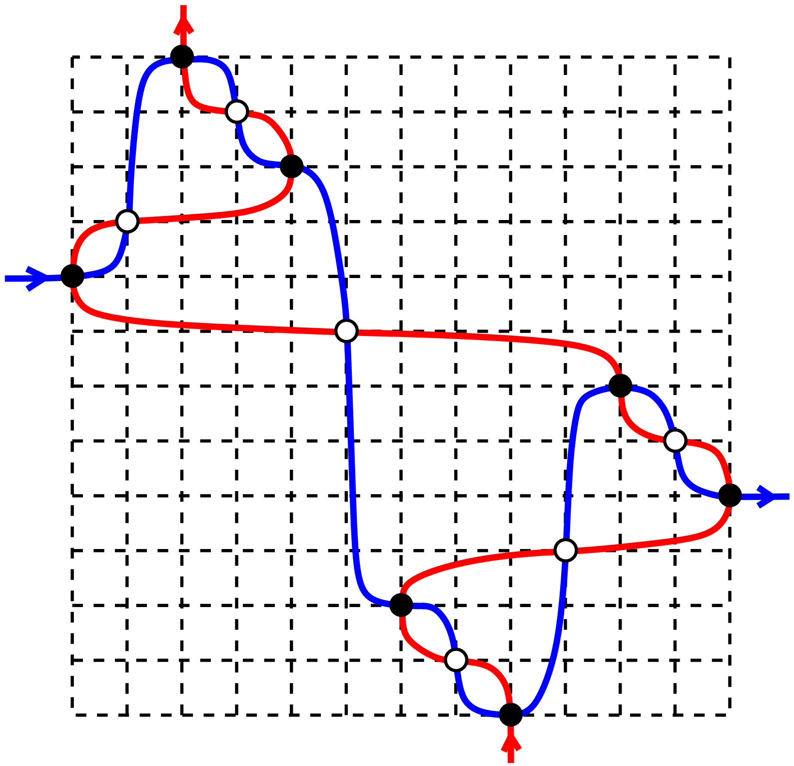
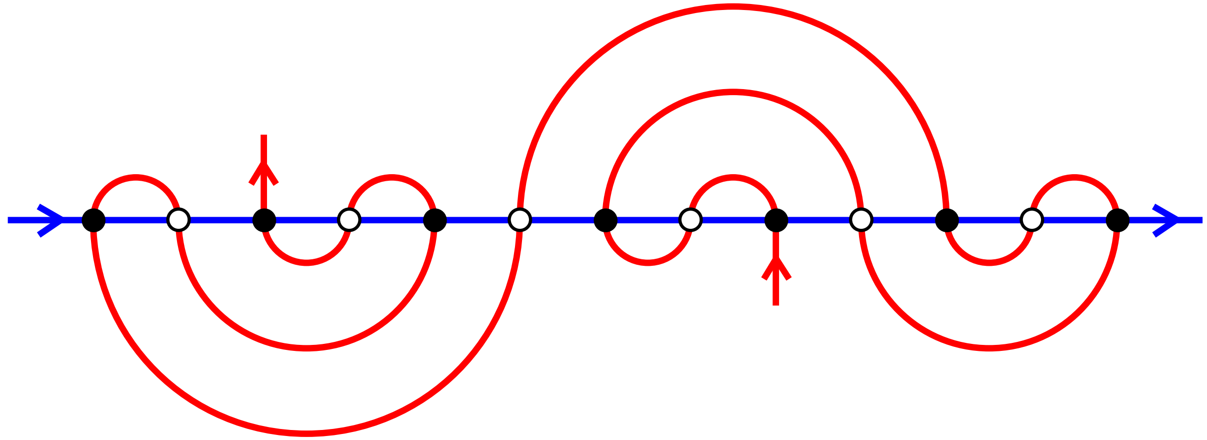


Local conditions for monotone 2-line meanders

 $\Leftrightarrow$ 

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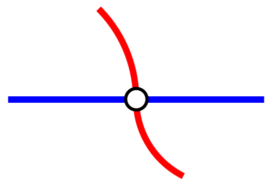
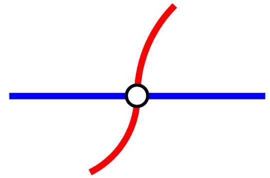
- white nodes are either  or 

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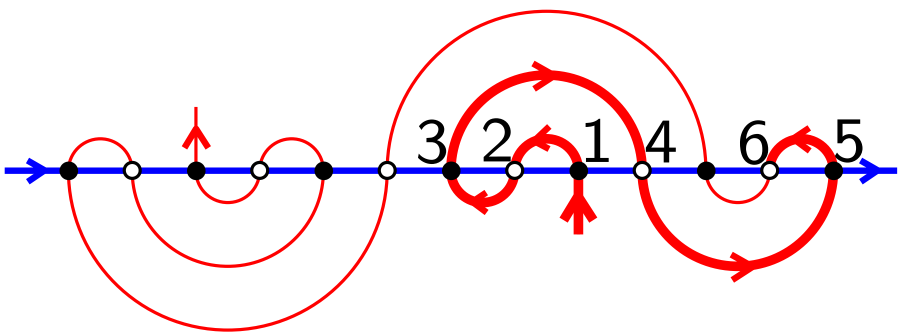

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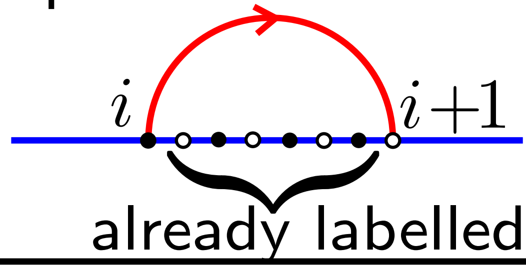

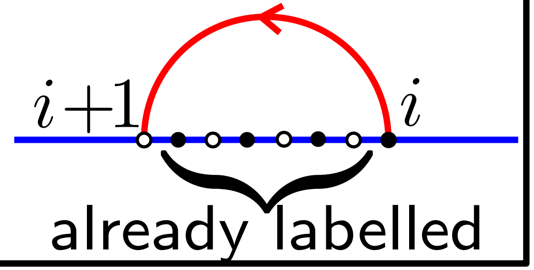

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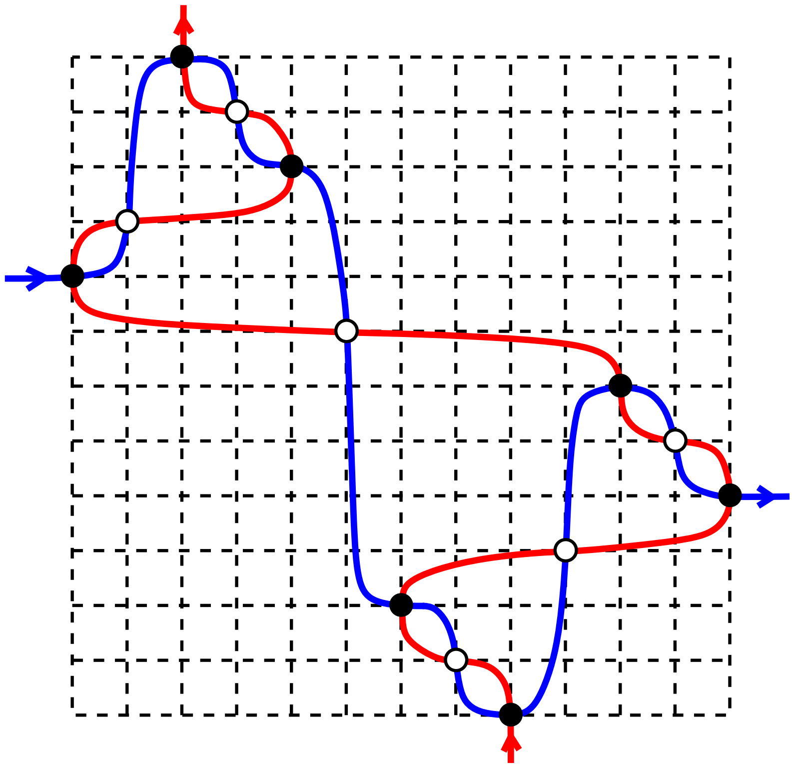

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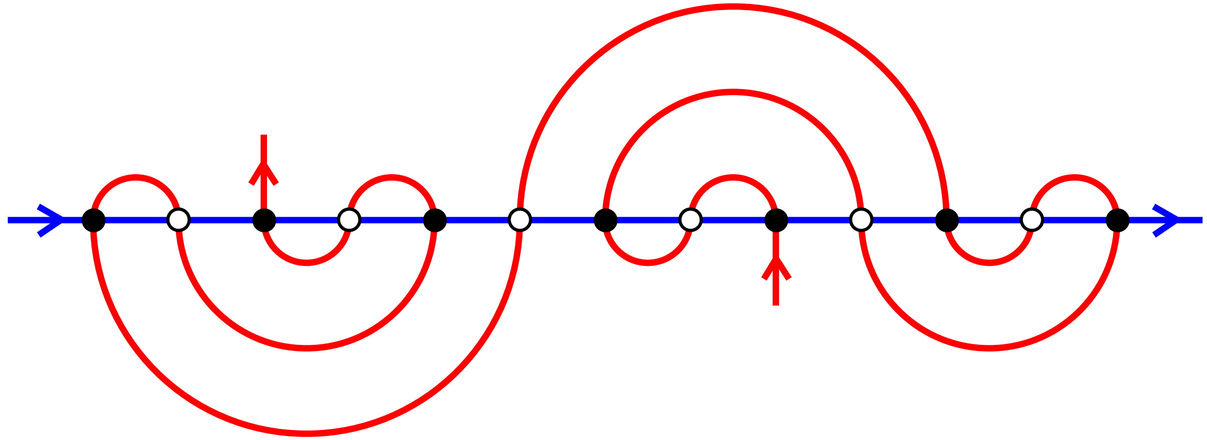

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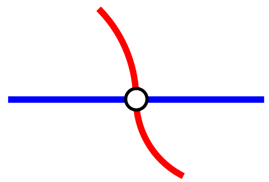
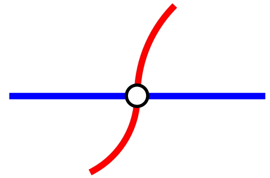
- white nodes are either  or 

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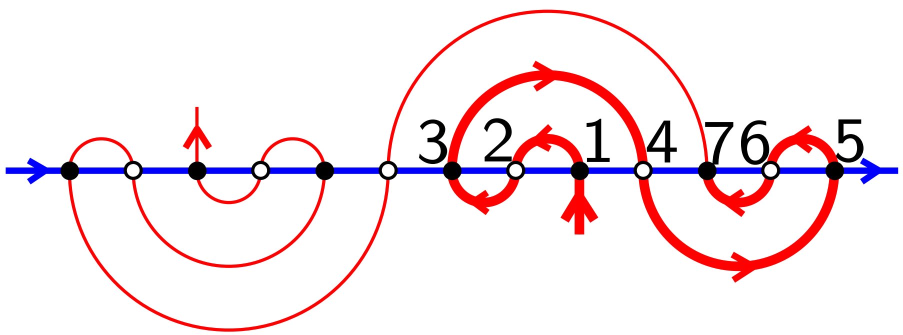

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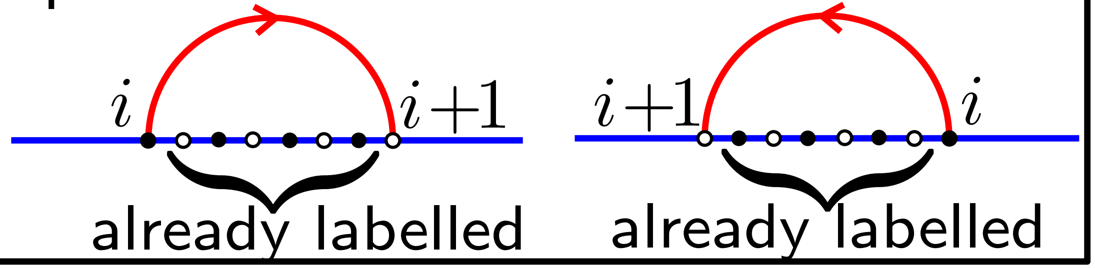
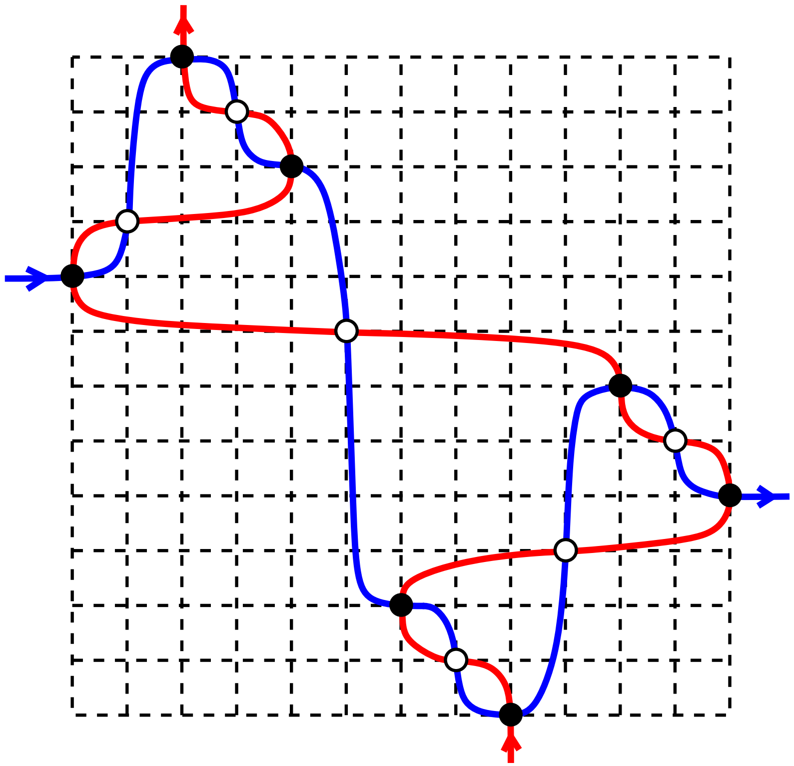


Local conditions for monotone 2-line meanders



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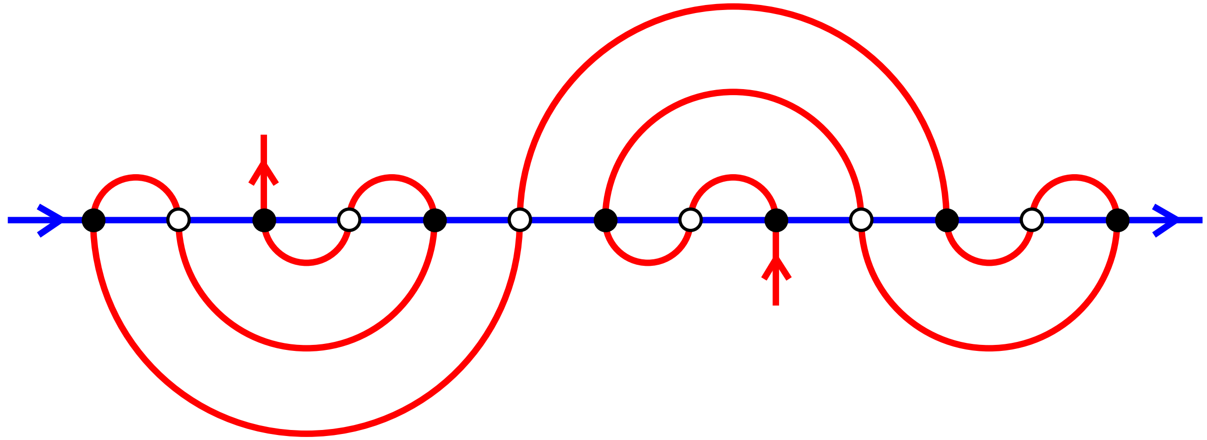

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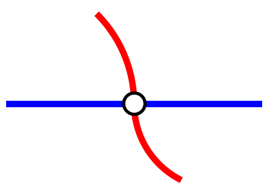
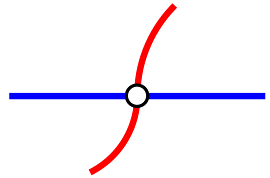
- white nodes are either  or 

