Baxter permutations and meanders

Éric Fusy (LIX, École Polytechnique)
Meanders on two lines

- A 2-line meander
Meanders on two lines

• A 2-line meander

encoded by a permutation

7 8 9 6 1 4 3 2 5
Meanders on two lines

- A 2-line meander encoded by a permutation

- Monotone 2-line meander:
can be obtained from two monotone lines (one in \( x \), the other in \( y \))
Meanders on two lines

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Which permutations can be obtained this way?
Meanders on two lines

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  can be obtained from two monotone lines (one in $x$, the other in $y$)

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• A 2-line meander

 encoded by a permutation

• Monotone 2-line meander:
  can be obtained from two monotone lines (one in $x$, the other in $y$)

Which permutations can be obtained this way?
Maps odd numbers to odd numbers, even numbers to even numbers
Permutations for monotone 2-line meanders
[Baxter’64, Boyce’67&’81]
Permutations for monotone 2-line meanders

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then has to go left
Permutations for monotone 2-line meanders

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Permutations for monotone 2-line meanders

[Baxter’64, Boyce’67&’81]

white points are either:

- rising
- descending

or
Permutations for monotone 2-line meanders

[Baxter’64, Boyce’67&’81]

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Permutations for monotone 2-line meanders

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Permutations for monotone 2-line meanders

[ Baxter’64, Boyce’67&’81]

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or
Permutations for monotone 2-line meanders
[Baxter’64, Boyce’67&’81]

Permutations mapping even to even, odd to odd, and satisfying condition shown on the right are called complete Baxter permutations
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**Theorem ([Boyce’81] reformulated bijectively):**
Monotone 2-line meanders with \(2n - 1\) crossings are in bijection with complete Baxter permutations on \(2n - 1\) elements.
Inverse construction
From a complete Baxter permutation to a monotone 2-line meander

white points are either:
Inverse construction
From a complete Baxter permutation to a monotone 2-line meander

white points are either:

1) Draw the blue curve

or
Inverse construction
From a complete Baxter permutation to a monotone 2-line meander

white points are either:

1) Draw the blue curve
Inverse construction
From a complete Baxter permutation to a monotone 2-line meander

white points are either:

1) Draw the blue curve
2) Draw the red curve
Inverse construction
From a complete Baxter permutation to a monotone 2-line meander

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Inverse construction
From a complete Baxter permutation to a monotone 2-line meander

white points are either:

1) Draw the blue curve
2) Draw the red curve

The two curves meet only at the permutation points (because of the empty area-property at white points)
Complete and reduced Baxter permutations
Complete and reduced Baxter permutations

- Complete one can be recovered from reduced one
Complete and reduced Baxter permutations

• complete one can be recovered from reduced one

Complete: [Diagram of complete Baxter permutations]
Reduced: [Diagram of reduced Baxter permutations]

Case of a descent
Complete and reduced Baxter permutations

- Complete one can be recovered from reduced one

Case of a descent
Complete and reduced Baxter permutations

- Complete one can be recovered from reduced one.

- Case of a descent.
Complete and reduced Baxter permutations

- complete one can be recovered from reduced one

**case of a rise**

**complete**

**reduced**
Complete and reduced Baxter permutations

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Case of a rise
Complete and reduced Baxter permutations

- complete one can be recovered from reduced one
Complete and reduced Baxter permutations

- Complete one can be recovered from reduced one.
- Reduced one is characterized by forbidden patterns $2 - 41 - 3$ and $3 - 14 - 2$. 
Complete and reduced Baxter permutations

• complete one can be recovered from reduced one
• reduced one is characterized by forbidden patterns $2 - 41 - 3$ and $3 - 14 - 2$
• permutation on white points (called anti-Baxter) is characterized by forbidden patterns $2 - 14 - 3$ and $3 - 41 - 2$
Counting results

- Baxter permutations
  - Number of reduced Baxter permutations with \( n \) elements
    
    \[
    b_n = \sum_{r=0}^{n-1} \frac{2}{n(n+1)^2} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2}
    \]

[Chung et al’78] [Mallows’79]
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    [Chung et al’78] [Mallows’79]
  - Bijective proof: [Viennot’81], [Dulucq-Guibert’98]

5 7 6 2 1 4 3 ⇔

\[
\begin{array}{c}
\begin{array}{cccccc}
5 & 7 & 6 & 2 & 1 & 4 & 3
\end{array}
\end{array}
\]
Counting results

- Baxter permutations
  - Number of reduced Baxter permutations with $n$ elements
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    b_n = \sum_{r=0}^{n-1} \frac{2}{n(n+1)^2} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2}
    \]
  - Bijective proof: [Viennot’81], [Dulucq-Guibert’98]

