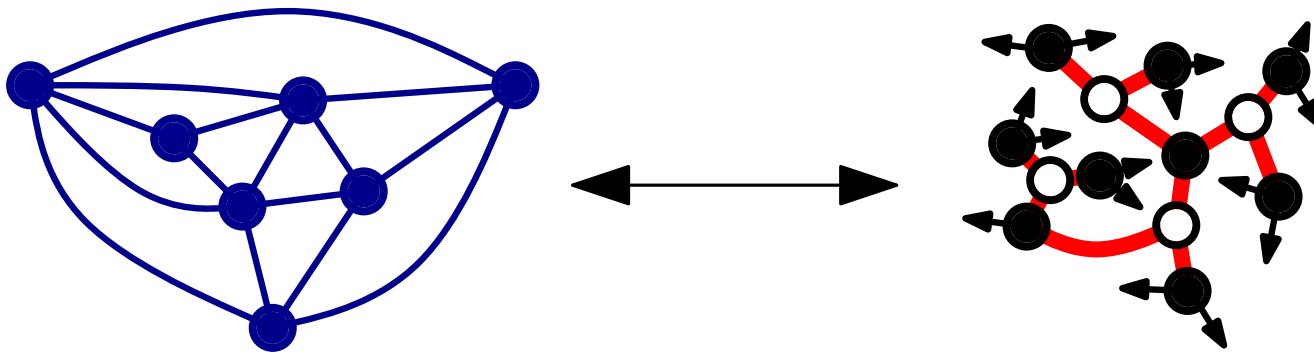


# A master bijection for planar maps and its applications

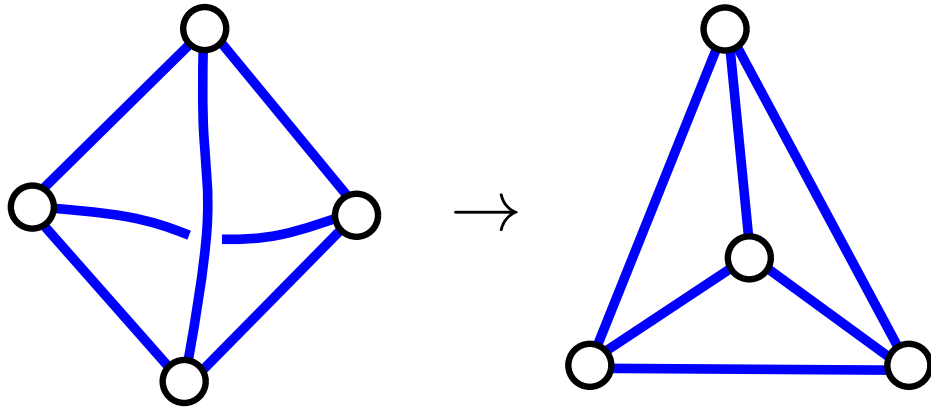
Éric Fusy (CNRS/LIX)  
Joint work with Olivier Bernardi (MIT)



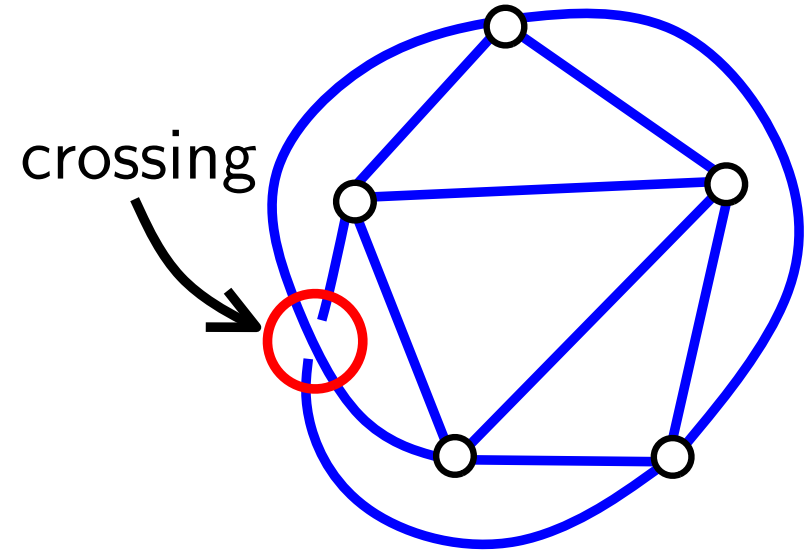
Madrid, June 2011

## Planar graphs. Definition

A **planar graph** is a graph that can be drawn in  $\mathbb{R}^2$  without edge-crossing



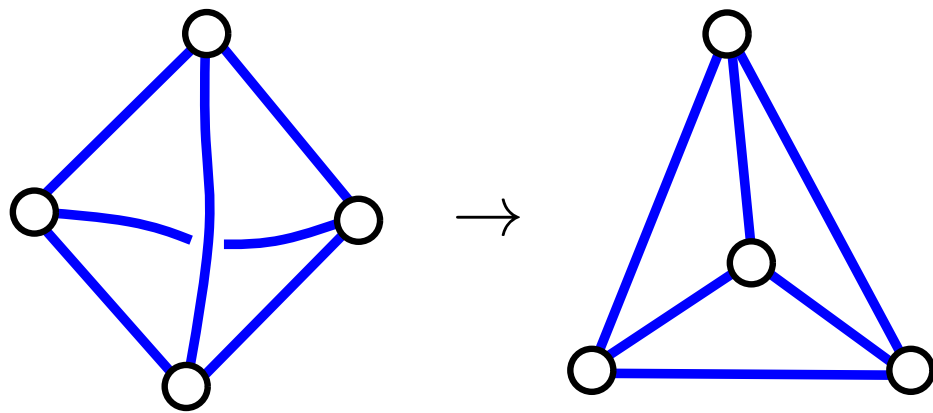
$K_4$  is planar



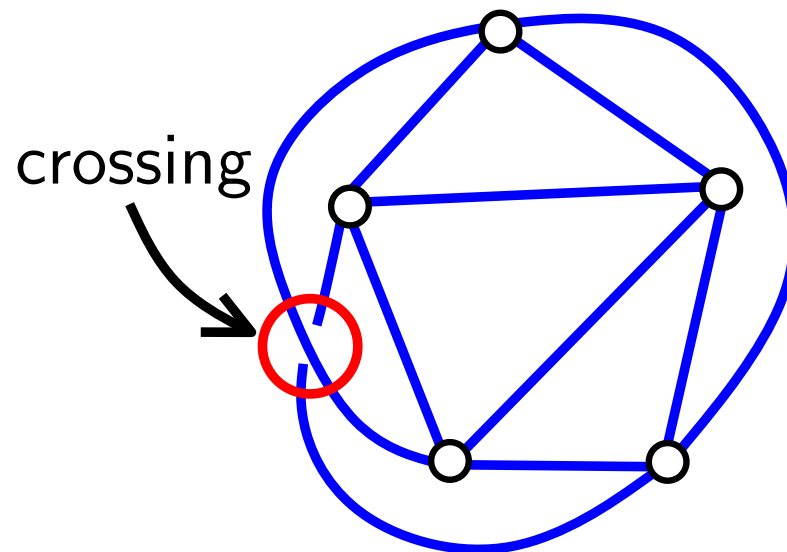
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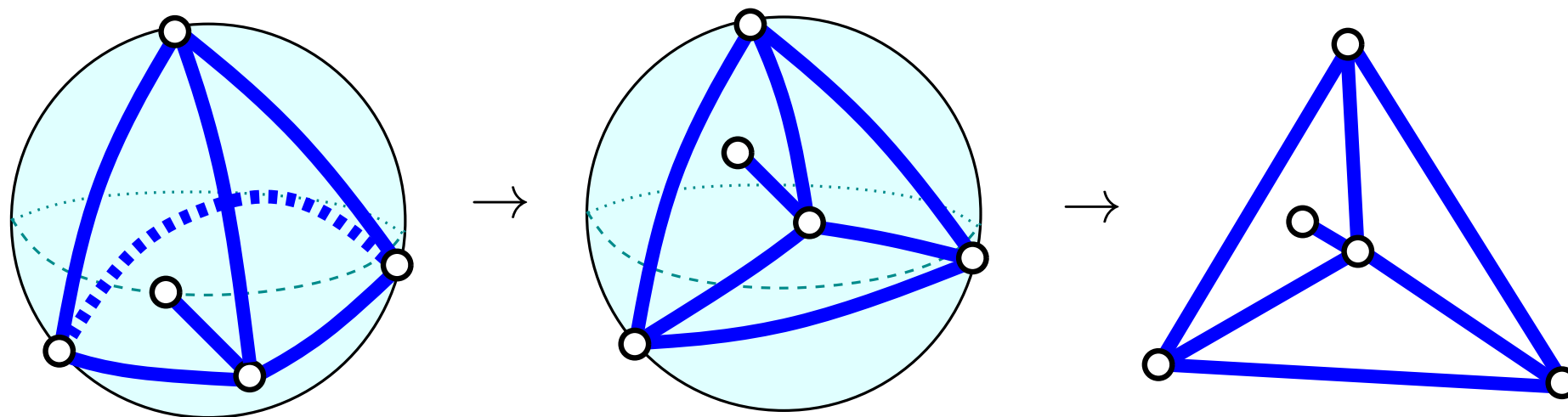


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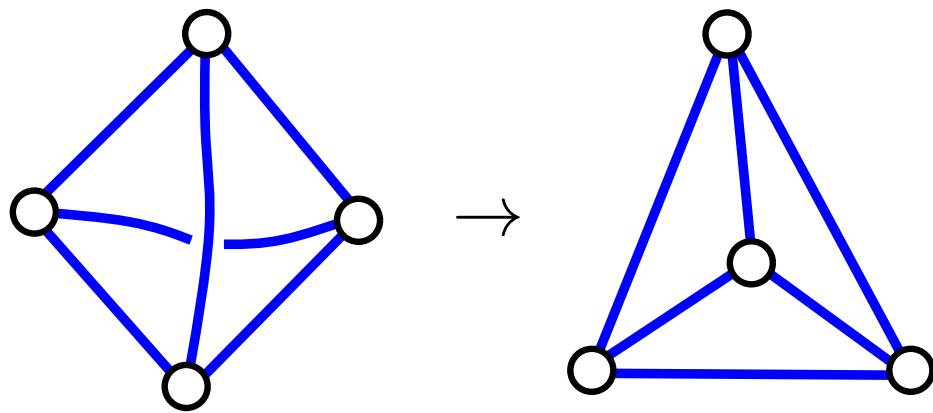
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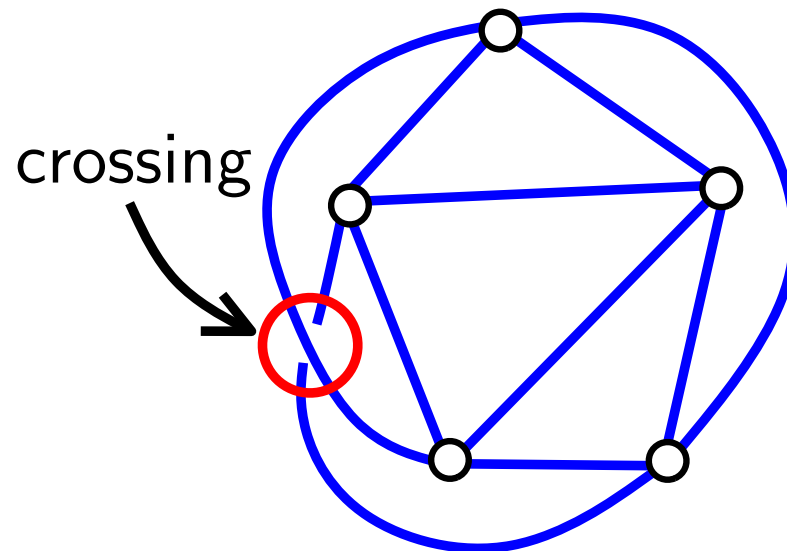


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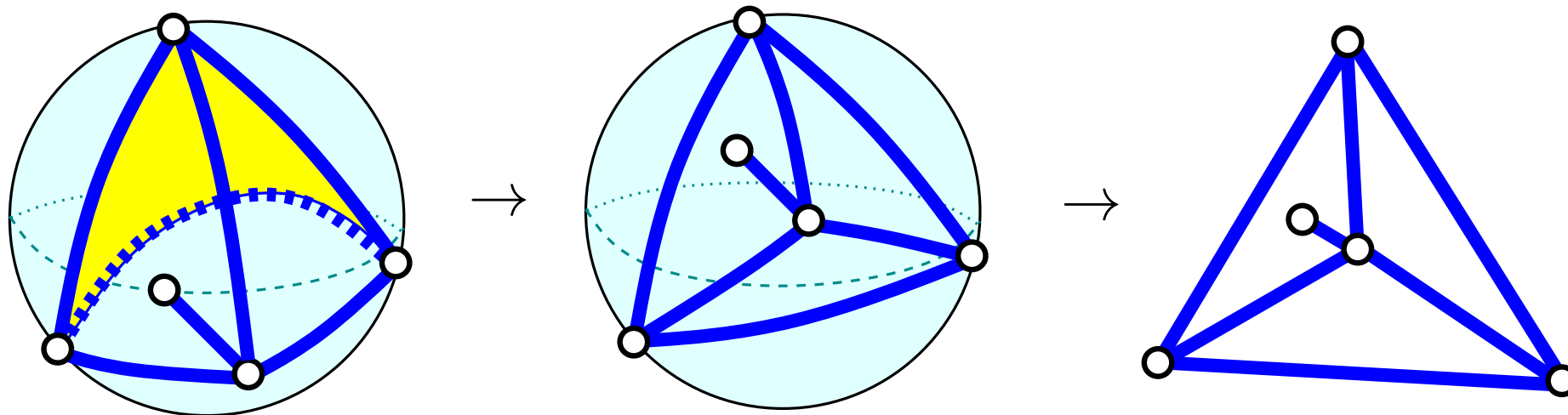


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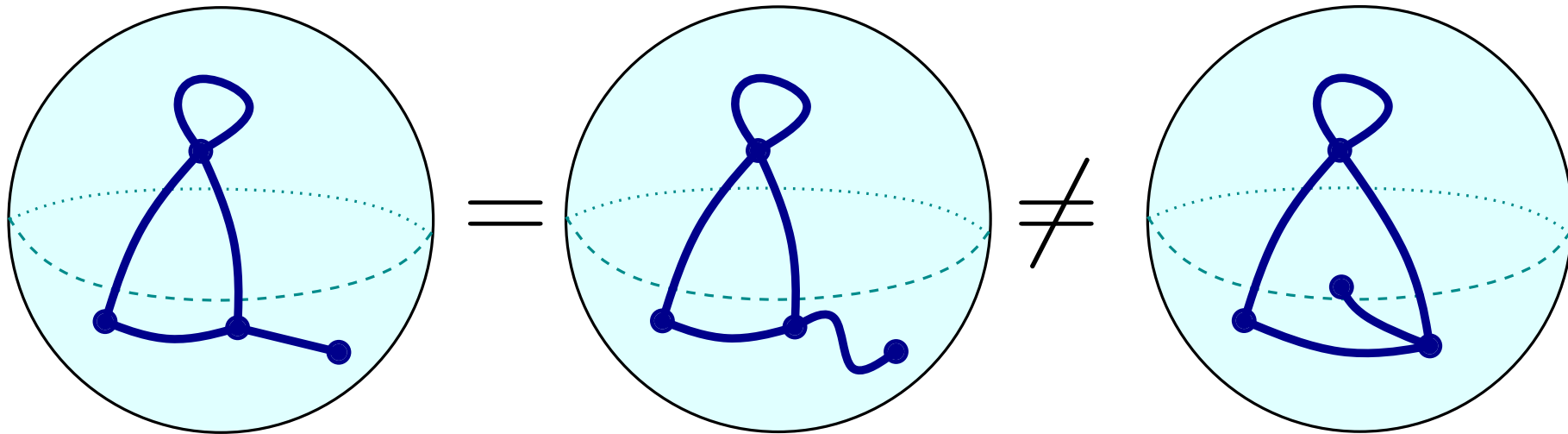
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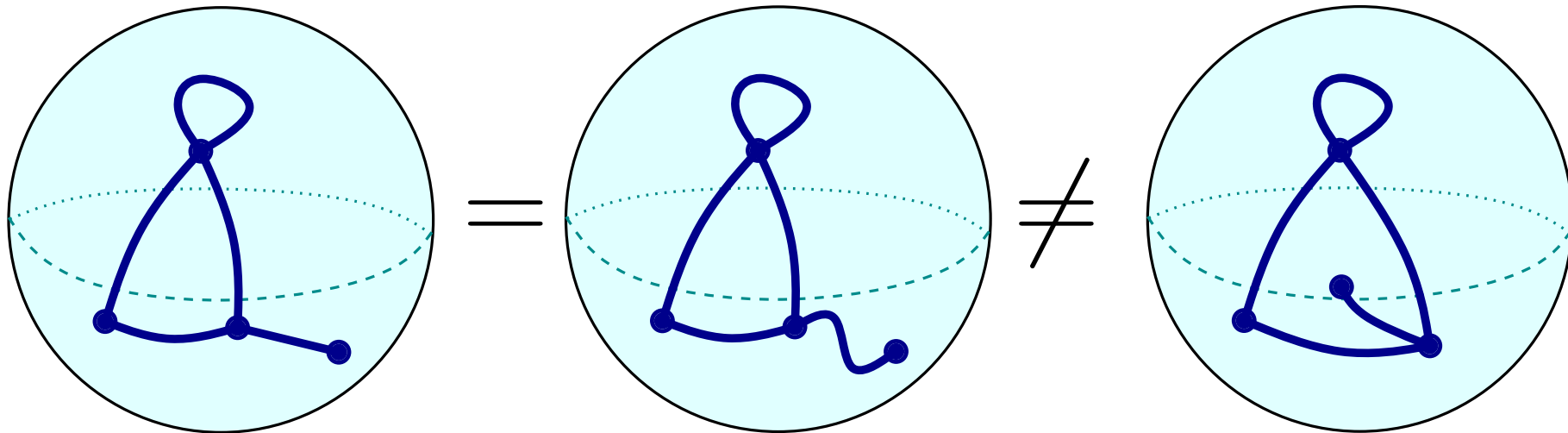
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- A **planar map** is a connected planar graph drawn in **the sphere** considered up to continuous deformation.



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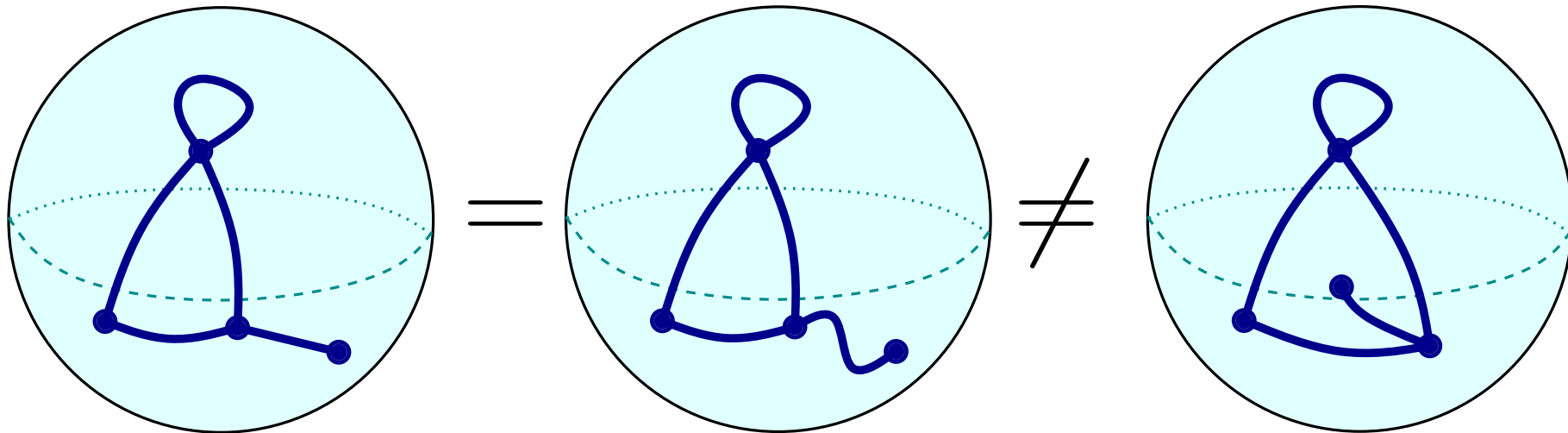
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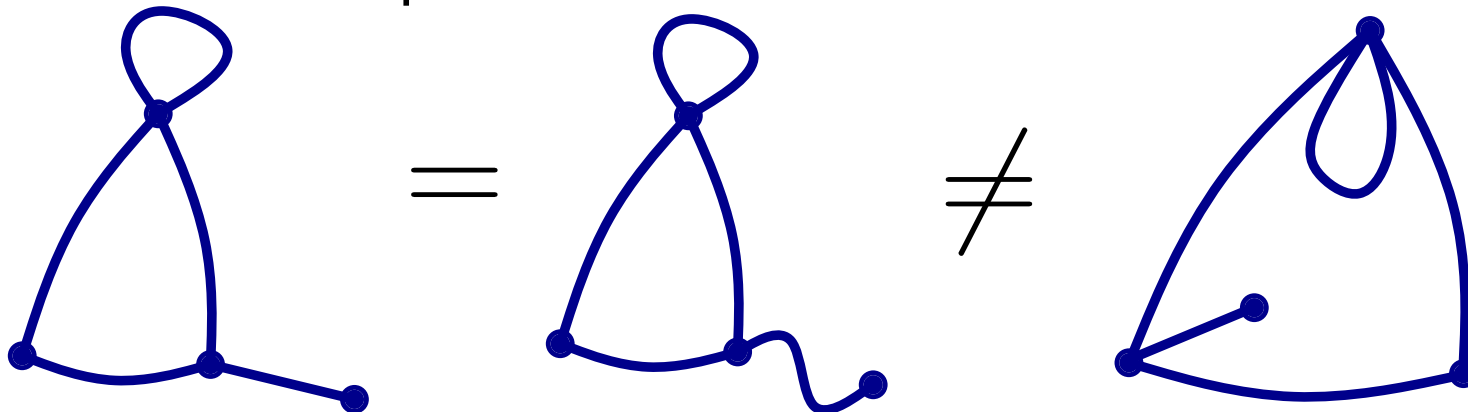
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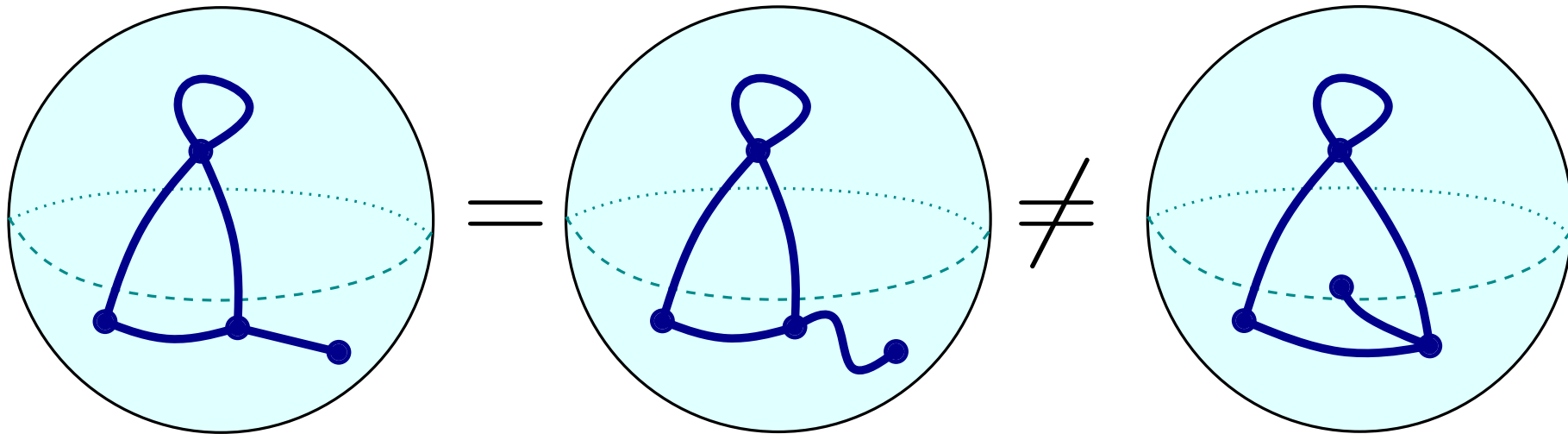
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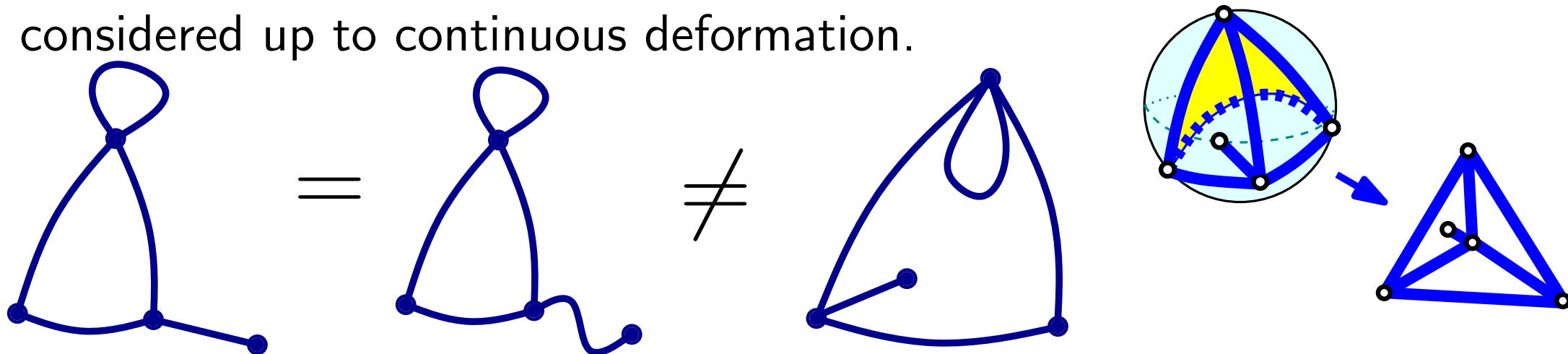
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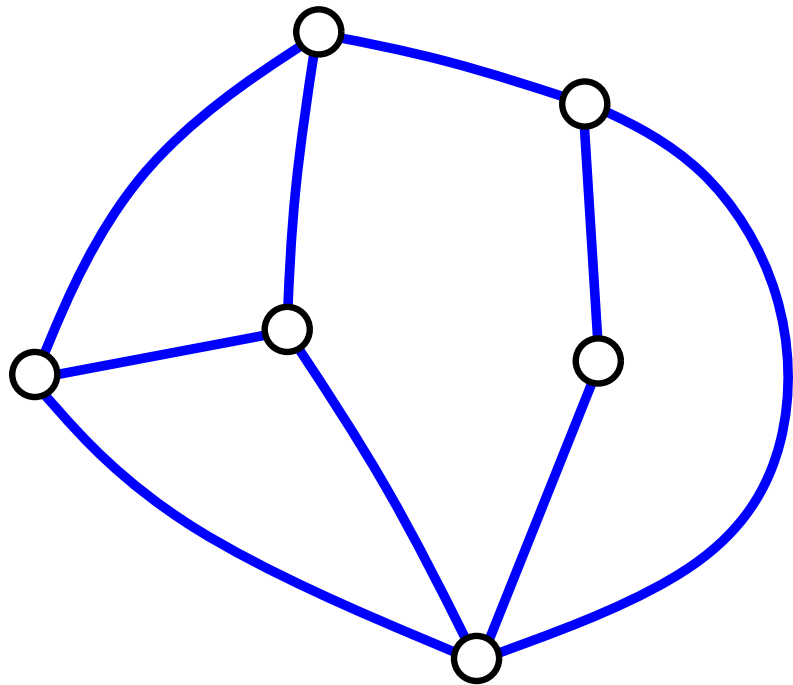


**Rk:** Plane map = planar map **with a marked face** (the outer face)

# The Euler relation

Let  $M = (V, E, F)$  be a planar map. Then

$$|V| - |E| + |F| = 2$$



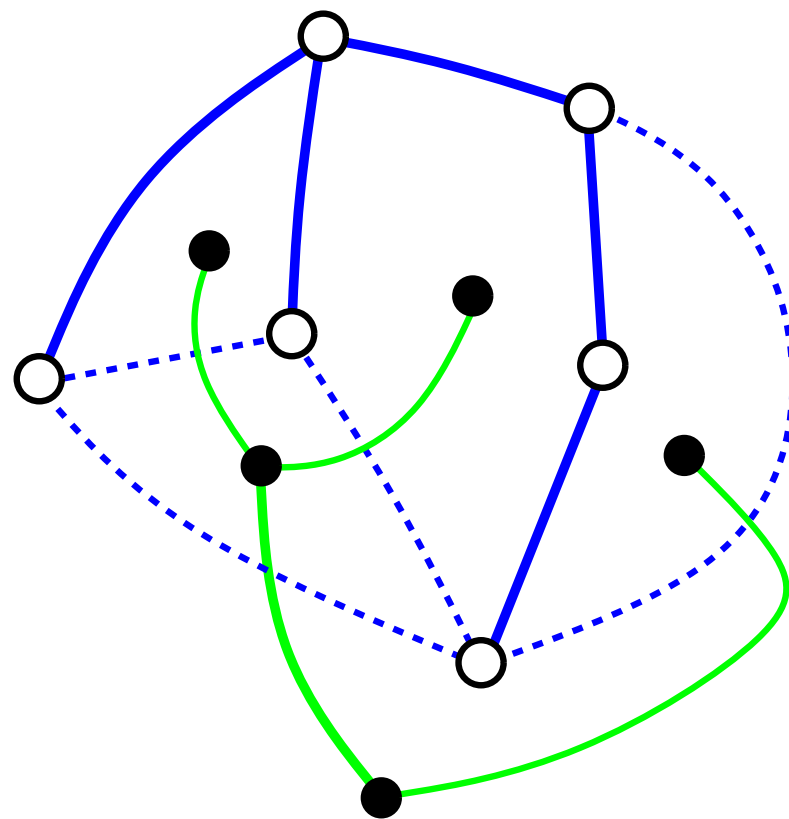
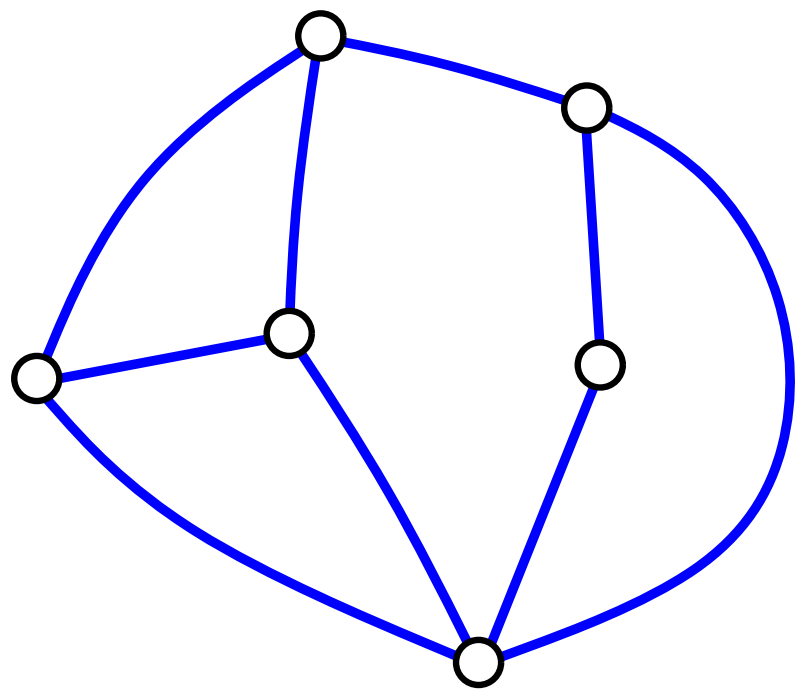
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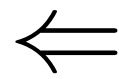
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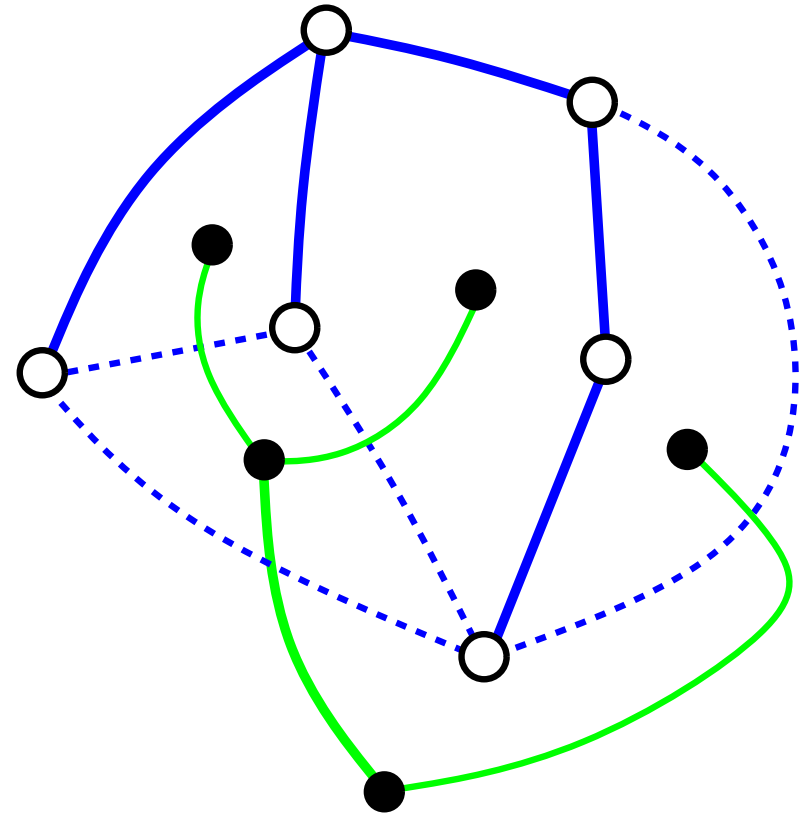
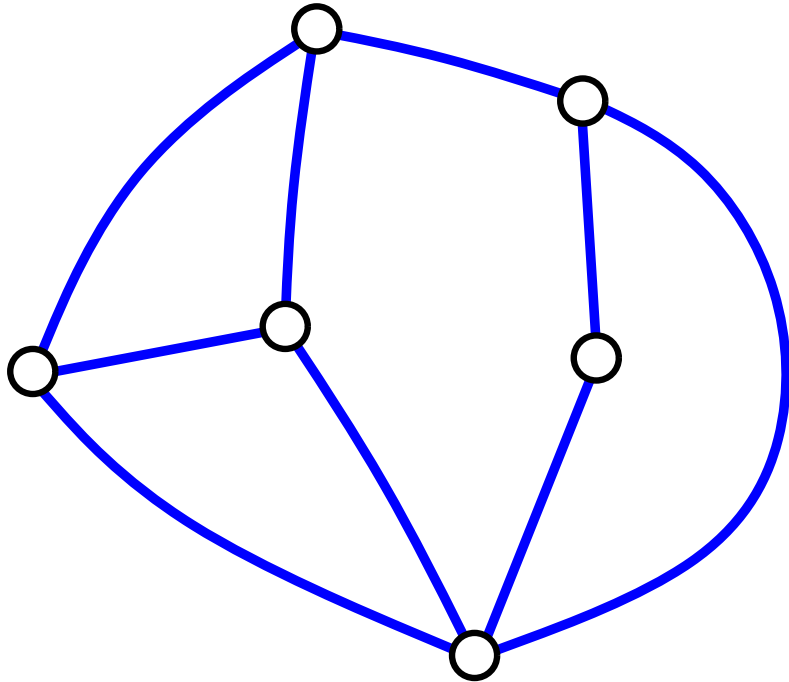
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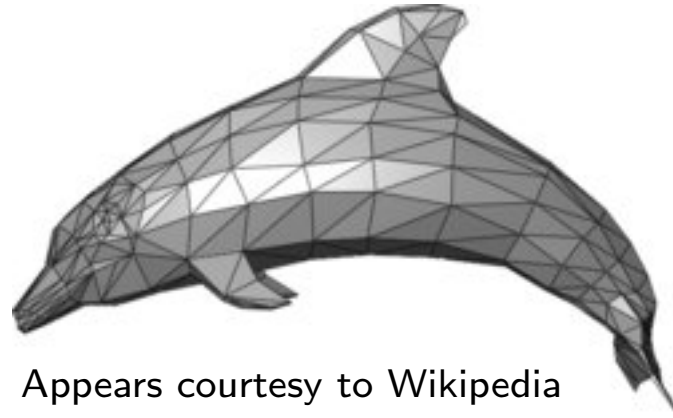
$$|E| = (|V| - 1) + (|F| - 1)$$



$\Rightarrow$  simple planar graph  $G = (V, E)$  satisfies  $|E| \leq 3|V| - 6$   
(hence  $K_5$  has too many edges to be planar)

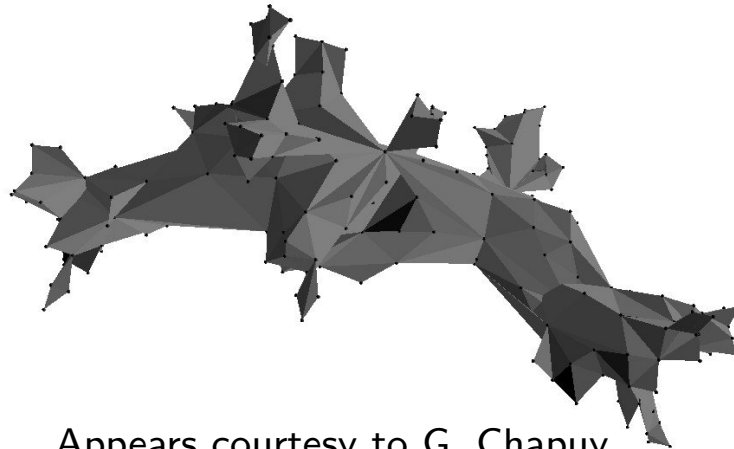
## Planar maps. Motivations

- Algorithmic applications: efficient encoding of meshed surfaces.