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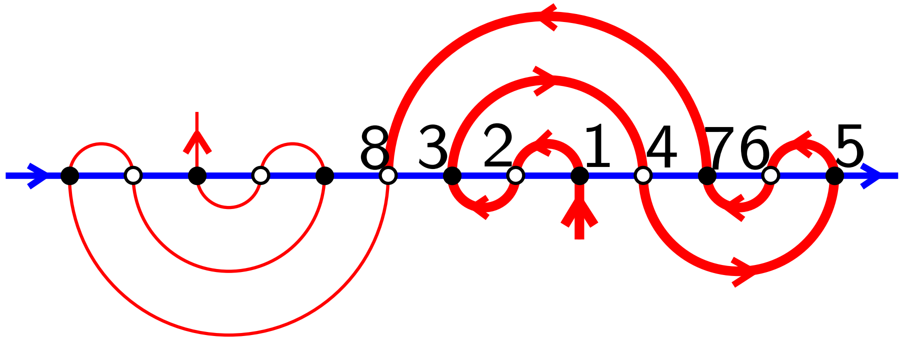

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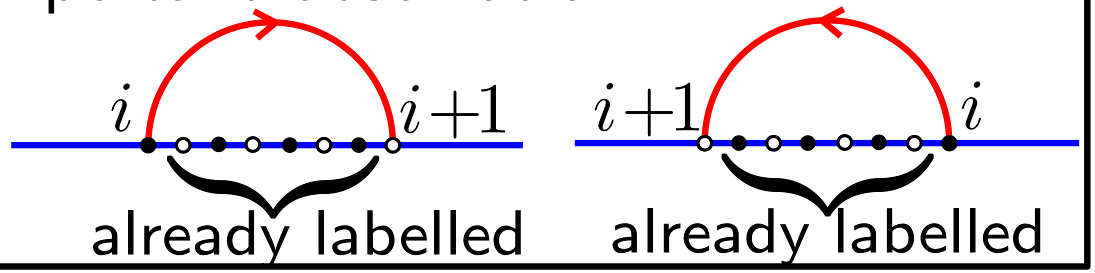

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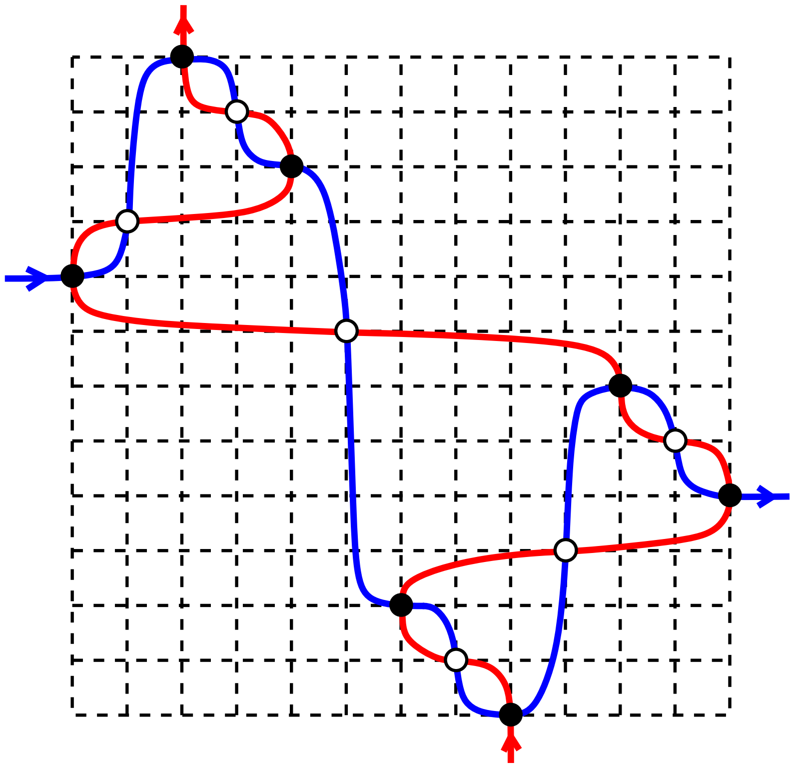

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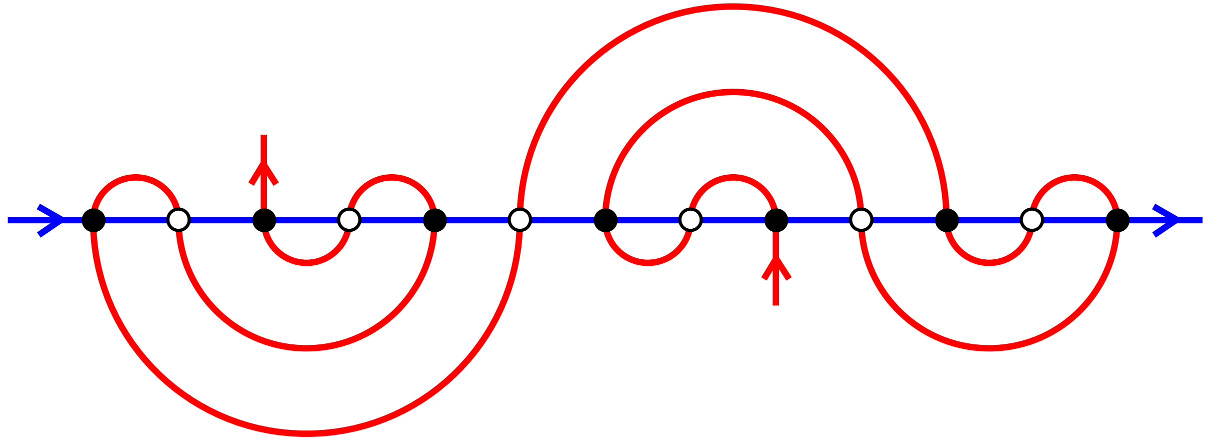

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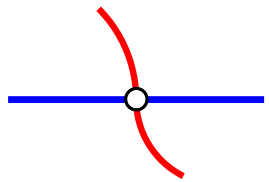
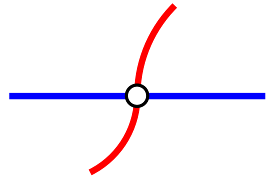
- white nodes are either  or 

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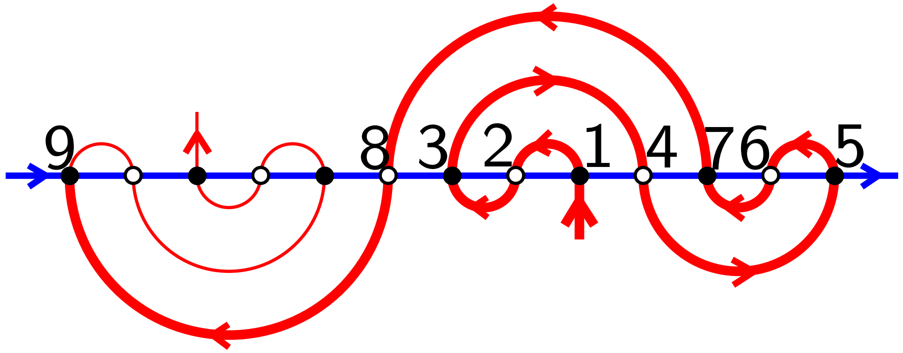

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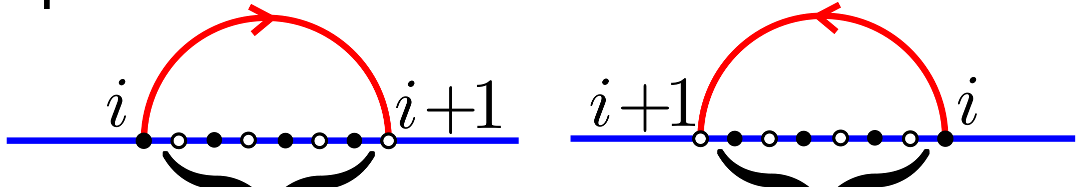

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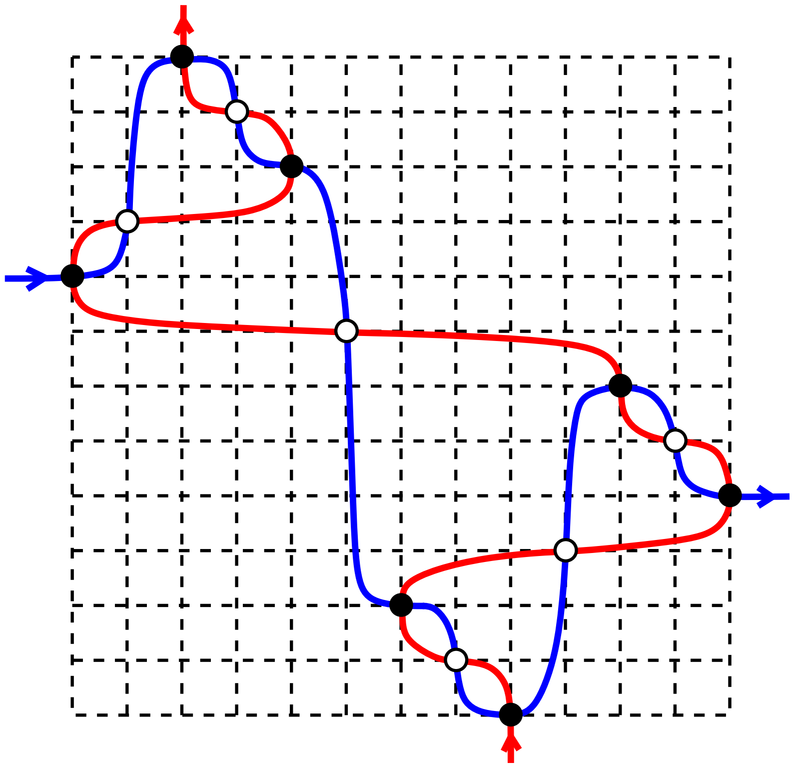

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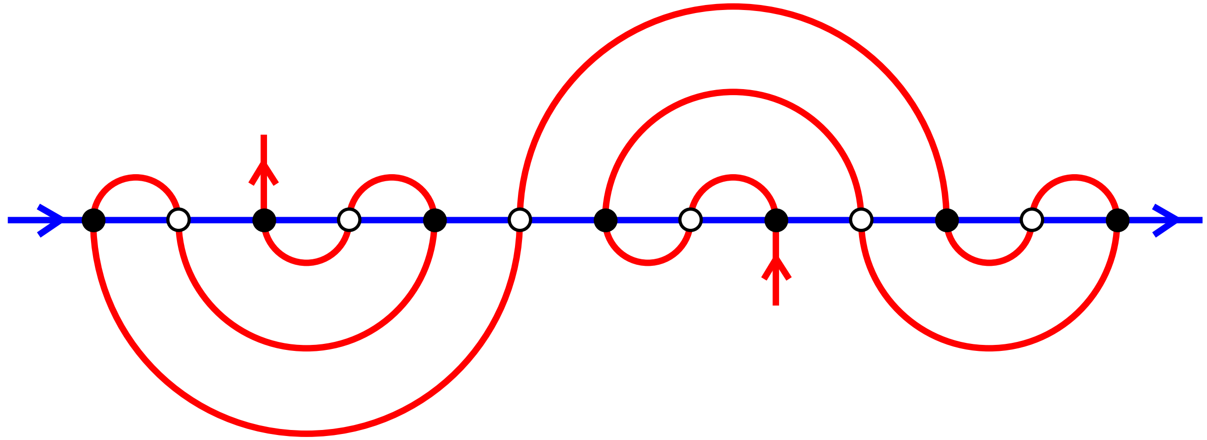

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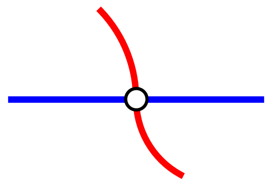
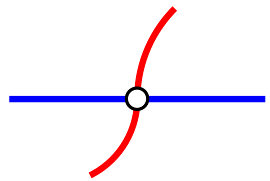
- white nodes are either  or 

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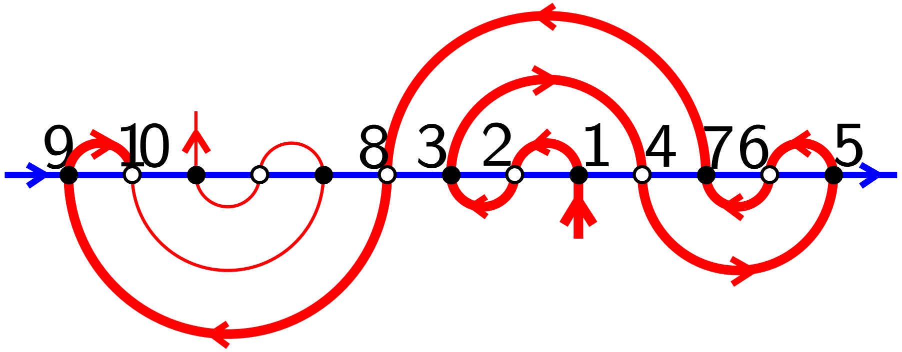

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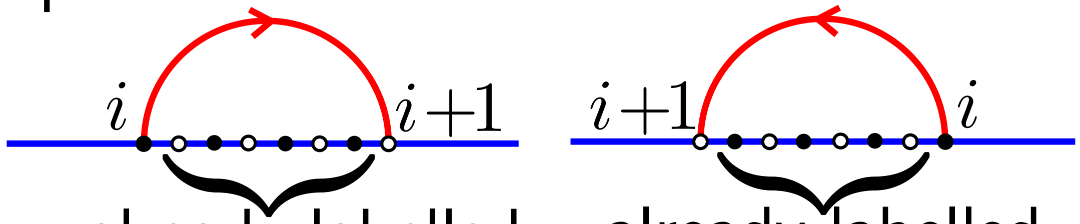

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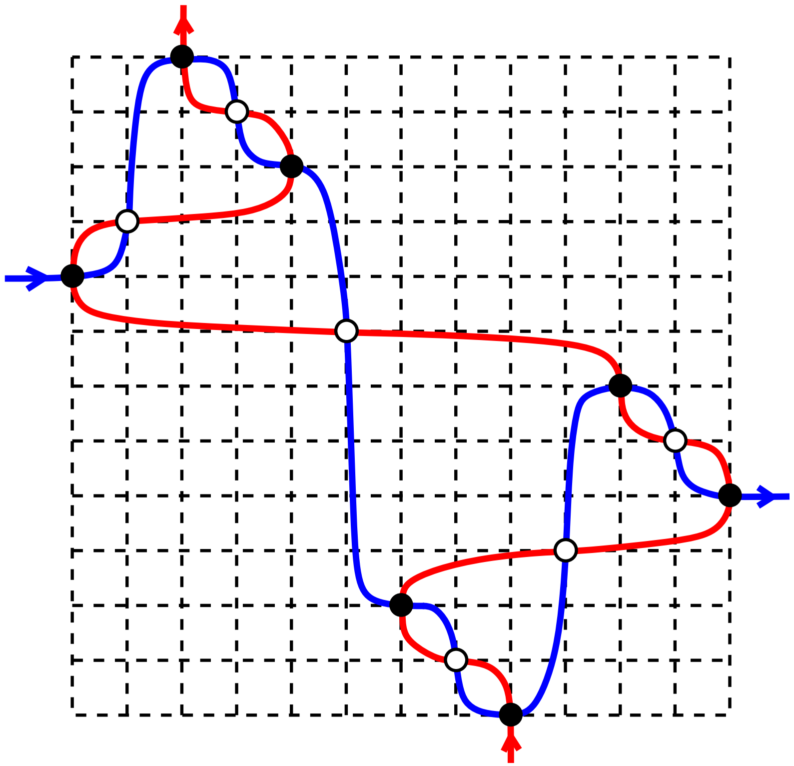

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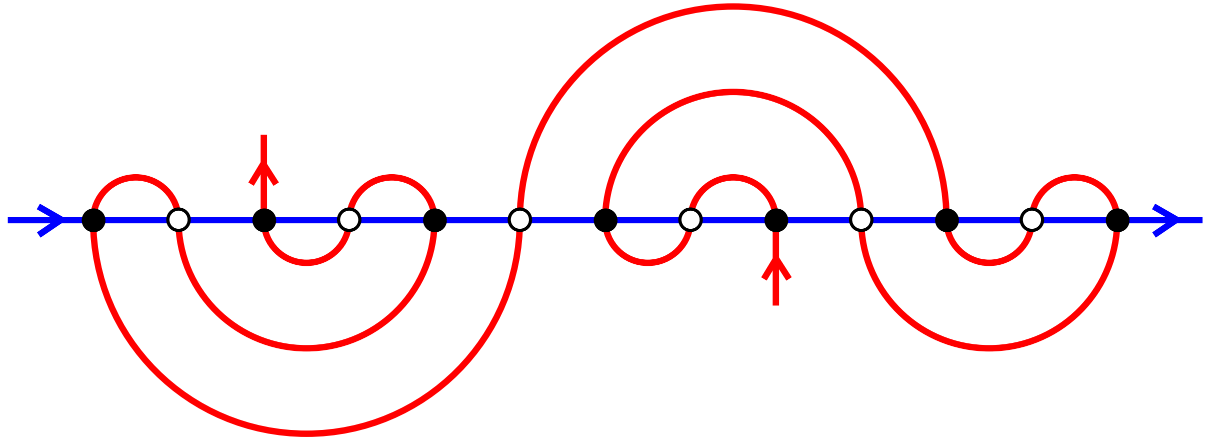