- Subfamilies
  - alternating [Cori-Dulucq-Viennot’86], [Dulucq-Guibert’98]
    \[
    \text{Cat}_k \text{Cat}_k \text{ if } n = 2k \quad \text{Cat}_k \text{Cat}_{k+1} \text{ if } n = 2k + 1
    \]
  - doubly alternating [Guibert-Linusson’00]
    \[
    \text{Cat}_k \text{ where } k = \lfloor n/2 \rfloor
    \]
Counting results

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  b_n = \sum_{r=0}^{n-1} \frac{2}{n(n+1)^2} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2}
  \]

  \[\text{[Chung et al'78] [Mallows'79]}\]
  - Bijective proof: \[\text{[Viennot'81], [Dulucq-Guibert'98]}\]

- Subfamilies
  - alternating \[\text{[Cori-Dulucq-Viennot'86], [Dulucq-Guibert'98]}\]
    
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    \text{Cat}_k \text{Cat}_k \text{ if } n = 2k
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    \[
    \text{Cat}_k \text{Cat}_{k+1} \text{ if } n = 2k + 1
    \]
  - doubly alternating \[\text{[Guibert-Linusson'00]}\]
    
    \[
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    \]

- anti-Baxter permutations \[\text{[Asinowski et al’10]}\]
  
  \[
  a_n = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} (-1)^i \binom{n+1}{i} b_{n+1-i}
  \]
Local conditions for monotone 2-line meanders
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Conditions
- two (bipartite) matchings missing a (black) point (one matching above, one below the blue line)
- white points are either rising or descending
Local conditions for monotone 2-line meanders

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Proof of $\Leftarrow$
Assume there is a red loop (say, clockwise):
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Proof of $\Leftarrow$
Assume there is a red loop (say, clockwise):

then the leftmost and the rightmost point on the loop are of different colors
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$\Rightarrow$ we have a 2-line meander
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- two (bipartite) matchings missing a (black) point
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Proof of $\Leftarrow$: construct permutation step by step
Local conditions for monotone 2-line meanders

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  \[\begin{array}{c}
  \text{or}
  \end{array}\]

Proof of \(\Leftarrow\): construct permutation step by step
Local conditions for monotone 2-line meanders

\[\iff\]

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\[\text{Proof of } \iff: \text{ construct permutation step by step}\]
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Proof of $\Leftarrow$: construct permutation step by step

important observation:

By similar argument as to show there is no red loop
Local conditions for monotone 2-line meanders

\[ \iff \]

**Conditions**
- two (bipartite) matchings missing a (black) point (one matching above, one below the blue line)
- white nodes are either [diagram showing two possible configurations]

**Proof of** \( \iff \): construct permutation step by step

**Important Observation:**

- already labelled
- already labelled

By similar argument as to show there is no red loop
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\[ \iff \]

\begin{itemize}
  \item two (bipartite) matchings missing a (black) point
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\end{itemize}

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important observation:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$i+1$</th>
<th>$i+1$</th>
<th>$i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>already labelled</td>
<td>already labelled</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Local conditions for monotone 2-line meanders

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- two (bipartite) matchings missing a (black) point (one matching above, one below the blue line)
- white nodes are either \[ \begin{array}{c}
\text{important observation:} \\
\begin{array}{c}
\text{already labelled} \\
i \\
\text{already labelled}
\end{array}
\quad \text{or} \\
\begin{array}{c}
\text{already labelled} \\
i+1 \\
\text{already labelled}
\end{array}
\end{array} \]

Proof of \( \Leftarrow \): construct permutation step by step
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Proof of \(\iff\): construct permutation step by step

important observation:
\[i\quad\quad\quad\quad i+1\quad\quad\quad\quad i+1\quad\quad\quad\quad i\]
\[\text{already labelled} \quad\quad\quad\quad \text{already labelled}\]

\[\emptyset\quad\quad\quad\quad \emptyset\]
Local conditions for monotone 2-line meanders

\[ \iff \]

- two (bipartite) matchings missing a (black) point
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**Conditions**

**Proof of** \( \iff \): construct permutation step by step

Similarly:

\[ \quad \text{or} \quad \]

\[ \quad \text{or} \quad \]
Encoding a monotone 2-line meander
Encoding a monotone 2-line meander
Encoding a monotone 2-line meander
Encoding a monotone 2-line meander
Encoding a monotone 2-line meander

\[ \begin{array}{cccccccc}
0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 0
\end{array} \]
Encoding a monotone 2-line meander

Each path has length $n - 1$
Encoding a monotone 2-line meander

close to encoding in [Viennot’81, Dulucq-Guibert’98]

each path has length $n - 1$
Encoding a monotone 2-line meander

Each path has length $n - 1$.