Appears courtesy to Wikipedia

- Probability and Physics: random lattices, random surfaces.



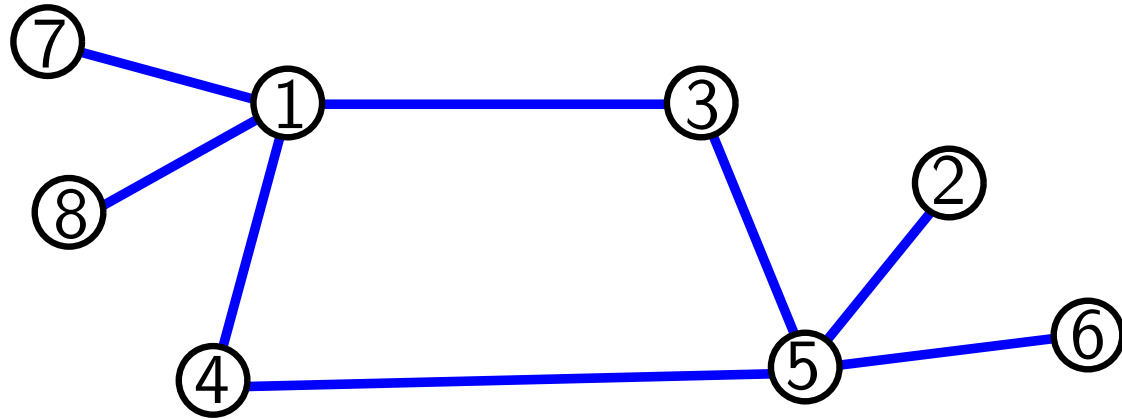
Appears courtesy to G. Chapuy

- Representation Theory: factorization problems.

# Symmetry issues.

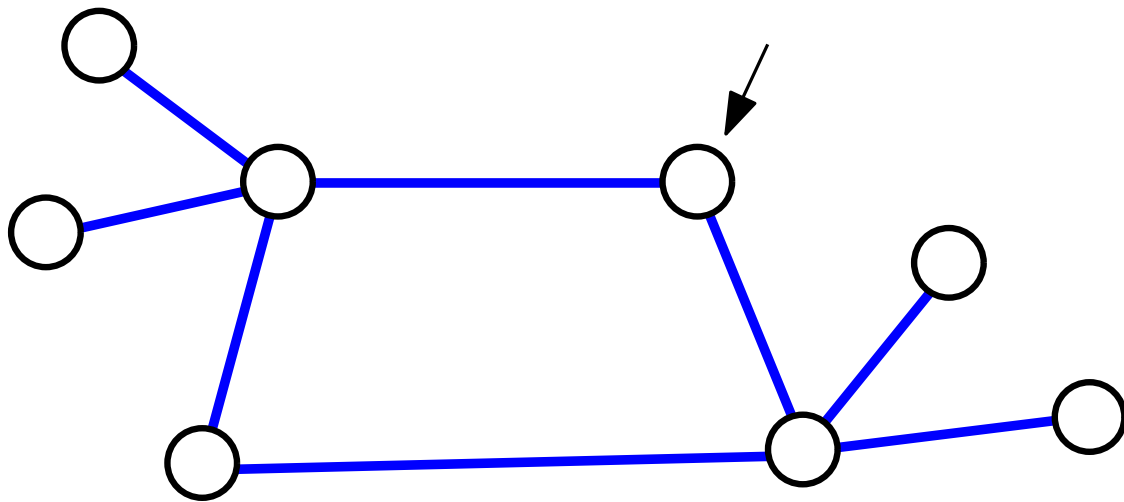
In order to identify vertices unambiguously (to **avoid symmetry issues**):

- Planar graphs: need to **label the vertices**



A labelled planar graph

- Planar maps: only need to **mark a corner**



A rooted planar map

# Asymptotic behaviour of planar maps/graphs

- **Asymptotic number:**

Labelled planar graphs  $n$  vertices:

$$\sim n! c n^{-7/2} \gamma^n \quad [\text{Giménez, Noy'05}]$$

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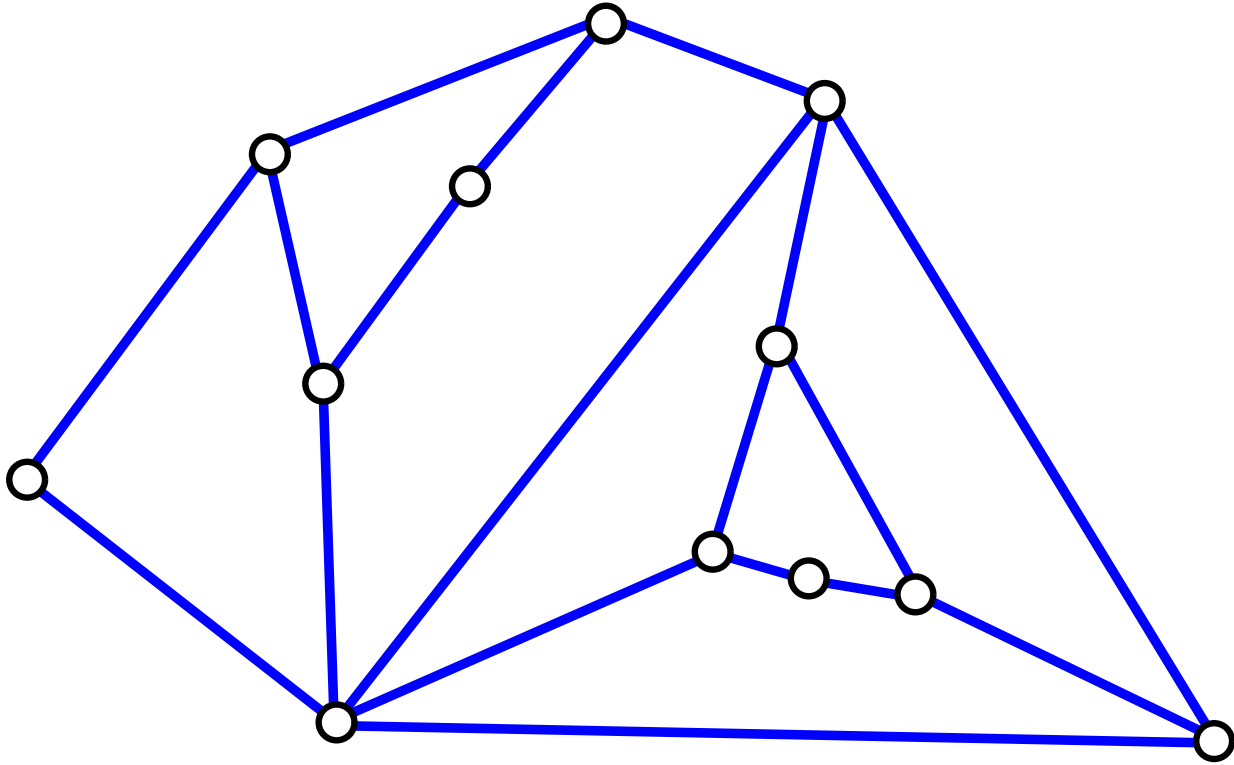
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- **Planar maps:**

- simpler enumeration formulas
- can control distance parameters
- **bijections!**

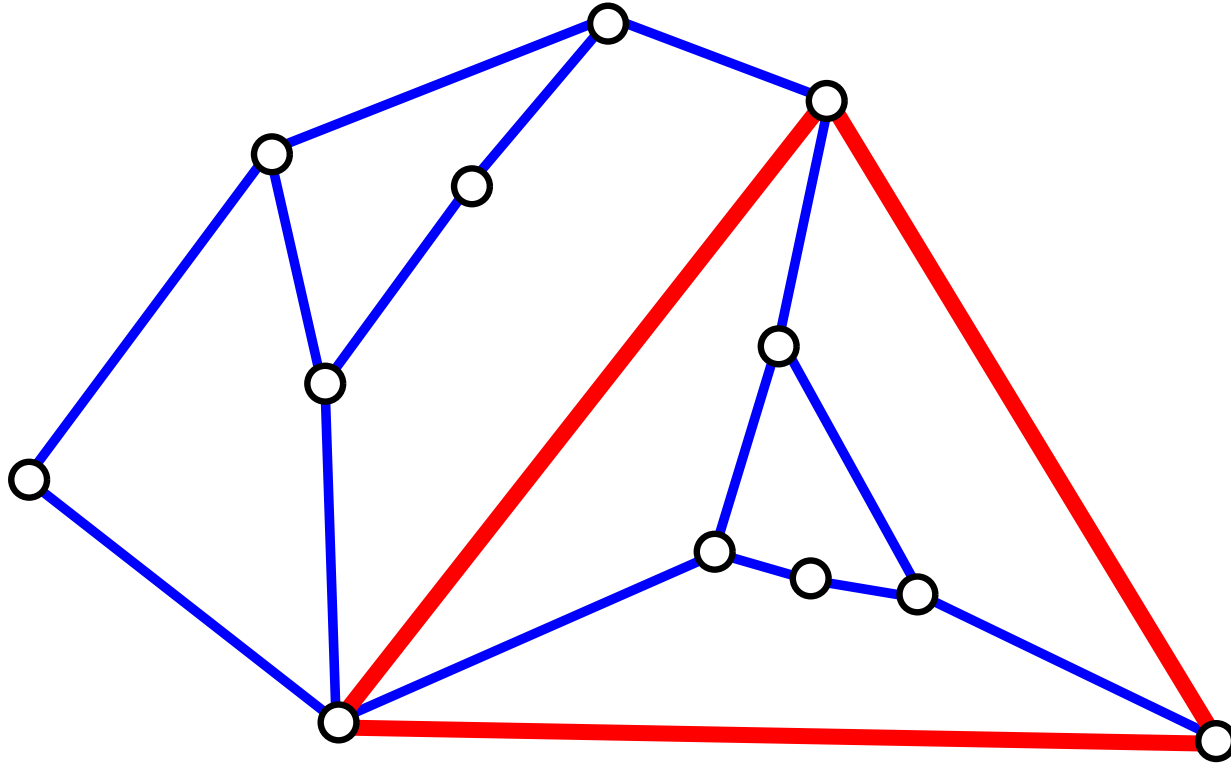
# The girth parameter

The **girth** of a graph is the length of a shortest cycle within the graph



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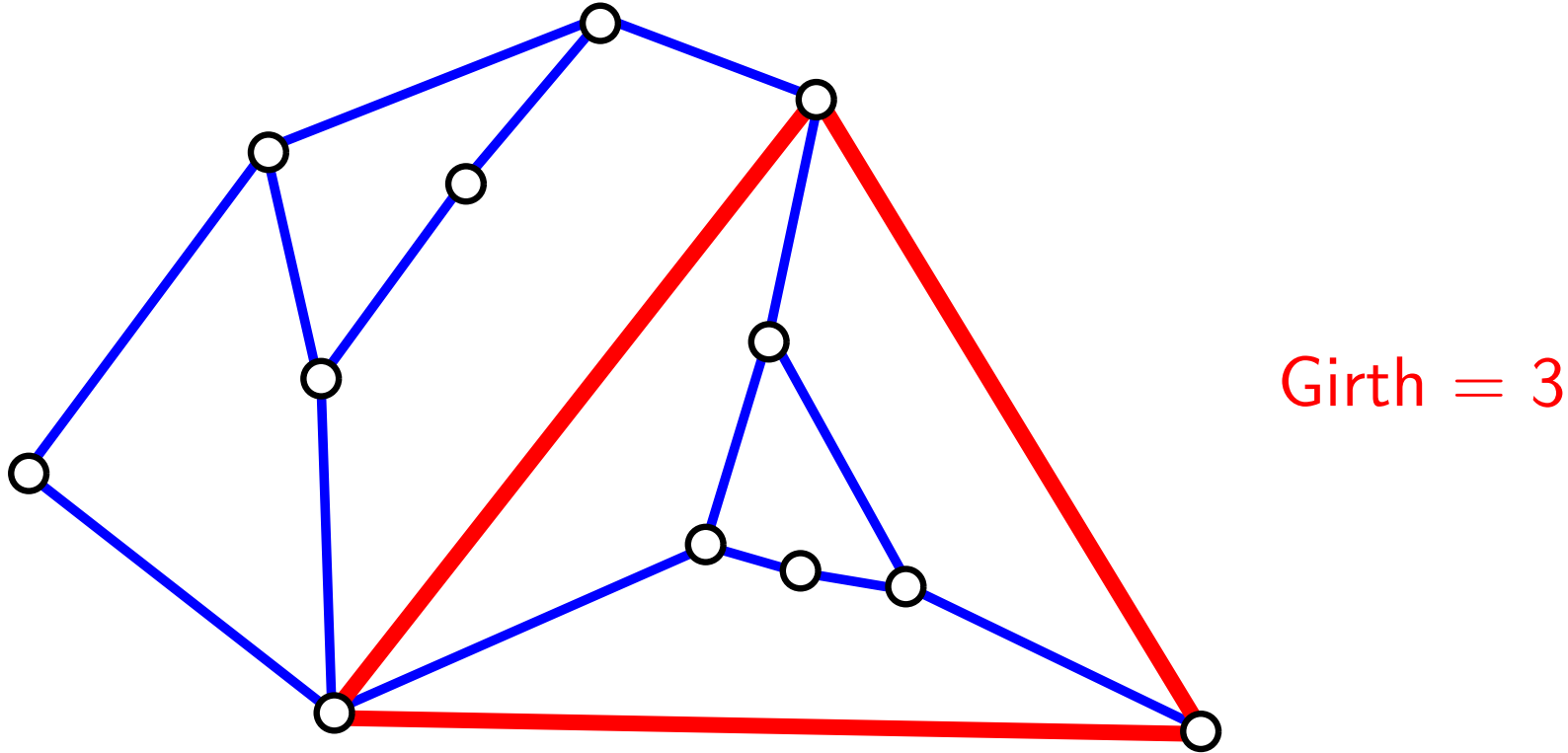
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Girth = 3

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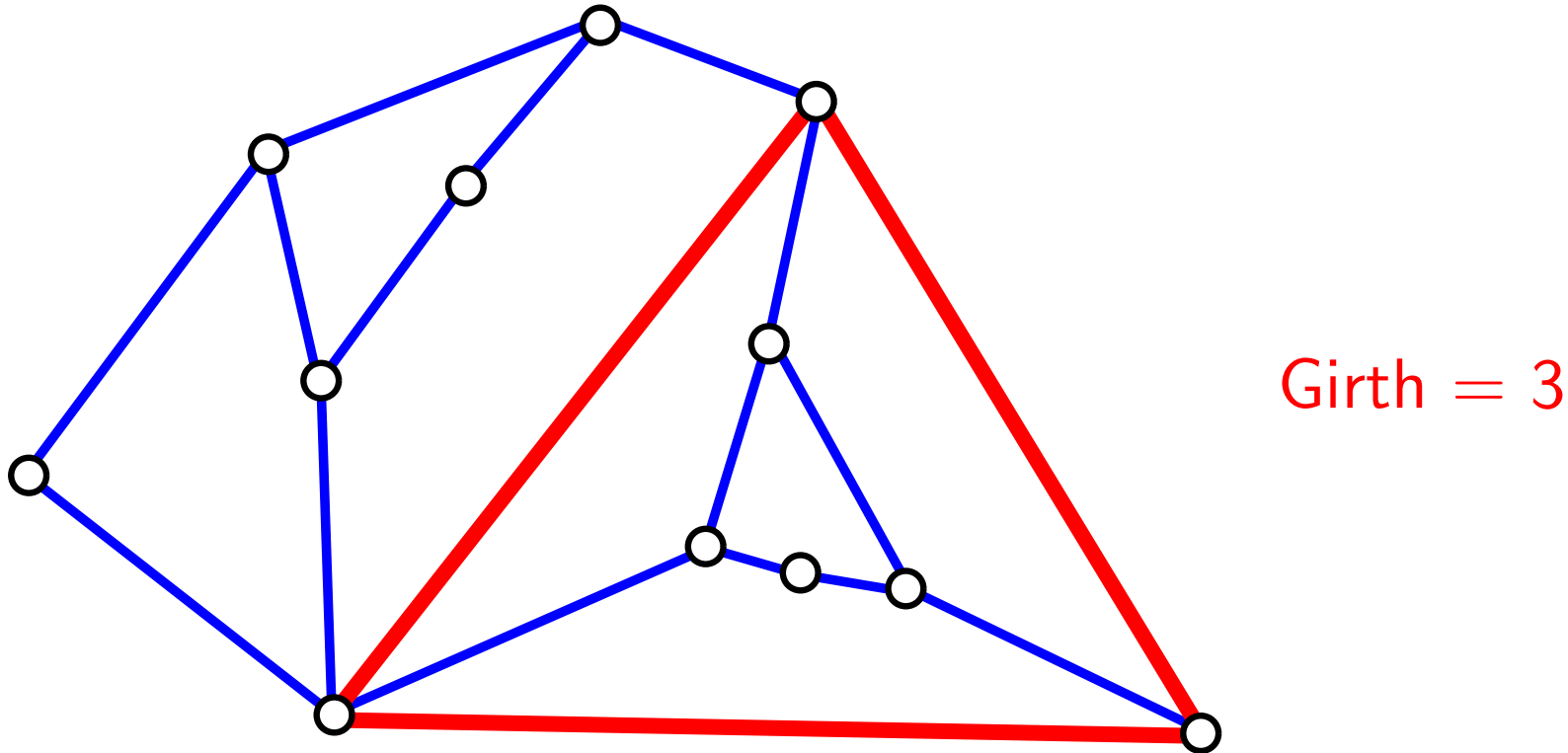


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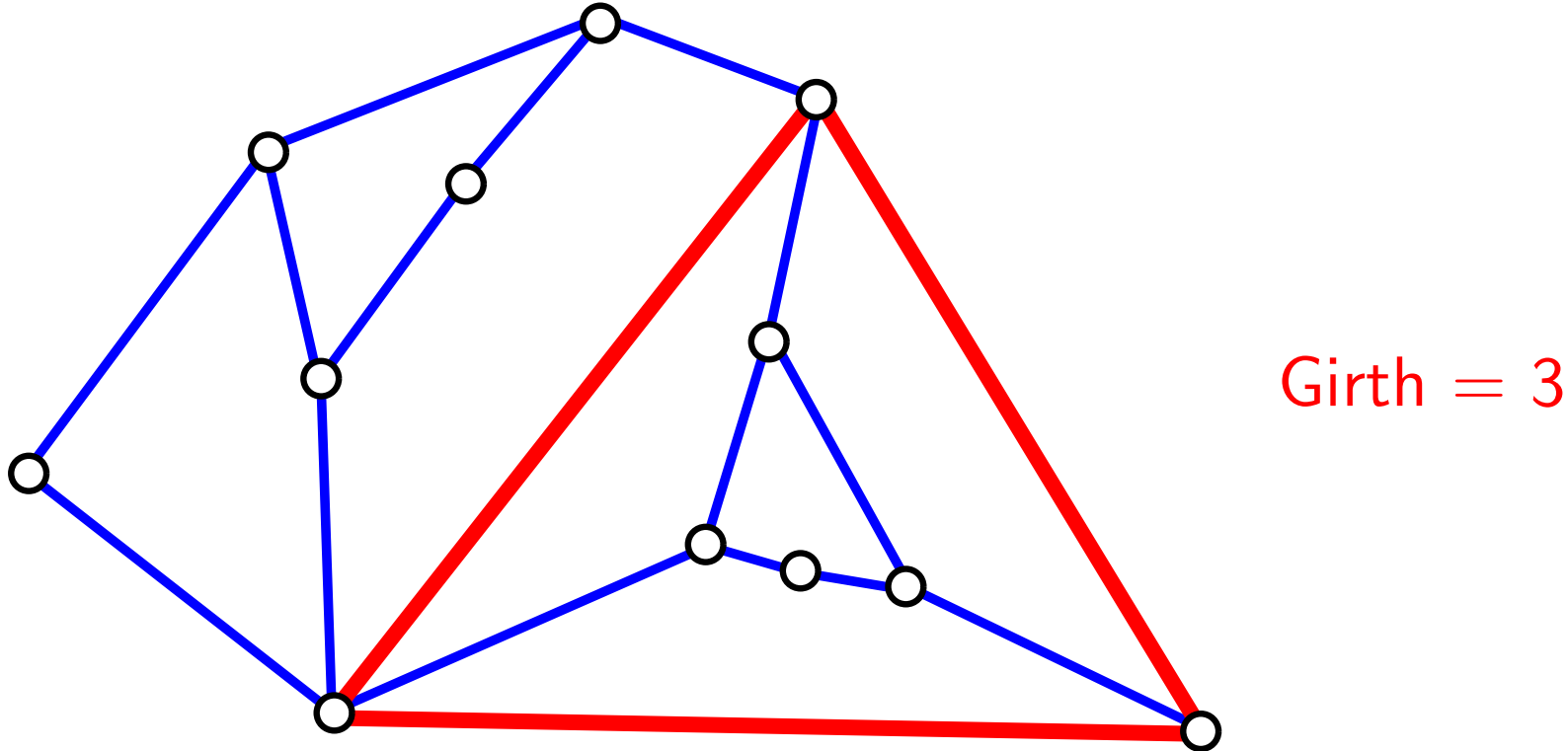
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Simple  $\Leftrightarrow girth \geq 3$

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Many natural map families are specified by constraints on the **girth** and on the **face-degrees** (loopless triangulations, simple quadrangulations,...)

# Planar maps. Exact counting results

- Triangulations ( $2n$  faces)

Loopless:  $\frac{2^n}{(n+1)(2n+1)} \binom{3n}{n}$

Simple:  $\frac{1}{n(2n-1)} \binom{4n-2}{n-1}$

- Quadrangulations ( $n$  faces)

General:  $\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$

Simple:  $\frac{2}{n(n+1)} \binom{3n}{n-1}$

- Bipartite maps ( $n_i$  faces of degree  $2i$ )

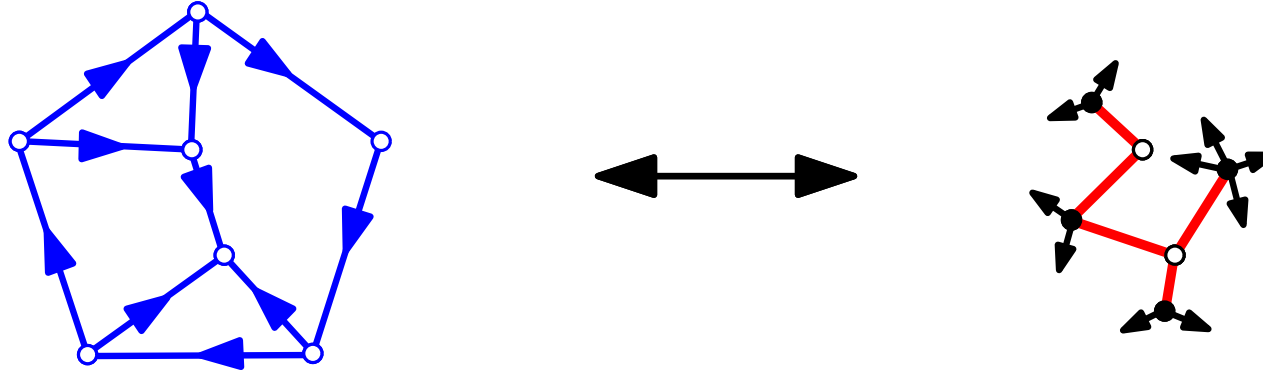
$$\frac{2 \cdot (\sum i n_i)!}{(2 + \sum (i-1)n_i)!} \prod_i \frac{1}{n_i!} \binom{2i-1}{i}^{n_i}$$

## Planar maps. **Counting methods**

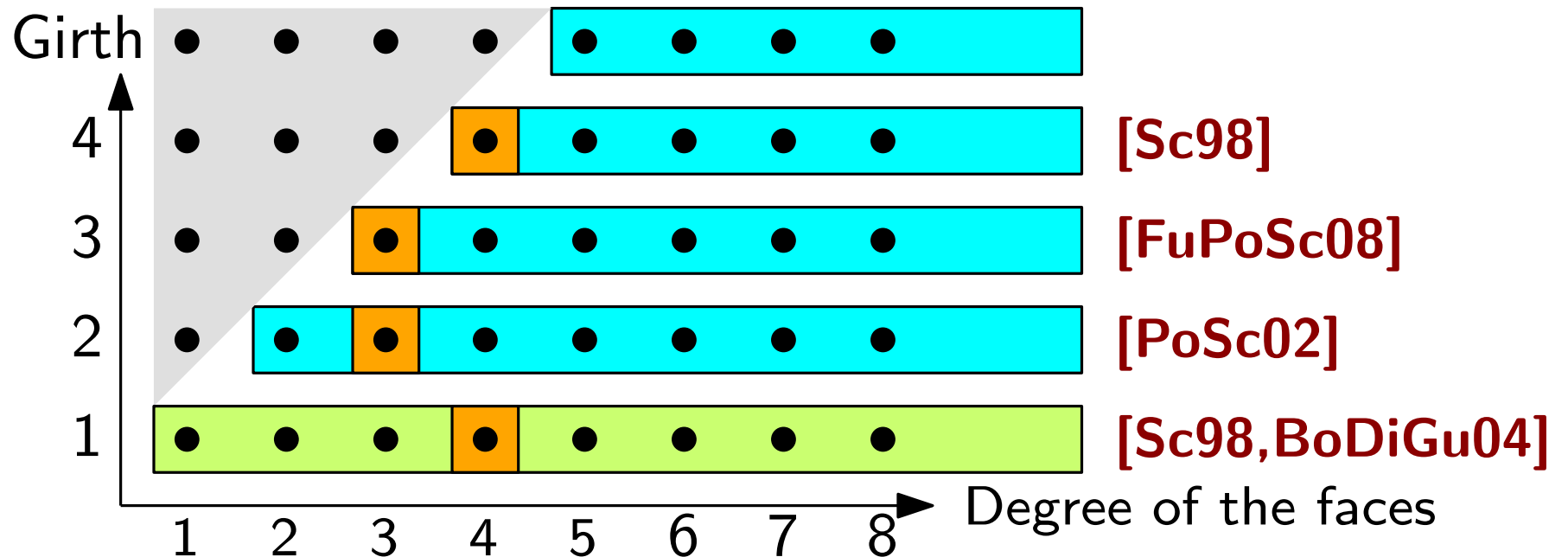
- **Generating functions** [Tutte 63]  
Recursive description of maps  $\rightsquigarrow$  recurrences.
- **Matrix Integrals** ['t Hooft 74, Brézin et al'78]  
Feynmann Diagram  $\approx$  maps.
- **Bijections** [Cori-Vauquelin 81, Schaeffer 98]  
Maps  $\rightsquigarrow$  decorated trees.

# Outline

1. **Master bijection** between a class of **oriented maps** and a class of bicolored **decorated trees** (which are called mobiles).

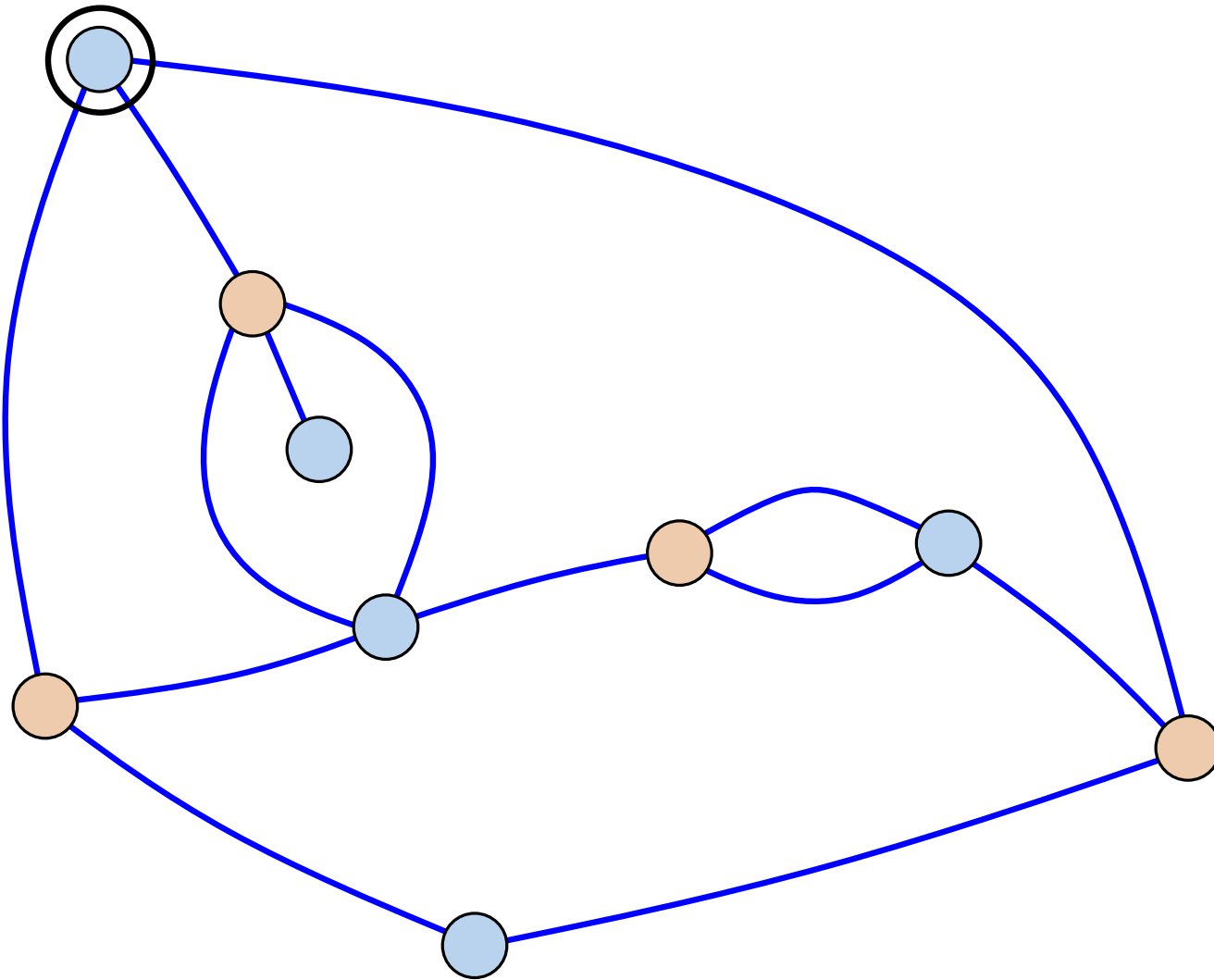


2. **Specializations** to classes of maps (via canonical orientations).

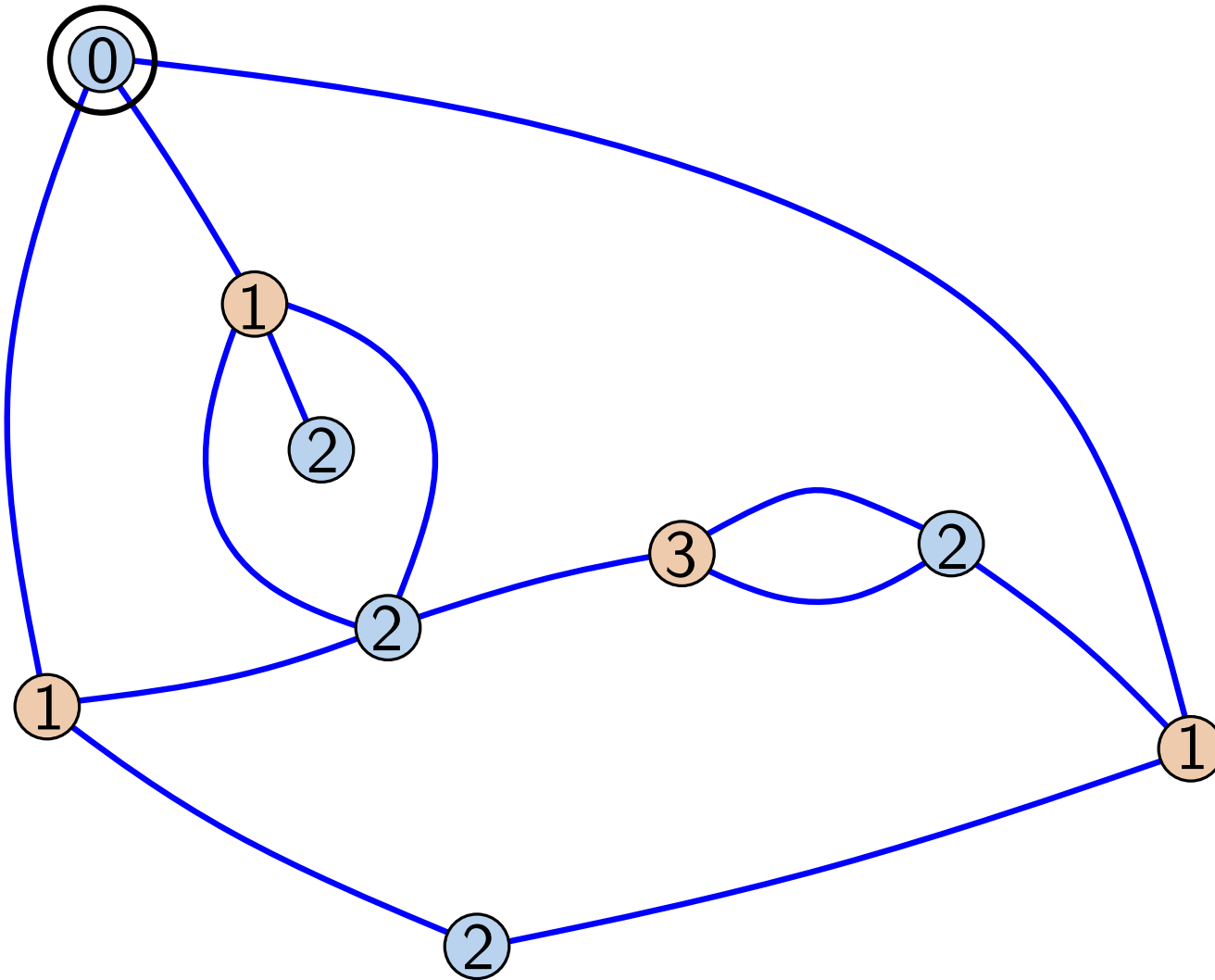


# **From oriented maps to mobiles**

Pointed bipartite map  $\rightarrow$  labelled mobile. [Sc98] [BoDiGu04]



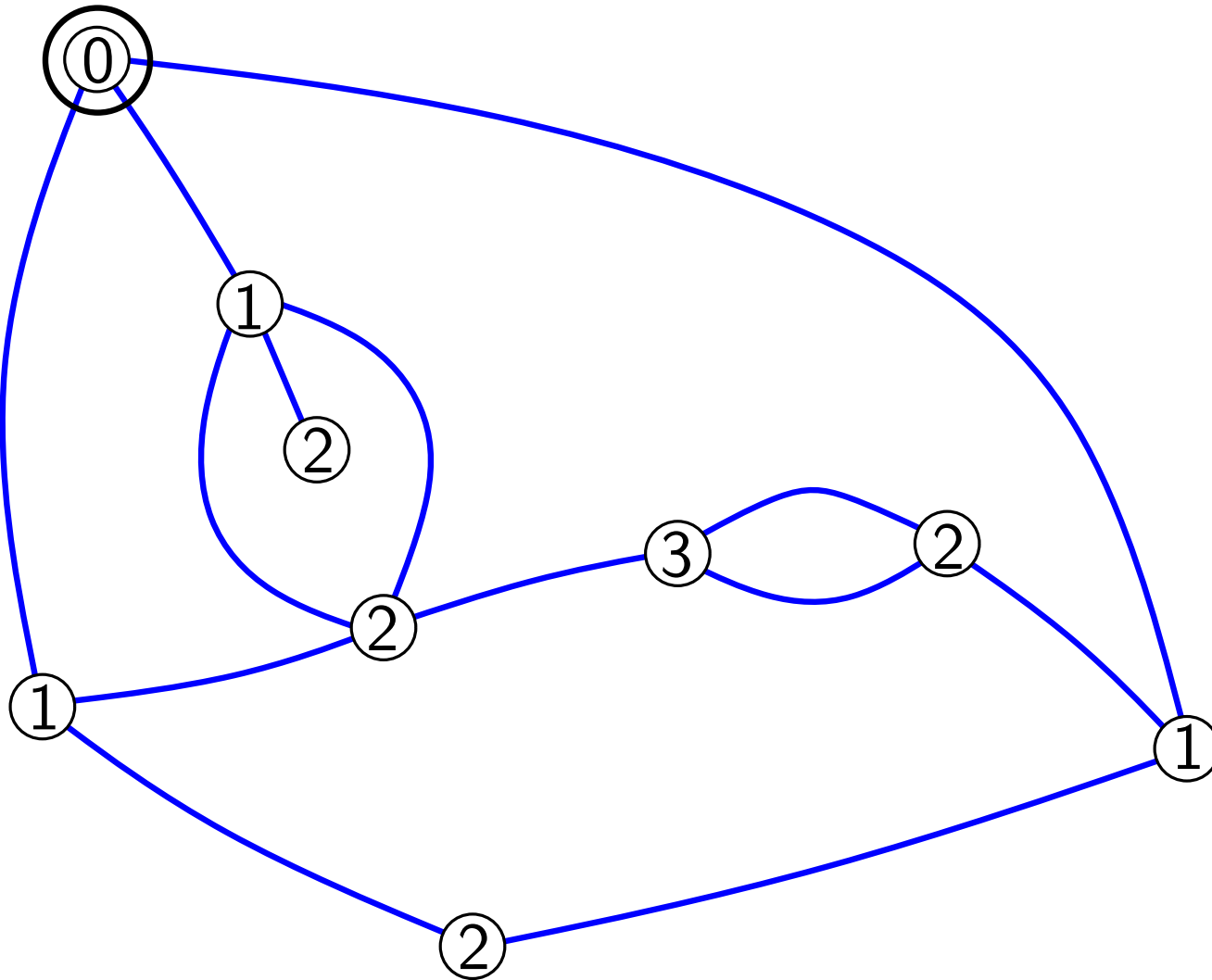
Pointed bipartite map  $\rightarrow$  labelled mobile. [Sc98] [BoDiGu04]