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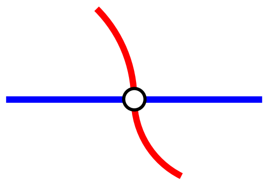
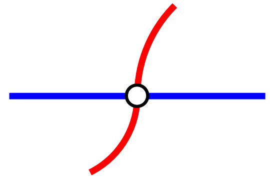
- white nodes are either  or 

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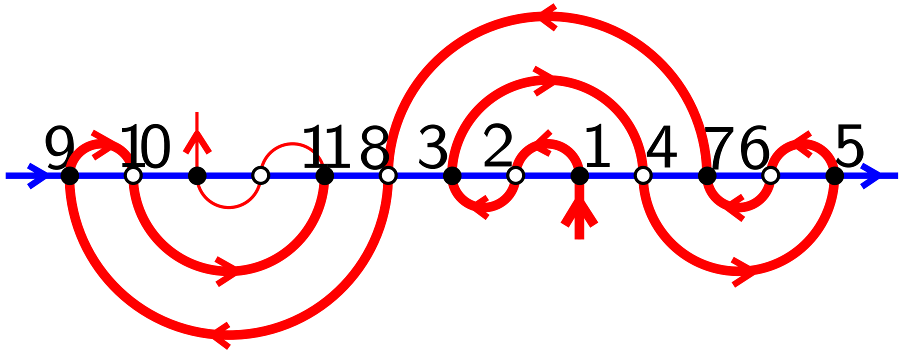

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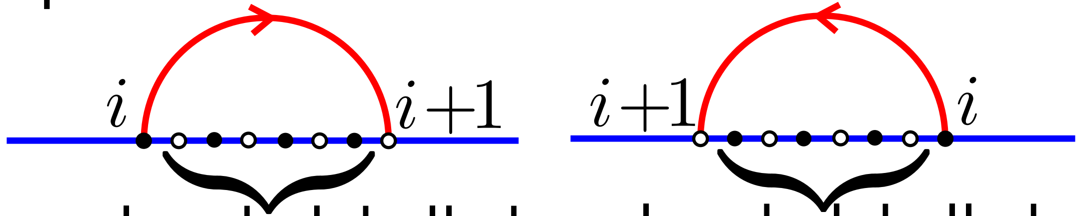

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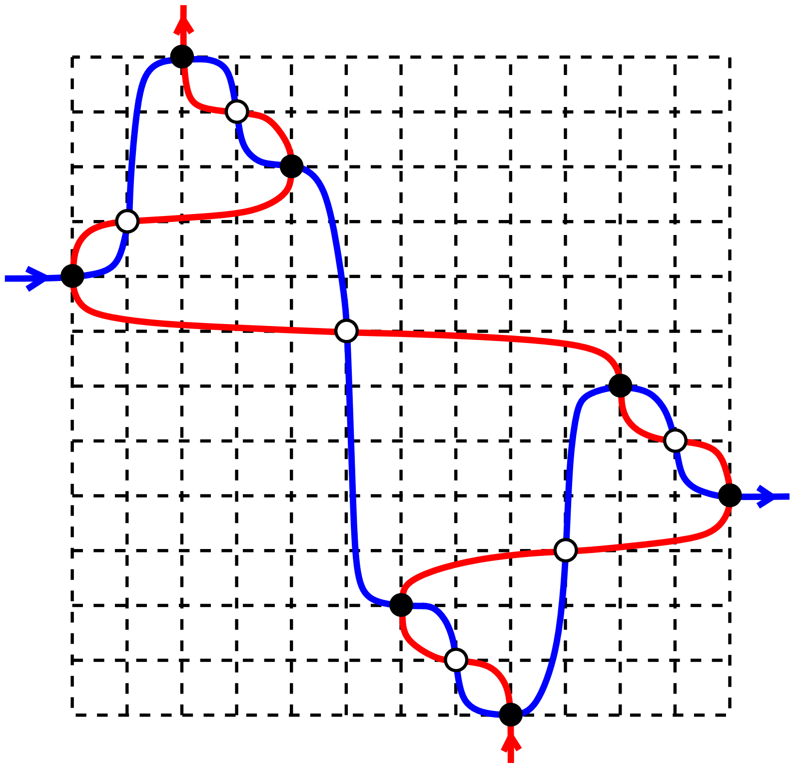

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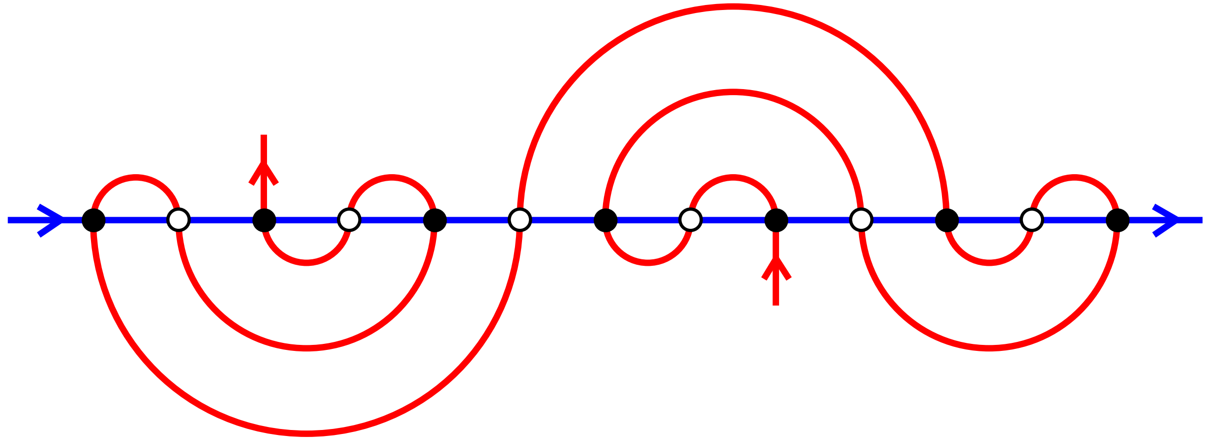

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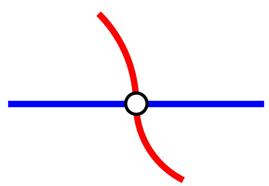
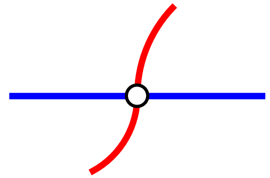
- white nodes are either  or 

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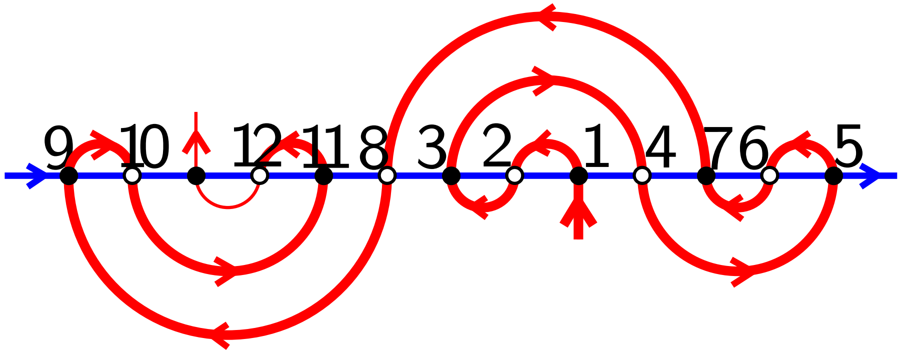

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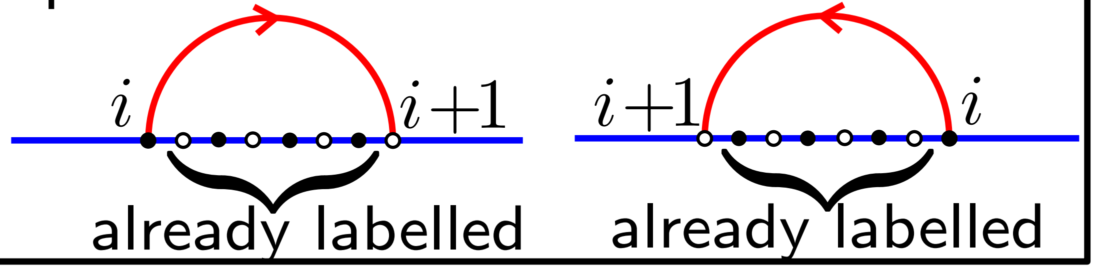

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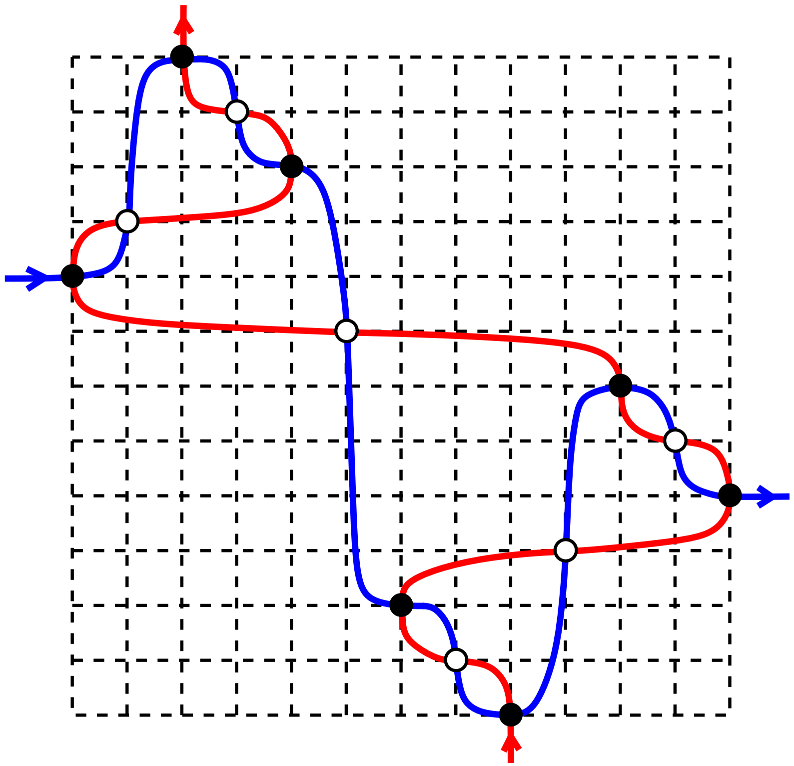

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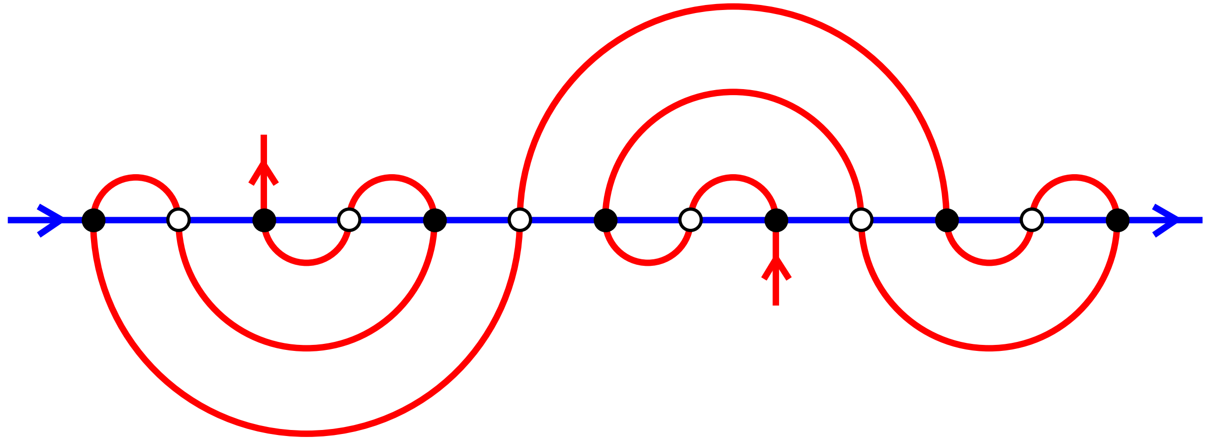

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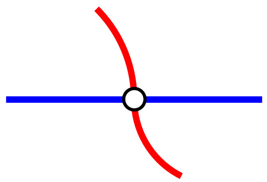
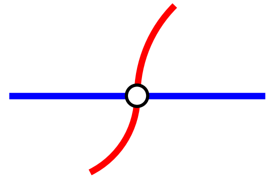
- white nodes are either  or 

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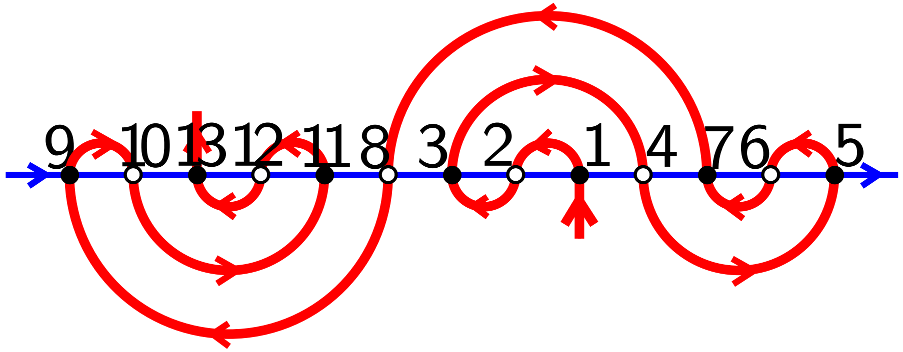

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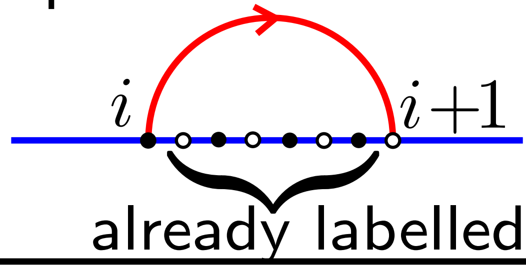

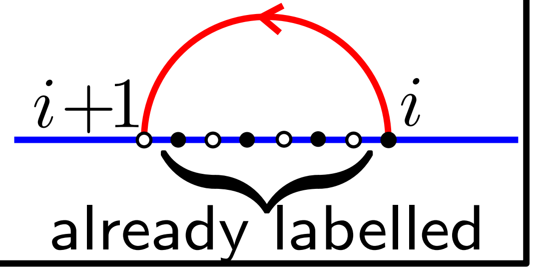

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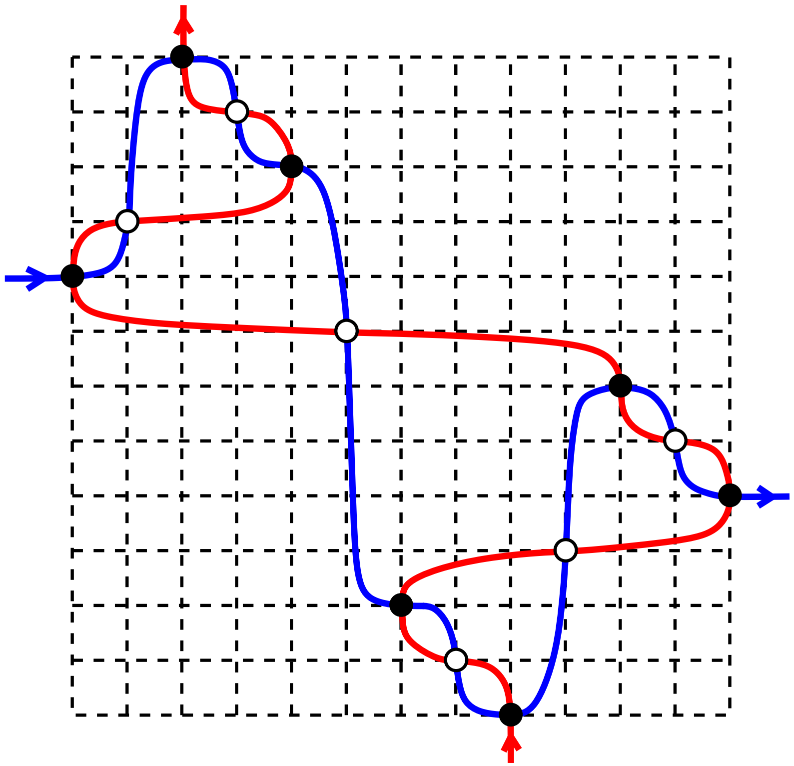