Close to encoding in [Viennot'81, Dulucq-Guibert'98]
Exactly coincides with encoding in [Felsner-F-Noy-Orden'11]
(uses "equatorial line" in separating decompositions of quadrangulations)
Enumeration using the LGV lemma

Each path has length $n - 1$
Enumeration using the LGV lemma

Let $a_{i,j} = \# (\text{upright lattice paths from } A_i \text{ to } B_j) = \binom{n-1}{x(B_j) - x(A_i)}$

By the Lindstroem-Gessel-Viennot Lemma (used in [Viennot’81]) the number $b_{n,r}$ of such nonintersecting triples of paths is

$$b_{n,r} = \text{Det}(a_{i,j}) = \begin{vmatrix} \binom{n-1}{r} & \binom{n-1}{r+1} & \binom{n-1}{r+2} \\ \binom{n-1}{r-1} & \binom{n-1}{r} & \binom{n-1}{r+1} \\ \binom{n-1}{r-2} & \binom{n-1}{r-1} & \binom{n-1}{r} \end{vmatrix} = \frac{2}{n(n+1)^2} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2}$$
Enumeration using the LGV lemma

Let \( a_{i,j} = \# \) (upright lattice paths from \( A_i \) to \( B_j \)) = \( \binom{n-1}{x(B_j) - x(A_i)} \)

By the Lindstroem-Gessel-Viennot Lemma (used in [Viennot’81]),
the number \( b_{n,r} \) of such nonintersecting triples of paths is

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\binom{n-1}{r-1} & \binom{n-1}{r} & \binom{n-1}{r+1} \\
\binom{n-1}{r-2} & \binom{n-1}{r-1} & \binom{n-1}{r}
\end{vmatrix} = \frac{2}{n(n+1)^2} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2}
\]

\( b_{n,r} \) is also the number of reduced Baxter permutations of size \( n \) with \( r \) rises
Alternating (reduced) Baxter permutations

[Cori-Dulucq-Viennot’86], [Dulucq-Guibert’98]

1) Case $n$ even, $n = 2k$

$\pi = 8 \ 9 \ 7 \ 10 \ 1 \ 4 \ 3 \ 5 \ 2 \ 6$
Alternating (reduced) Baxter permutations

[Cori-Dulucq-Viennot’86], [Dulucq-Guibert’98]

1) Case $n$ even, $n = 2k$

$\pi = 8 \ 9 \ 7 \ 10 \ 1 \ 4 \ 3 \ 5 \ 2 \ 6$

alternation $\iff$ middle word is $010101\ldots0$
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$\pi = 8 \ 9 \ 7 \ 10 \ 1 \ 4 \ 3 \ 5 \ 2 \ 6$

alternation $\Leftrightarrow$ middle word is $010101\ldots0$

middle path is

...
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alternation $\Leftrightarrow$ middle word is 010101...0

middle path is

There are $\text{Cat}_k \text{Cat}_k$ alternating (reduced) Baxter permutations of size $n$
Alternating (reduced) Baxter permutations

[Cori-Dulucq-Viennot’86], [Dulucq-Guibert’98]

2) Case $n$ odd, $n = 2k + 1$

There are $\text{Cat}_k \text{Cat}_{k+1}$ alternating (reduced) Baxter permutations of size $n$
Doubly alternating (reduced) Baxter permutations
[Guibert-Linusson’00]

1) Case \( n \) even, \( n = 2k \)

\[ \pi = 5 \ 7 \ 6 \ 8 \ 3 \ 4 \ 1 \ 2 \]

alternation of \( \pi \): middle word is 010101...0

middle path is

alternation of \( \pi^{-1} \): black points are \( \bullet \) or \( \circ \)
Doubly alternating (reduced) Baxter permutations
[Guibert-Linusson’00]

1) Case $n$ even, $n = 2k$

$\pi = 5 \ 7 \ 6 \ 8 \ 3 \ 4 \ 1 \ 2$

alternation of $\pi$: middle word is $010101 \ldots 0$

middle path is $\Rightarrow$

alternation of $\pi^{-1}$: black points are $\bullet$ or $\circ$

There are $\text{Cat}_k$ doubly alternating (reduced) Baxter permutations of size $n$
2) Case $n$ odd, $n = 2k + 1$

There are $\text{Cat}_k$ doubly alternating (reduced) Baxter permutations of size $n$