Label the vertices  
by the distance  
from pointed vertex

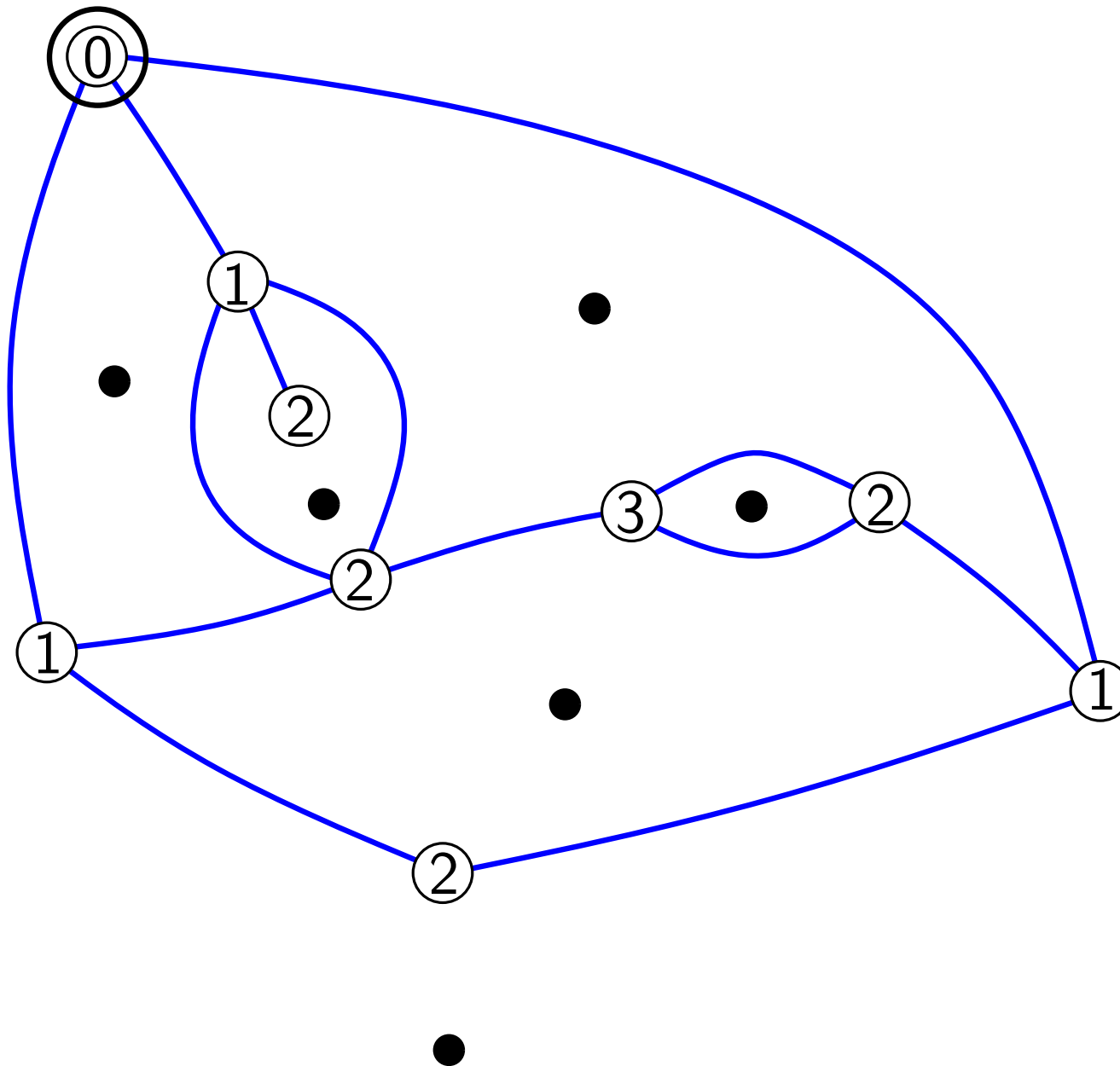
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Construct the  
labelled mobile



Pointed bipartite map  $\rightarrow$  labelled mobile.

[Sc98] [BoDiGu04]

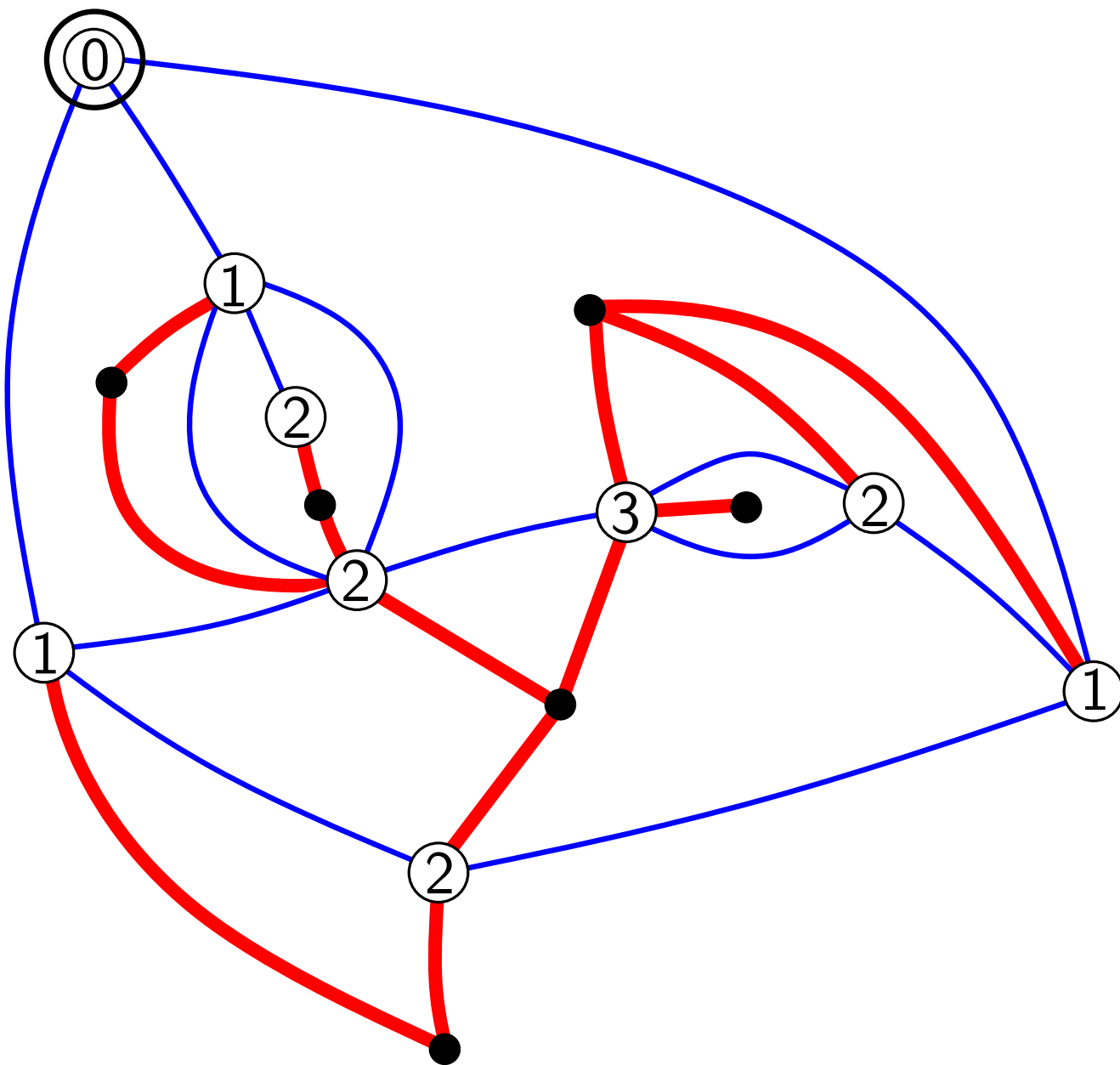


**Construct the  
labelled mobile**

(i) put one black  
vertex in each face

Pointed bipartite map  $\rightarrow$  labelled mobile.

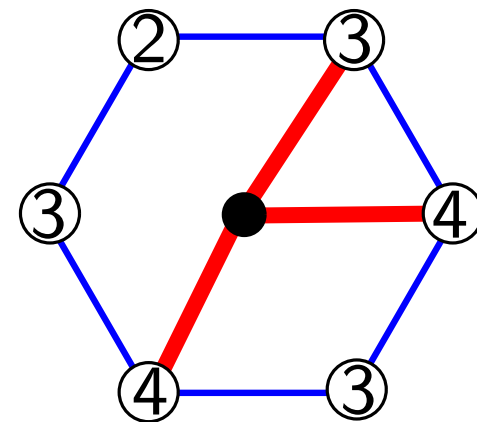
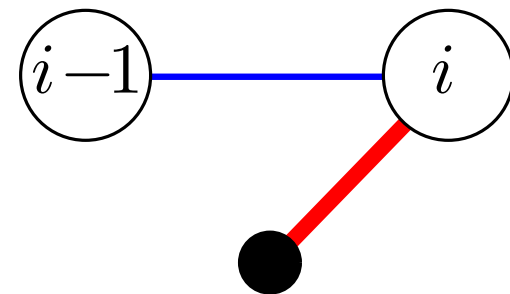
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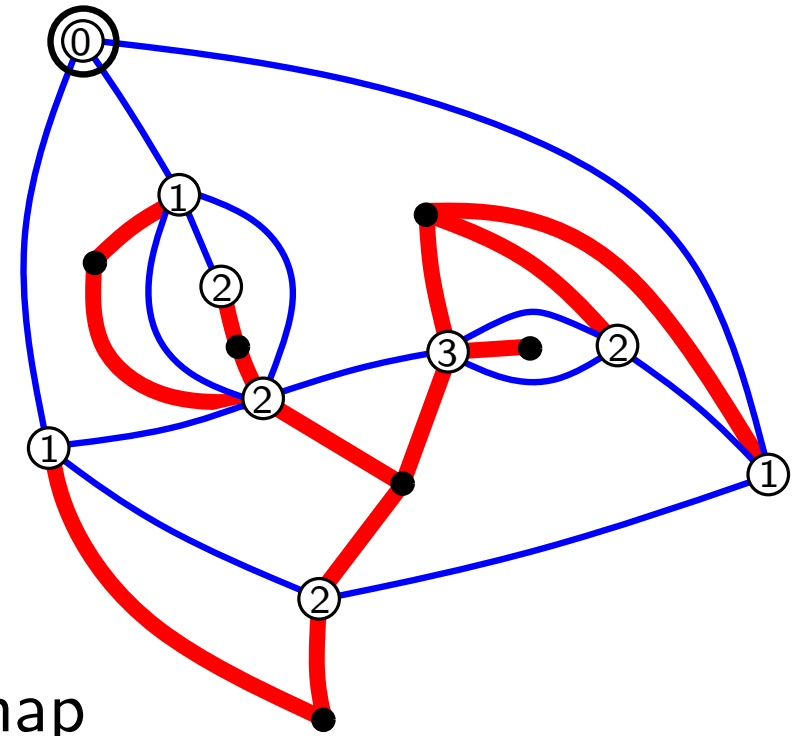
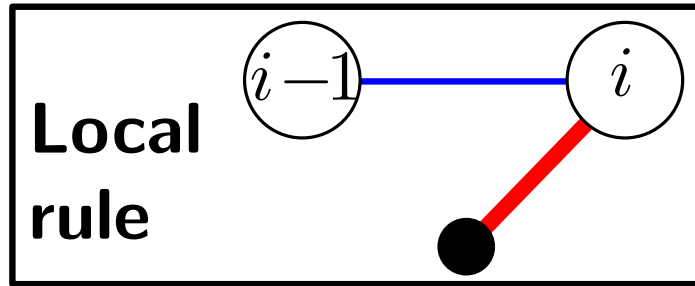
(i) put one black vertex in each face

(ii) each edge of the map gives one edge in the mobile



Pointed bipartite map  $\rightarrow$  labelled mobile.

[Sc98] [BoDiGu04]



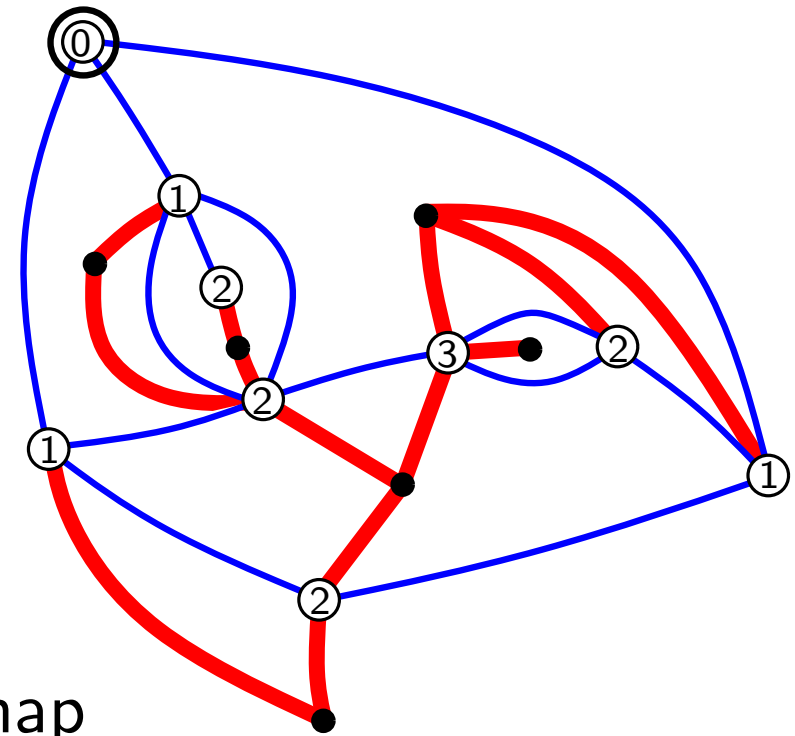
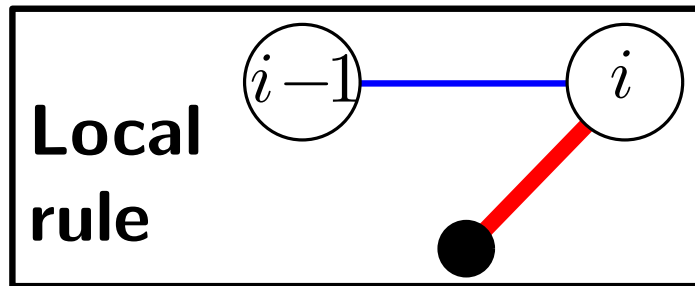
**Proof that the mobile is a tree**

Let  $G = (V, E, F)$  be a pointed bipartite map

Let  $T$  be the associated mobile

Pointed bipartite map  $\rightarrow$  labelled mobile.

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**Proof that the mobile is a tree**

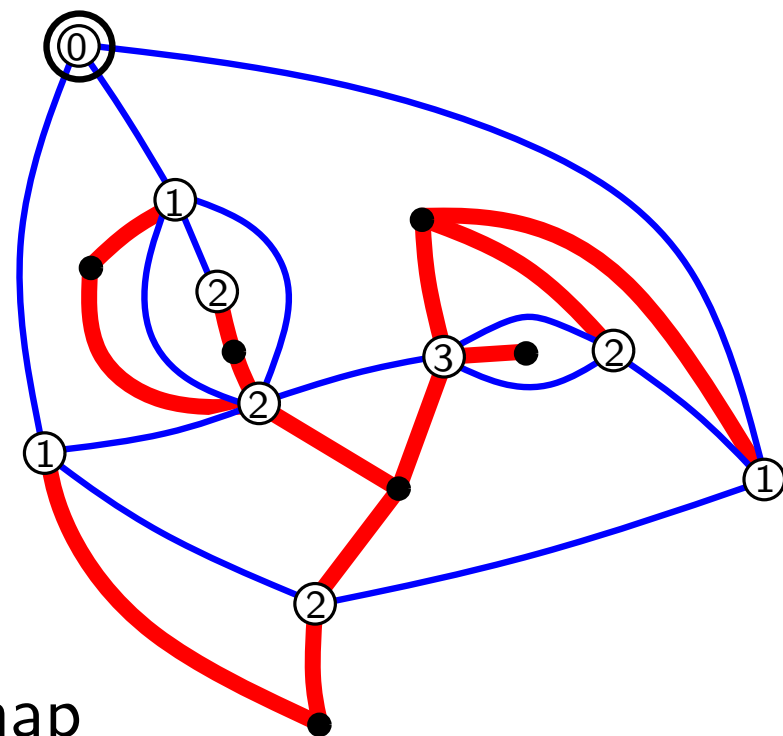
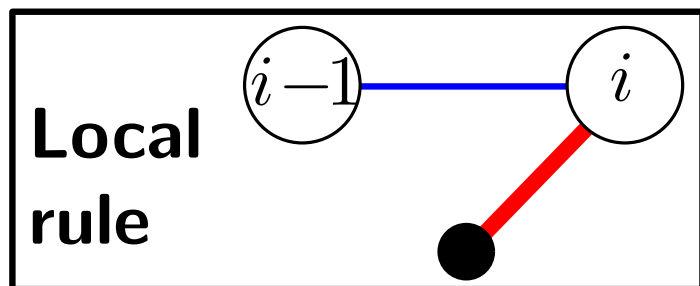
Let  $G = (V, E, F)$  be a pointed bipartite map

Let  $T$  be the associated mobile

$T$  has  $|E|$  edges, and has  $|V| + |F| - 1 = |E| + 1$  vertices (Euler relation)

Pointed bipartite map  $\rightarrow$  labelled mobile.

[Sc98] [BoDiGu04]



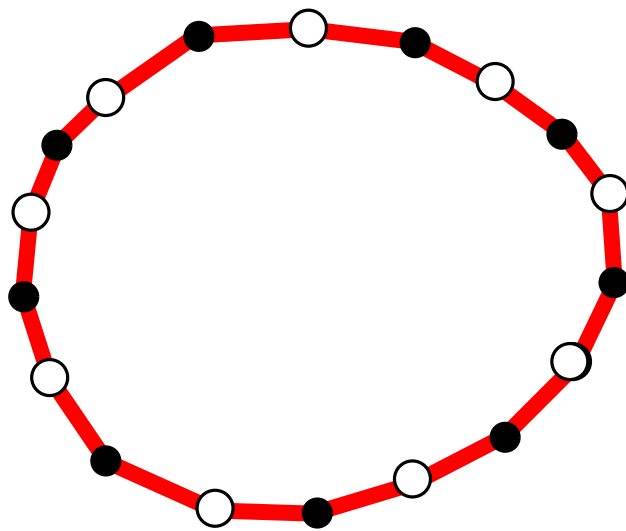
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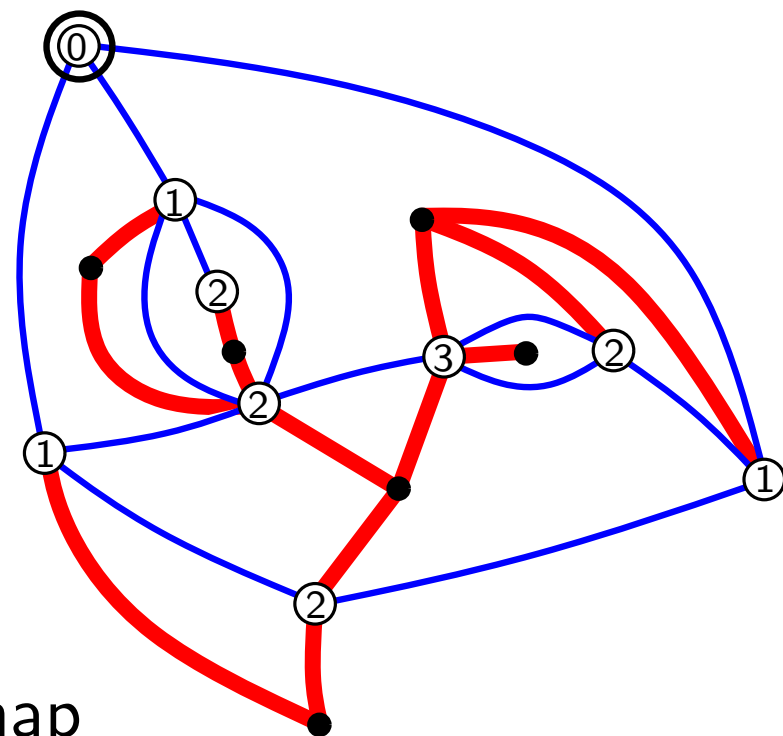
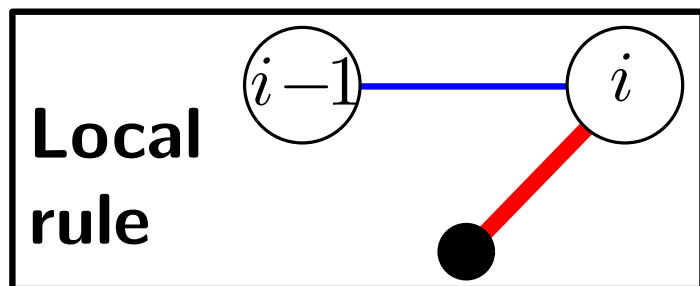
⊙



Assume that  $T$  has a cycle  $C$

Pointed bipartite map  $\rightarrow$  labelled mobile.

[Sc98] [BoDiGu04]

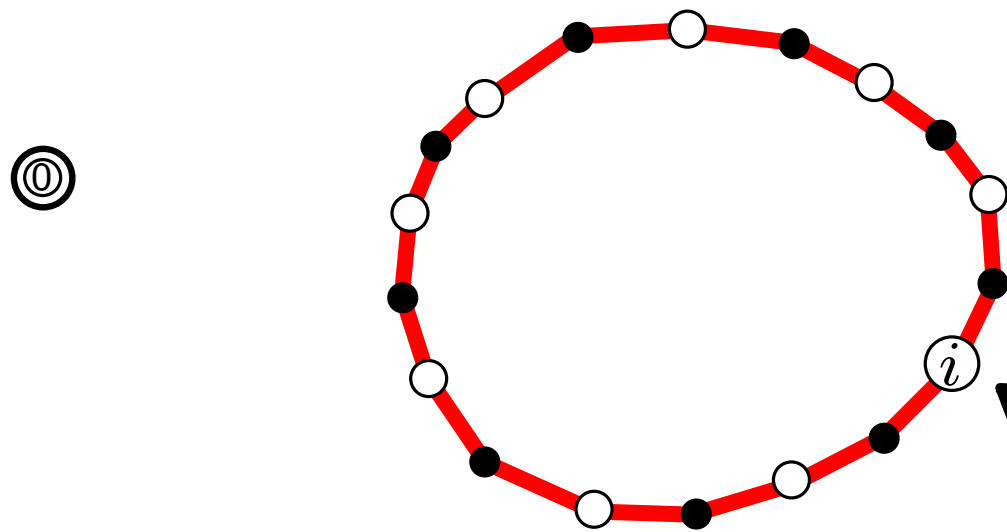


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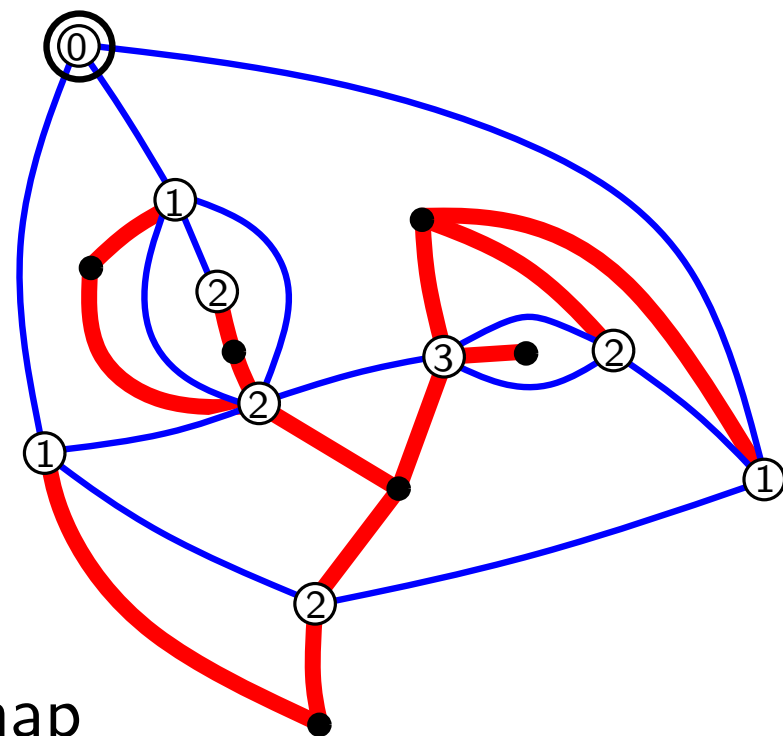
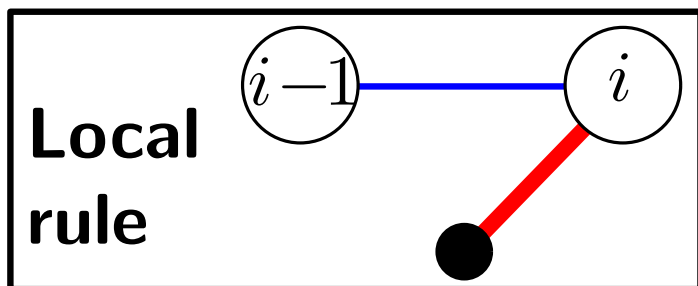


Assume that  $T$  has a cycle  $C$

smallest label on  $C$

Pointed bipartite map  $\rightarrow$  labelled mobile.

[Sc98] [BoDiGu04]

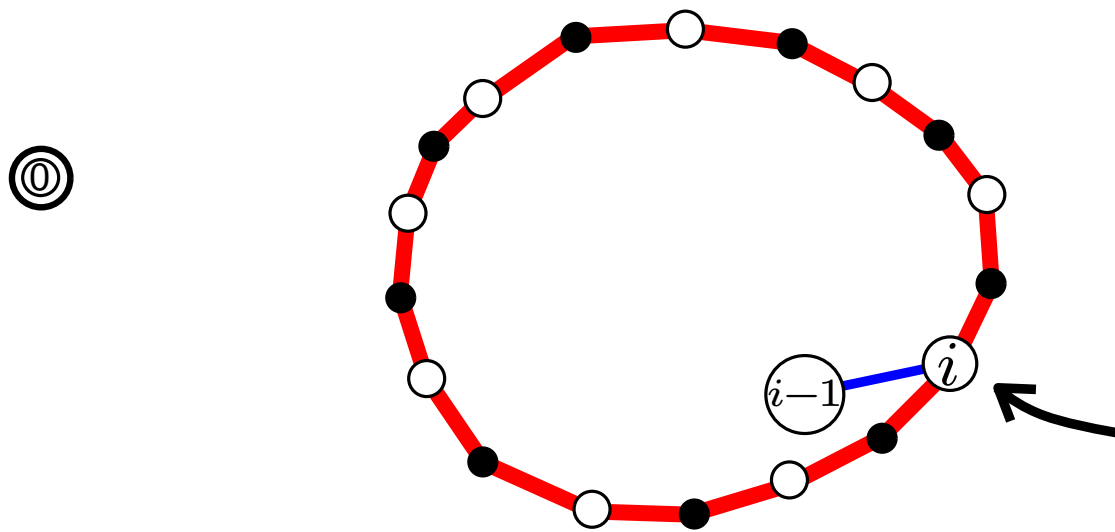


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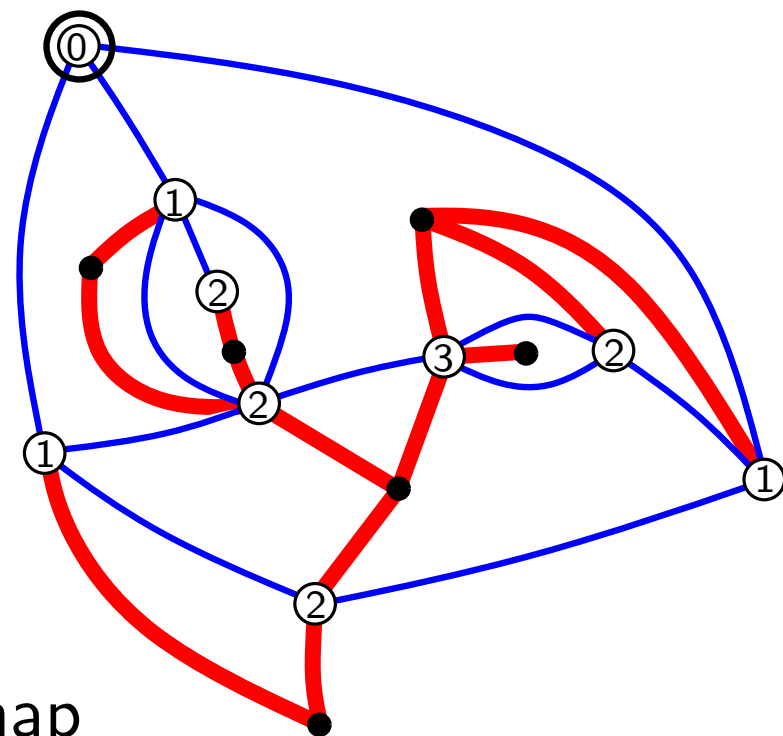
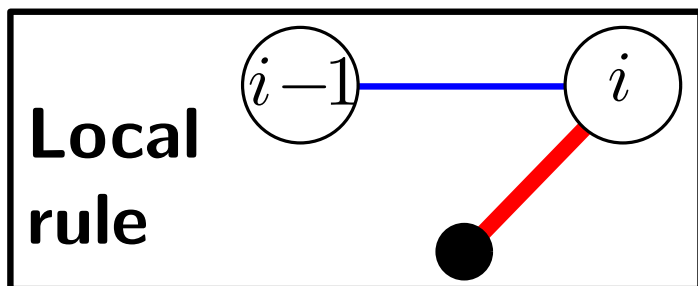


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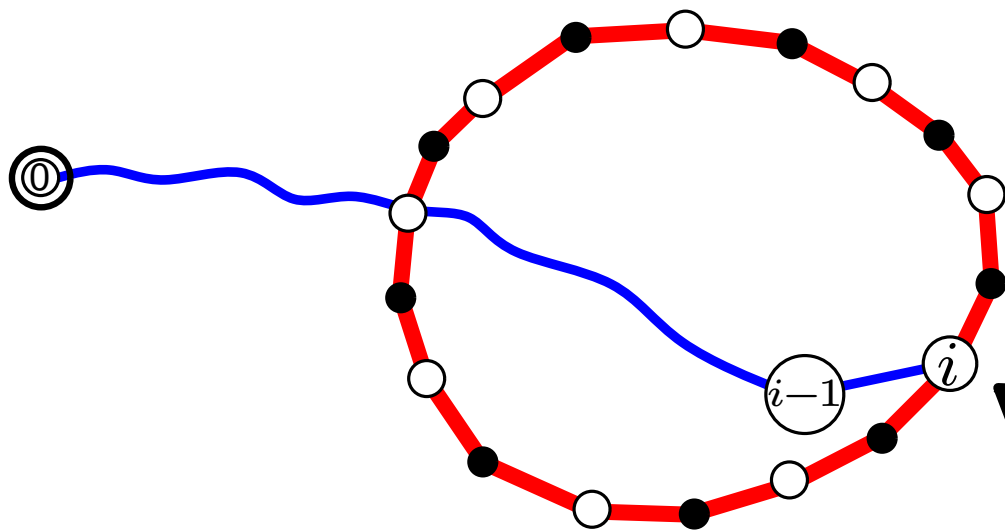


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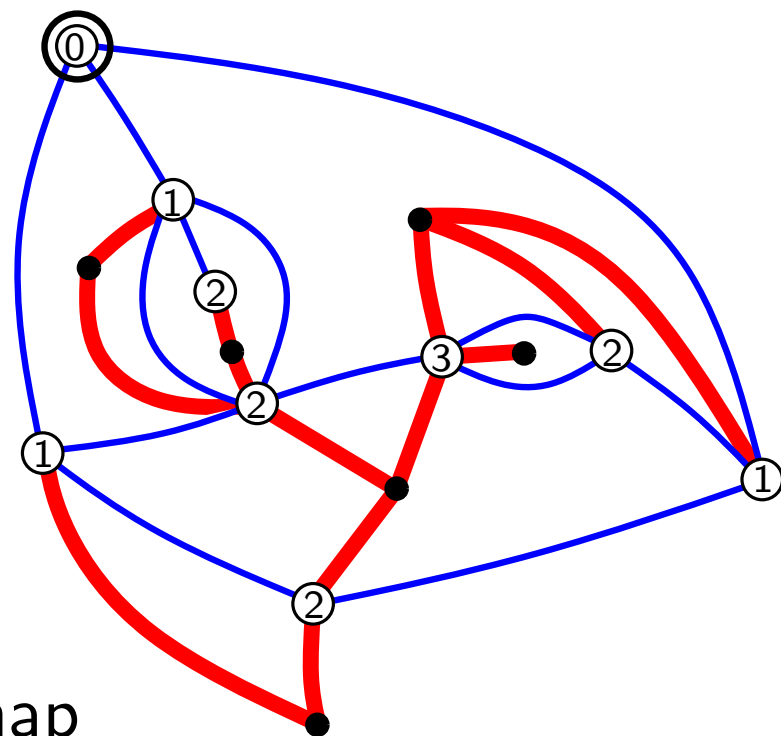
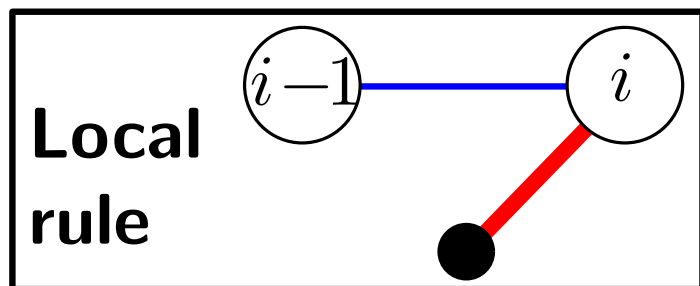


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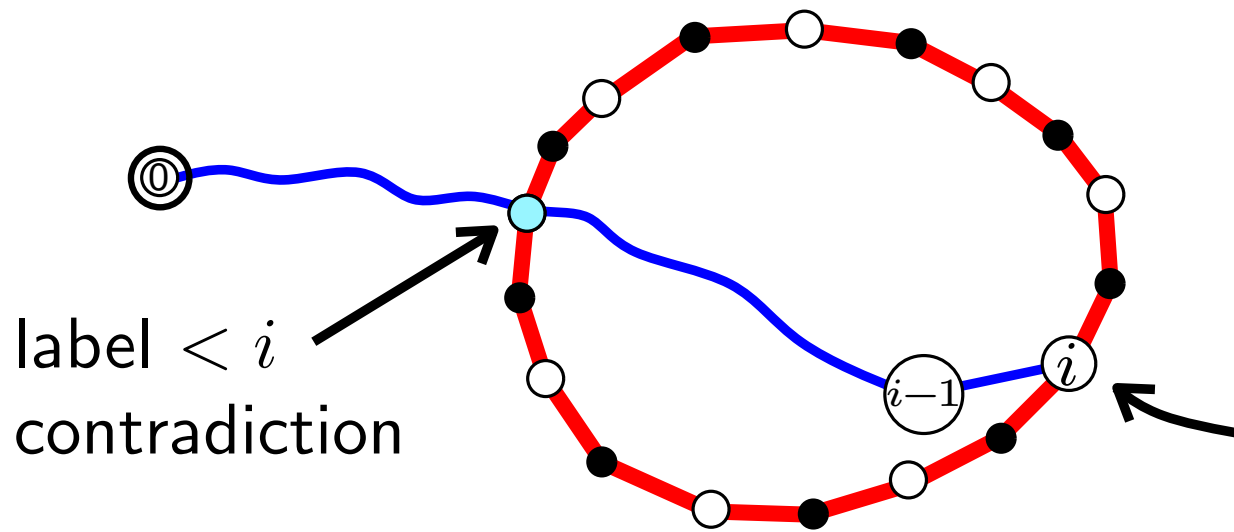