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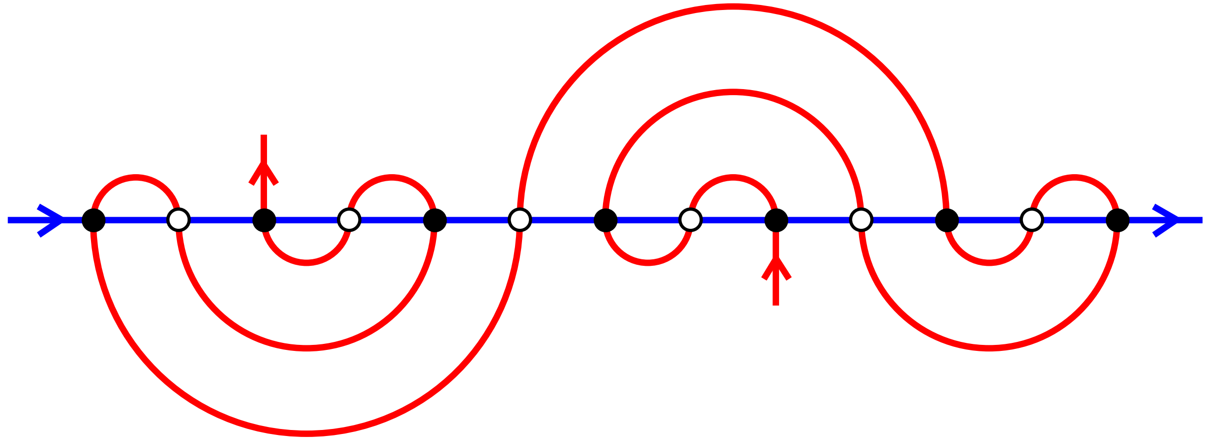
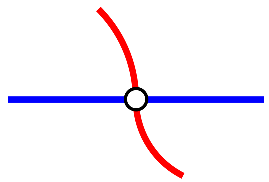
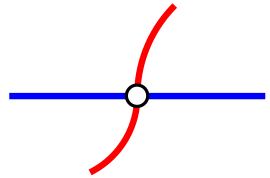
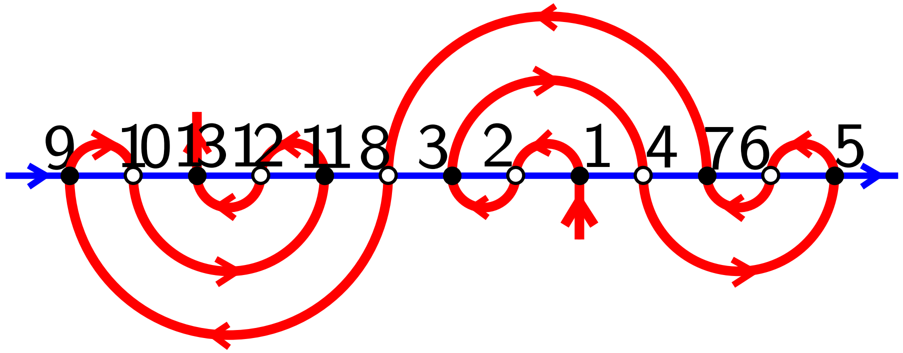
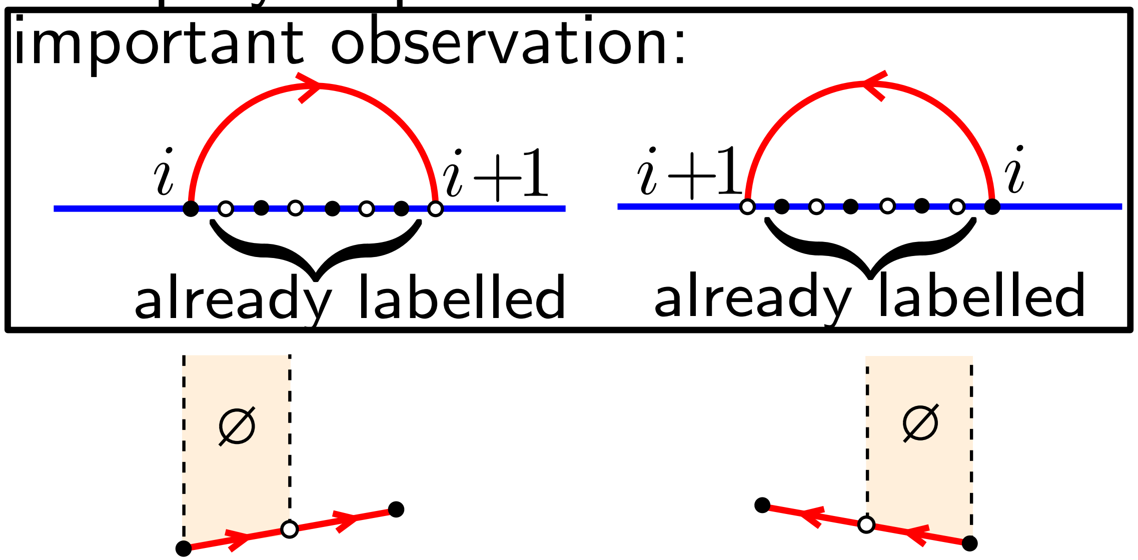


Conditions

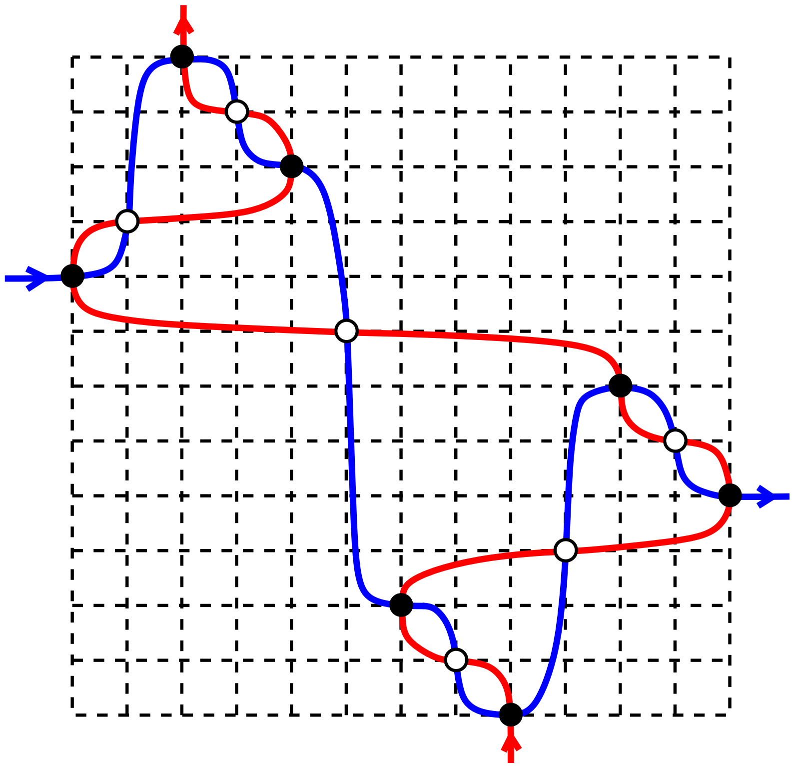
- two (bipartite) matchings missing a (black) point (one matching above, one below the blue line)
- white nodes are either  or 

Proof of $\Leftarrow$: construct permutation step by step

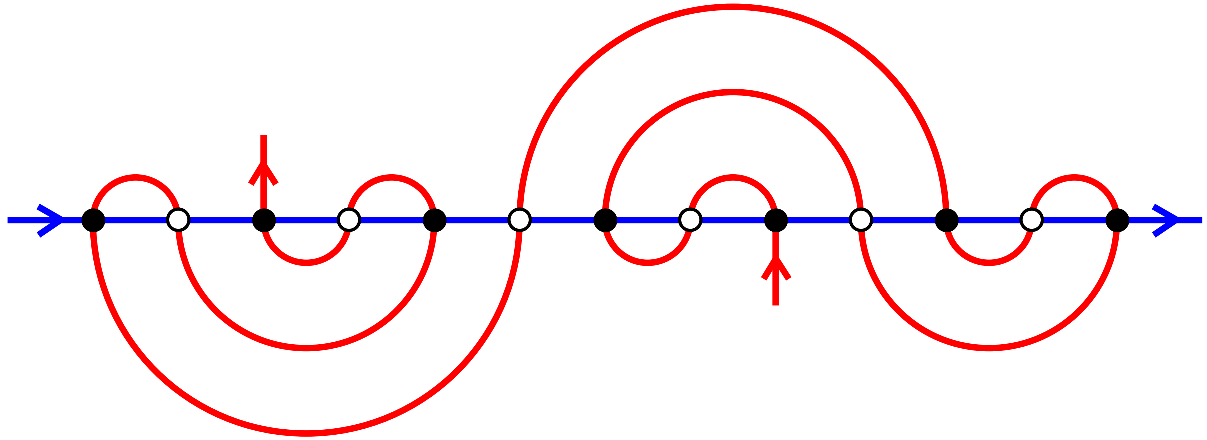




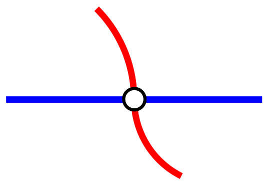
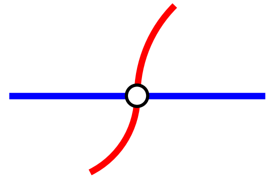
Local conditions for monotone 2-line meanders



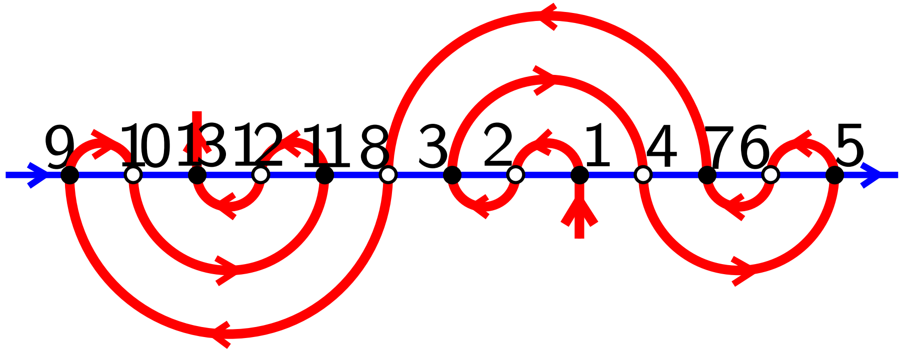
$\Leftrightarrow$



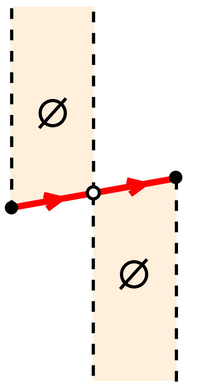
Conditions

- two (bipartite) matchings missing a (black) point (one matching above, one below the blue line)
- white nodes are either  or 

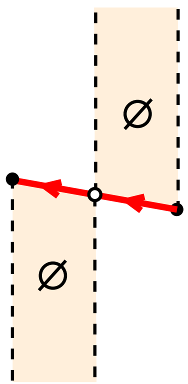
Proof of $\Leftarrow$: construct permutation step by step



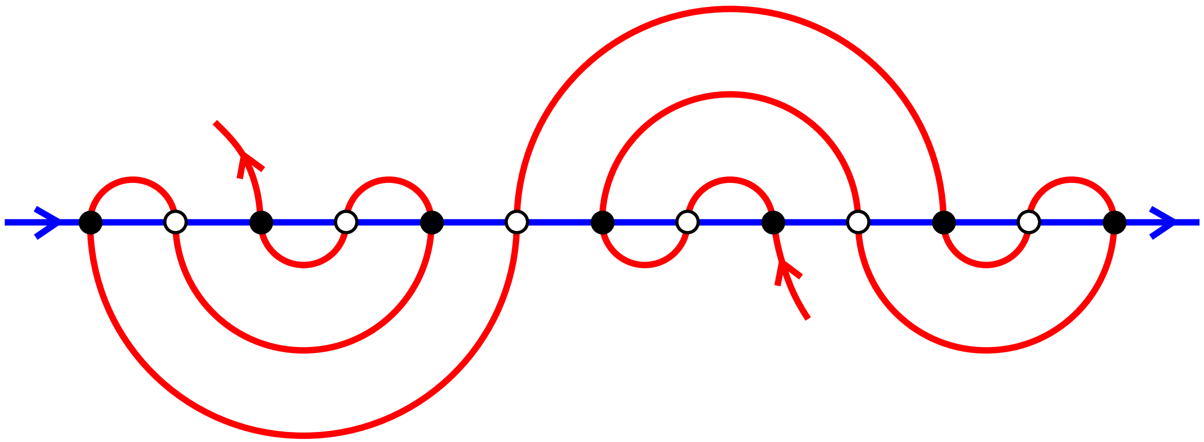
Similarly:



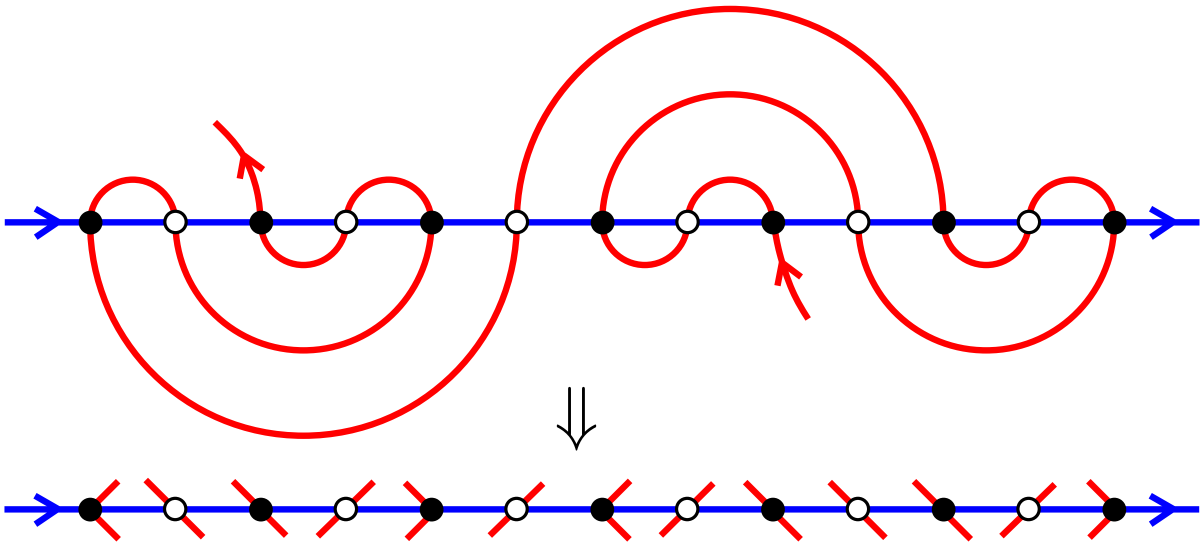
or



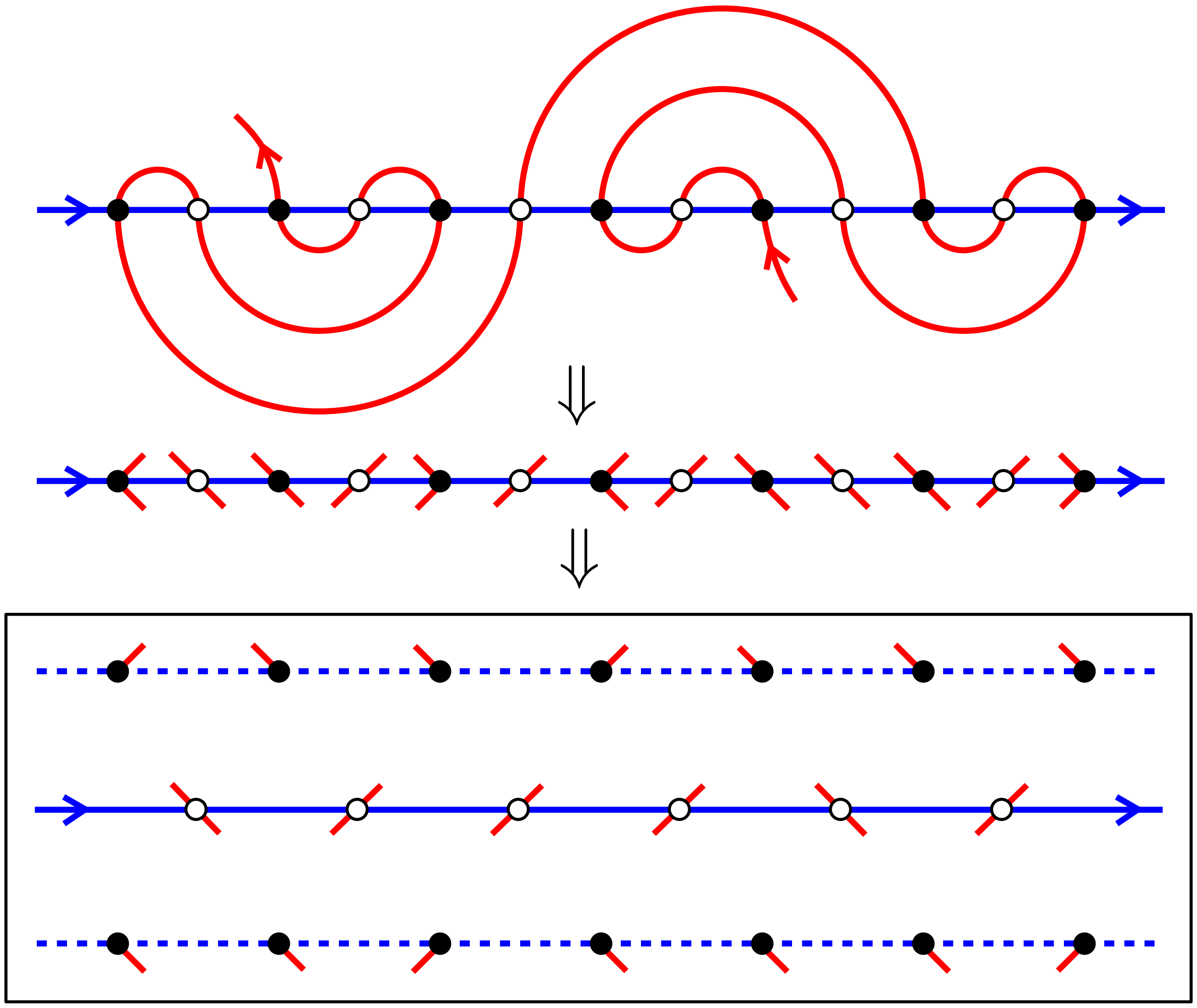
Encoding a monotone 2-line meander



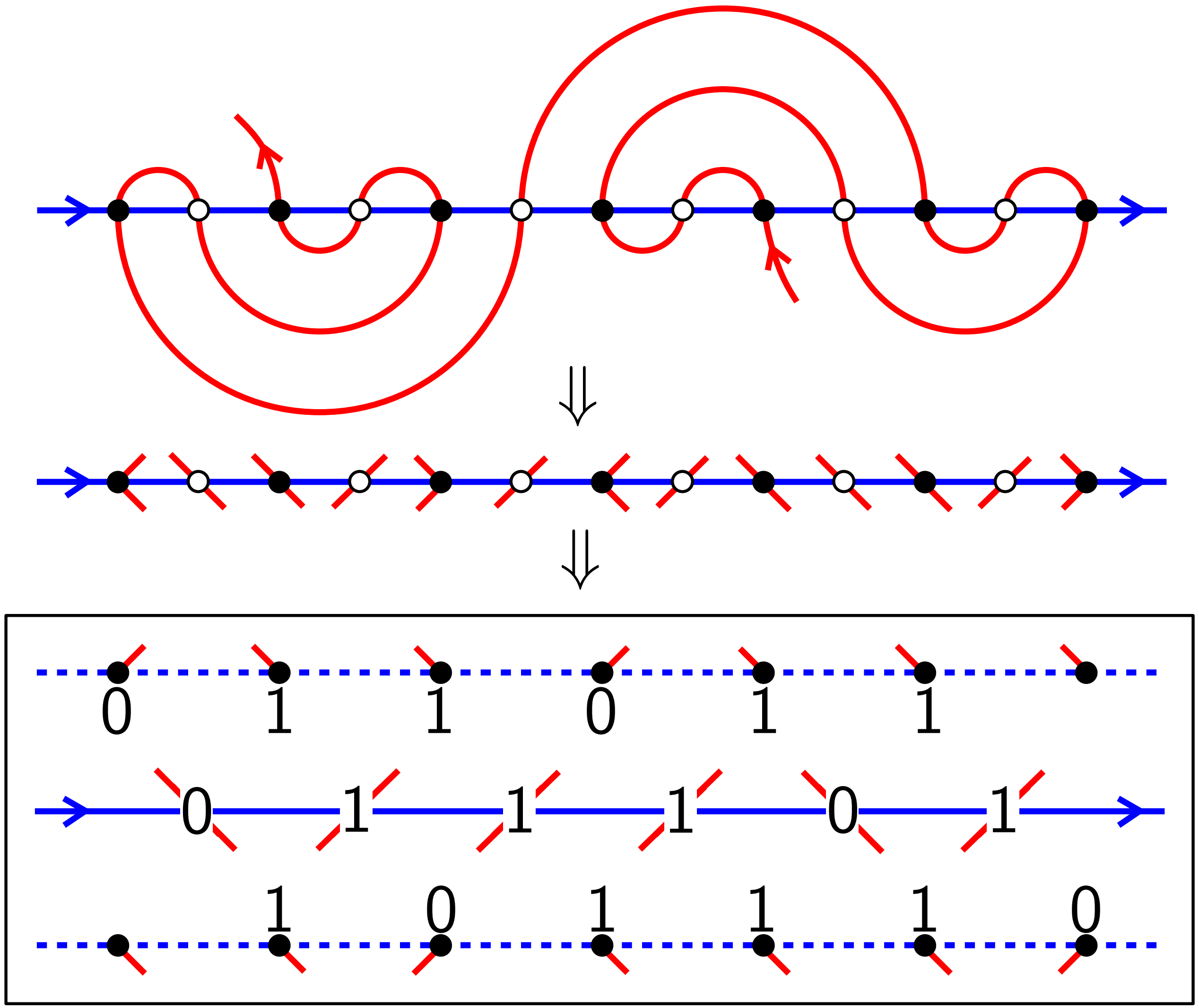
Encoding a monotone 2-line meander



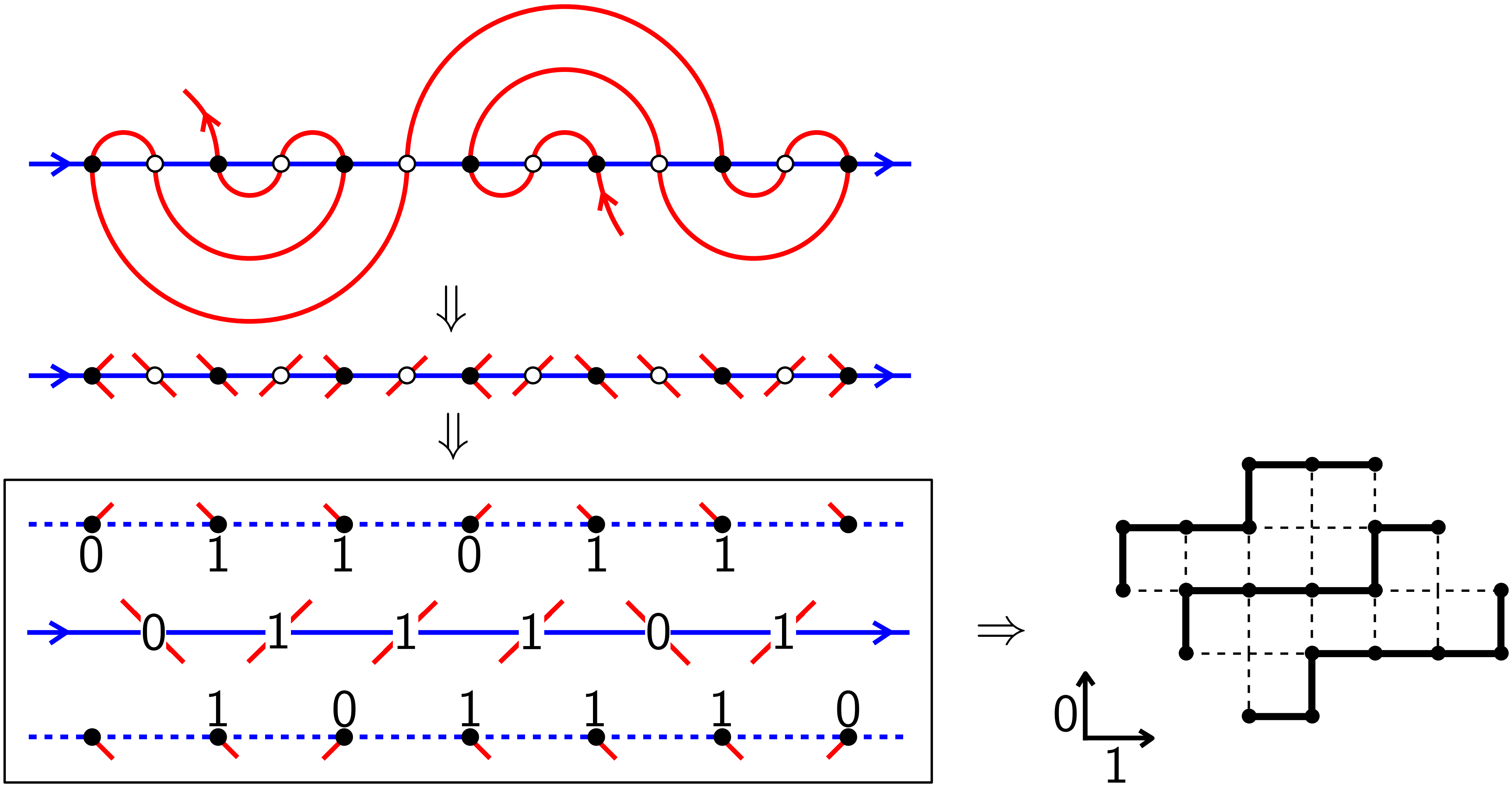
Encoding a monotone 2-line meander



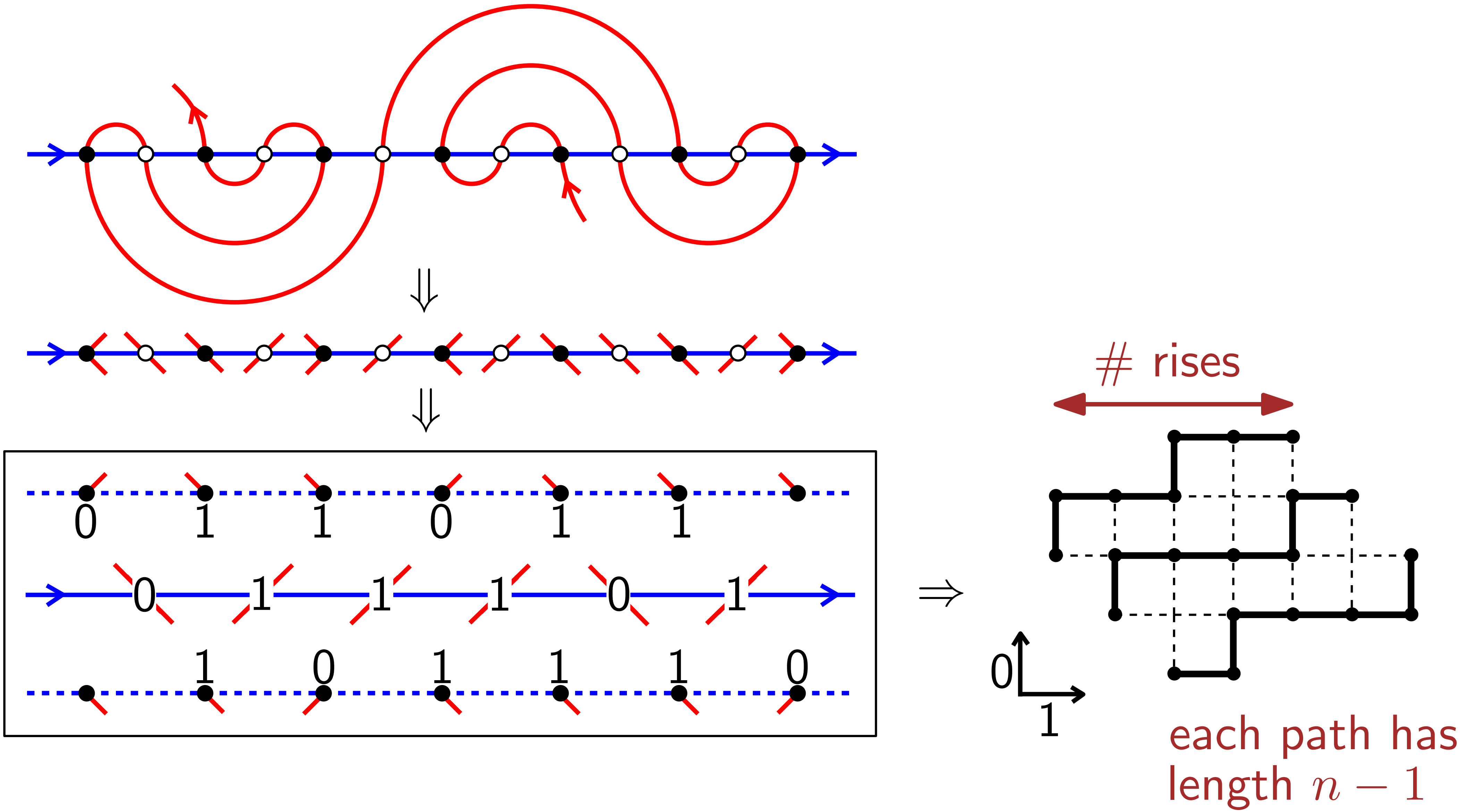
Encoding a monotone 2-line meander



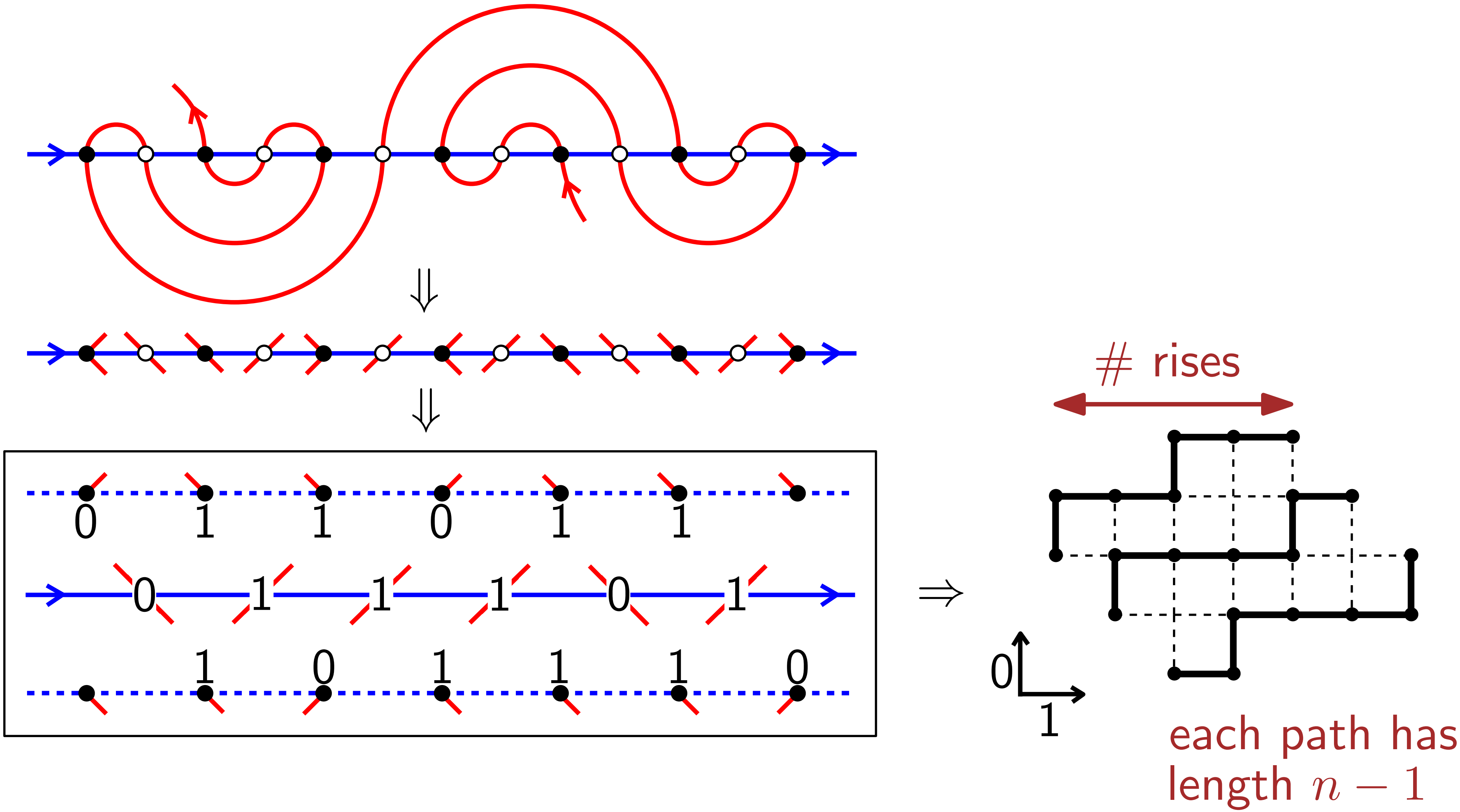
Encoding a monotone 2-line meander



Encoding a monotone 2-line meander

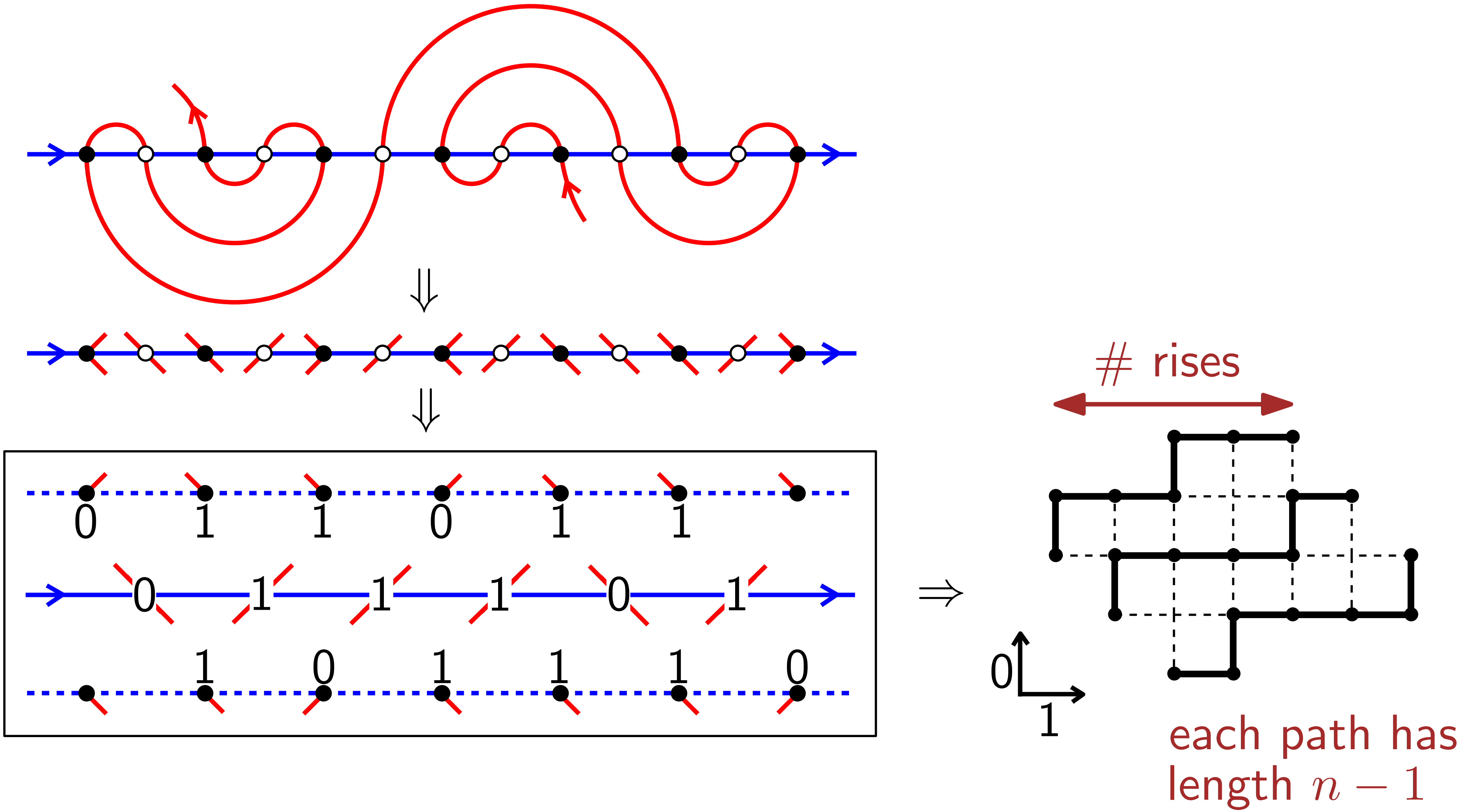


Encoding a monotone 2-line meander



close to encoding in **[Viennot'81,Dulucq-Guibert'98]**

Encoding a monotone 2-line meander

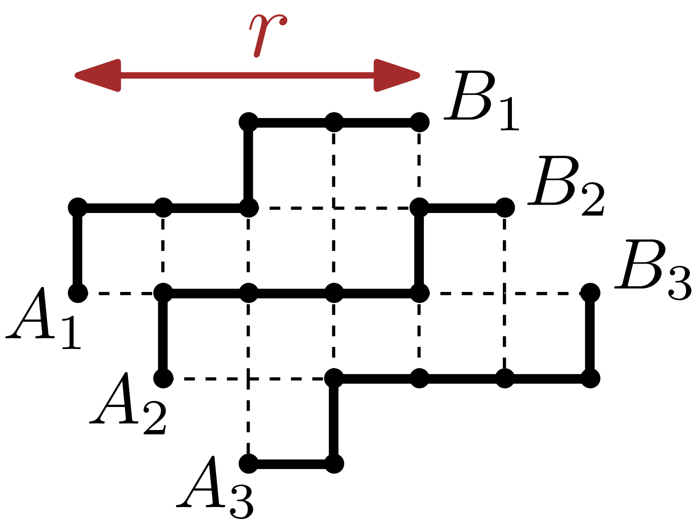


close to encoding in **[Viennot'81,Dulucq-Guibert'98]**

exactly coincides with encoding in **[Felsner-F-Noy-Orden'11]**

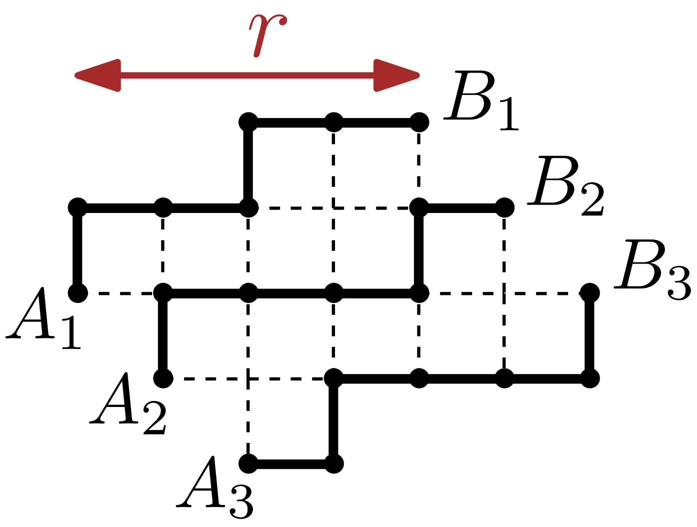
(uses "equatorial line" in separating decompositions of quadrangulations)

Enumeration using the LGV lemma



each path has length $n-1$

Enumeration using the LGV lemma



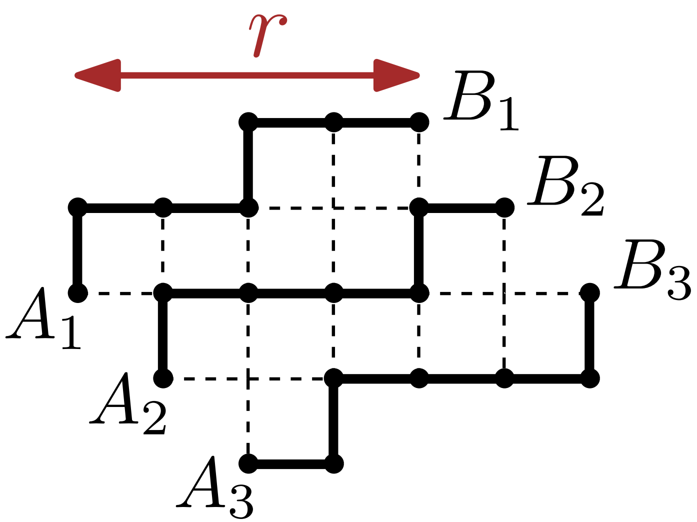
each path has length $n-1$

Let $a_{i,j} = \#$ (upright lattice paths from A_i to B_j) $= \binom{n-1}{x(B_j)-x(A_i)}$