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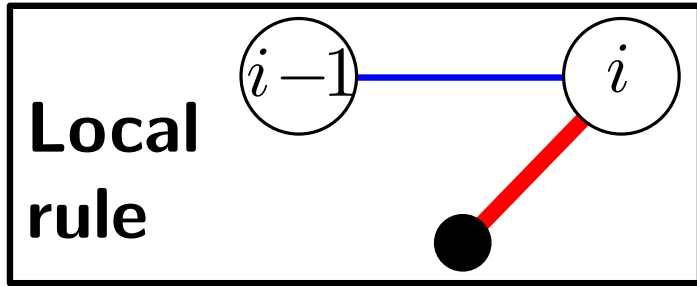
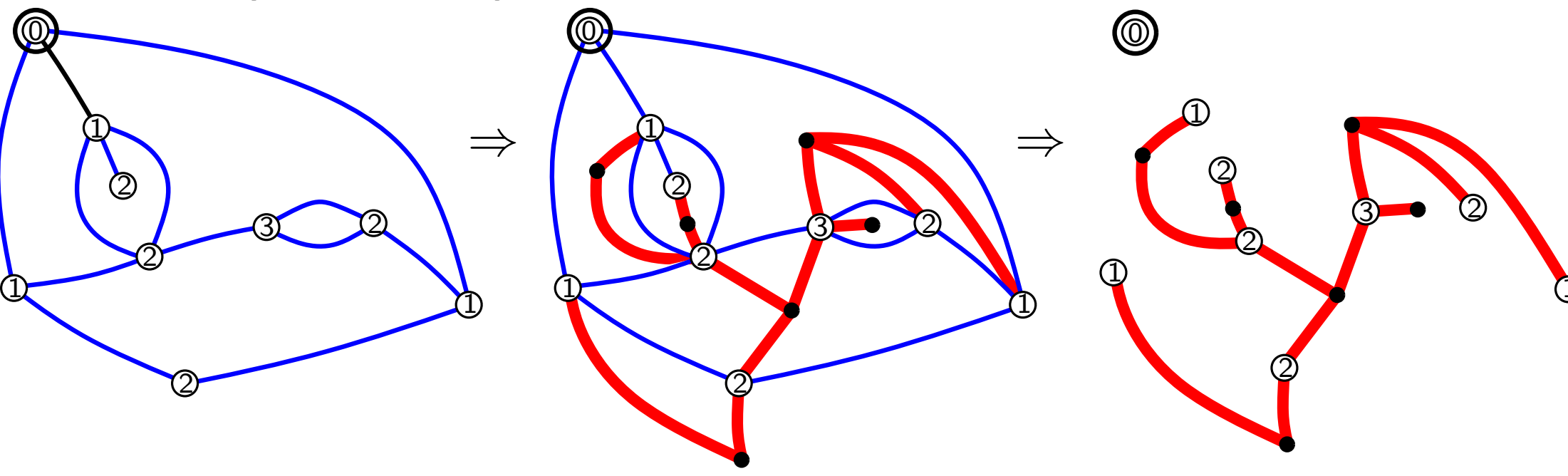
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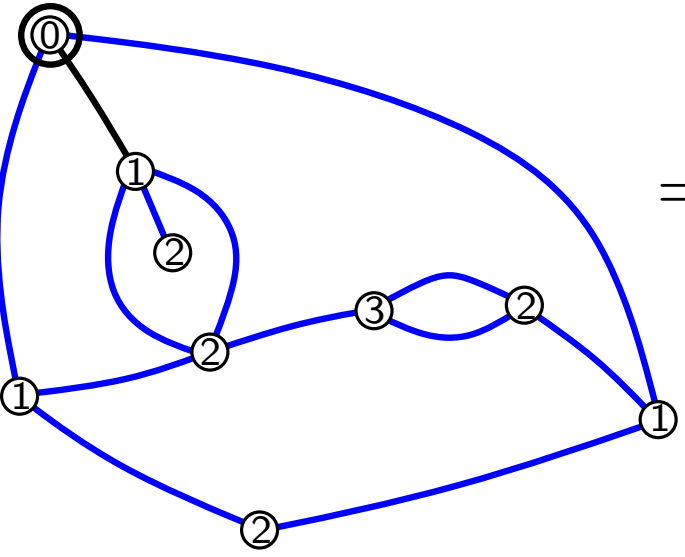


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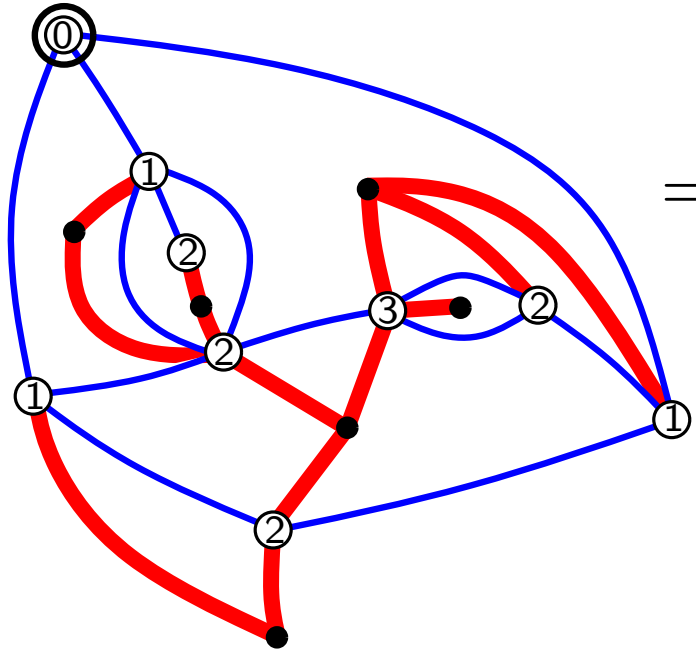
Pointed bipartite map  $\rightarrow$  labelled mobile. [Sc98] [BoDiGu04]



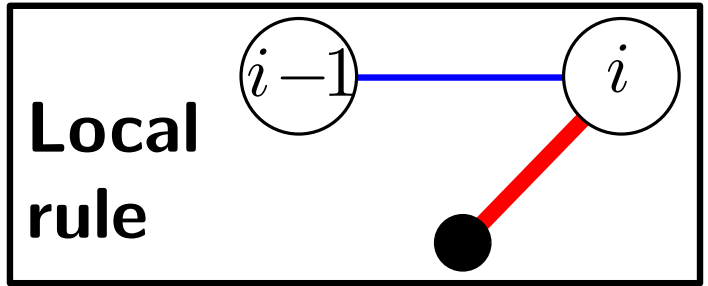
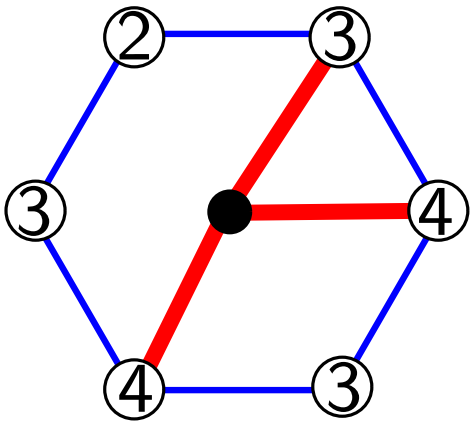
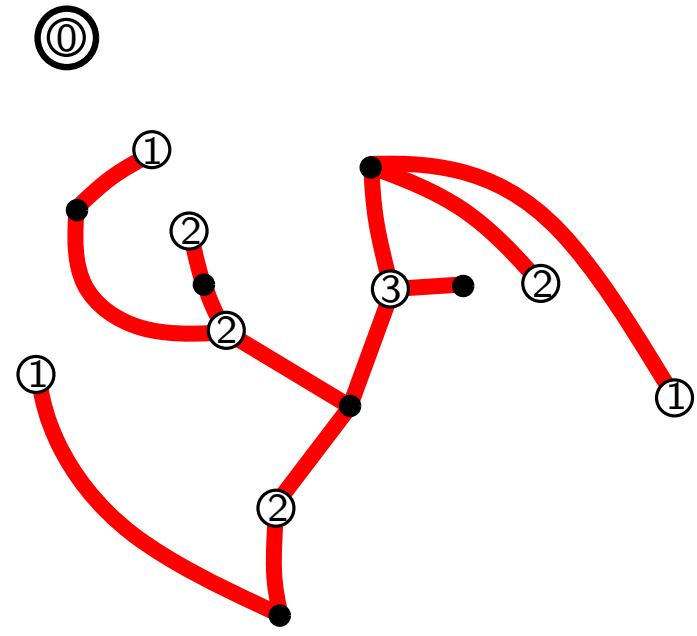
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$\Rightarrow$



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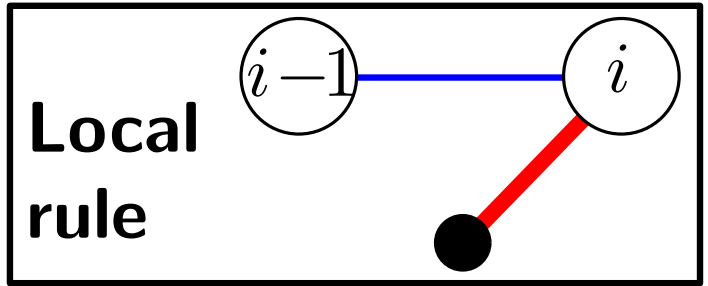
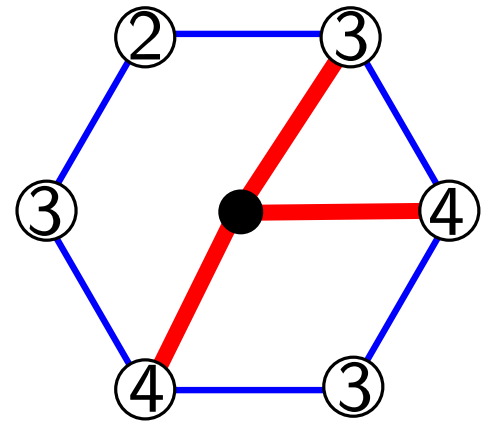
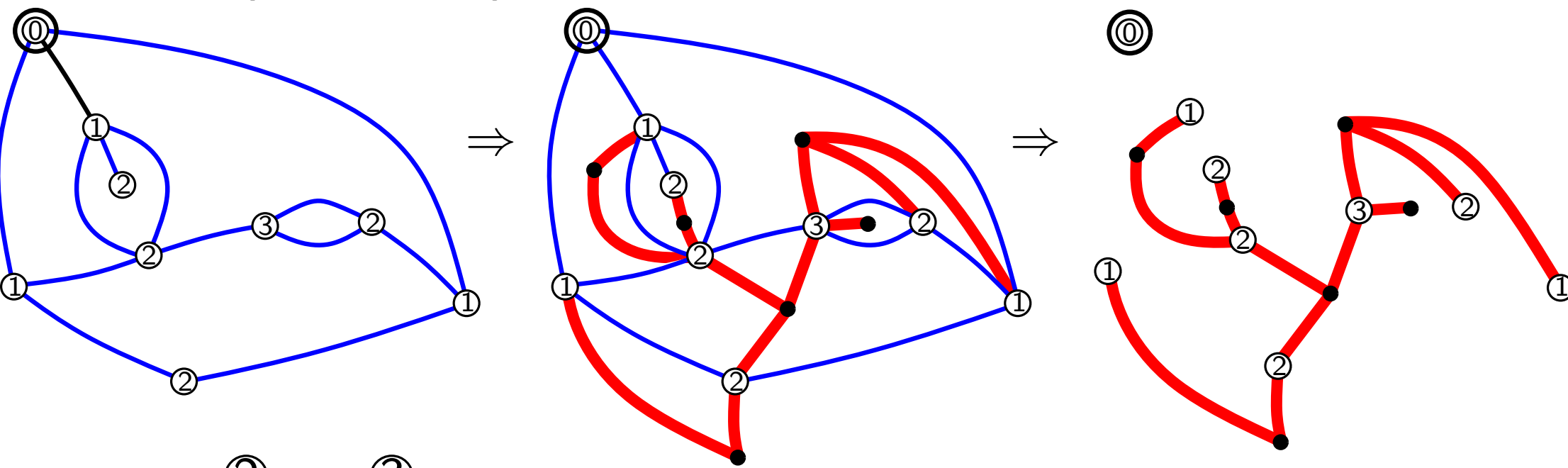


**Conditions:**

(i)  $\exists$  vertex label 1

(ii)  $j \leq i+1$

Pointed bipartite map  $\rightarrow$  labelled mobile. [Sc98] [BoDiGu04]

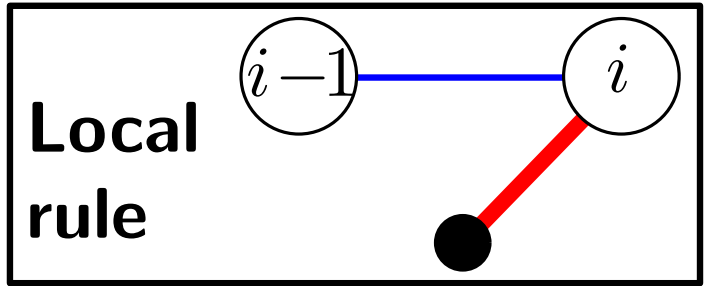
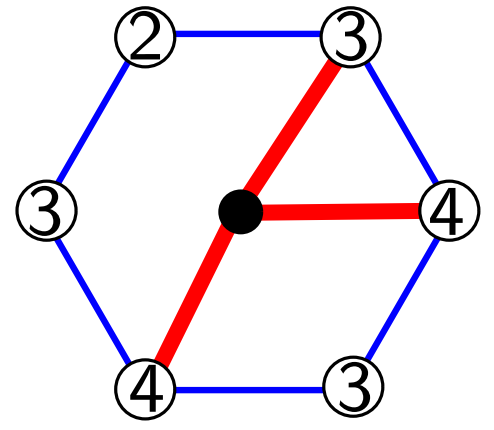
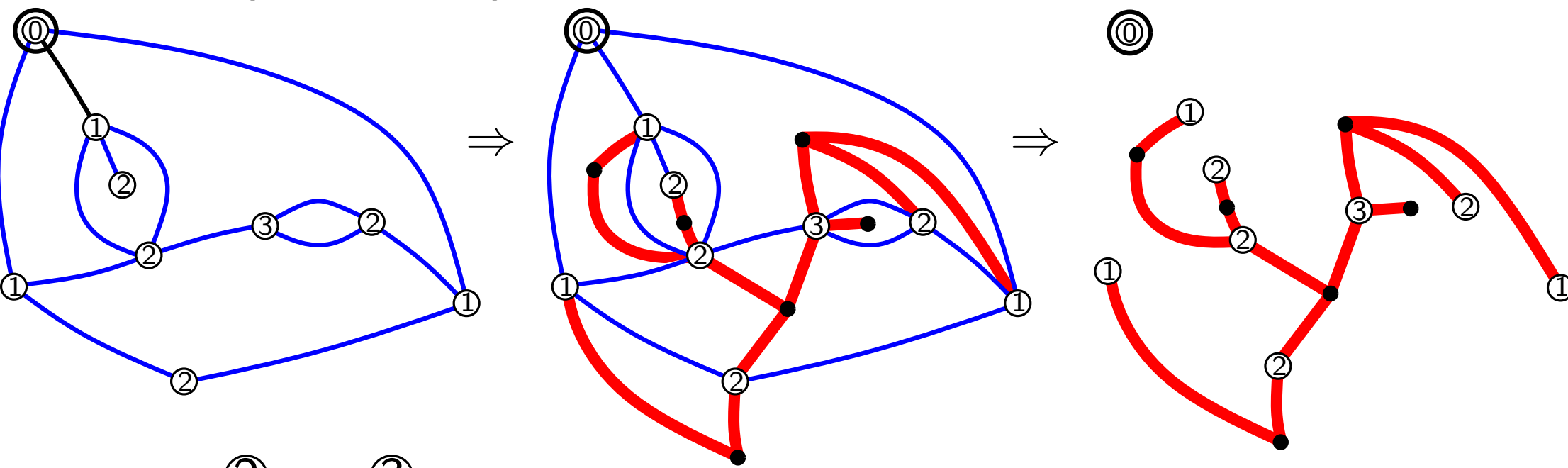


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**Theorem:** The mapping is a **bijection**. Each **face of degree  $2i$**  of the bipartite map corresponds to a **black vertex of degree  $i$**  in the mobile

Pointed bipartite map  $\rightarrow$  labelled mobile. [Sc98] [BoDiGu04]



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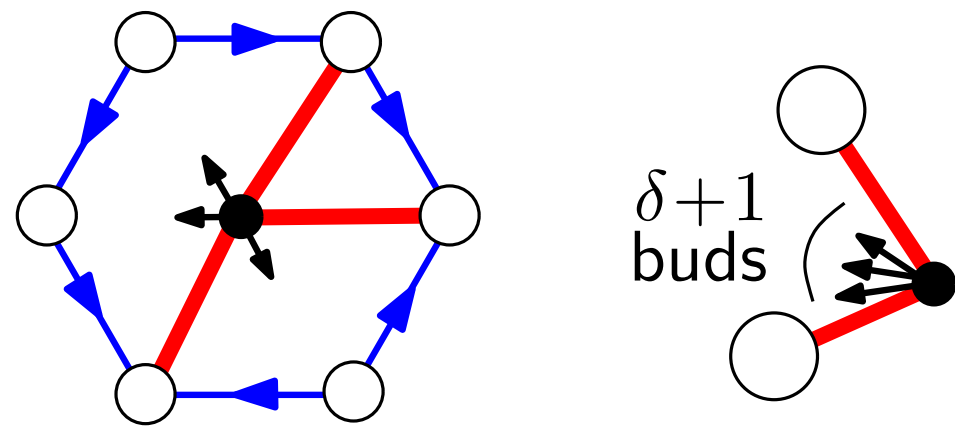
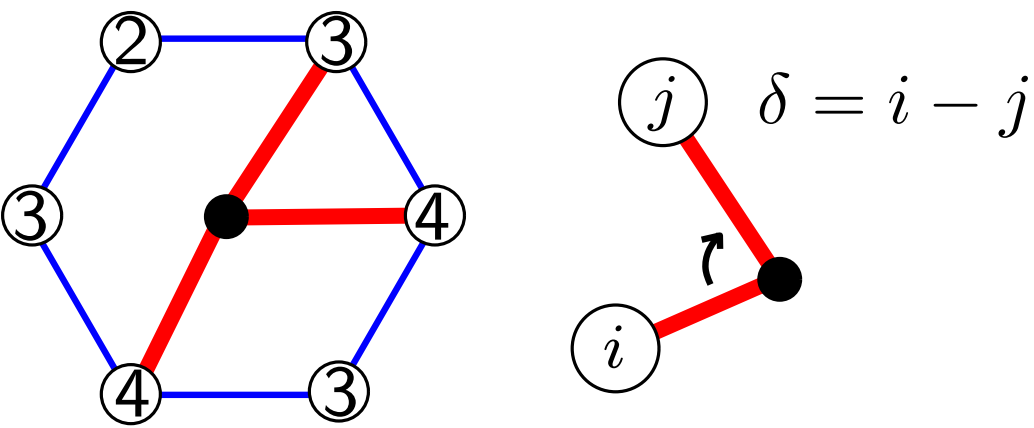
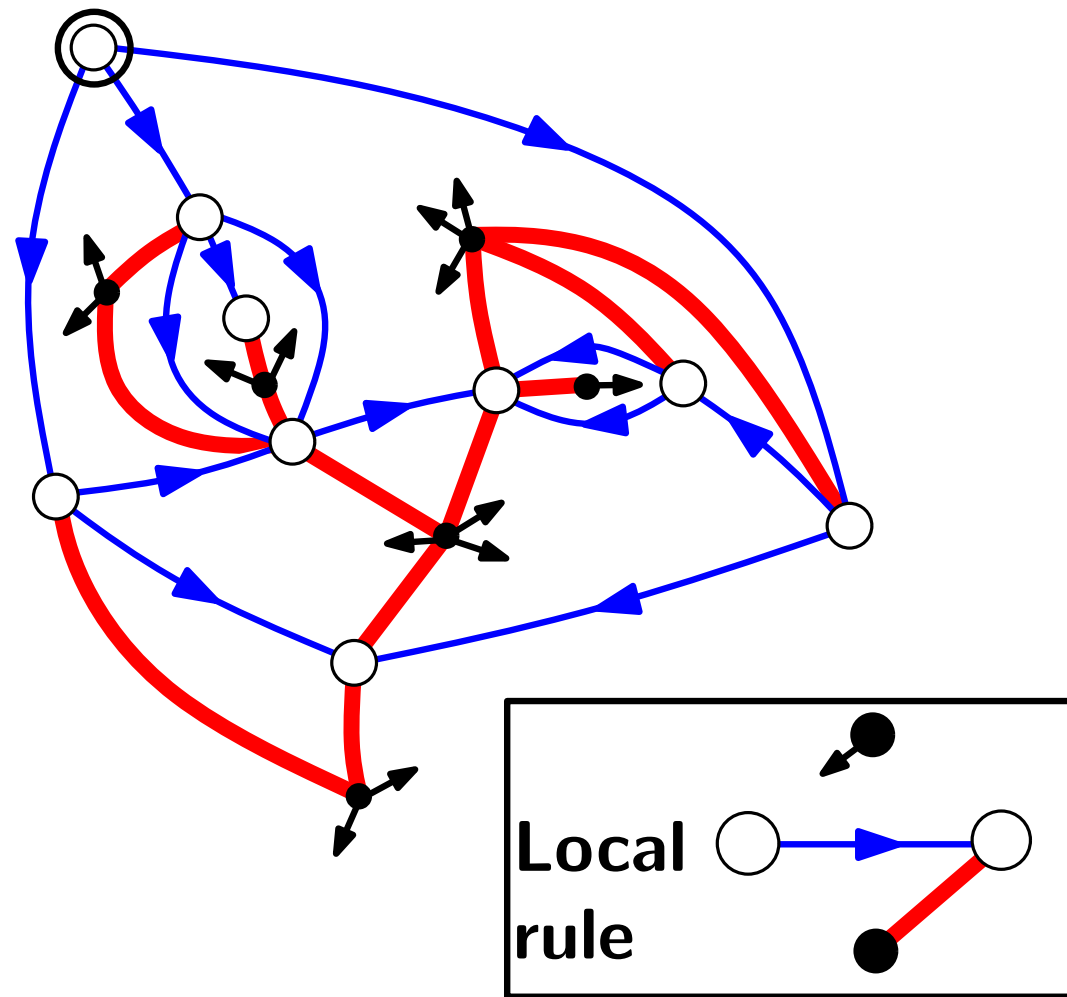
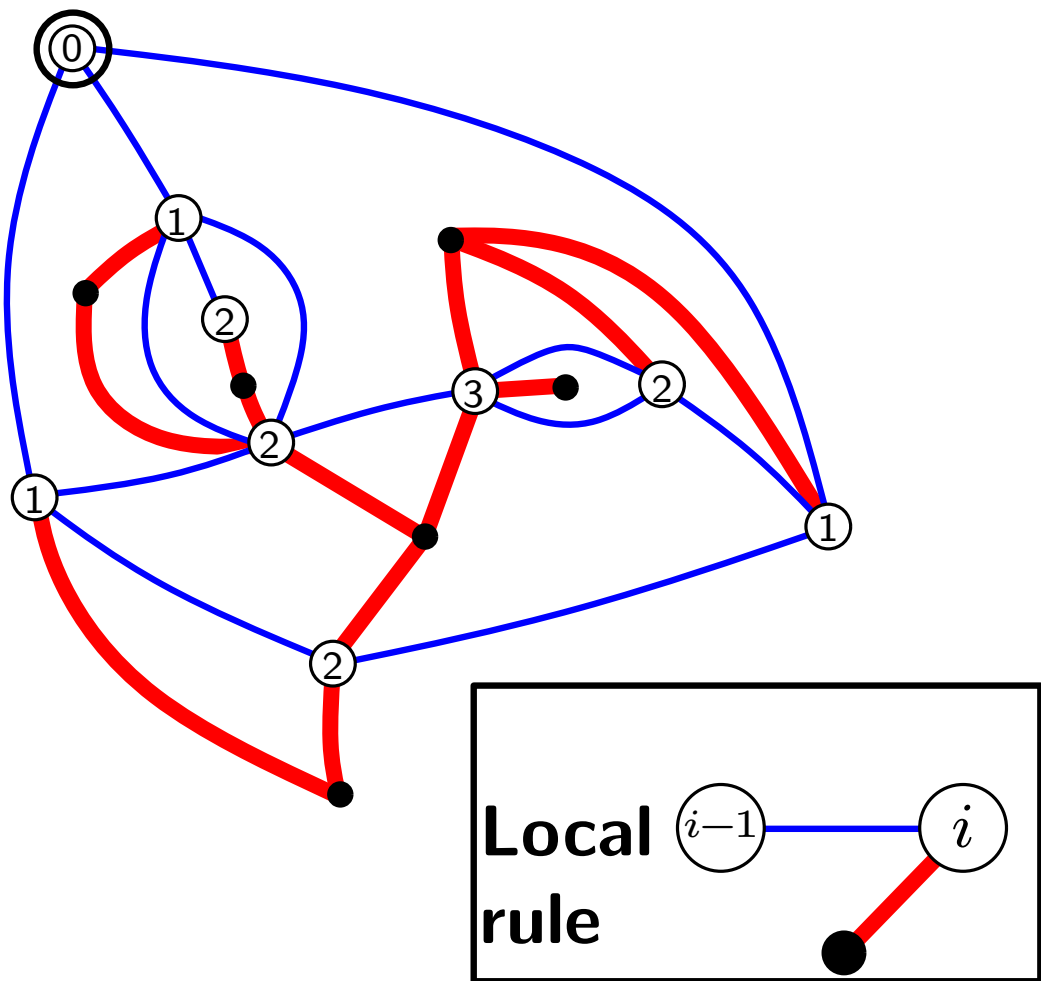
$\Rightarrow$  # rooted bipartite maps with  $n_i$  faces of degree  $2i$  is

$$\frac{2 \cdot (\sum i n_i)!}{(2 + \sum (i-1)n_i)!} \prod_i \frac{1}{n_i!} \binom{2i-1}{i}^{n_i}$$

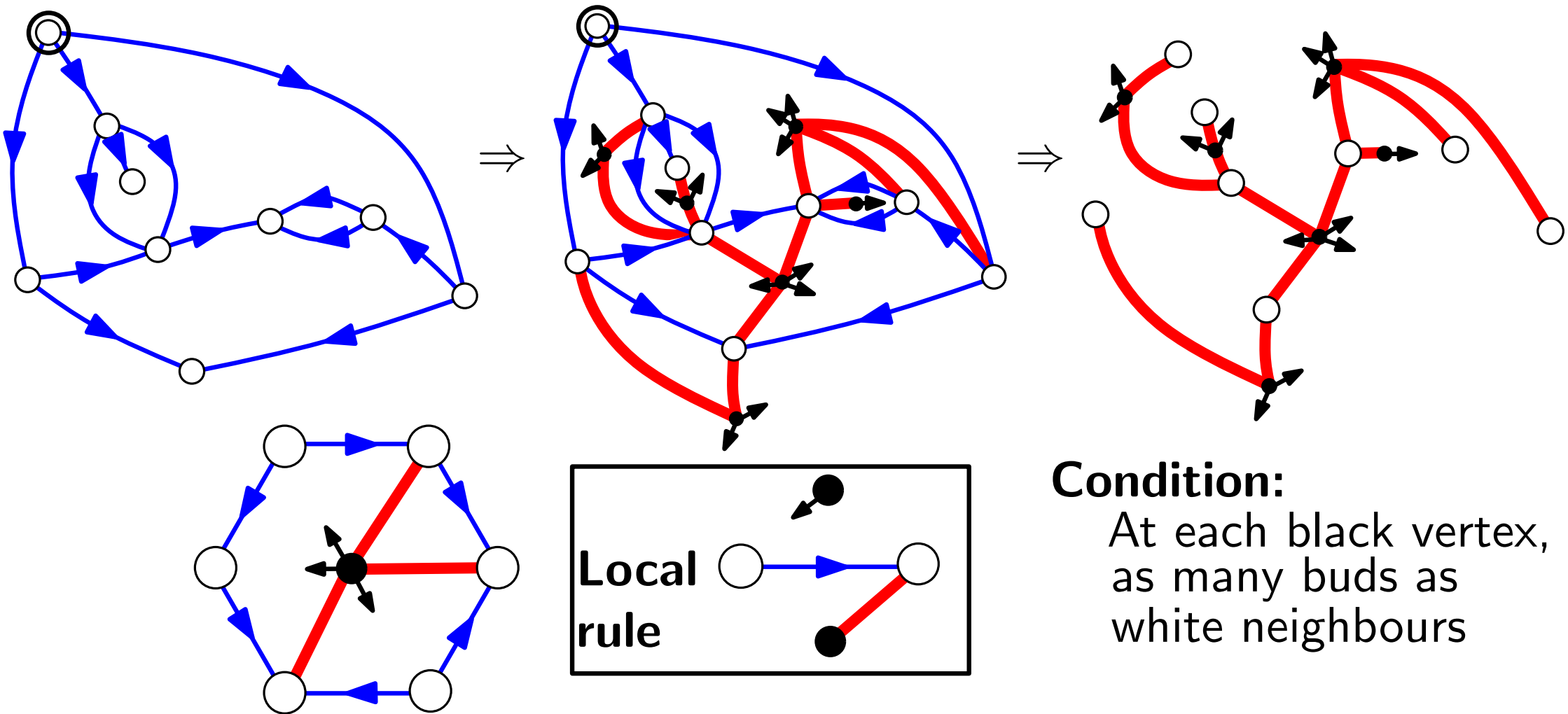
# Reformulation with orientations.

Distance labelling

Geodesic orientation

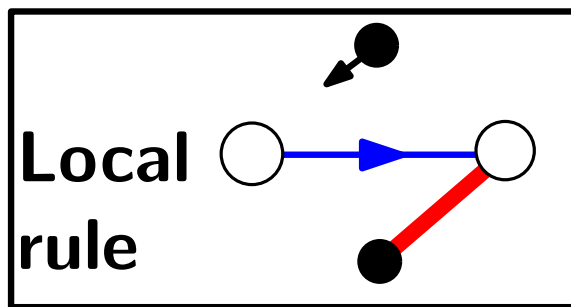
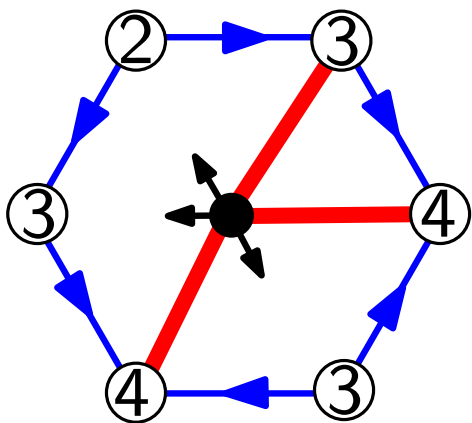
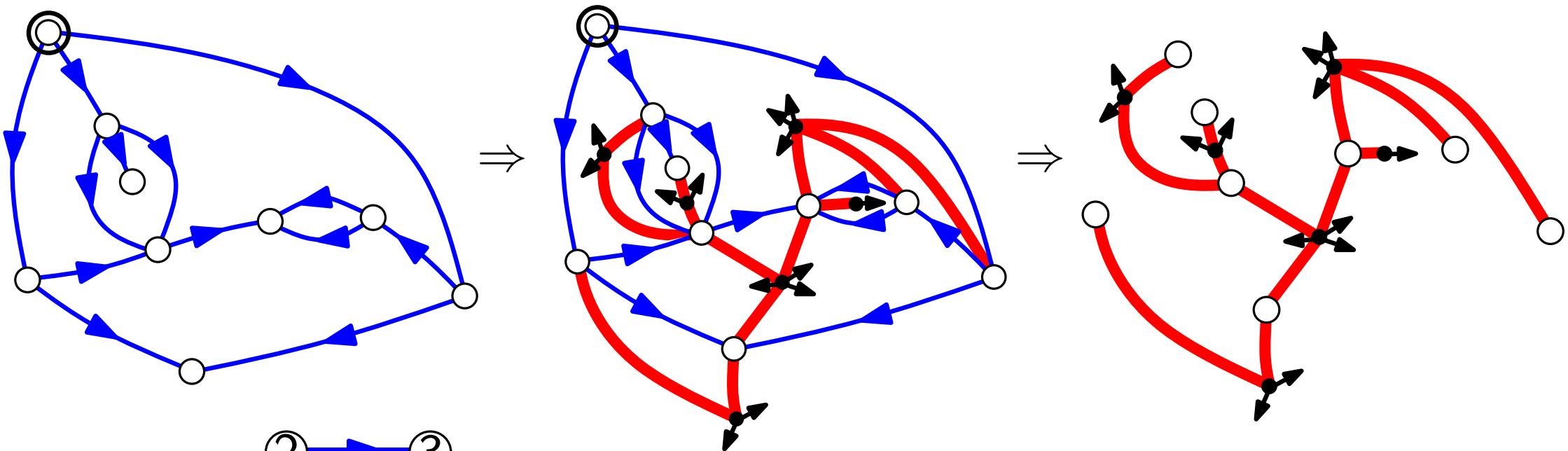


# Reformulation with orientations.



**Theorem:** The mapping is a **bijection**. Each **face of degree  $2i$**  of the bipartite map corresponds to a **black vertex of degree  $2i$**  in the mobile

# Reformulation with orientations.



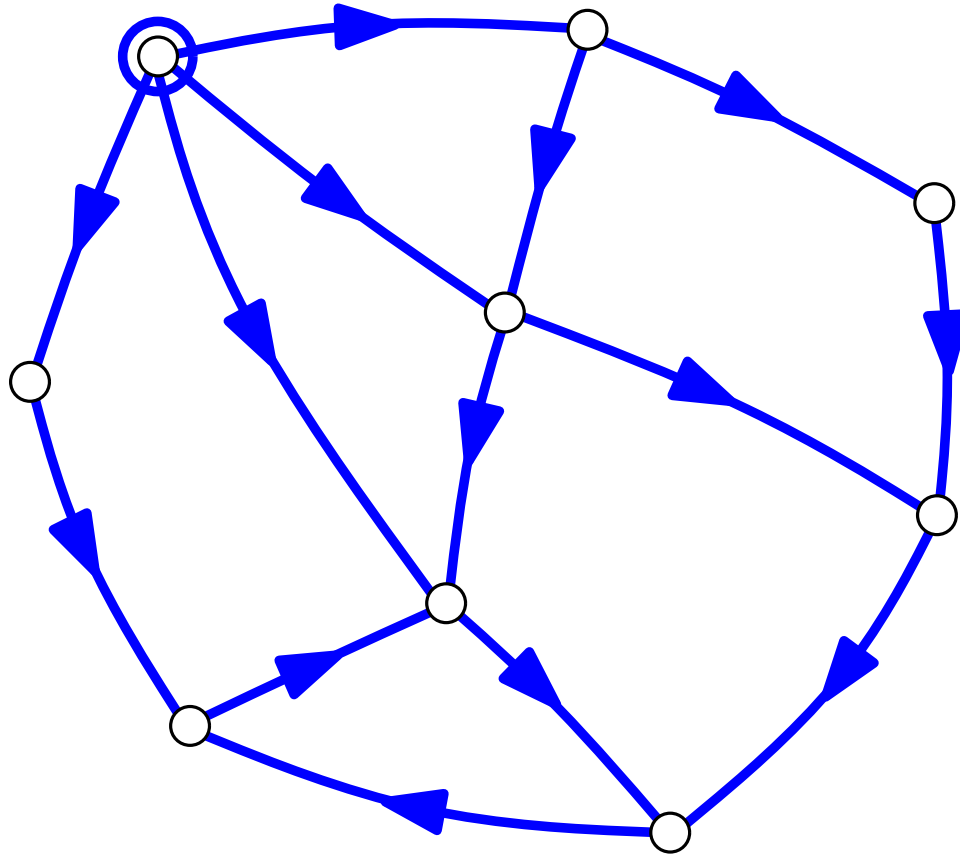
## Condition:

At each black vertex,  
as many buds as  
white neighbours

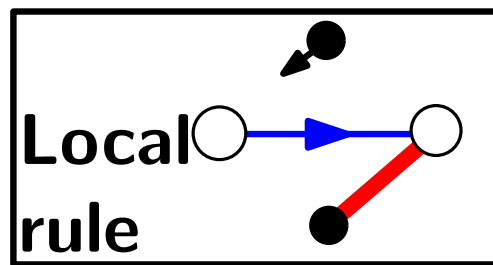
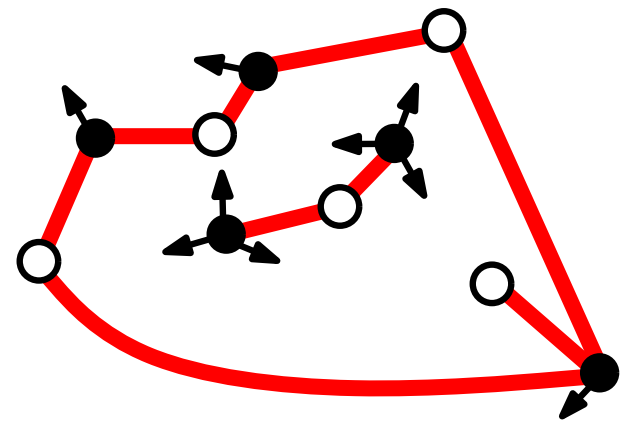
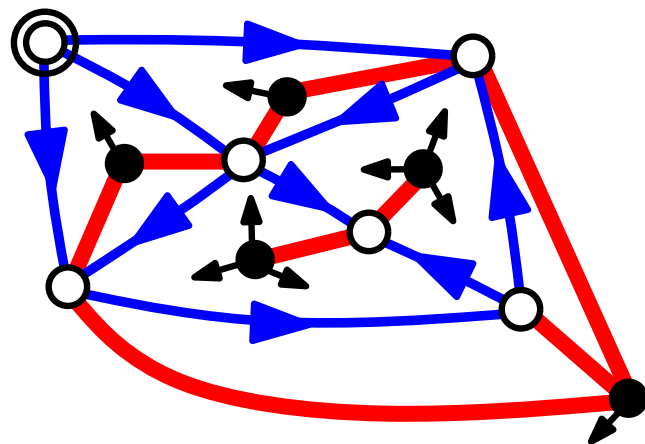
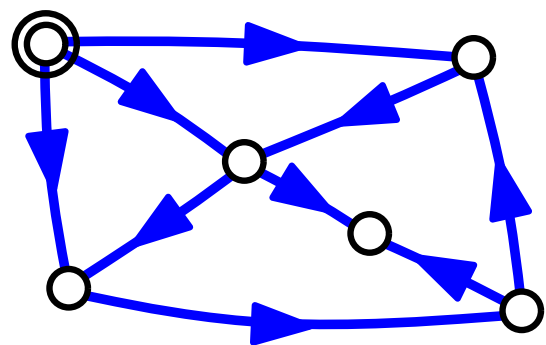
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# Source-orientations

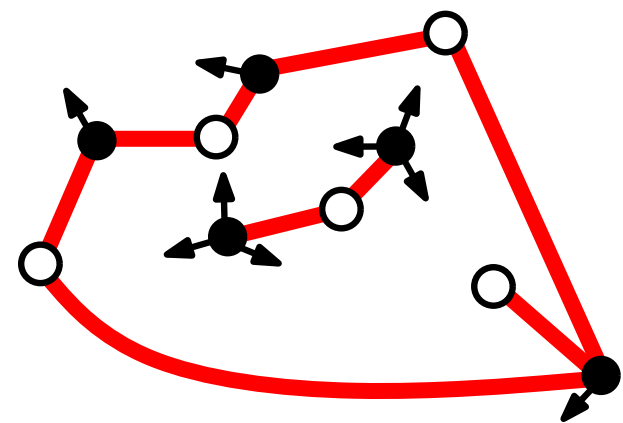
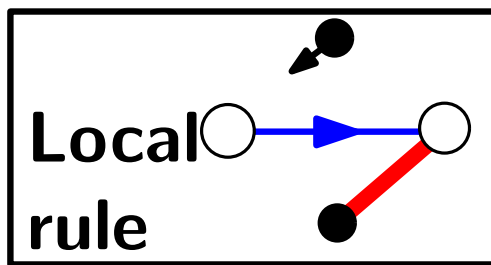
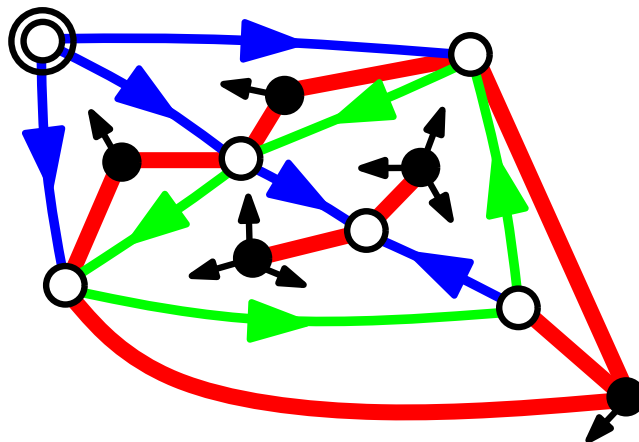
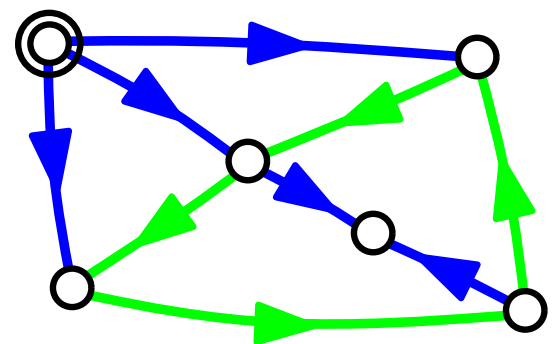
- A **source-orientation** is an orientation of a pointed map such that
- The pointed vertex (called the source) has **only outgoing edges**
  - **Accessibility**: Each vertex can be reached from the source



# Mobile construction for source-orientations

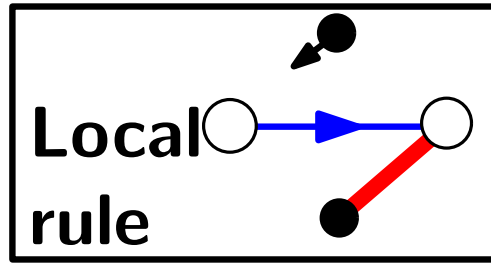
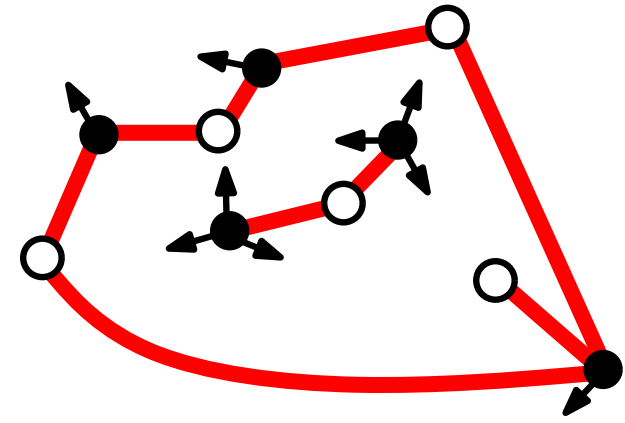
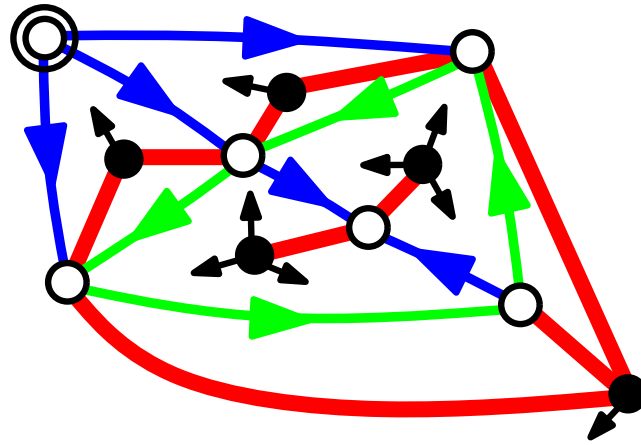
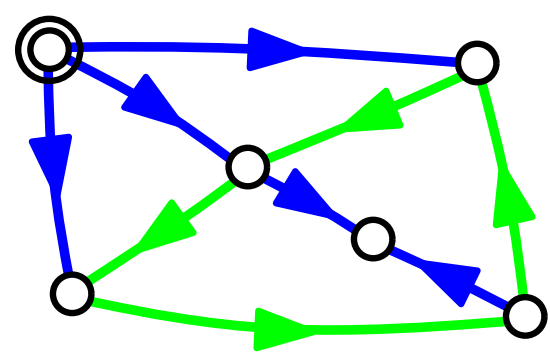


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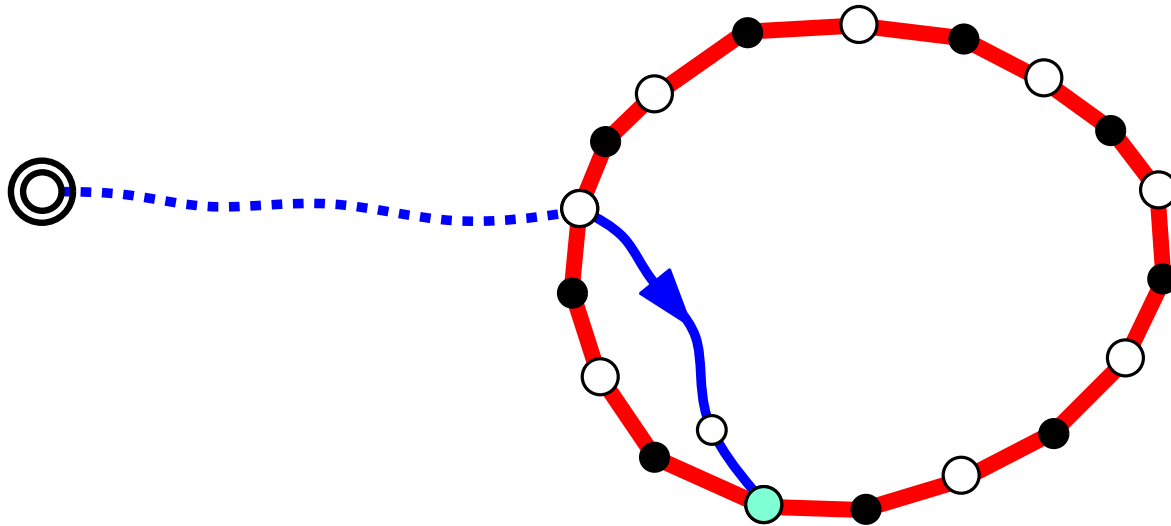




# Mobile construction for source-orientations

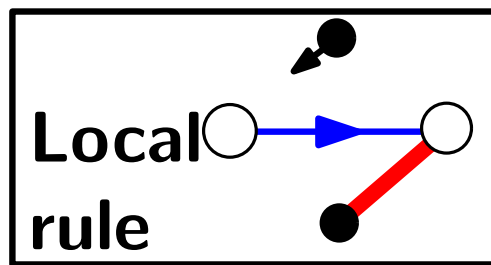
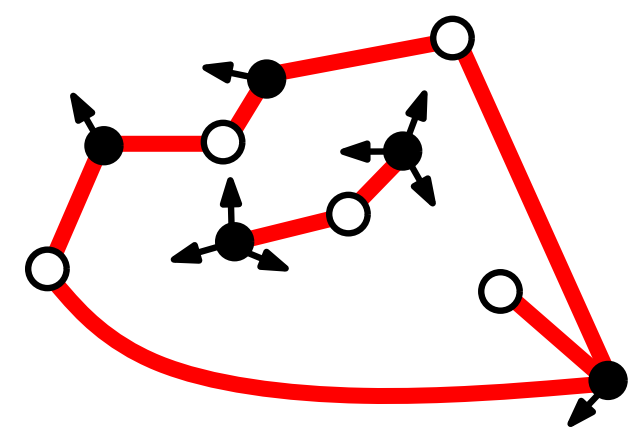
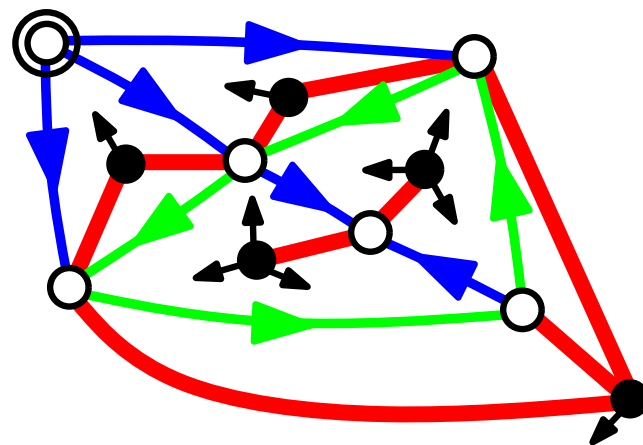
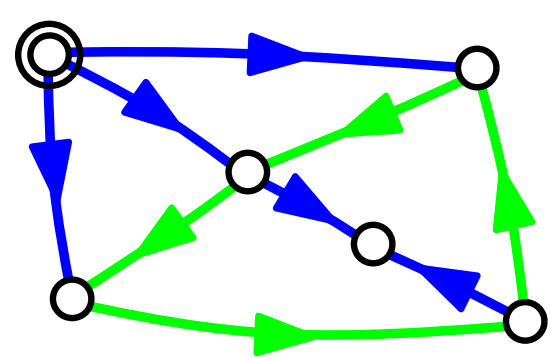


Cycle in mobile  $\Rightarrow$  ccw circuit in the source-orientation

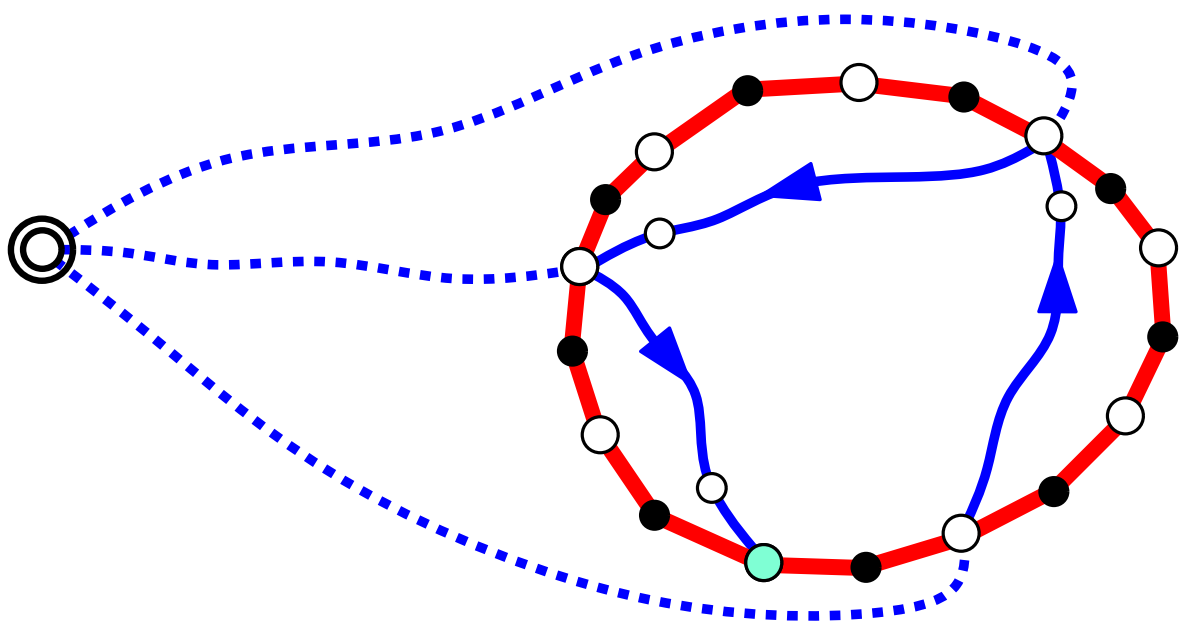




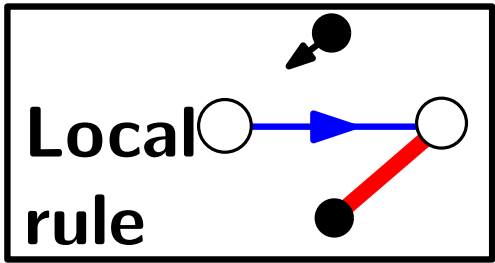
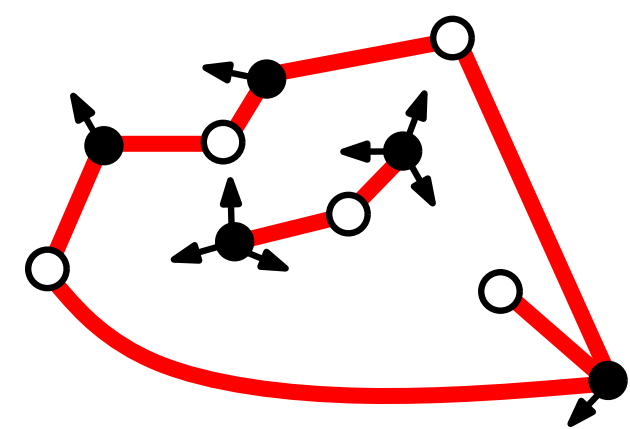
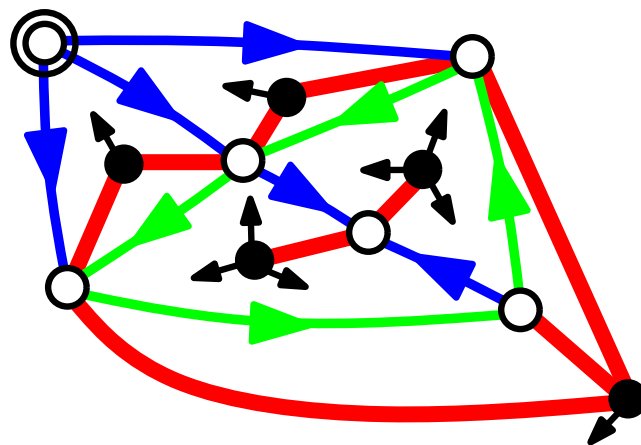
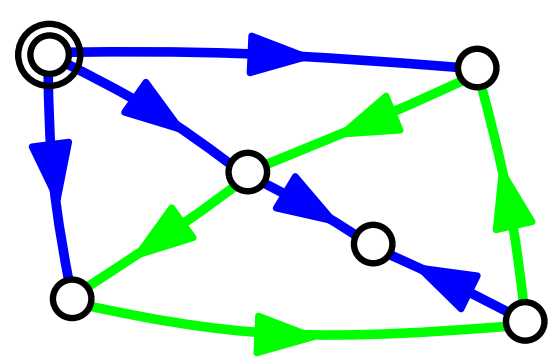
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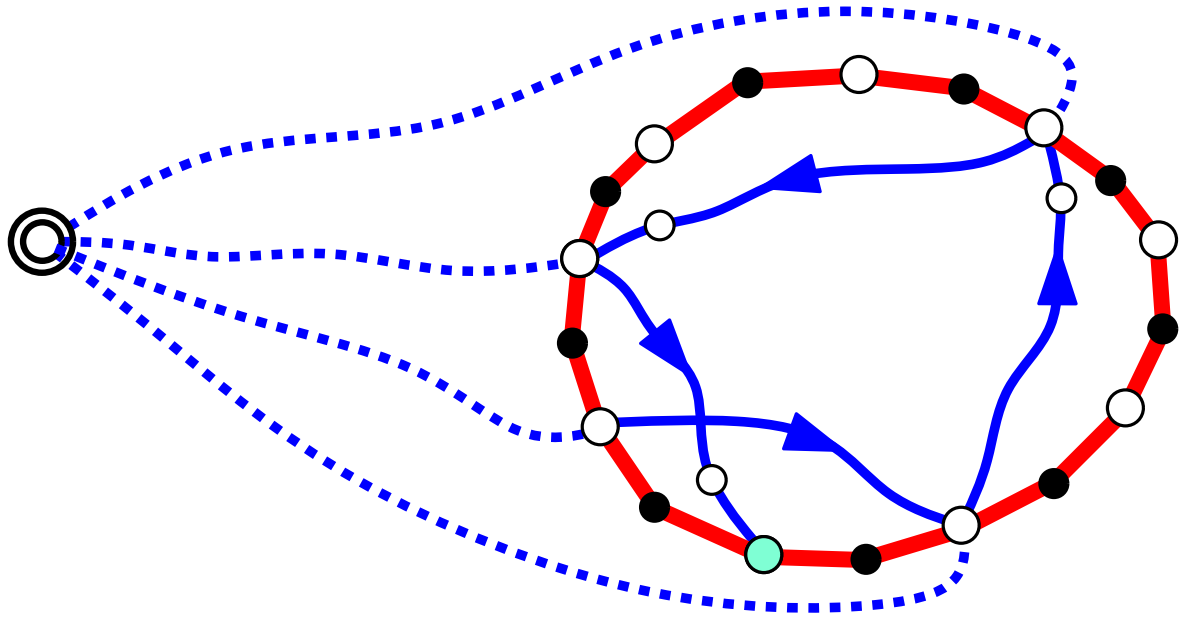
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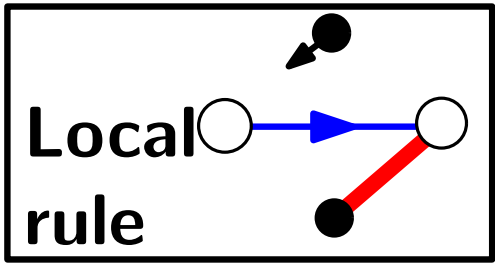
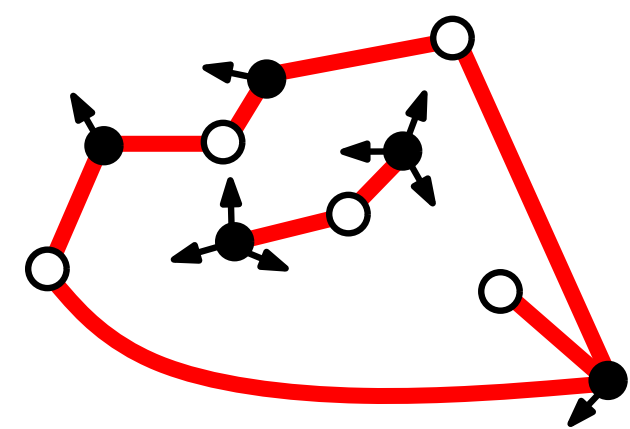
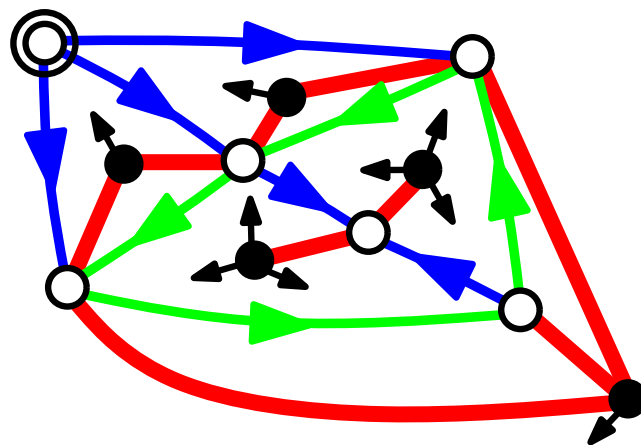
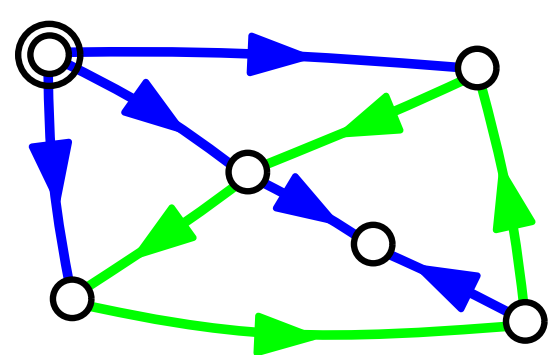
# Mobile construction for source-orientations



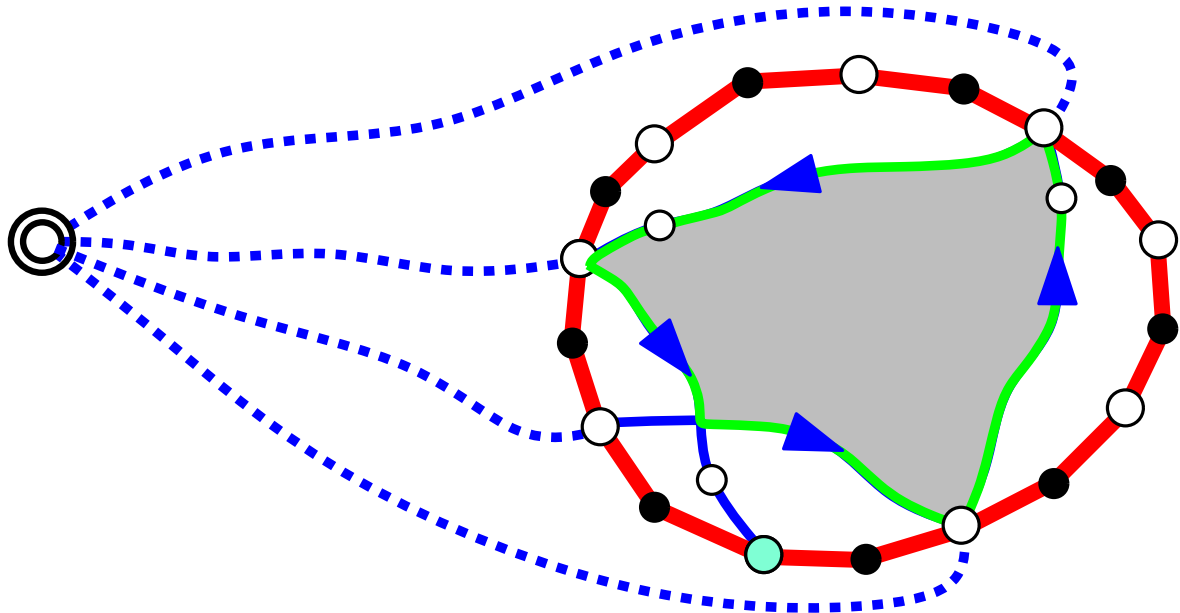
Cycle in mobile  $\Rightarrow$  ccw circuit in the source-orientation



# Mobile construction for source-orientations

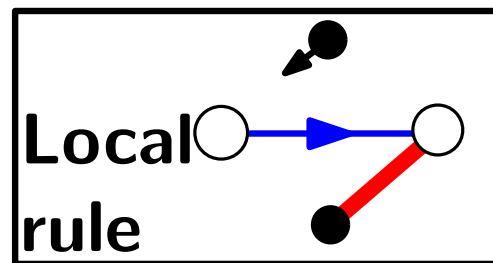
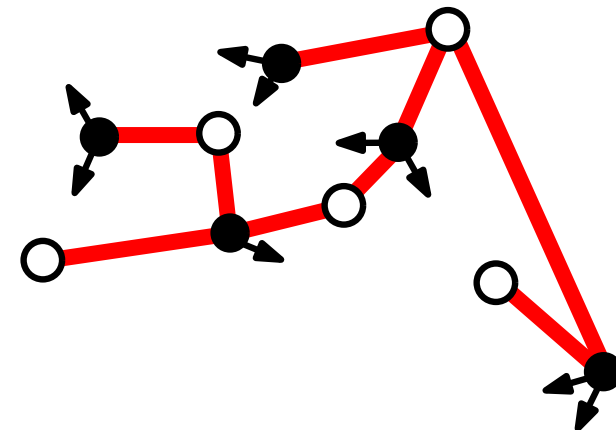
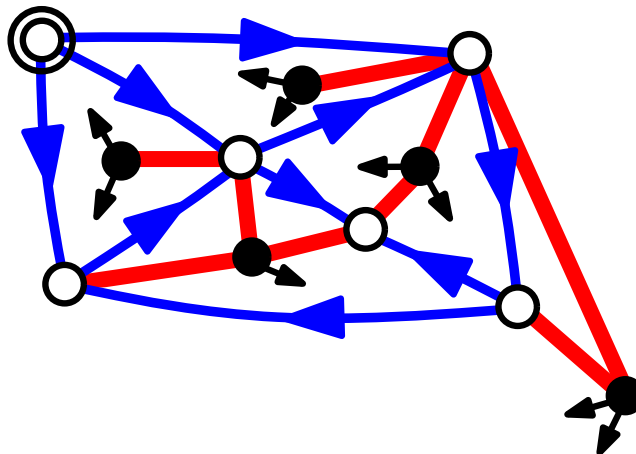
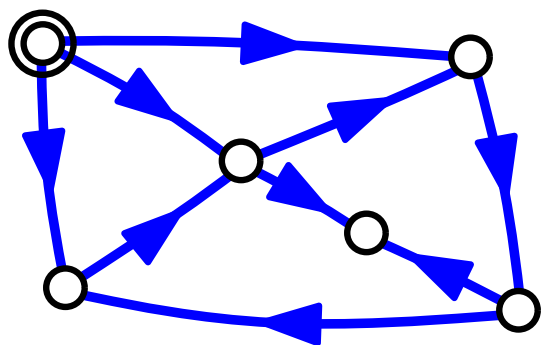


## Cycle in mobile $\Rightarrow$ ccw circuit in the source-orientation

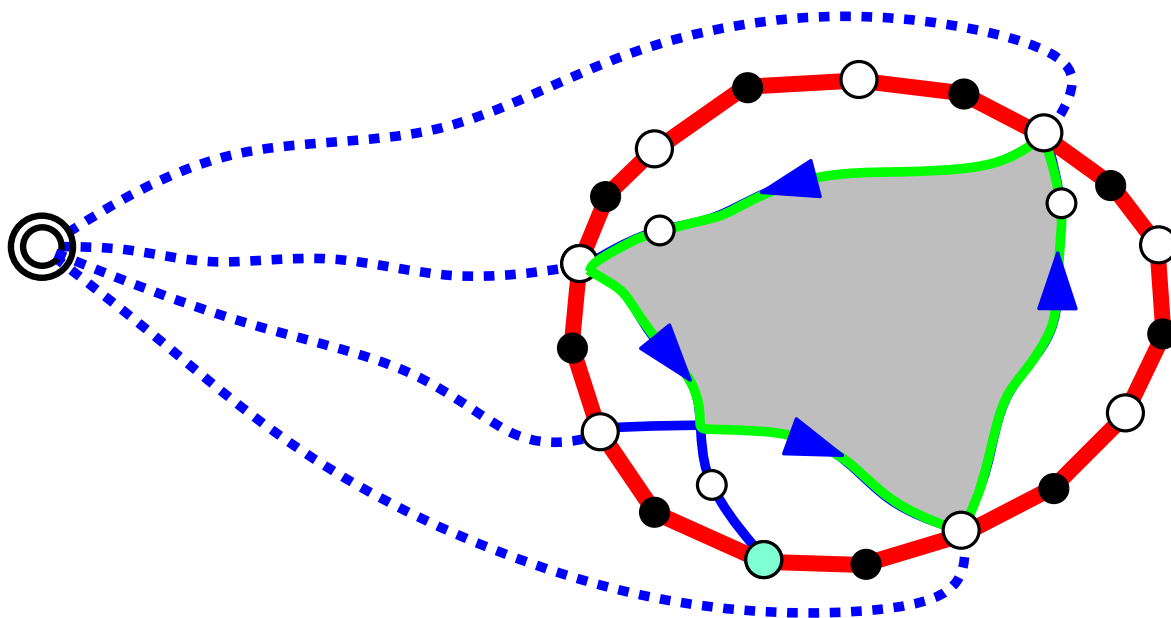


Prisoner  
cycle lemma

# Mobile construction for source-orientations



Cycle in mobile  $\Rightarrow$  ccw circuit in the source-orientation

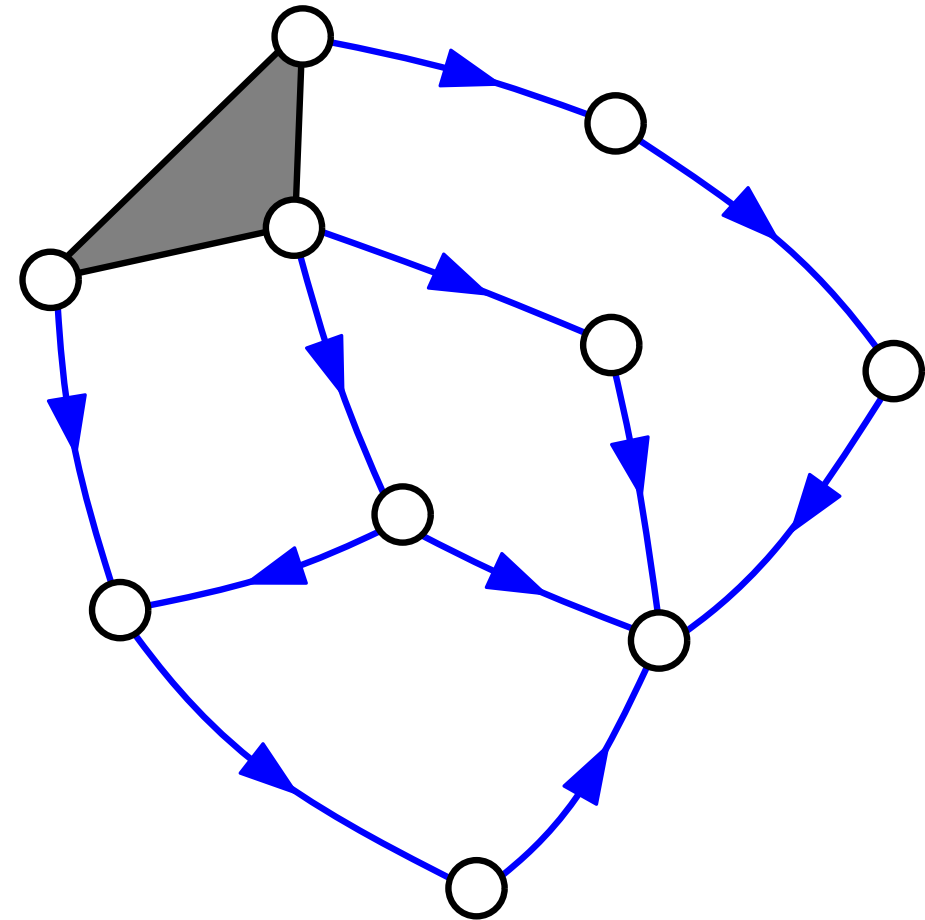


Prisoner  
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## **d-gonal source-orientations**

We allow the source of the orientation to be a  $d$ -gon, with  $d \geq 0$

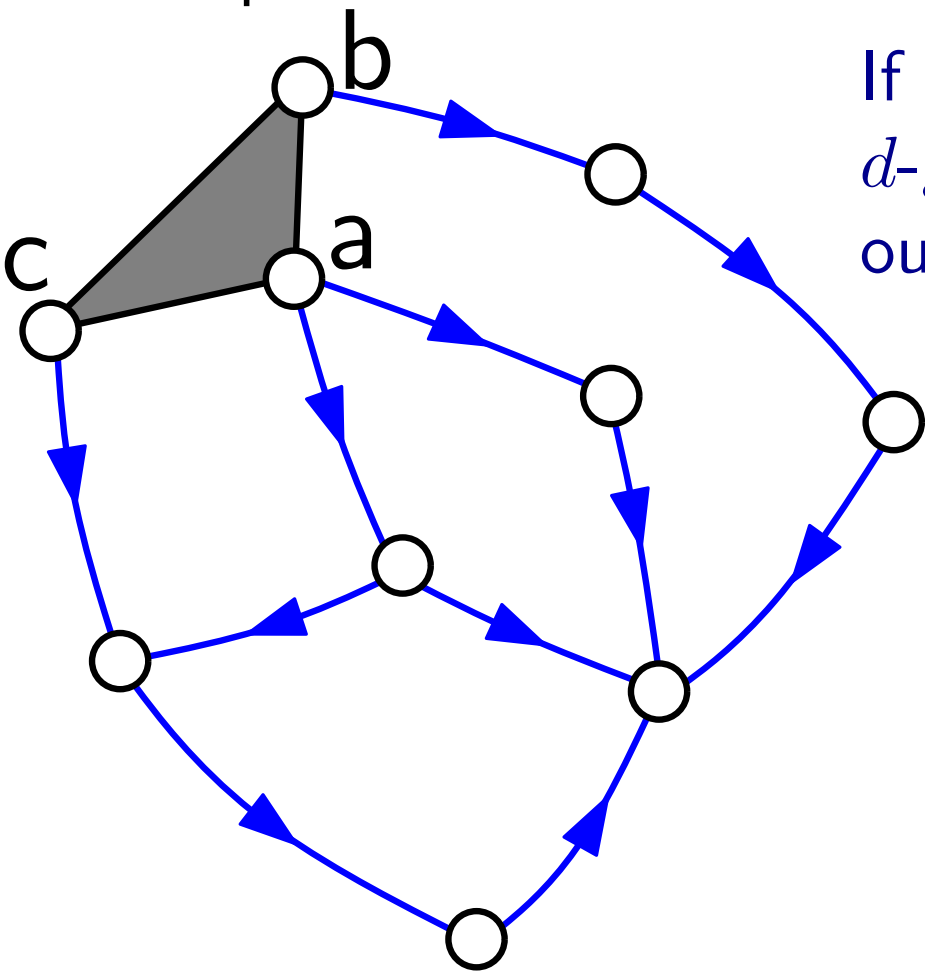
Example for  $d = 3$



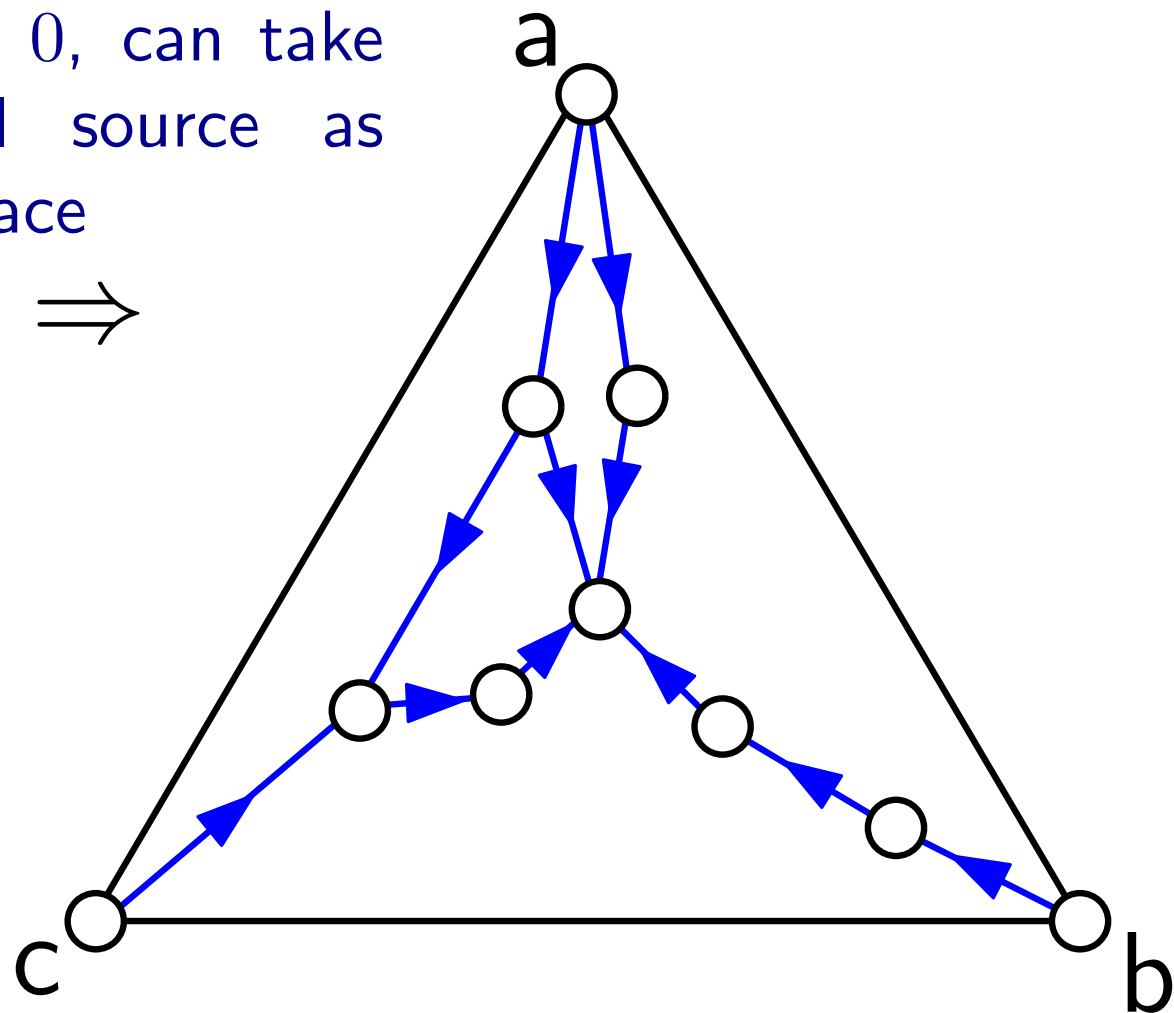
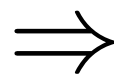
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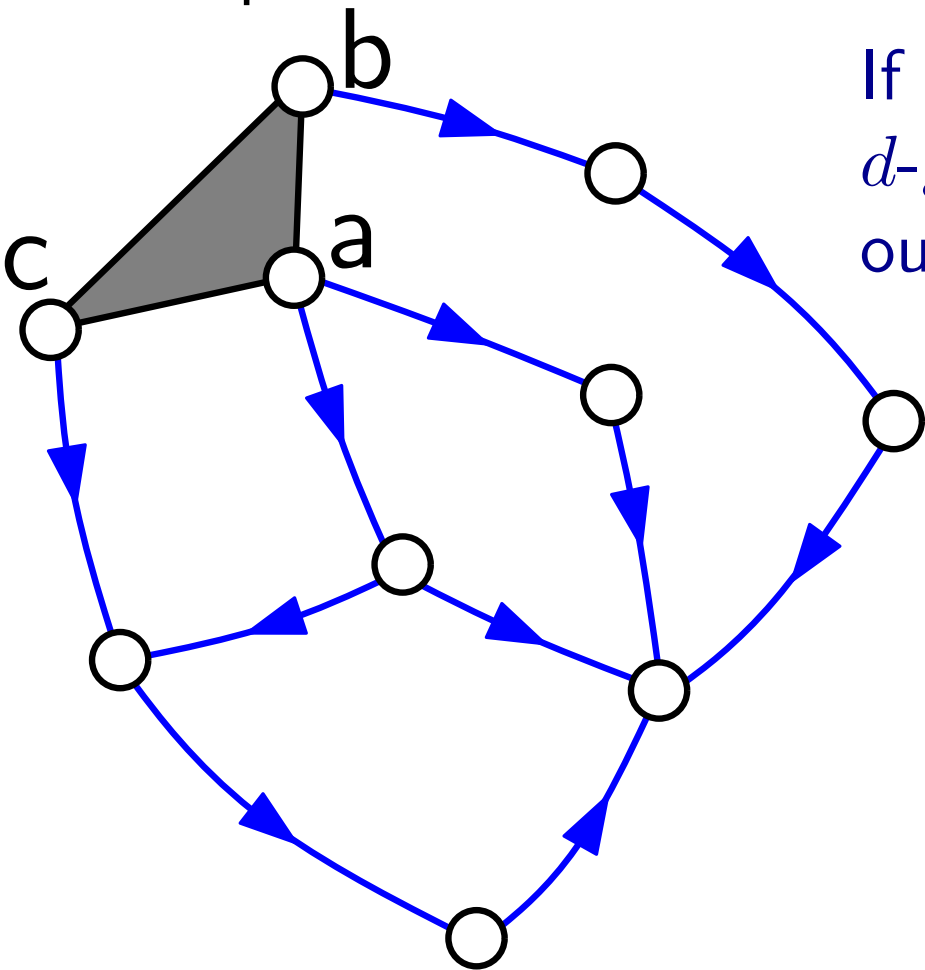
If  $d > 0$ , can take  $d$ -gonal source as outer face



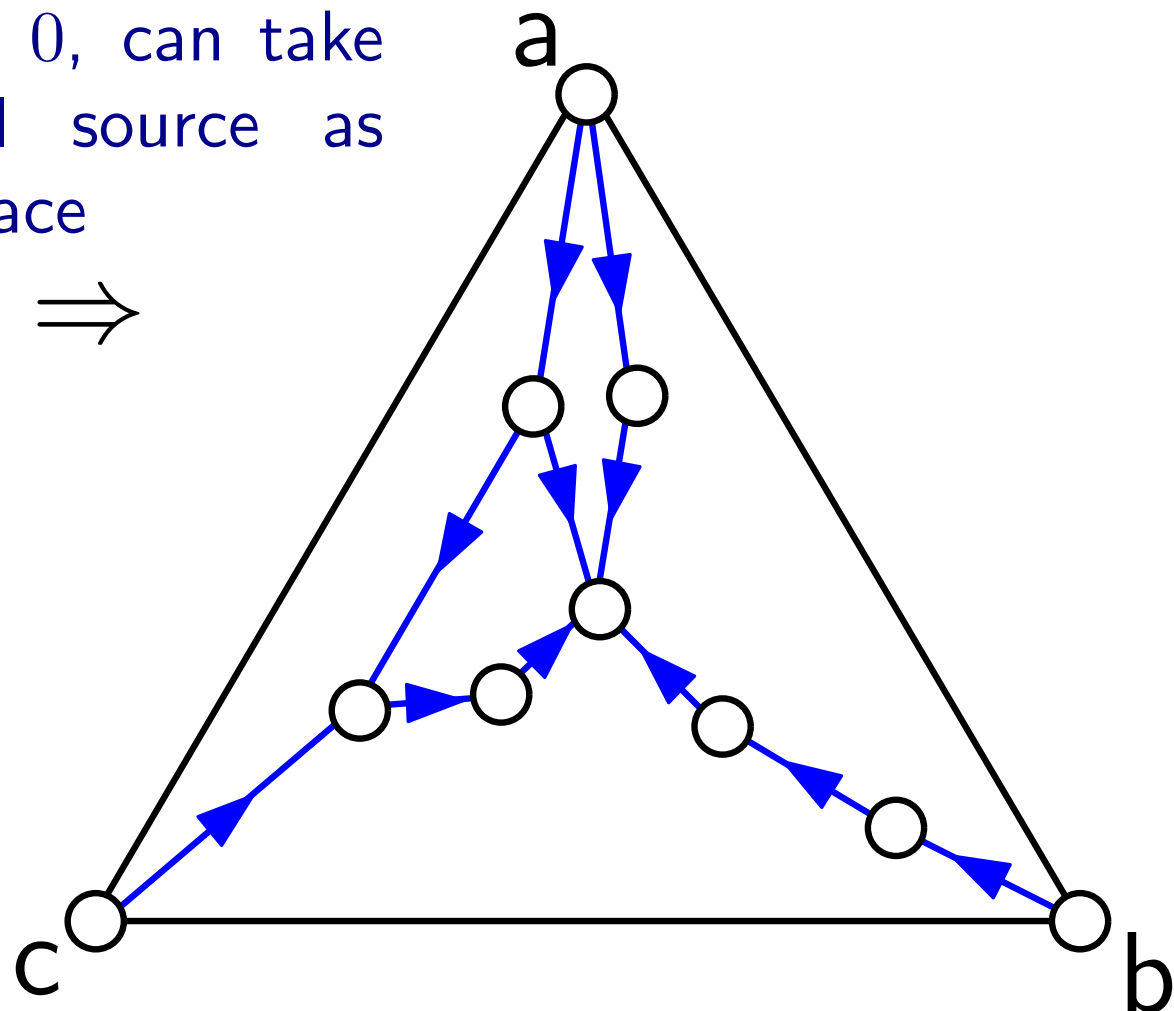
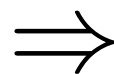
# **d-gonal source-orientations**

We allow the source of the orientation to be a  $d$ -gon, with  $d \geq 0$

Example for  $d = 3$



If  $d > 0$ , can take  $d$ -gonal source as outer face

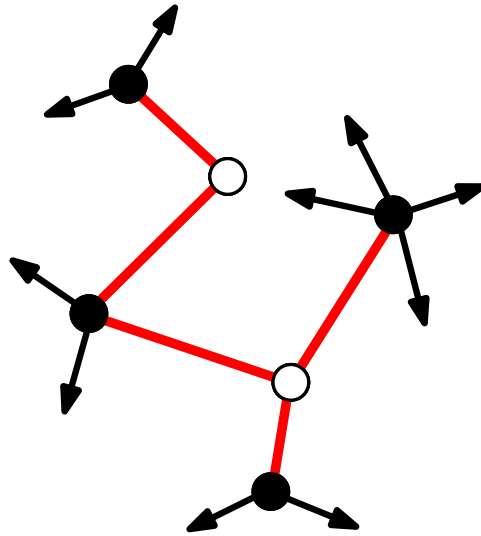


Let  $\mathcal{O}_d$  be the set of  $d$ -gonal source-orientations with no ccw circuit

Let  $\mathcal{O} = \cup_{d \geq 0} \mathcal{O}_d$

# Mobiles

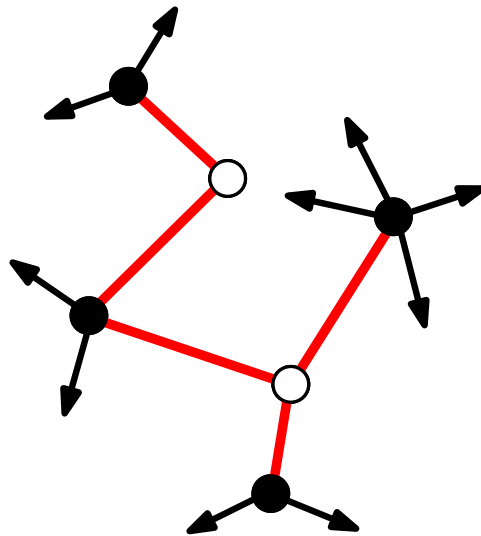
A **mobile** is a plane tree with vertices properly colored in black and white, together with **buds** (half-edges) incident to black vertices.



The **excess** is the number of buds minus the number of edges.

# Mobiles

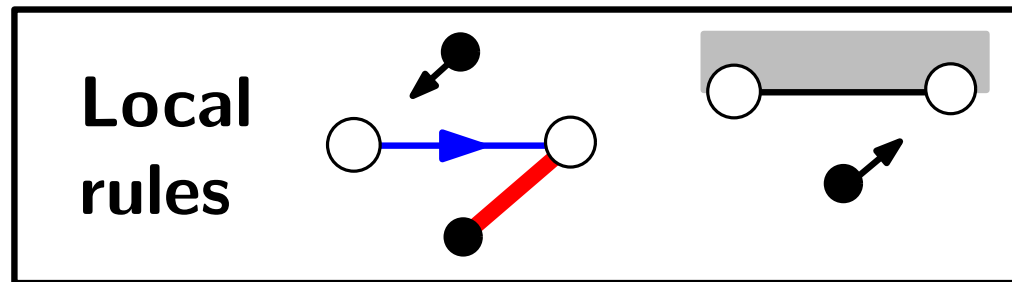
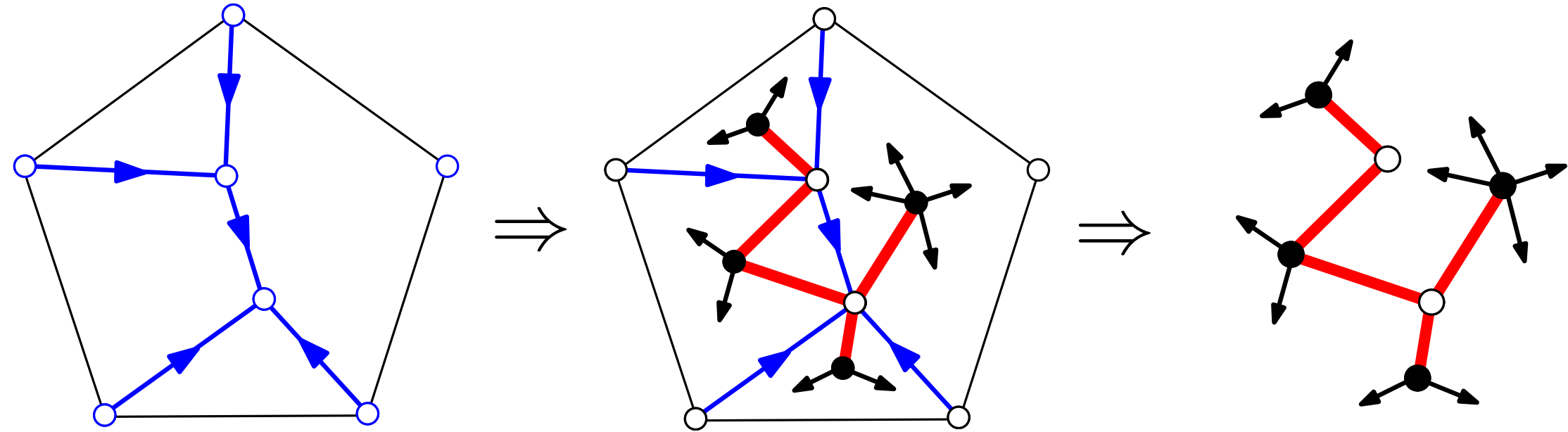
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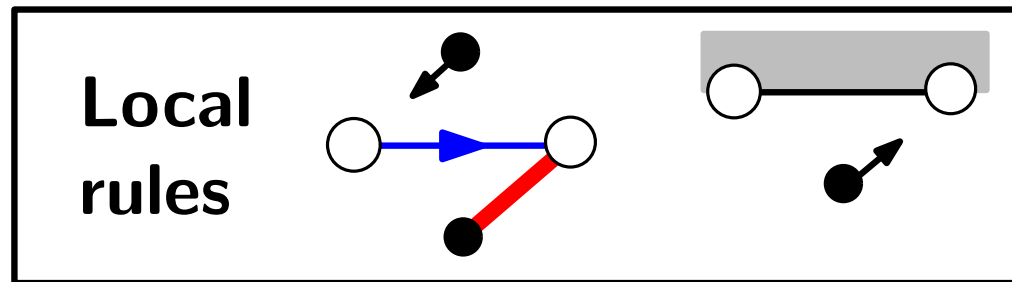
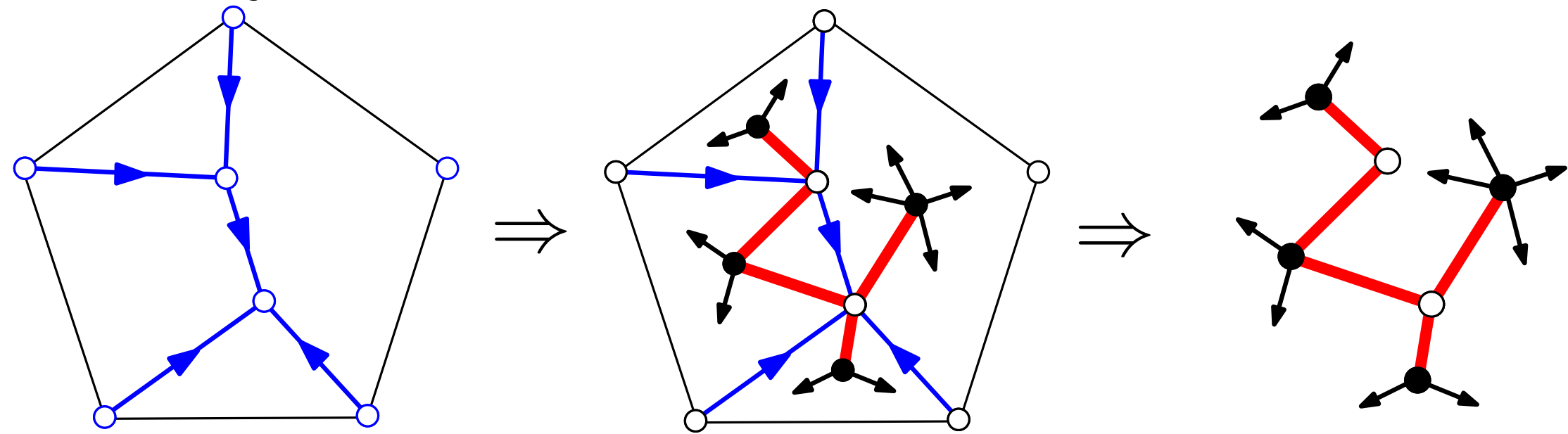
The **excess** is the number of buds minus the number of edges.

Let  $\mathcal{M}$  be the set of mobiles of nonnegative excess

# Master bijection $\Phi$



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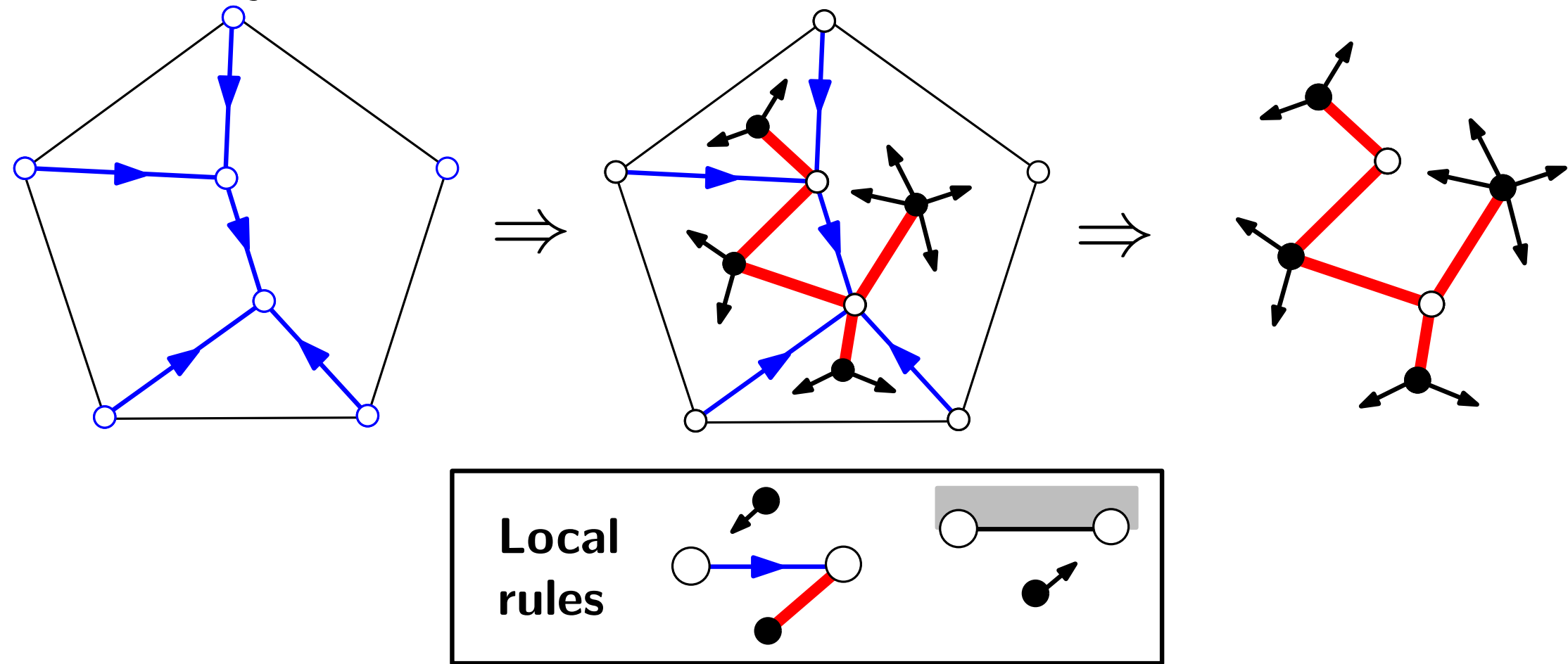


**Theorem [Bernardi-F'10]:**  $\Phi$  is a **bijection** between  $\mathcal{O}$  and  $\mathcal{M}$ .

Moreover,

- degree of external face  $\longleftrightarrow$  excess
- degree of internal faces  $\longleftrightarrow$  degree of black vertices
- indegree of internal vertices  $\longleftrightarrow$  degree of white vertices

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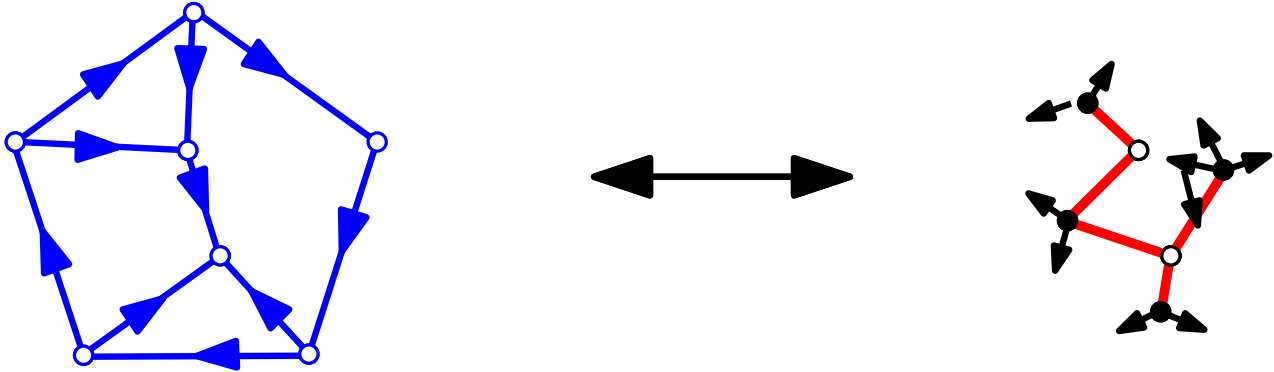
$\longleftrightarrow$  degree of white vertices

cf [Bernardi'07], [Bernardi-Chapuy'10]

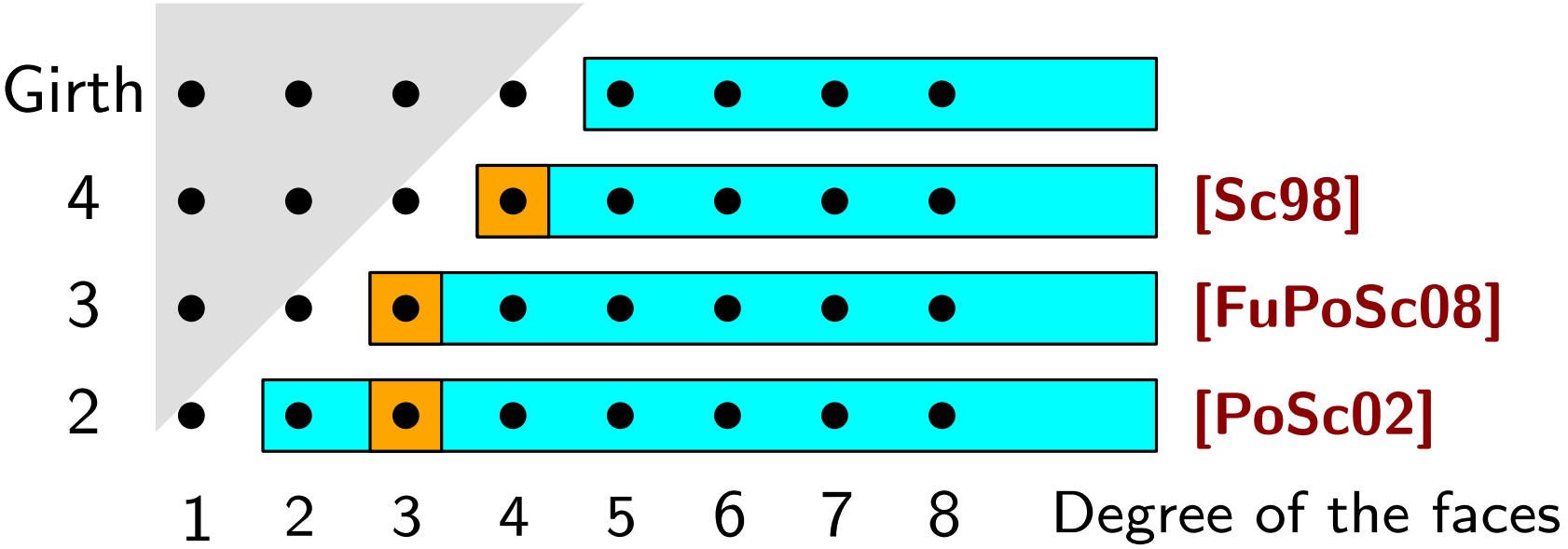
# Using the master bijection for map enumeration

# Main new results

The **Master bijection** between  $\mathcal{O}$  (orientations) and  $\mathcal{M}$  (mobiles)



allows to count maps by girth & face-degrees (via canonical orientations).



## Scheme for the strategy

(1) Map family  $\mathcal{C}$  identifies with a **subfamily**  $\mathcal{O}_{\mathcal{C}}$  of  $\mathcal{O}$  with conditions on:

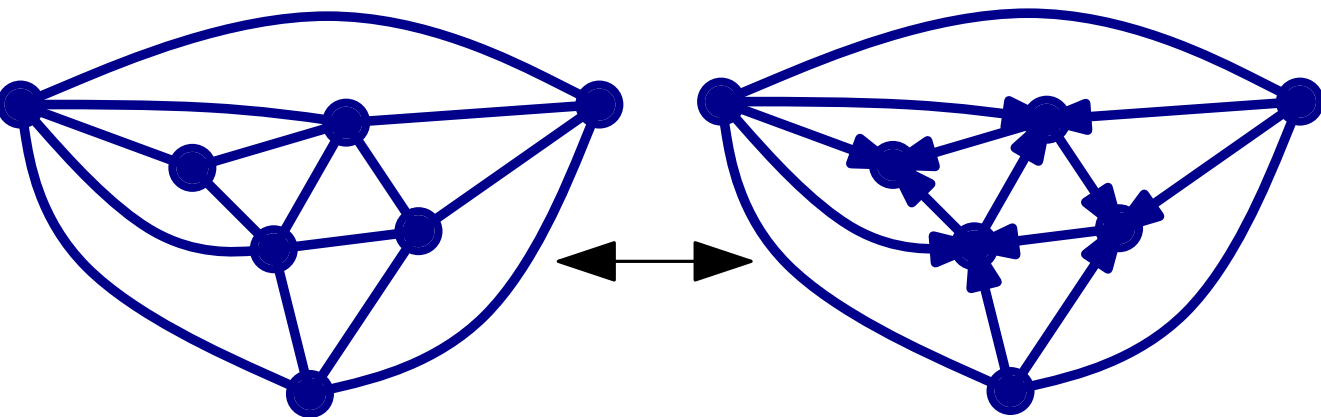
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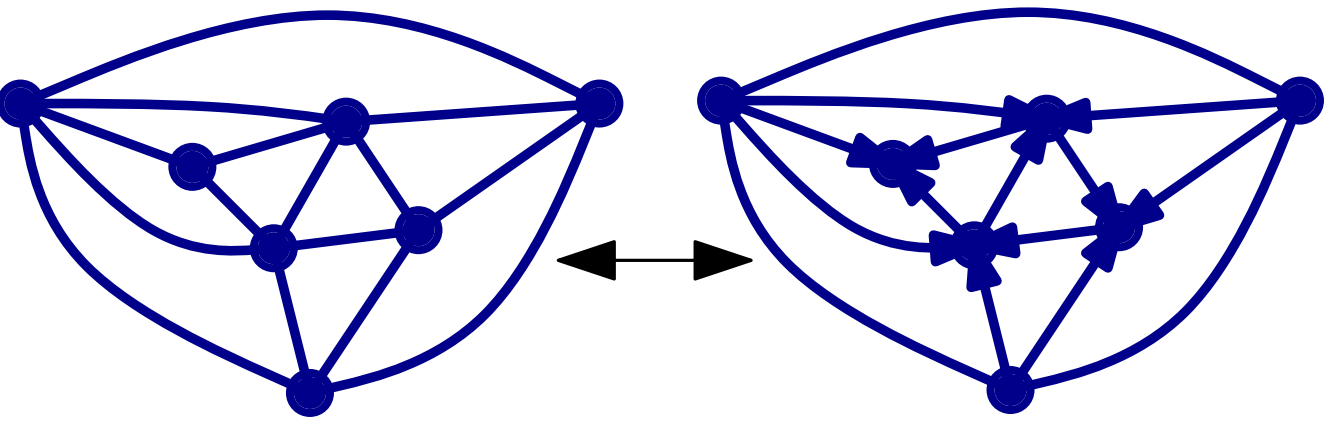
- Face-degree = 3
- Vertex-indegree = 3

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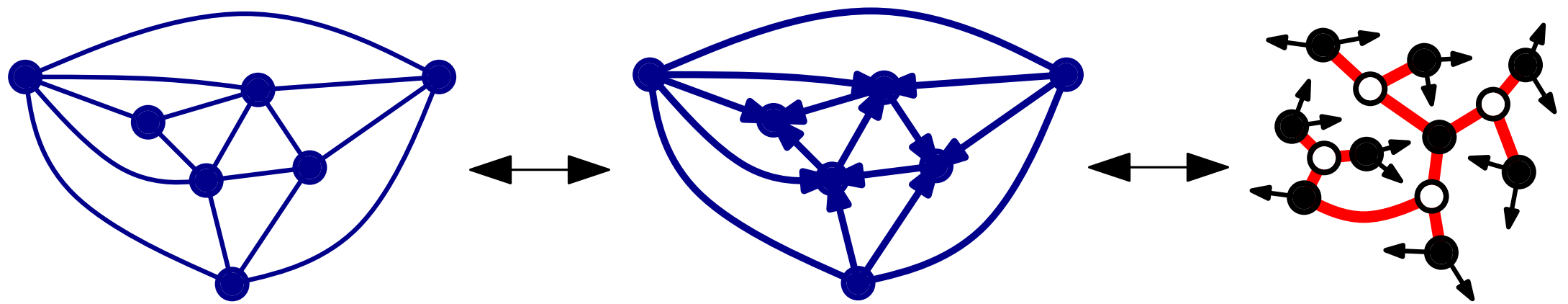
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(2) **Specialize** the master bijection to the subfamily  $\mathcal{O}_C$

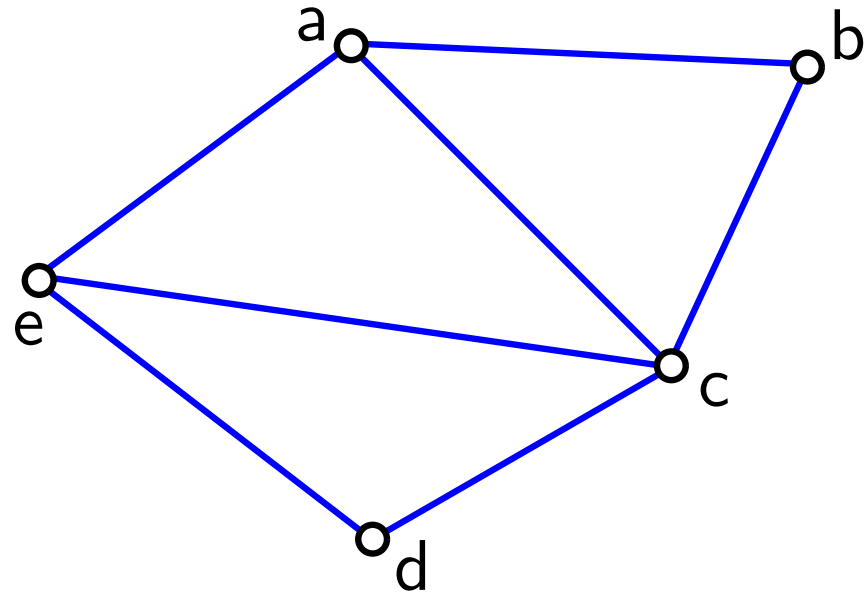


degree of internal faces  $\longleftrightarrow$  degree of black vertices  
 indegree of internal vertices  $\longleftrightarrow$  degree of white vertices