By the **Lindstroem-Gessel-Viennot Lemma** (used in **[Viennot'81]**) the number $b_{n,r}$ of such nonintersecting triples of paths is

$$b_{n,r} = \mathrm{Det}(a_{i,j}) = \begin{vmatrix} \binom{n-1}{r} & \binom{n-1}{r+1} & \binom{n-1}{r+2} \\ \binom{n-1}{r-1} & \binom{n-1}{r} & \binom{n-1}{r+1} \\ \binom{n-1}{r-2} & \binom{n-1}{r-1} & \binom{n-1}{r} \end{vmatrix} = \frac{2}{n(n+1)^2}\binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2}$$

Enumeration using the LGV lemma



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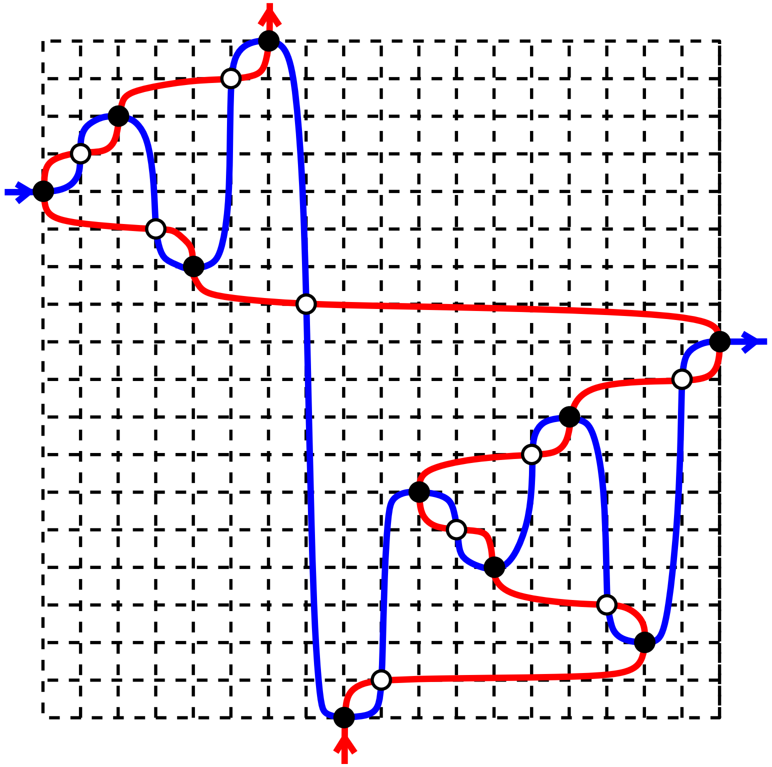
$b_{n,r}$ is also the number of reduced Baxter permutations of size n with r rises

Alternating (reduced) Baxter permutations

[Cori-Dulucq-Viennot'86], [Dulucq-Guibert'98]

1) Case n even, $n = 2k$

$\pi =$ 8 9 7 10 1 4 3 5 2 6



Alternating (reduced) Baxter permutations

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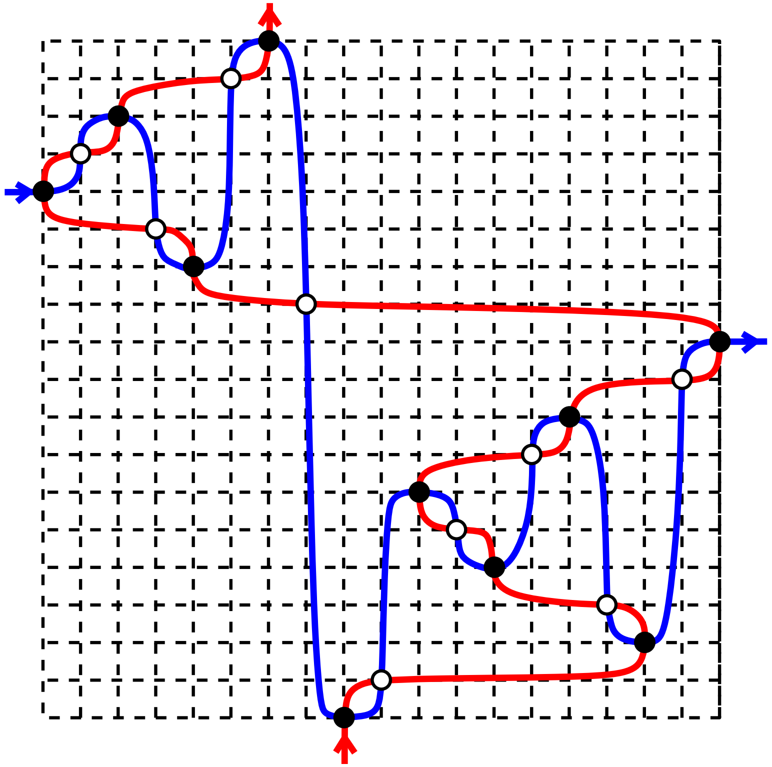

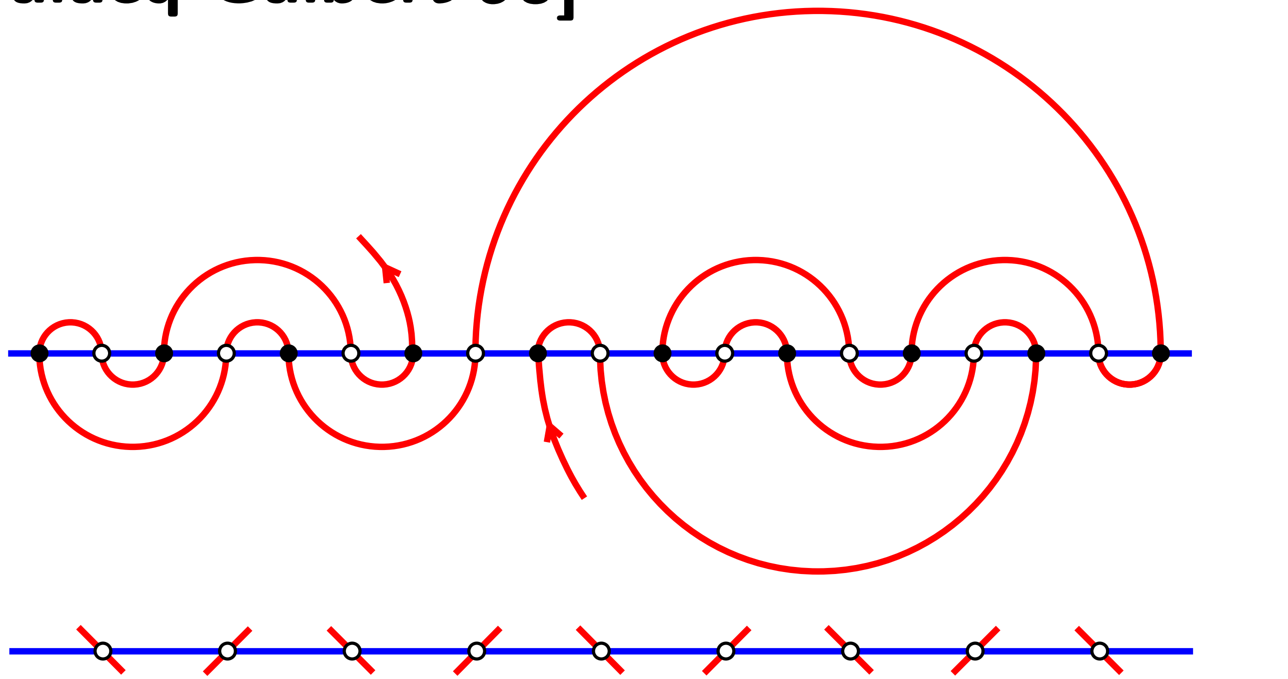

alternation $\Leftrightarrow$ middle word is $010101\ldots0$

Alternating (reduced) Baxter permutations

[Cori-Dulucq-Viennot'86], [Dulucq-Guibert'98]