## $\alpha$ -orientations

Let  $G = (V, E)$  be a graph

Let  $\alpha$  be a function from  $V$  to  $\mathbb{N}$

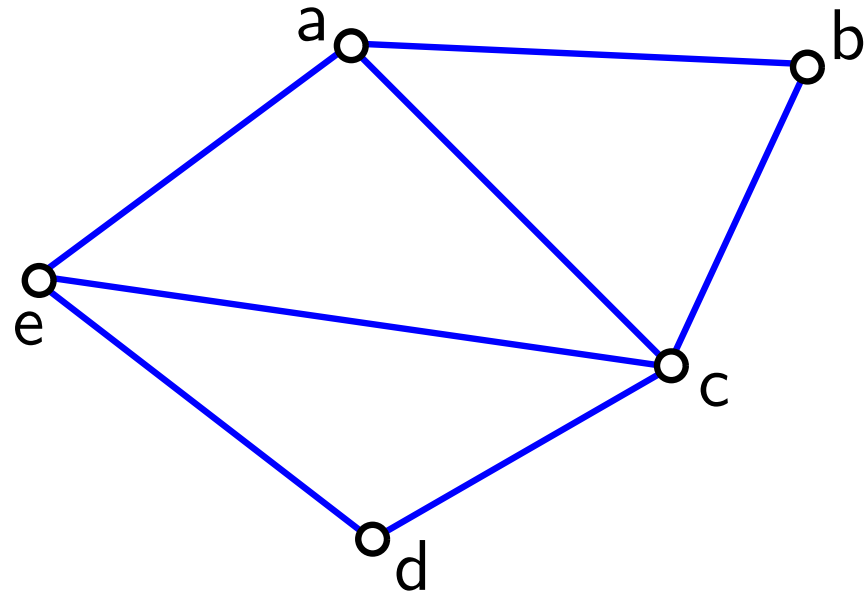


$\alpha :$	a	$\rightarrow$	2
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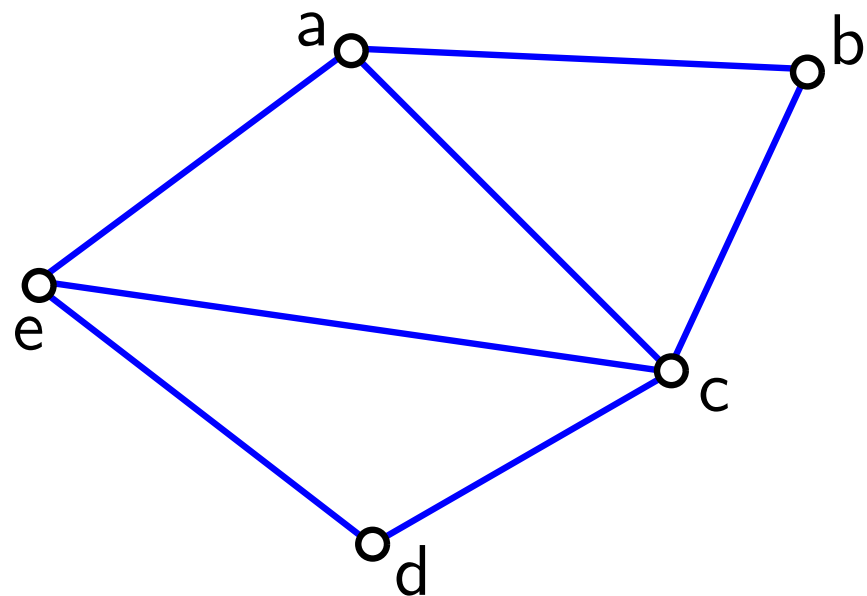
**Def:** An  $\alpha$ -orientation is an orientation of  $G$  where for each  $v \in V$

$$\text{indegree}(v) = \alpha(v)$$

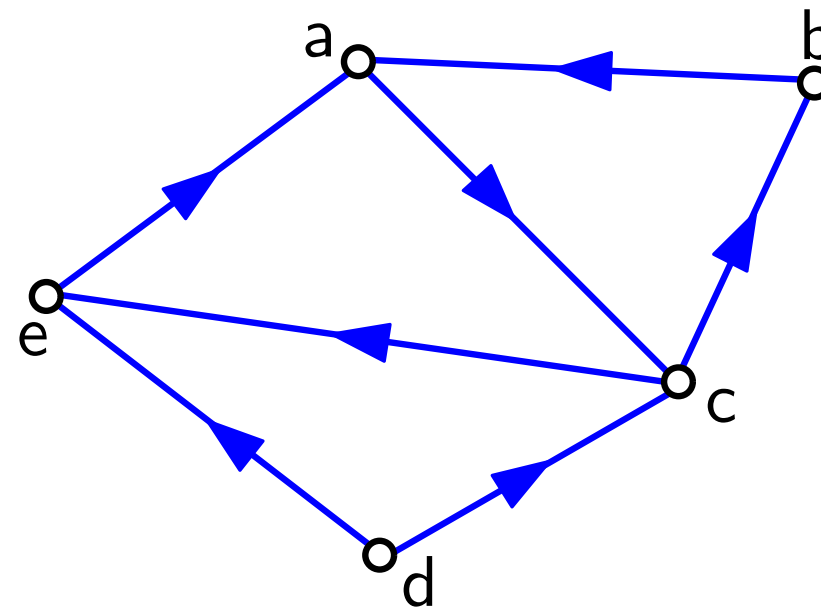
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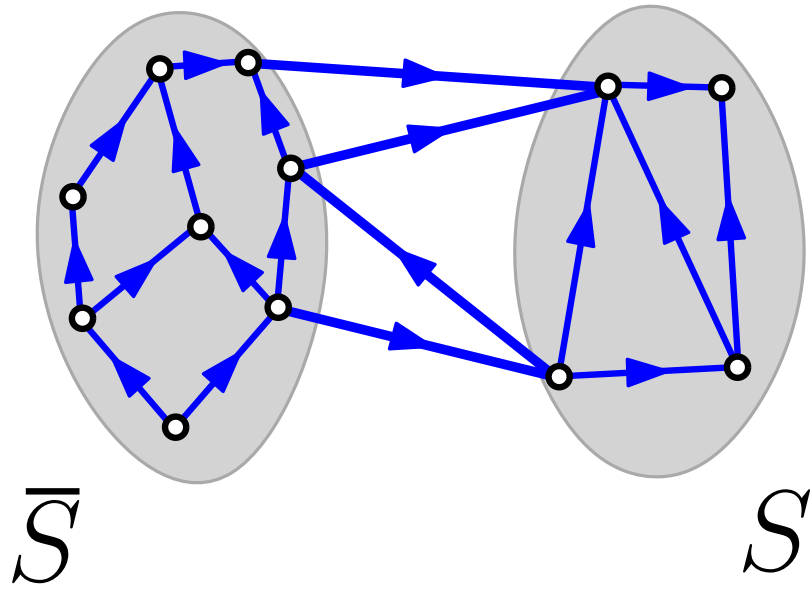


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# $\alpha$ -orientations: criteria for existence and accessibility

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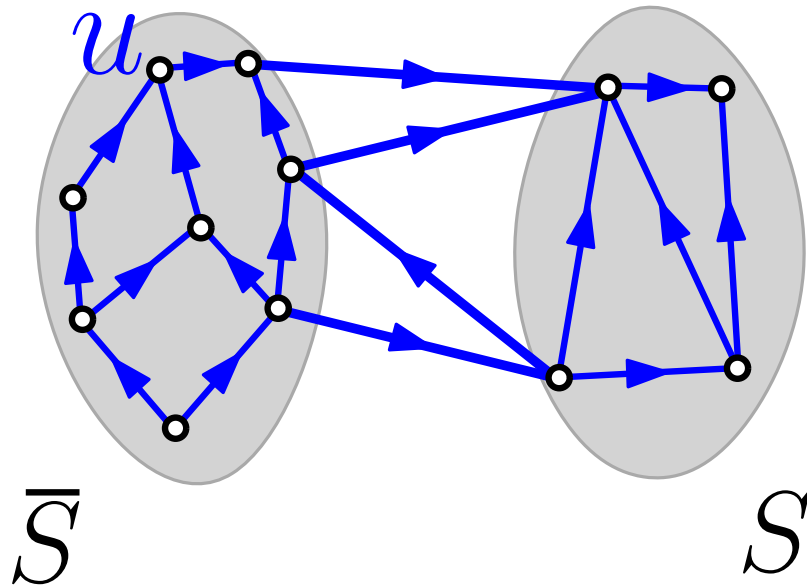


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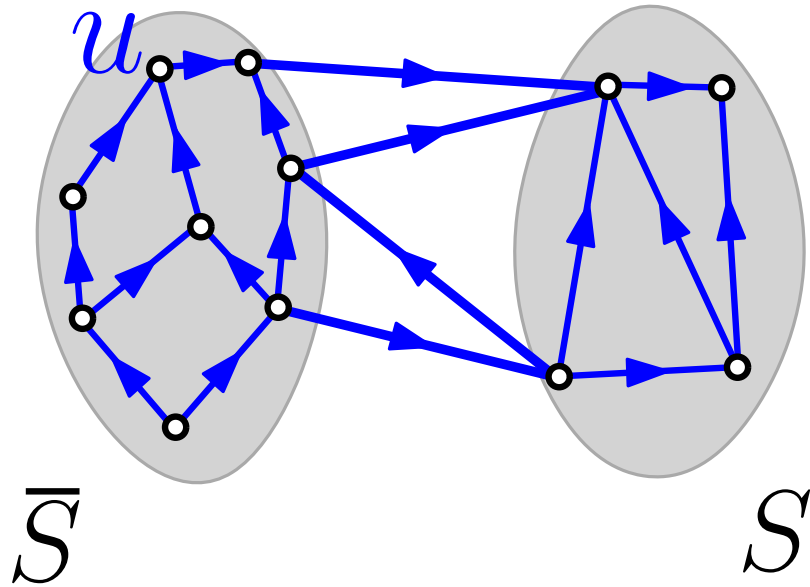
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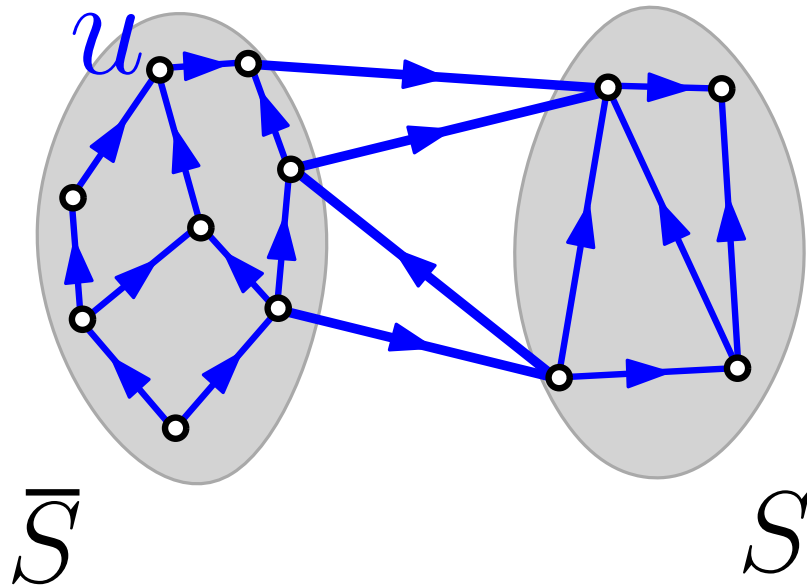
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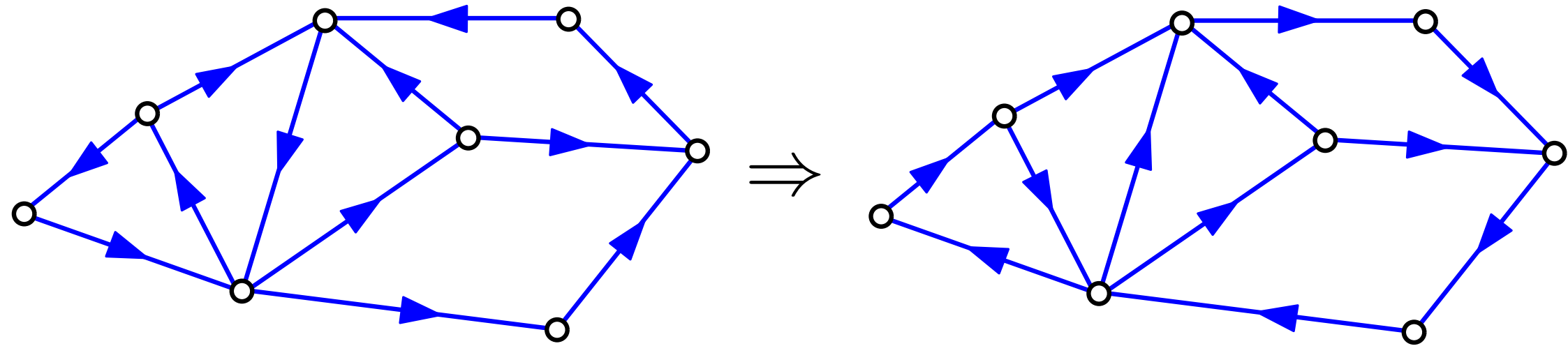
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$\Rightarrow$  accessibility from  $u \in V$  just depends on  $\alpha$  (not on which  $\alpha$ -orientation)

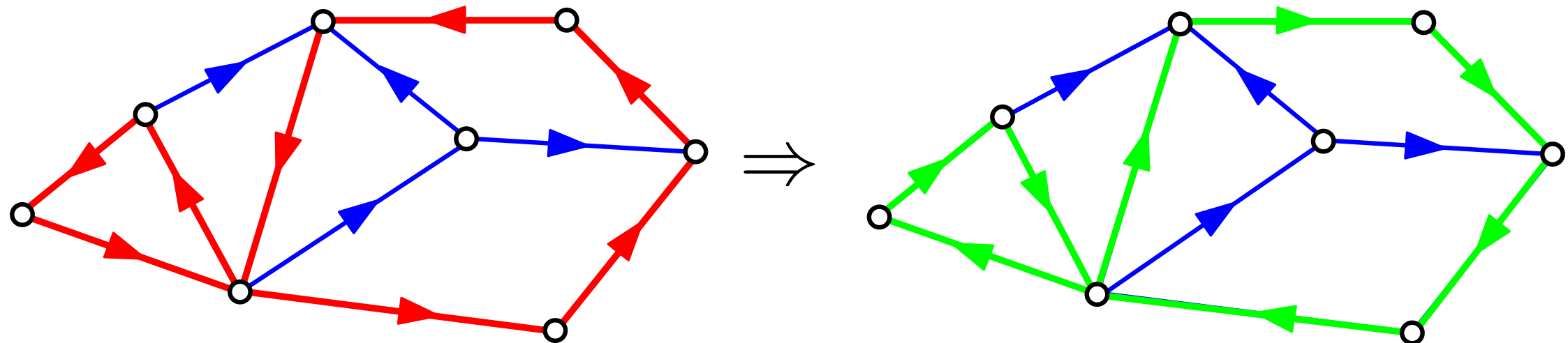
## $\alpha$ -orientations for plane maps

**Fundamental lemma:** If a plane map admits an  $\alpha$ -orientation, then it admits a **unique**  $\alpha$ -orientation **without ccw circuit**, called **minimal**



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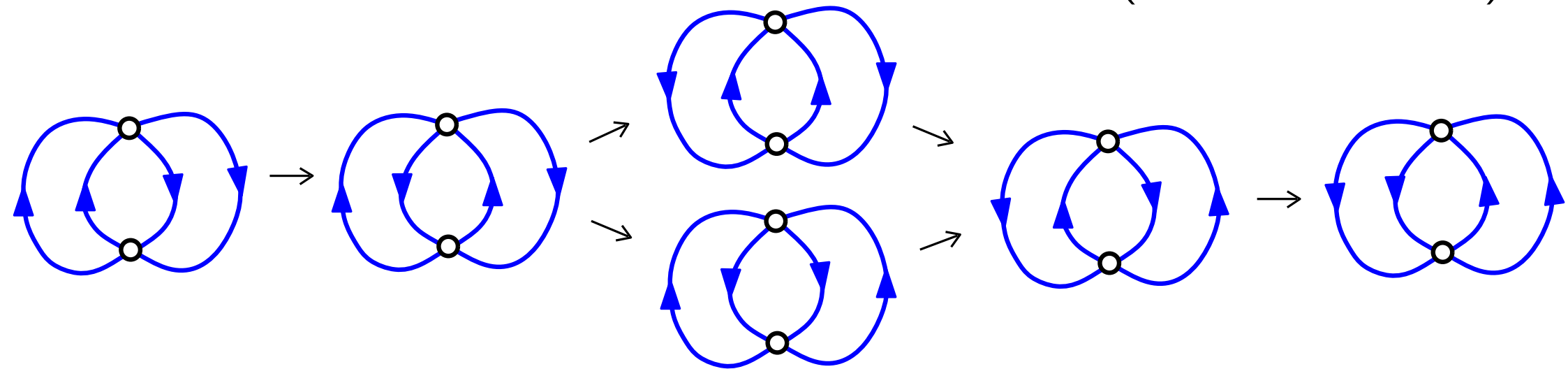
**Uniqueness proof:** if  $O_1 \neq O_2$ , edges where  $O_1$  and  $O_2$  **disagree** form an **eulerian suborientation** of  $O_1 \Rightarrow$  contains a circuit (ccw in  $O_1$  or  $O_2$ )

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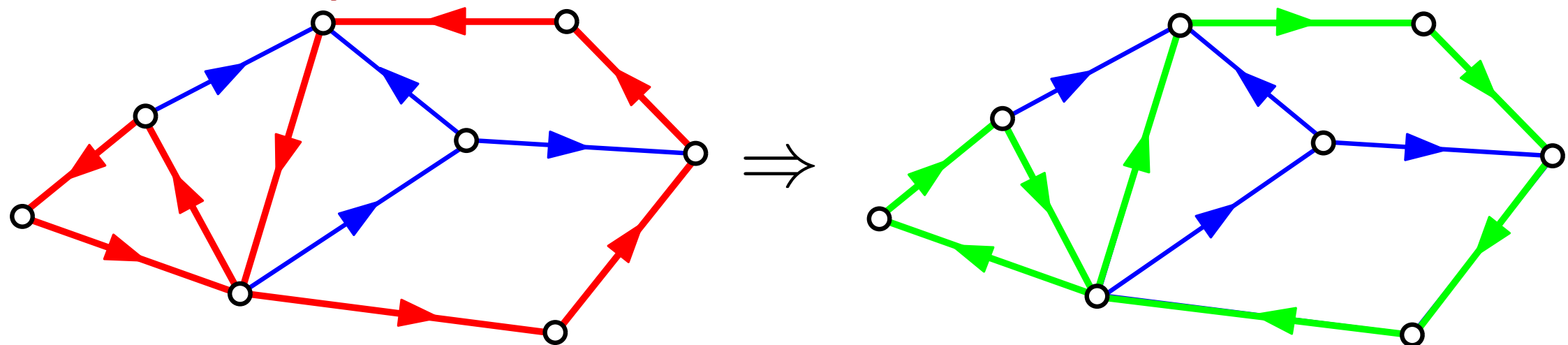


Set of  $\alpha$ -orientations = **distributive lattice**

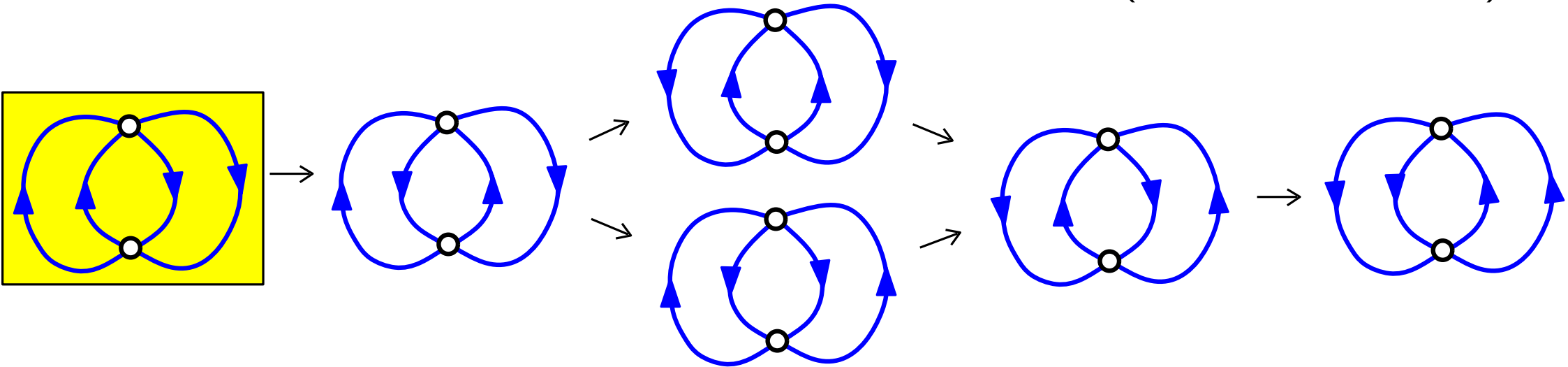
[Khueler et al'93], [Propp'93], [O. de Mendez'94], [Felsner'03]

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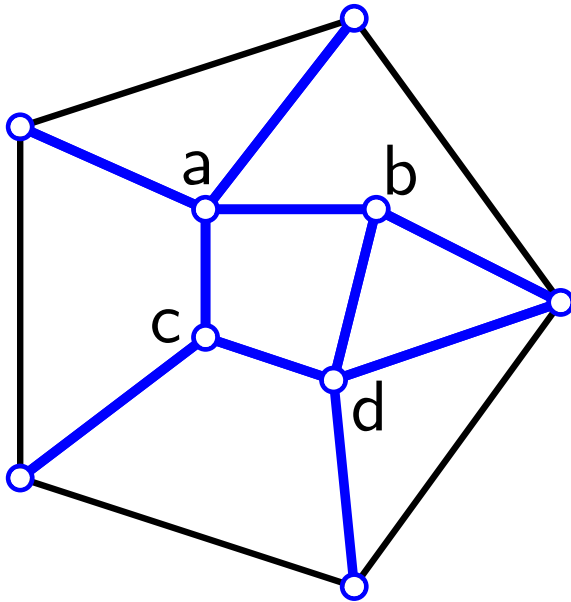
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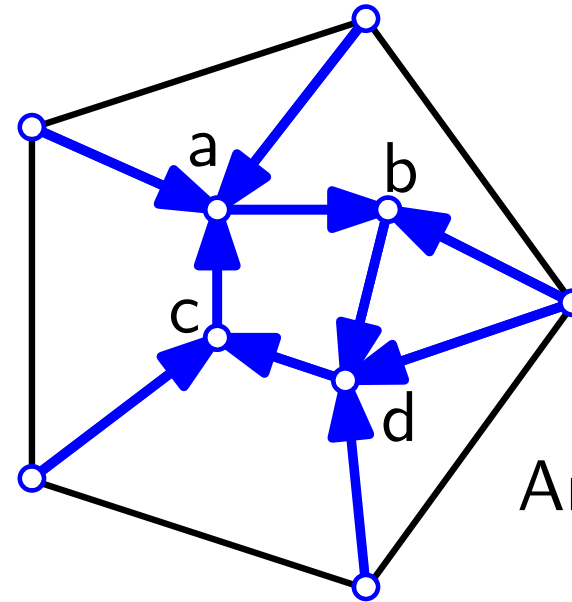
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## $\alpha$ -orientations for plane maps in our setting

- **External polygon** (the source) of the plane map is **unoriented**
- **Indegrees** are only on the **internal vertices**



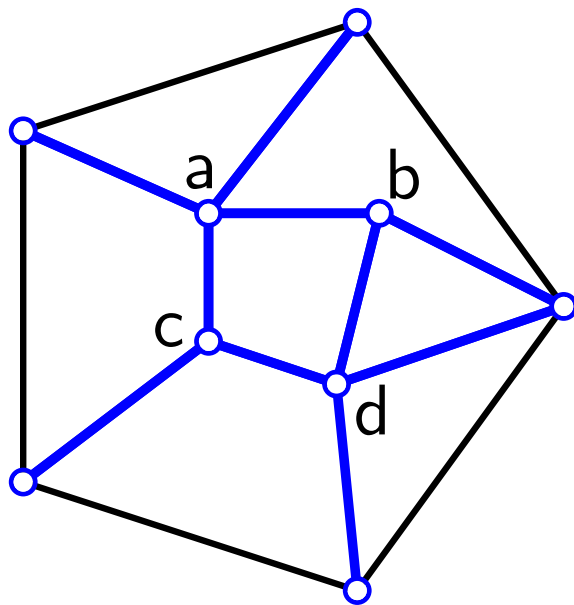
$$\alpha : \begin{array}{l} a \rightarrow 3 \\ b \rightarrow 2 \\ c \rightarrow 2 \\ d \rightarrow 3 \end{array}$$



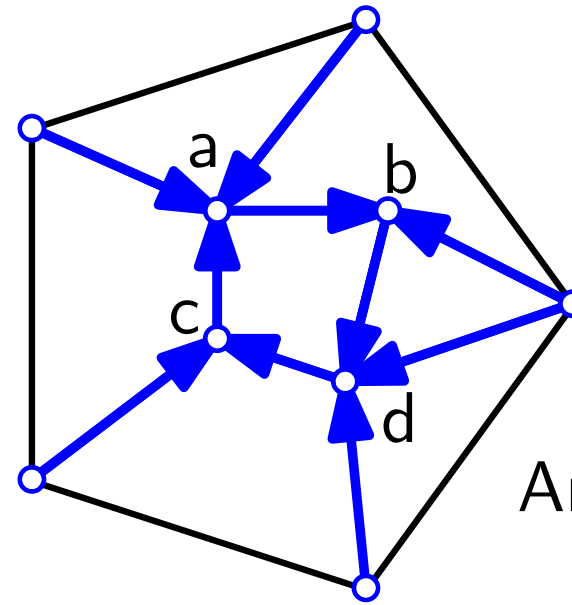
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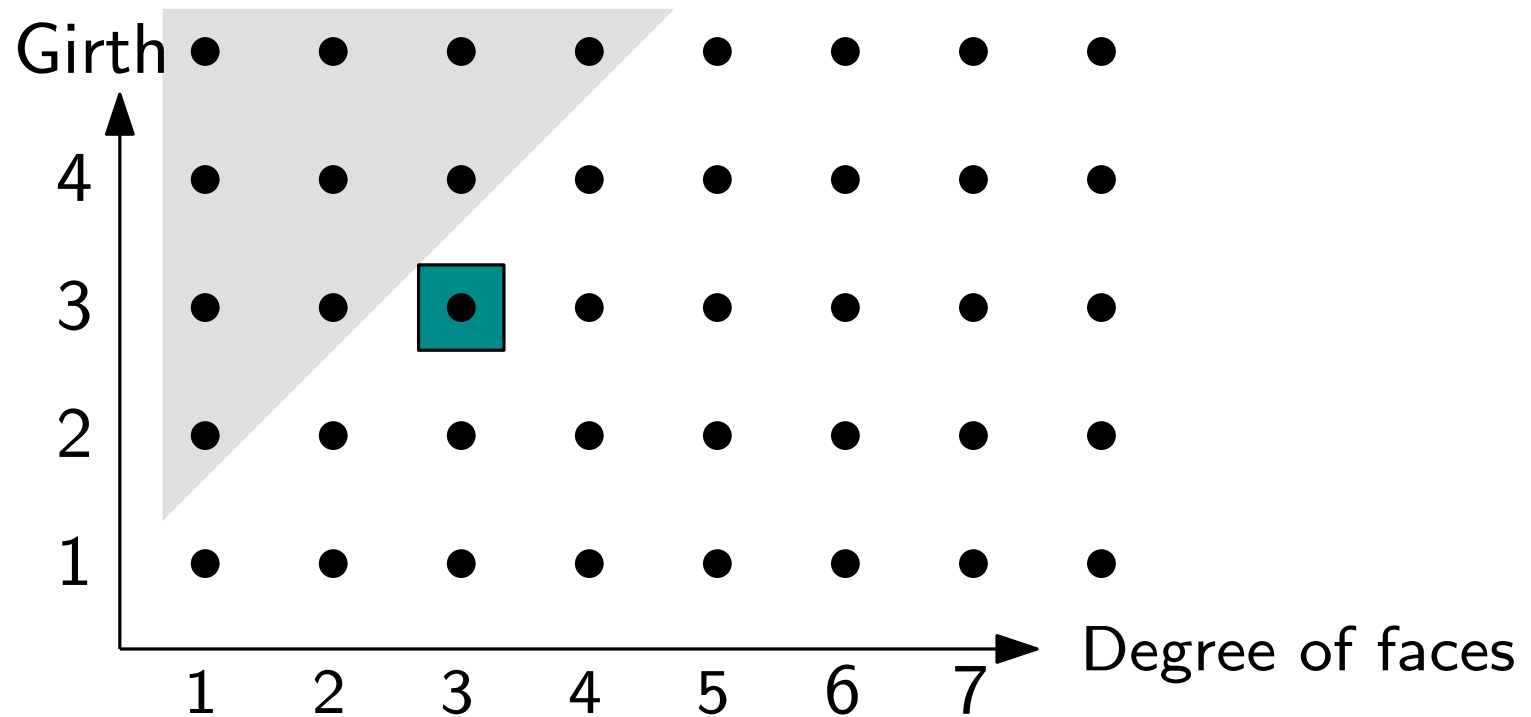
Partition  $V$  (vertex-set) as  $V_i \cup V_e$  and  $E$  (edge-set) as  $E_i \cup E_e$

- **Existence:** 
$$\begin{aligned} \text{(i)} \quad & \sum_{v \in V_i} \alpha(v) = |E_i| \\ \text{(ii)} \quad & \forall S \subseteq V, \quad \sum_{v \in S \cap V_i} \alpha(v) \geq |E_S \cap E_i| \end{aligned}$$

- **Accessibility from outer face:** 
$$\text{(iii)} \quad \forall S \subseteq V_i, \quad \sum_{v \in S \cap V_i} \alpha(v) > |E_S \cap E_i|$$
  
 $S \neq \emptyset$

- **Distributive lattice** structure

# Example: simple triangulations



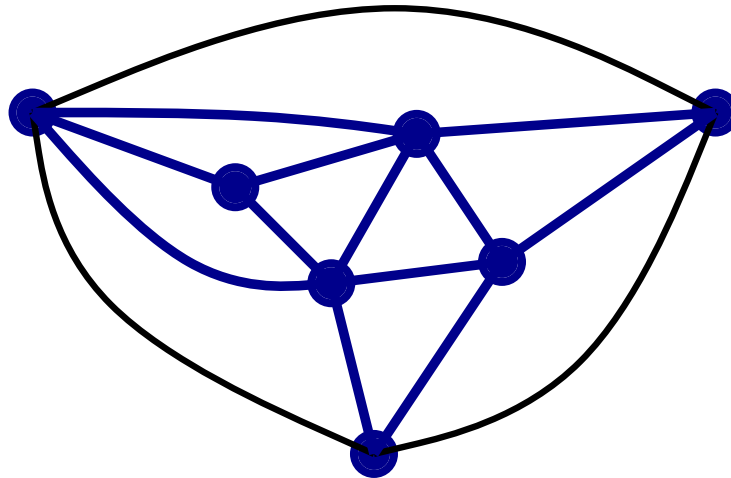
# Triangulations

**Fact:** A triangulation with  $n$  internal vertices has  $3n$  internal edges.

**Proof:** The numbers  $v$ ,  $e$ ,  $f$  of vertices edges and faces satisfy:

- Incidence relation:  $3f = 2e$ .
- Euler relation:  $v - e + f = 2$ .

□



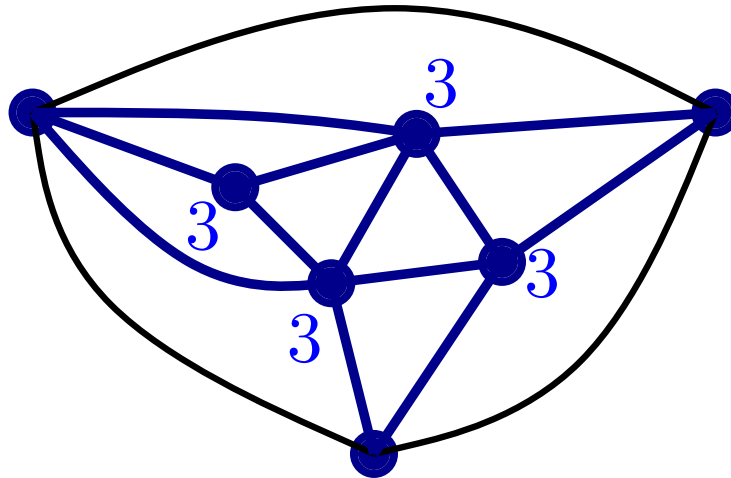
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**Natural candidate for indegree function:**

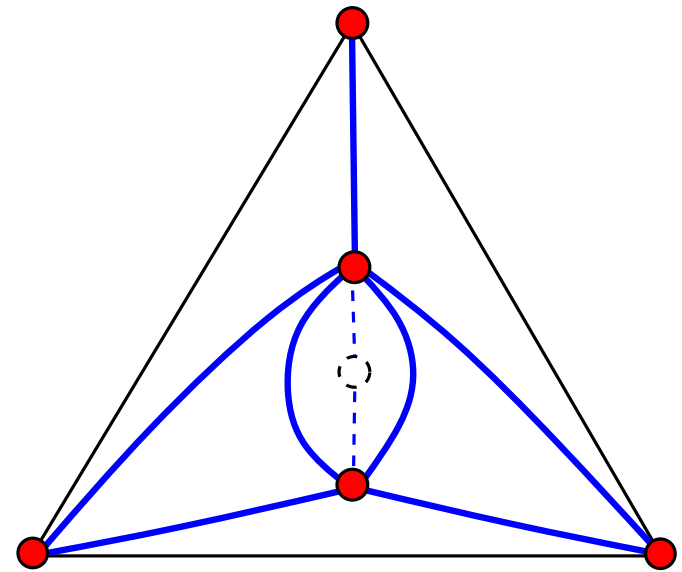
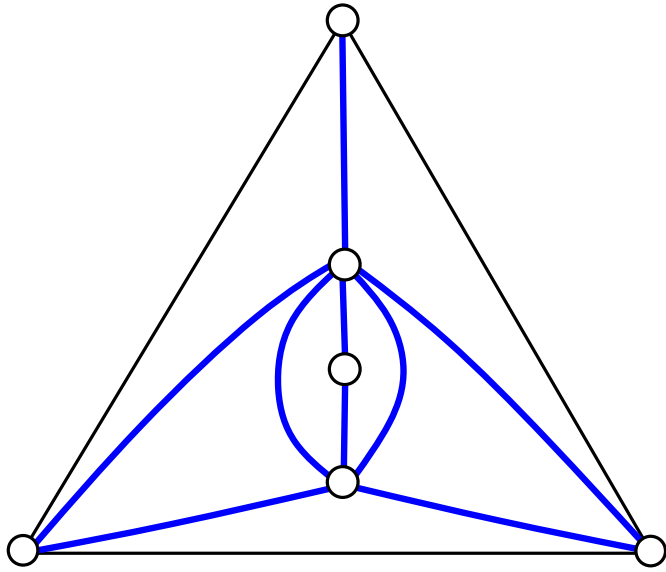
$\alpha : v \mapsto 3$  for each internal vertex  $v$ .

call **3-orientation** such an  $\alpha$ -orientation



# Triangulations

**Fact:** A triangulation admitting a 3-orientation is simple



$k$  internal vertices  
 $3k + 1$  internal edges

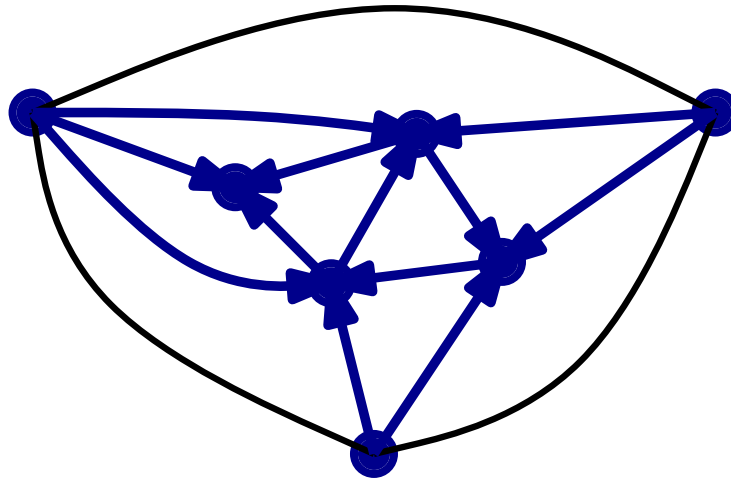
# Triangulations

**Thm [Schnyder 89]:** A simple triangulation admits a 3-orientation.

**New (easier) proof:** Any simple planar graph  $G = (V, E)$  satisfies

$$\frac{|E| - 3}{|V| - 3} \geq 3 \quad (\text{Euler relation})$$

hence the existence/accessibility conditions are satisfied.  $\square$

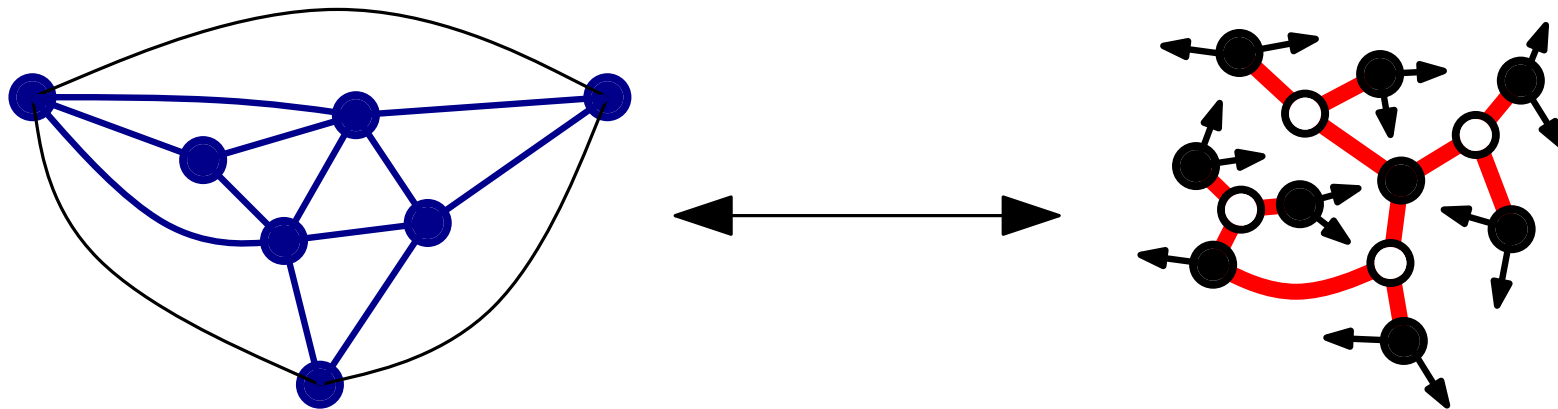


# Triangulations

⇒ The class  $\mathcal{T}$  of simple triangulations is identified with the class of plane orientation  $\mathcal{O}_{\mathcal{T}} \subset \mathcal{O}$  with faces of degree 3, and internal vertices of indegree 3.