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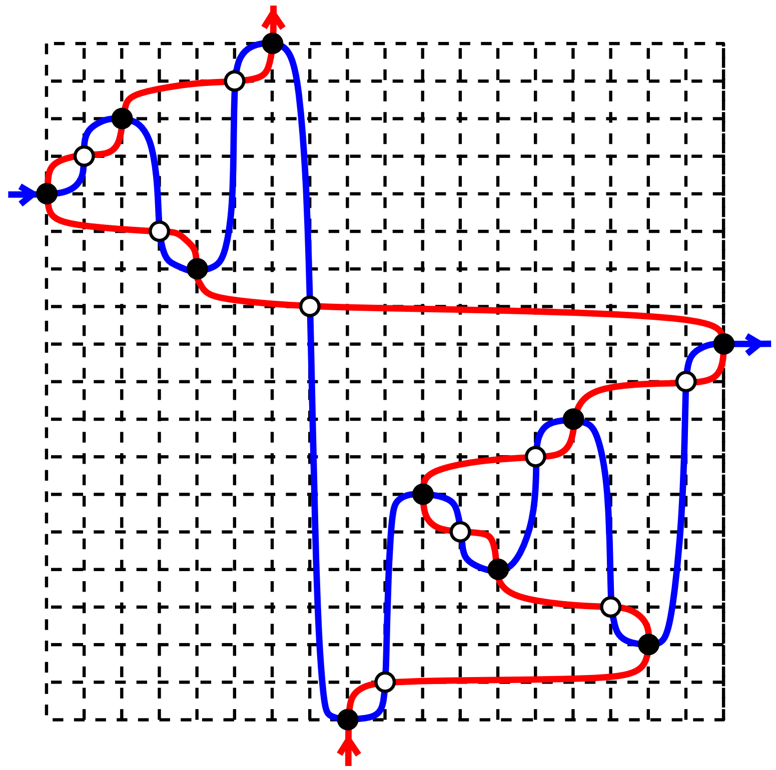
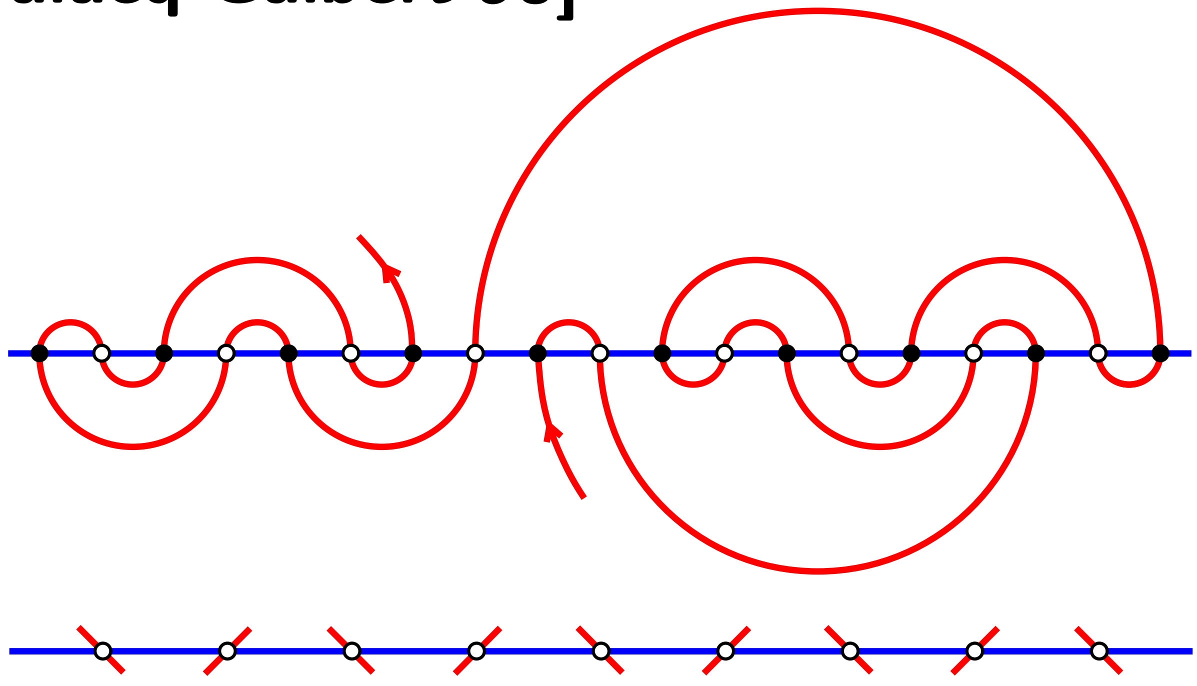
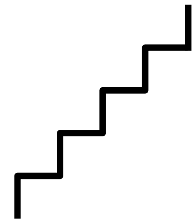
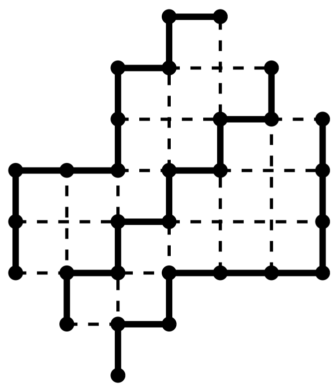




alternation $\Leftrightarrow$ middle word is $010101\ldots0$

middle path is





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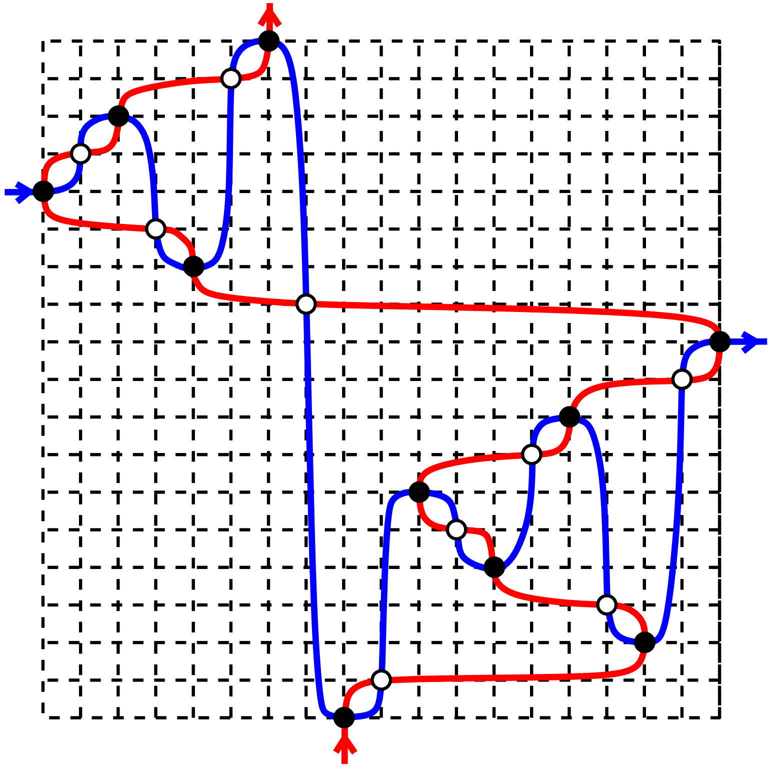

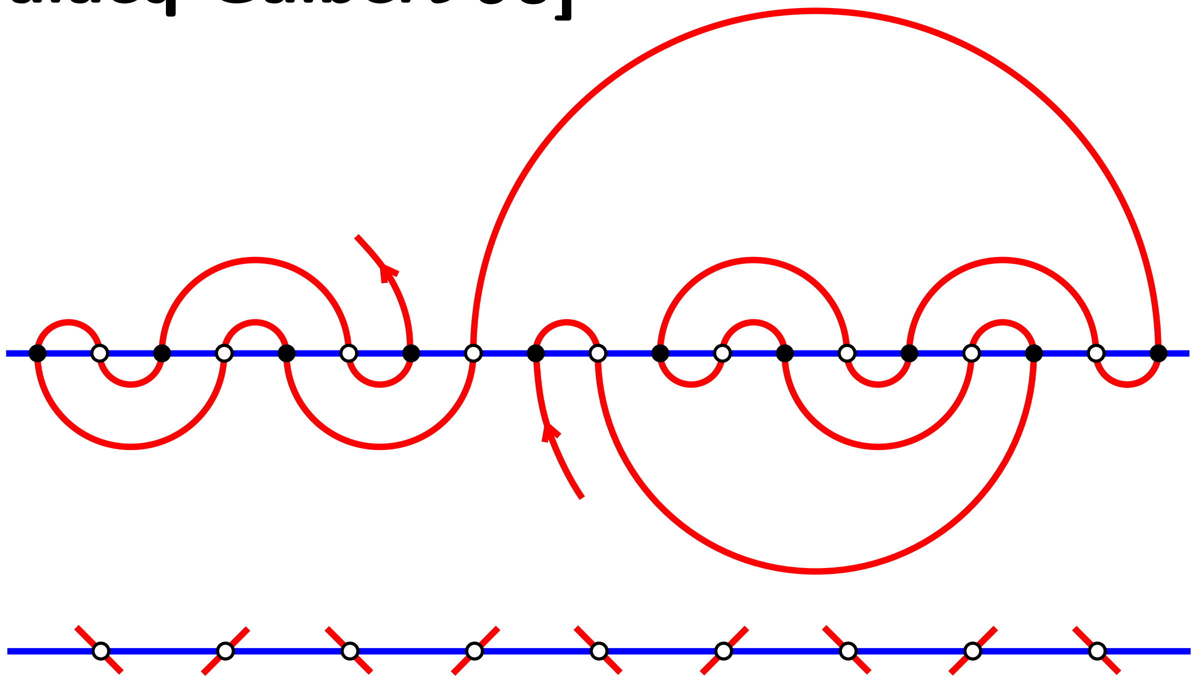

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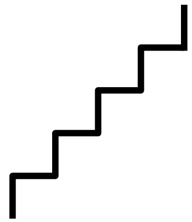
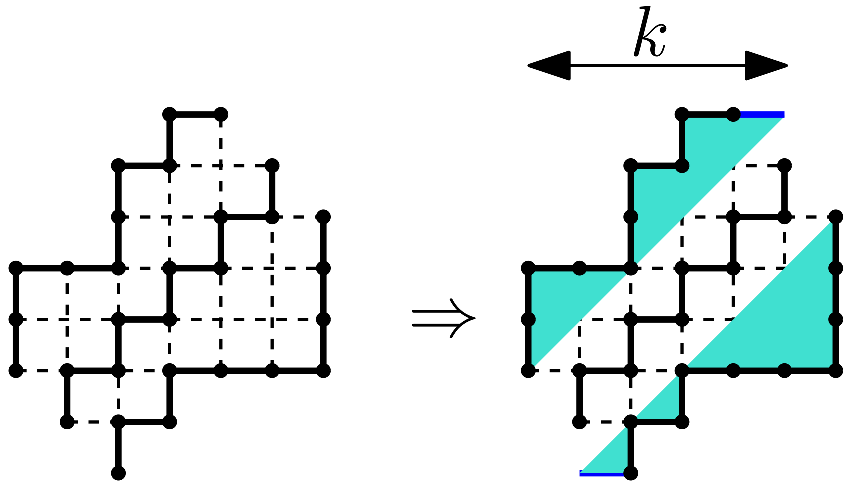




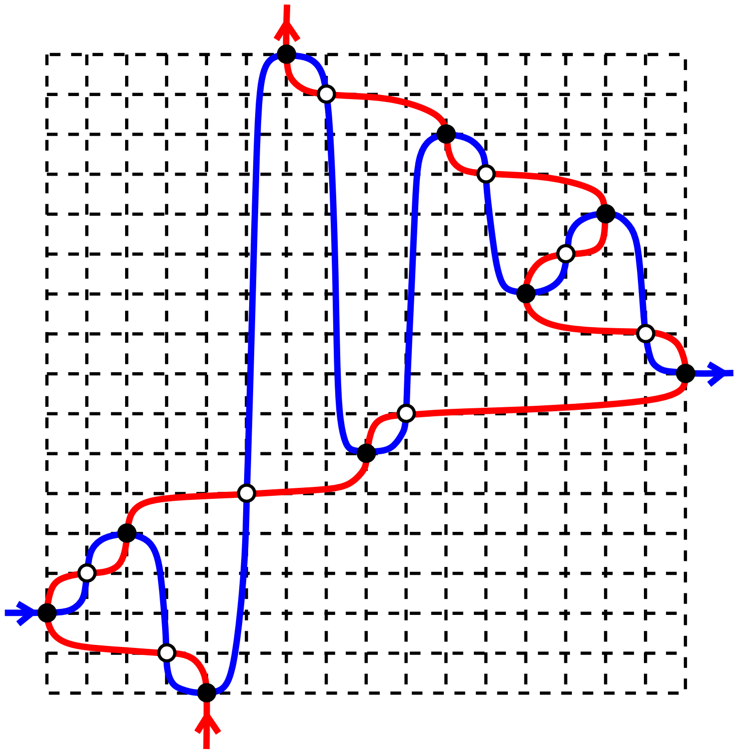
There are $\mathrm{Cat}_k\mathrm{Cat}_k$ alternating (reduced) Baxter permutations of size n

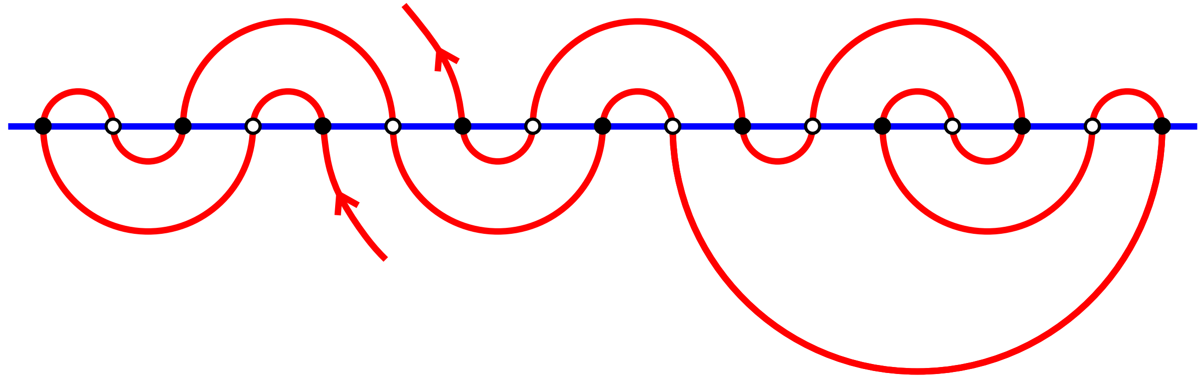
Alternating (reduced) Baxter permutations

[Cori-Dulucq-Viennot'86], [Dulucq-Guibert'98]

2) Case n odd, $n = 2k + 1$

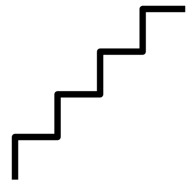
$\pi =$ 2 3 1 9 4 8 6 7 5

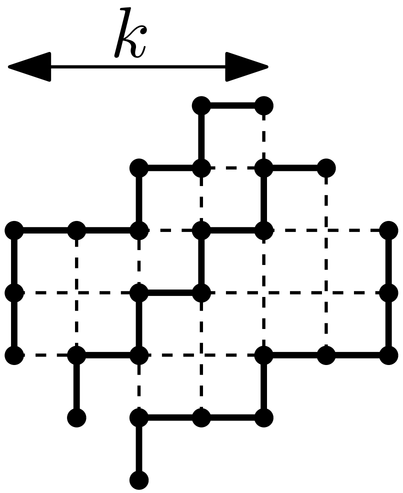




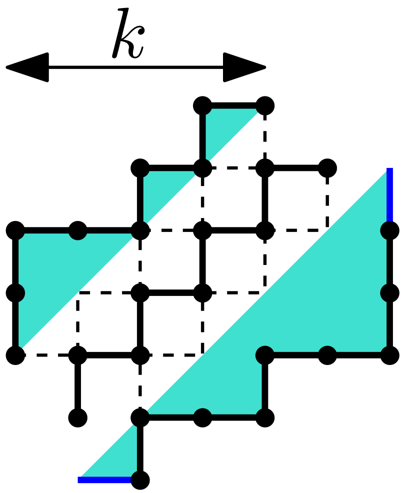
alternation $\Leftrightarrow$ middle word is $010101\dots1$

middle path is





$\Rightarrow$



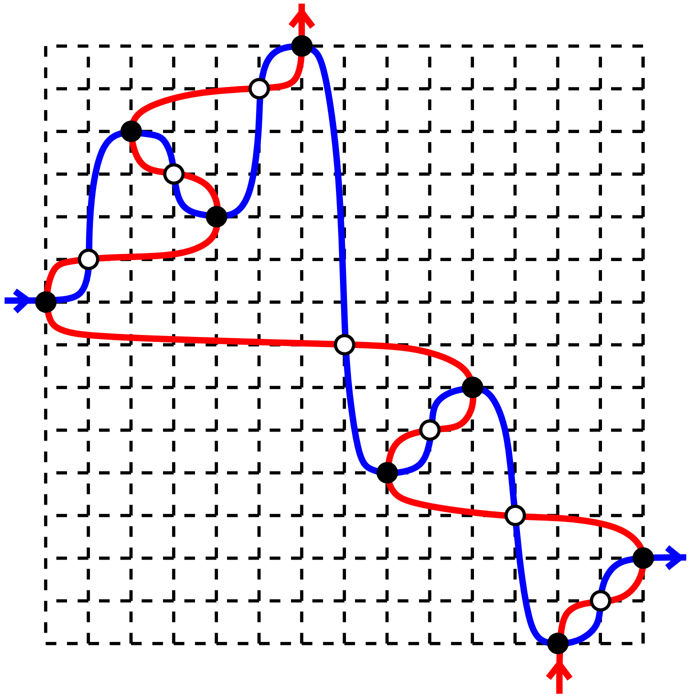
There are $\mathrm{Cat}_k\mathrm{Cat}_{k+1}$ alternating (reduced) Baxter permutations of size n

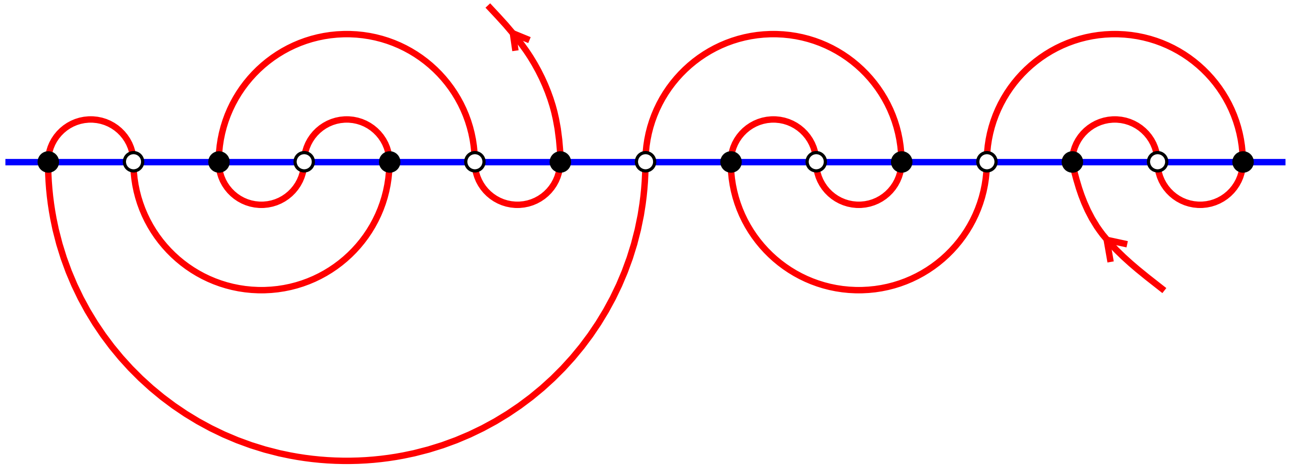
Doubly alternating (reduced) Baxter permutations

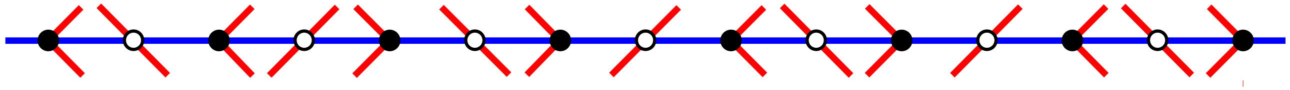
[Guibert-Linusson'00]

1) Case n even, $n = 2k$

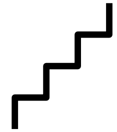
$\pi = 5\ 7\ 6\ 8\ 3\ 4\ 1\ 2$

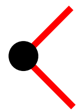
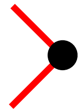






alternation of π: middle word is $010101\ldots0$

middle path is 

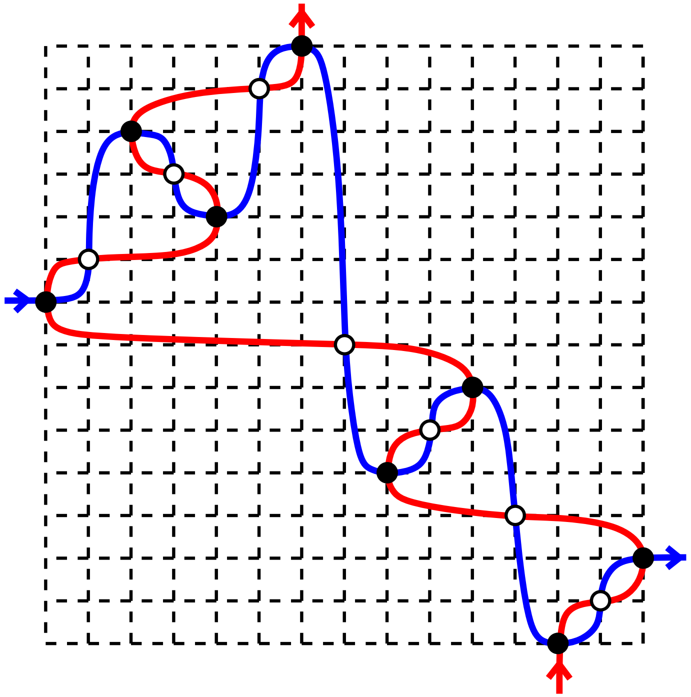
alternation of π^{-1}: black points are  or 

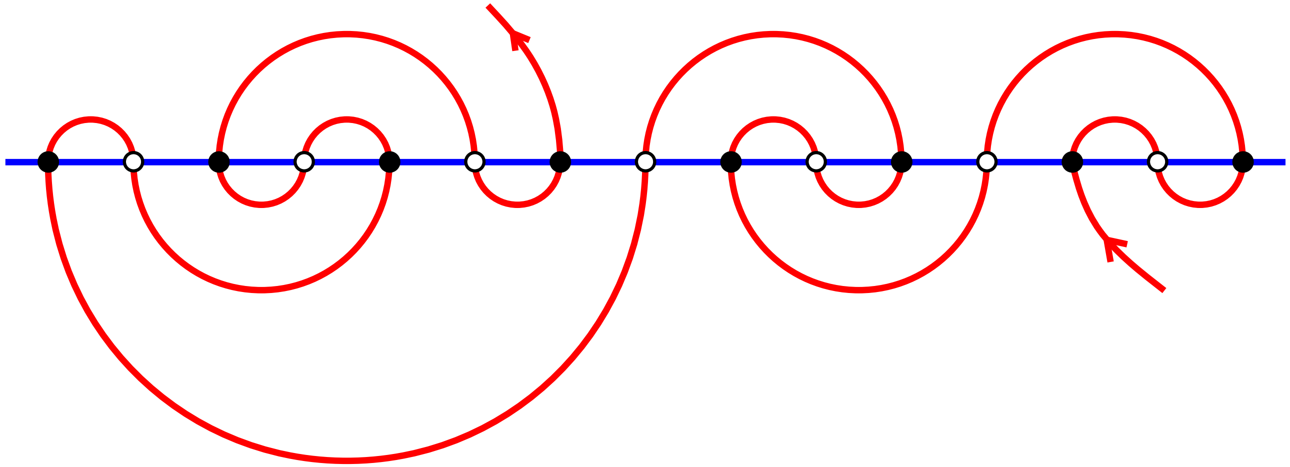
Doubly alternating (reduced) Baxter permutations

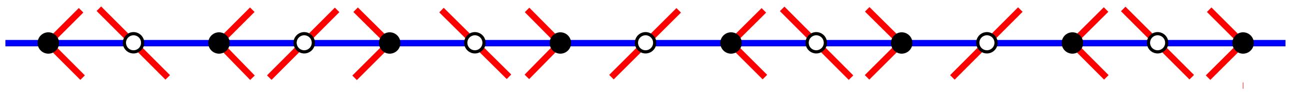
[Guibert-Linusson'00]

1) Case n even, $n = 2k$

$\pi = 5\ 7\ 6\ 8\ 3\ 4\ 1\ 2$

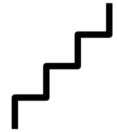




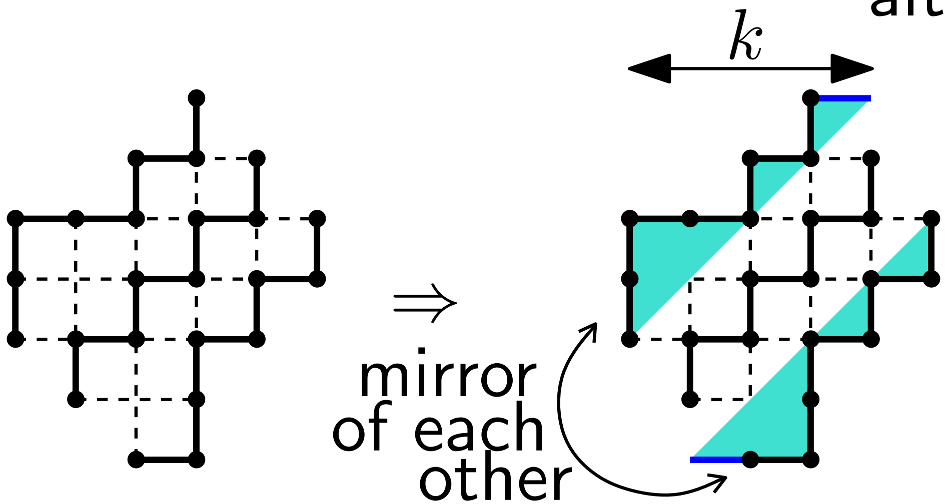


alternation of π: middle word is $010101\ldots0$

middle path is



alternation of π^{-1}: black points are or



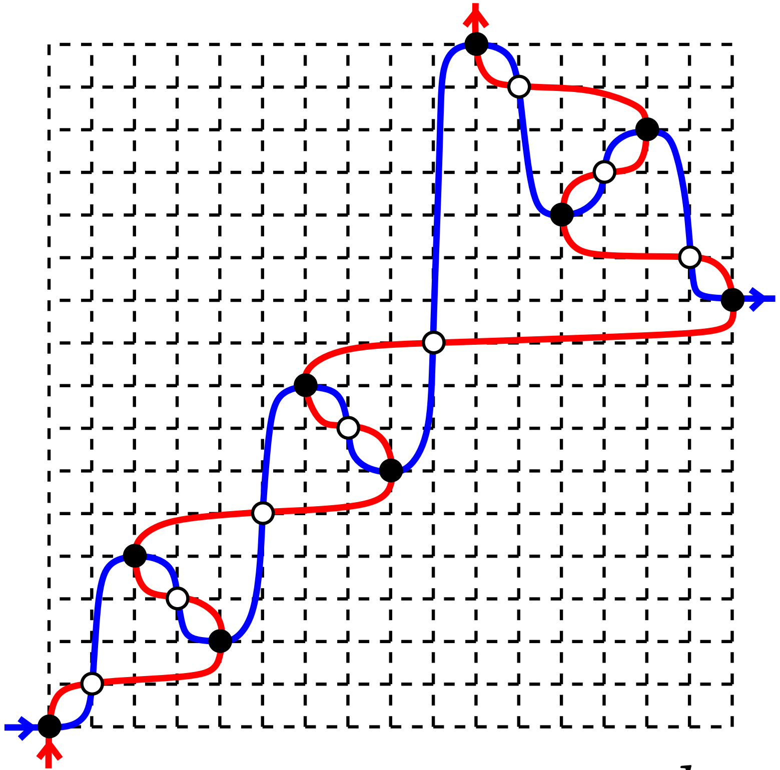
There are Cat_k doubly alternating (reduced) Baxter permutations of size n

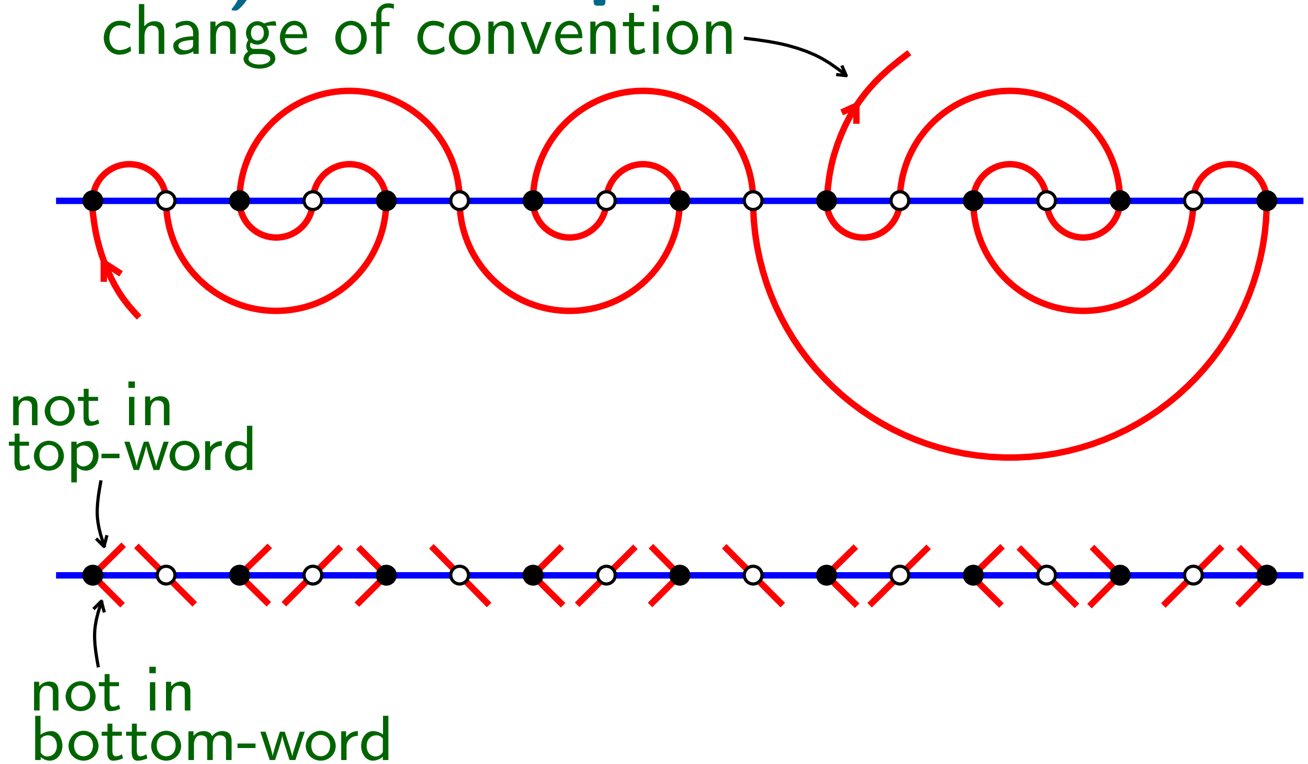
Doubly alternating (reduced) Baxter permutations

[Guibert-Linusson'00]

2) Case n odd, $n = 2k+1$

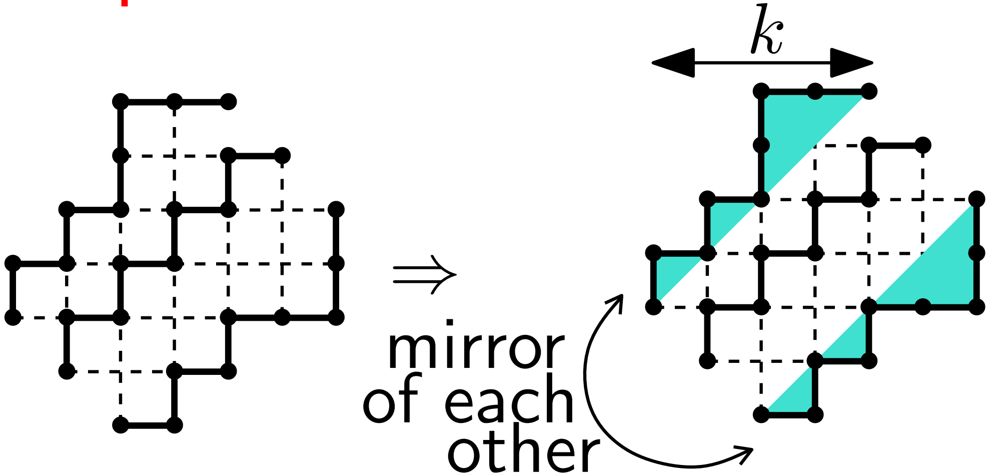
$\pi = 1\ 3\ 2\ 5\ 4\ 9\ 7\ 8\ 6$





alternation of π: middle word is $010101\ldots1$

alternation of π^{-1}: black points are or



There are Cat_k doubly alternating (reduced) Baxter permutations of size n