**Thm [recovering FuPoSc08]:** By specializing the master bijection  $\Phi$  to  $\mathcal{O}_{\mathcal{T}}$  one obtains a bijection between simple triangulations and mobiles such that

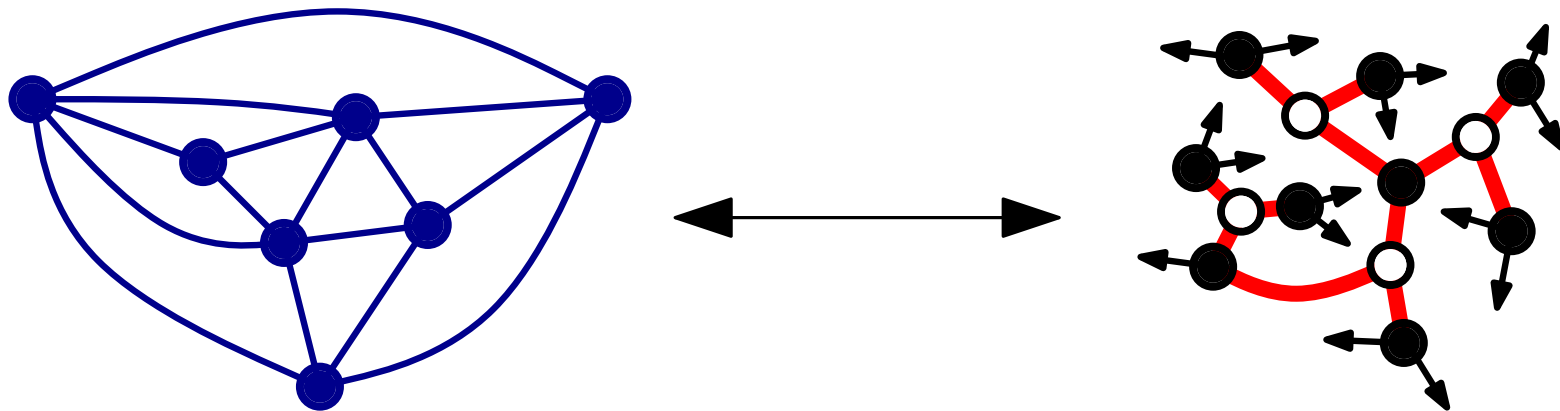
- black vertices have degree 3
- white vertices have degree 3
- the excess is +3 (redundant).



# Triangulations

**Counting:** The generating function of mobiles with vertices of degree 3 rooted on a white corner is  $T(x) = U(x)^3$ , where  $U(x) = 1 + xU(x)^4$ .

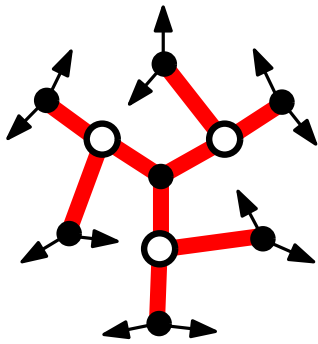
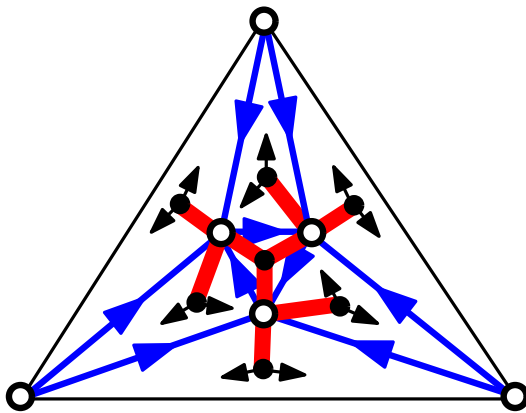
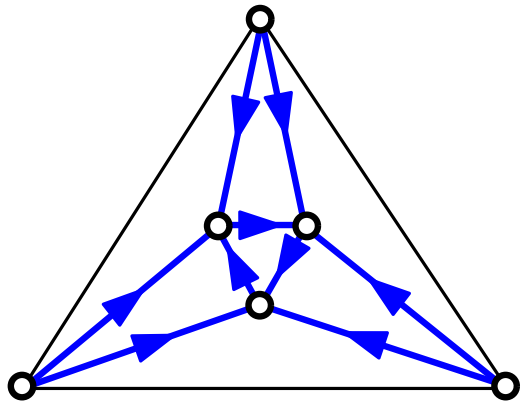
Consequently, the number of (rooted) simple triangulations with  $2n$  faces is  $\frac{1}{n(2n-1)} \binom{4n-2}{n-1}$ .



# Triangulations: two constructions

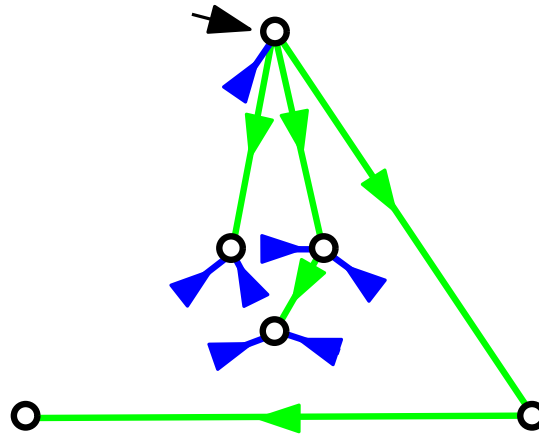
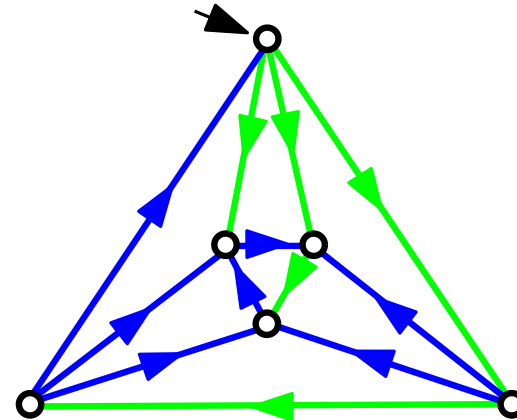
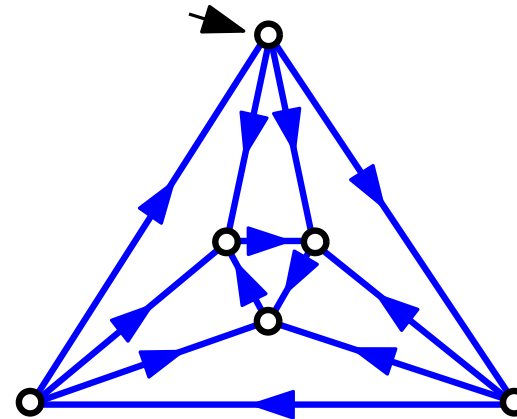
## mobiles

[FuPoSc'08], [Bernardi-F'10]



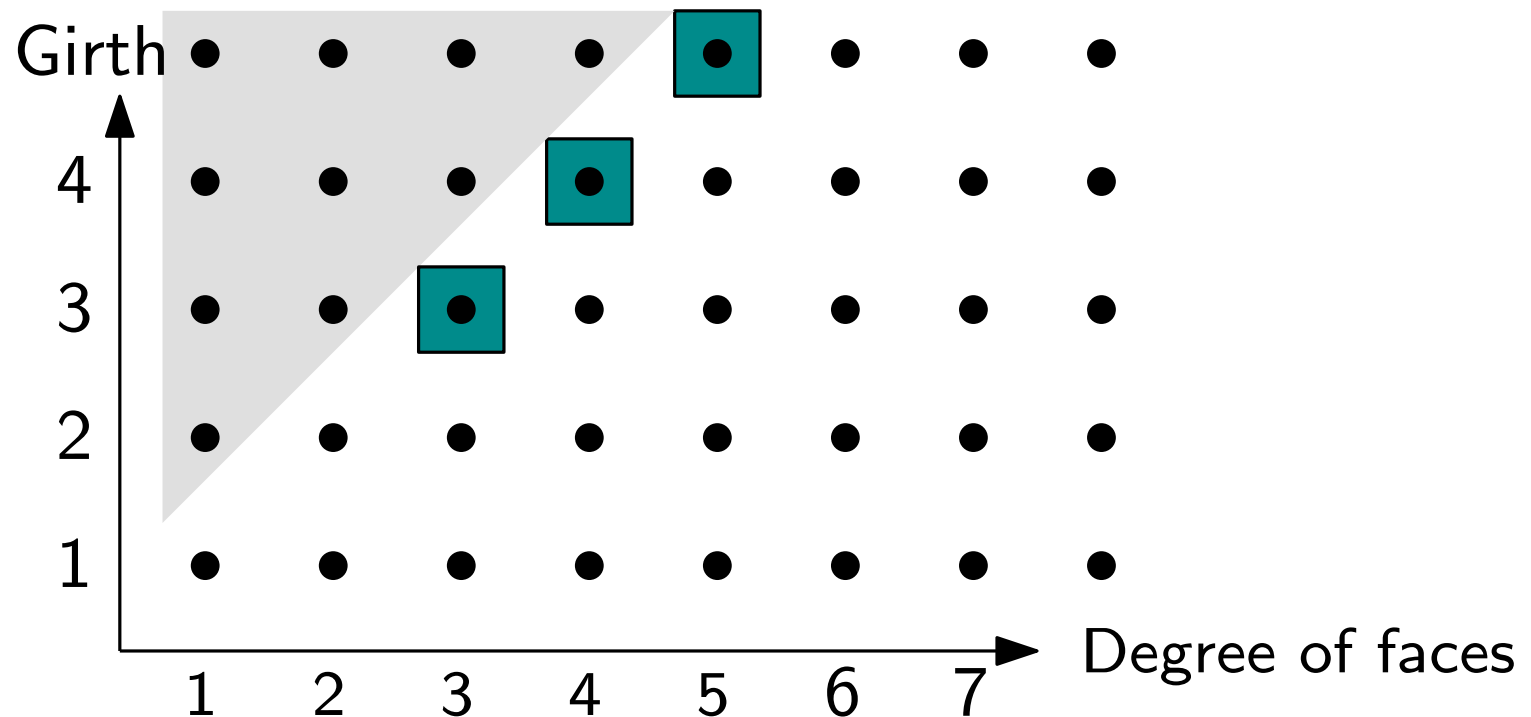
## blossoming trees

[PoSc'03], [AlPo'11]



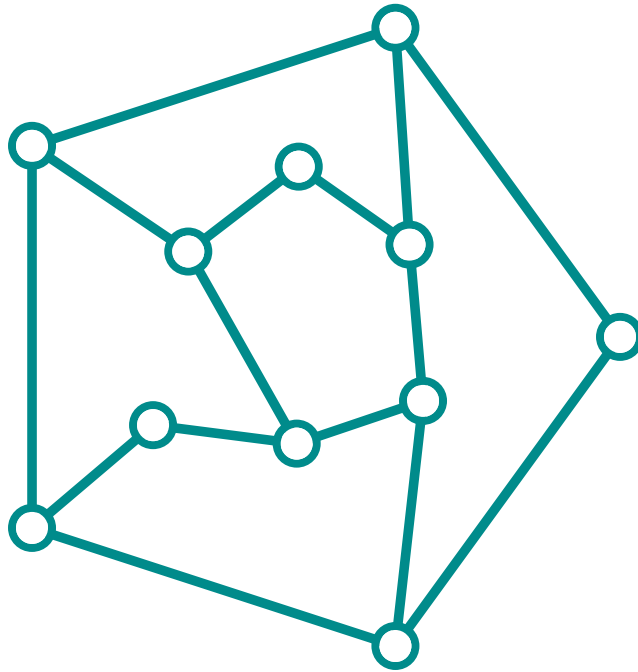
# More specializations

$d$ -angulations of girth  $d$ .



## $d$ -angulations of girth $d$

**Fact:** A  $d$ -angulation with  $(d-2)n$  internal vertices has  $dn$  internal edges.

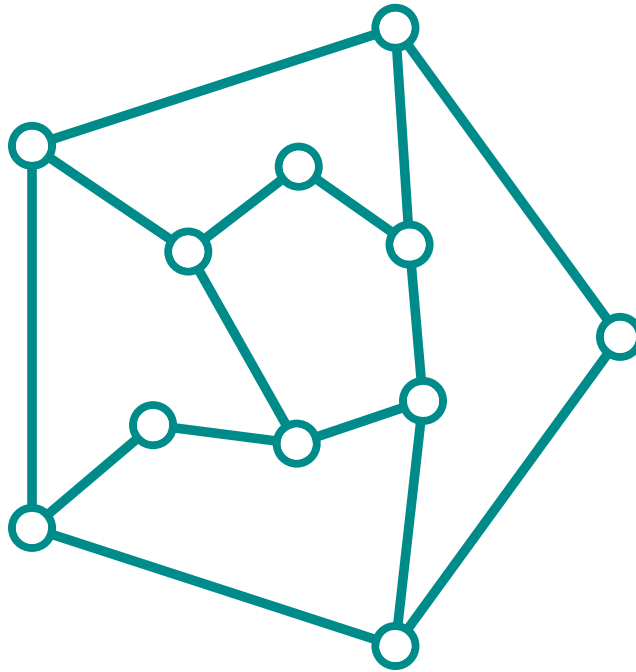


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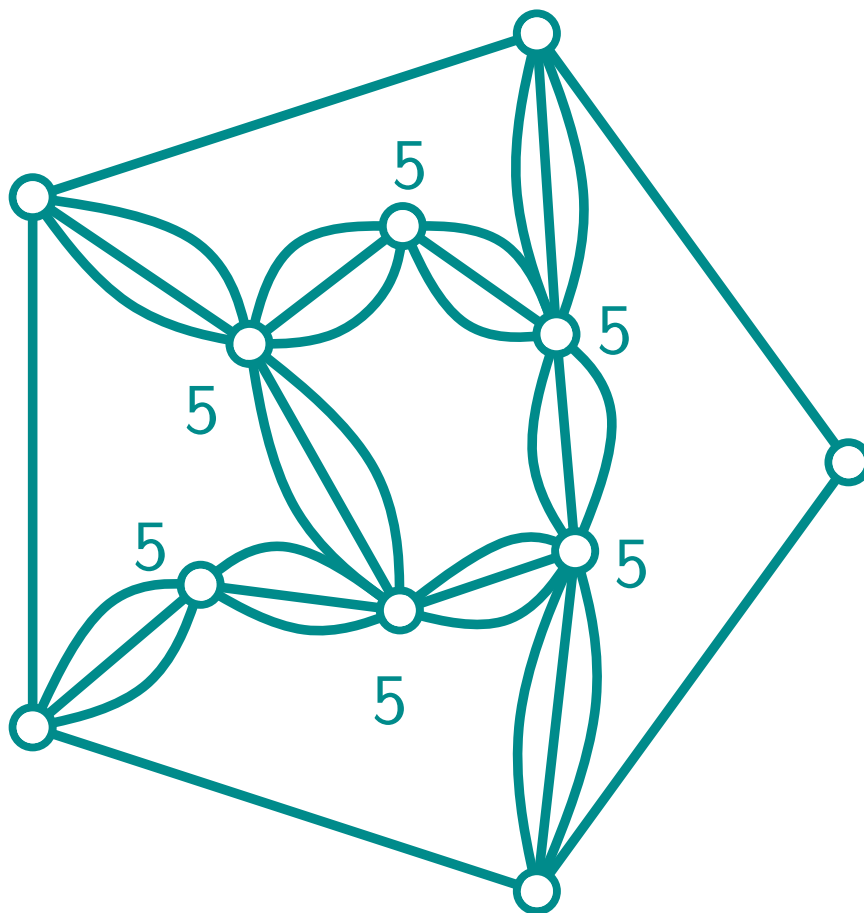
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## $d$ -angulations of girth $d$

**Fact:** A  $d$ -angulation with  $(d-2)n$  internal vertices has  $dn$  internal edges.

**Idea:** We can look for an orientation of  $(d-2)G$  with indegree function  $\alpha : v \mapsto d$  for each internal vertex  $v$ .

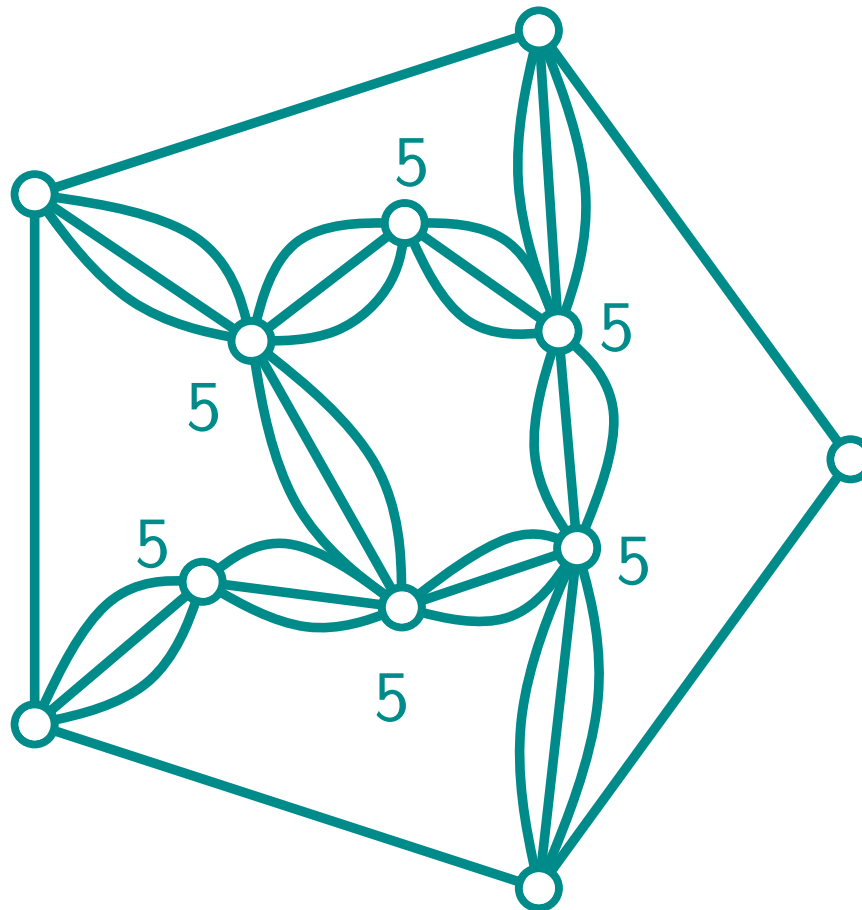


## $d$ -angulations of girth $d$

**Fact:** A  $d$ -angulation with  $(d-2)n$  internal vertices has  $dn$  internal edges.

**Idea:** We can look for an orientation of  $(d-2)G$  with indegree function  $\alpha : v \mapsto d$  for each internal vertex  $v$ .

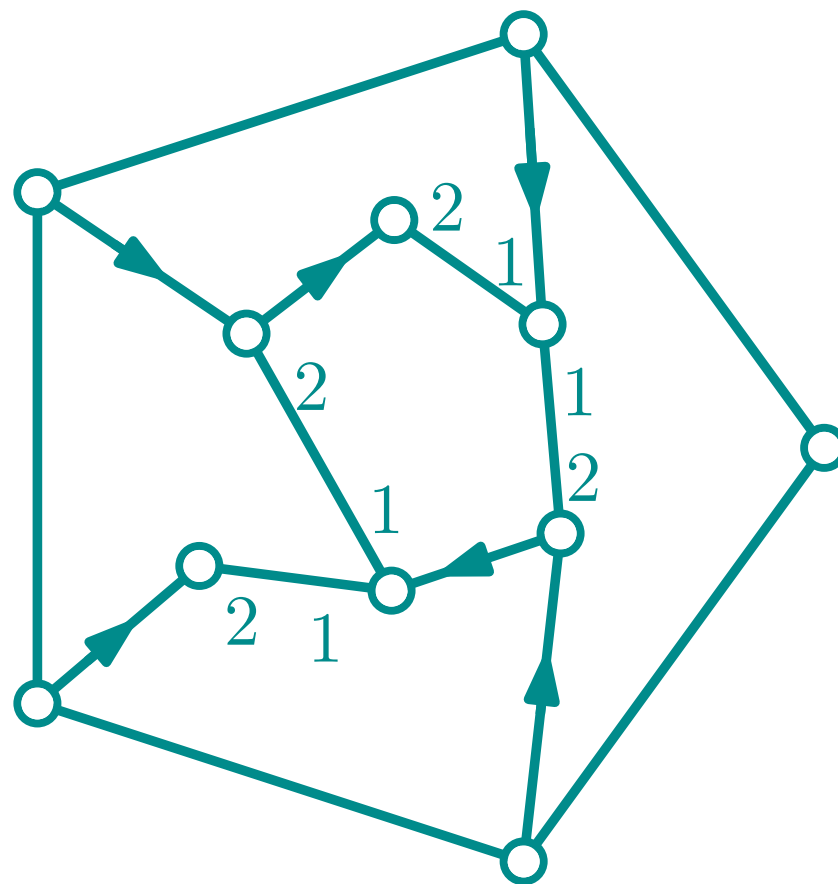
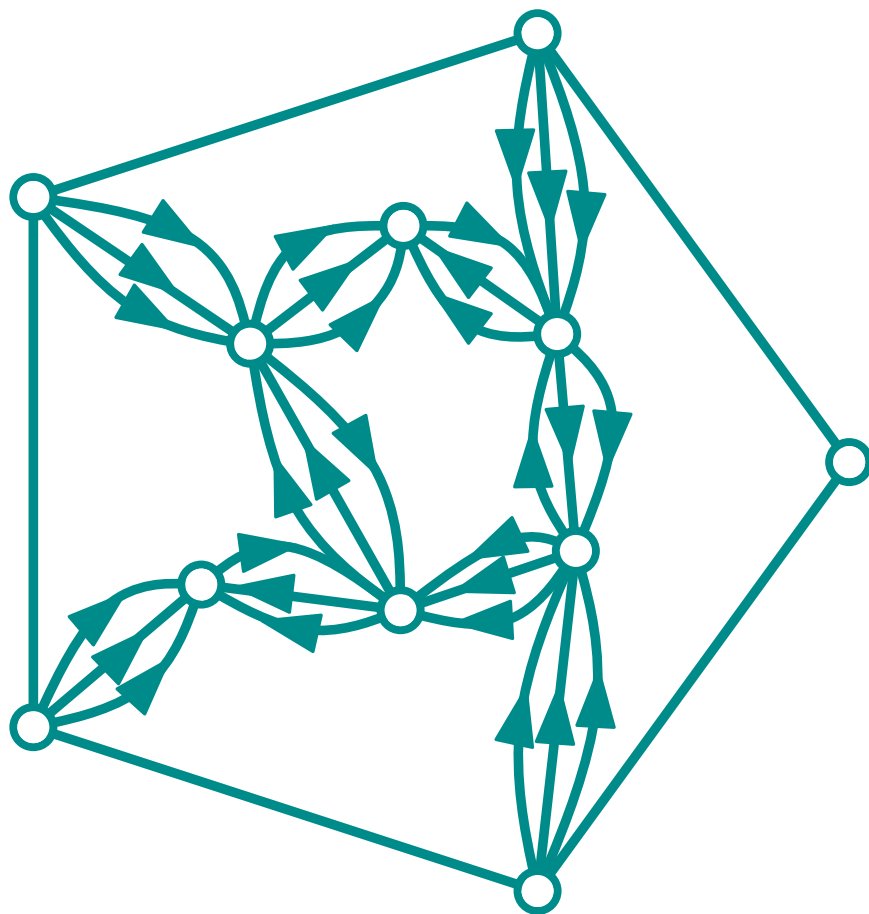
call  $d/(d-2)$ -orientation such an orientation



## $d$ -angulations of girth $d$

**Thm [Bernardi-F'10]:** Let  $G$  be a  $d$ -angulation. Then  $(d-2)G$  admits a  $d/(d-2)$ -orientation if and only if  $G$  has girth  $d$ .

$d = 5$

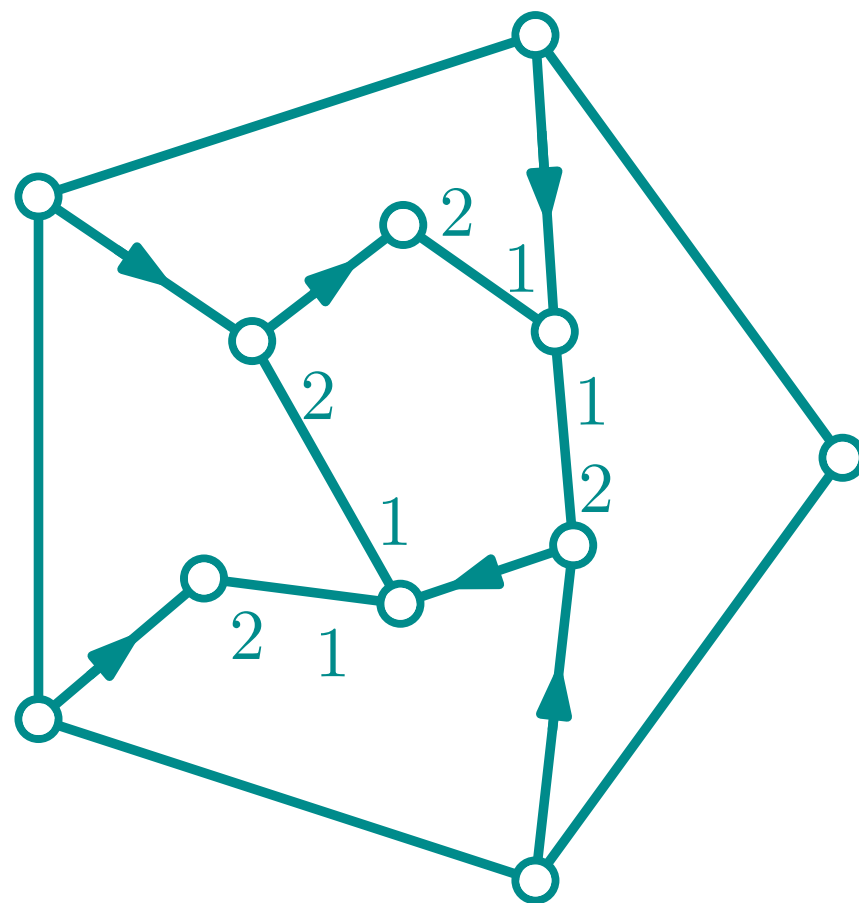
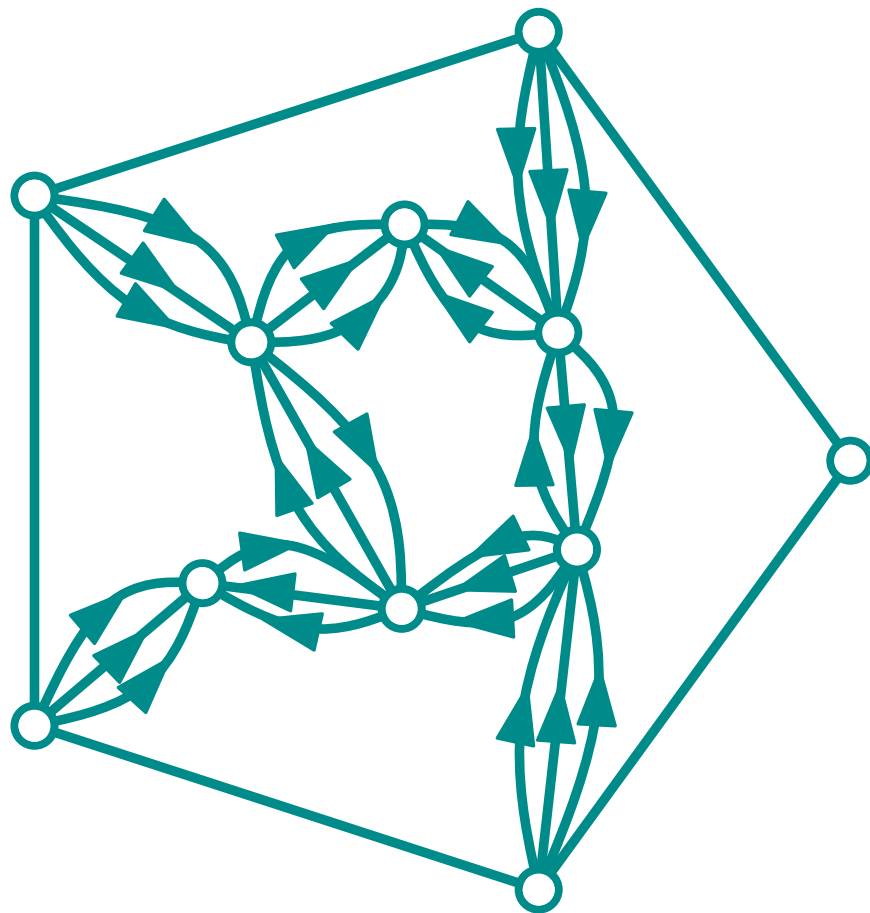


## $d$ -angulations of girth $d$

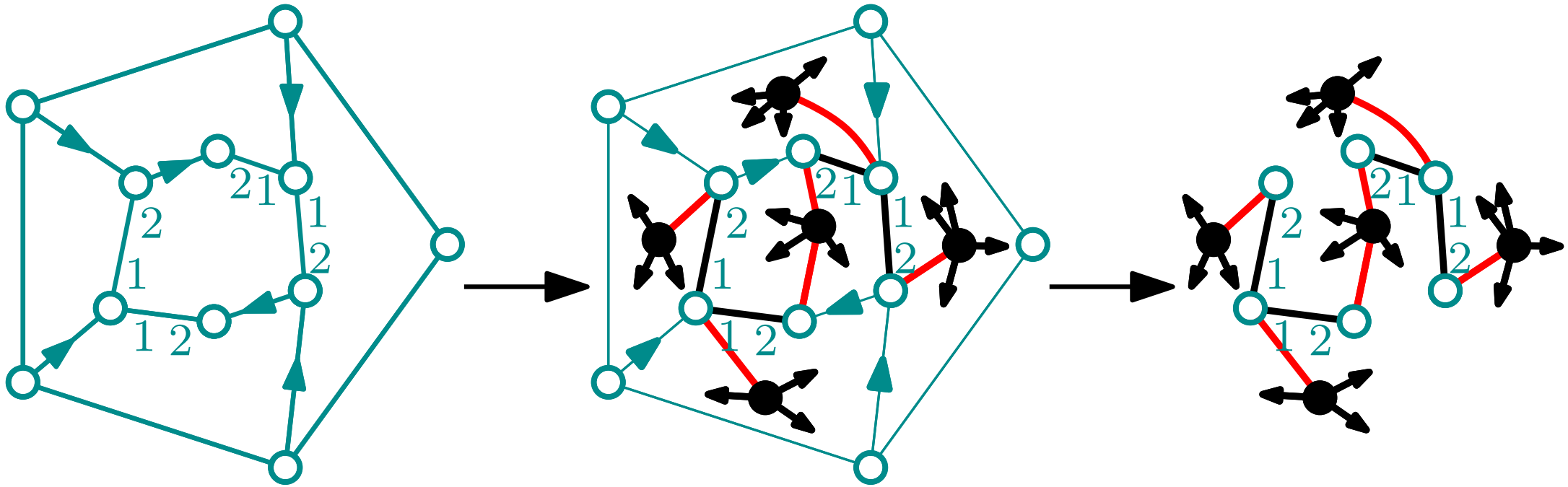
**Thm [Bernardi-F'10]:** Let  $G$  be a  $d$ -angulation. Then  $(d-2)G$  admits a  $d/(d-2)$ -orientation if and only if  $G$  has girth  $d$ .

**Proof:** Similar to  $d = 3$ . Uses the fact that a planar graph  $G = (V, E)$  of girth at least  $d$  satisfies  $\frac{|E| - d}{|V| - d} \geq d$

$d = 5$

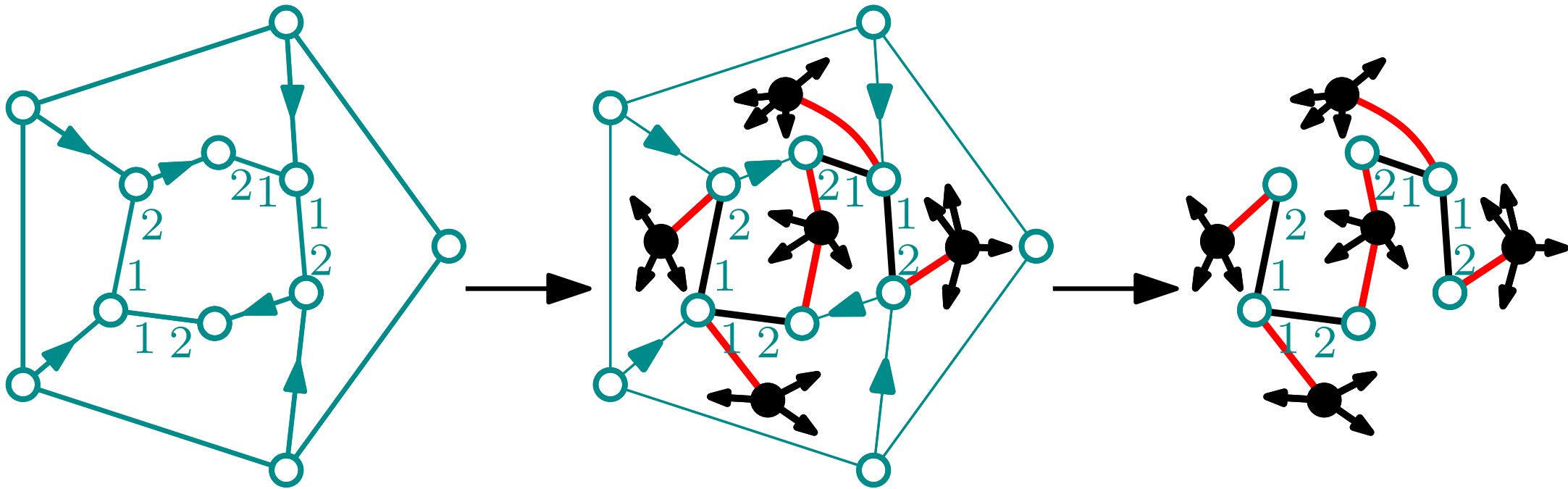


# Master bijection for weighted orientations



There are now white-white edges in the mobile, with two positive weights summing to  $d - 2$ .

# Master bijection for weighted orientations



**Theorem [Bernardi-F'10]:** The master bijection can be expressed in the weighted setting:

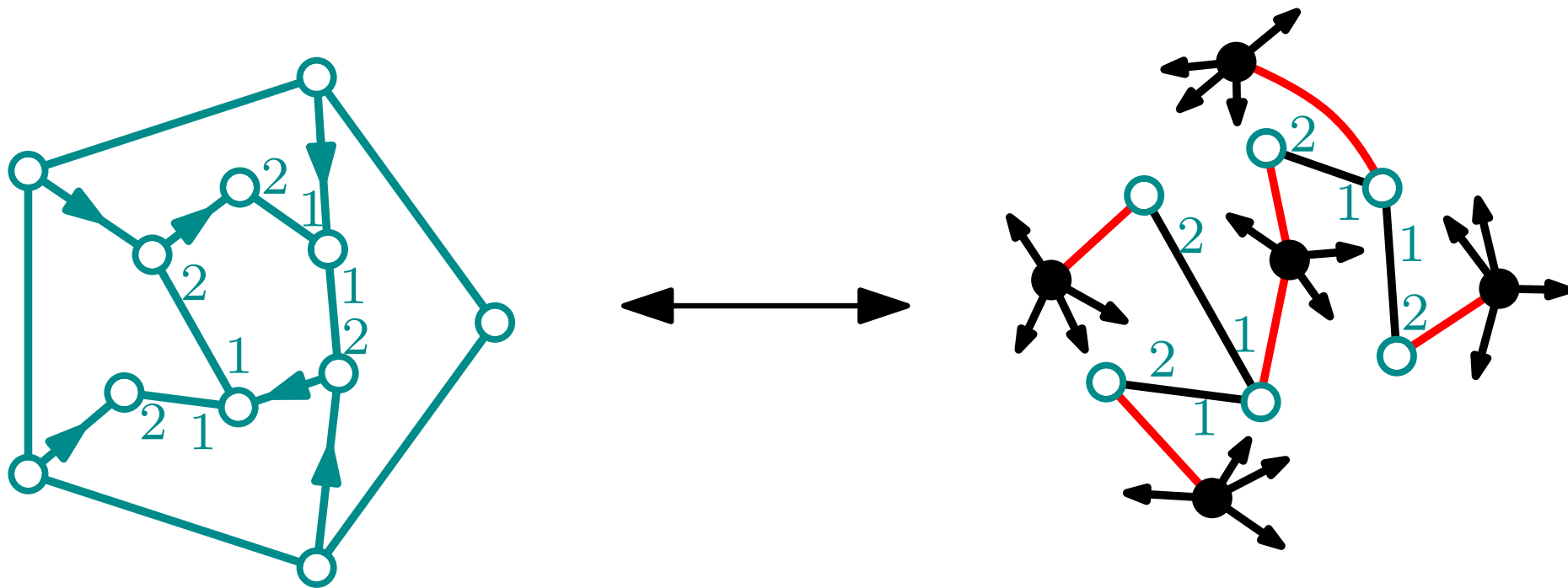
Moreover,

degree of internal faces	$\longleftrightarrow$	degree of black faces
indegree of internal vertices	$\longleftrightarrow$	indegree of white vertices
weights of internal edges	$\longleftrightarrow$	weights of edges
degree of external face	$\longleftrightarrow$	excess

## $d$ -angulations of girth $d$

**Thm [Bernardi-F'10]:** A  $d$ -angulation  $G$  admits a  $d/(d-2)$ -orientation if and only if  $G$  has girth  $d$ .

$\Rightarrow$  The class  $\mathcal{T}_d$  of  $d$ -angulations of girth  $d$  can be identified with the class of weighted orientations in  $\mathcal{O}$ , with faces of degree  $d$ , edges of weight  $d-2$ , and internal vertices of indegree  $d$ .

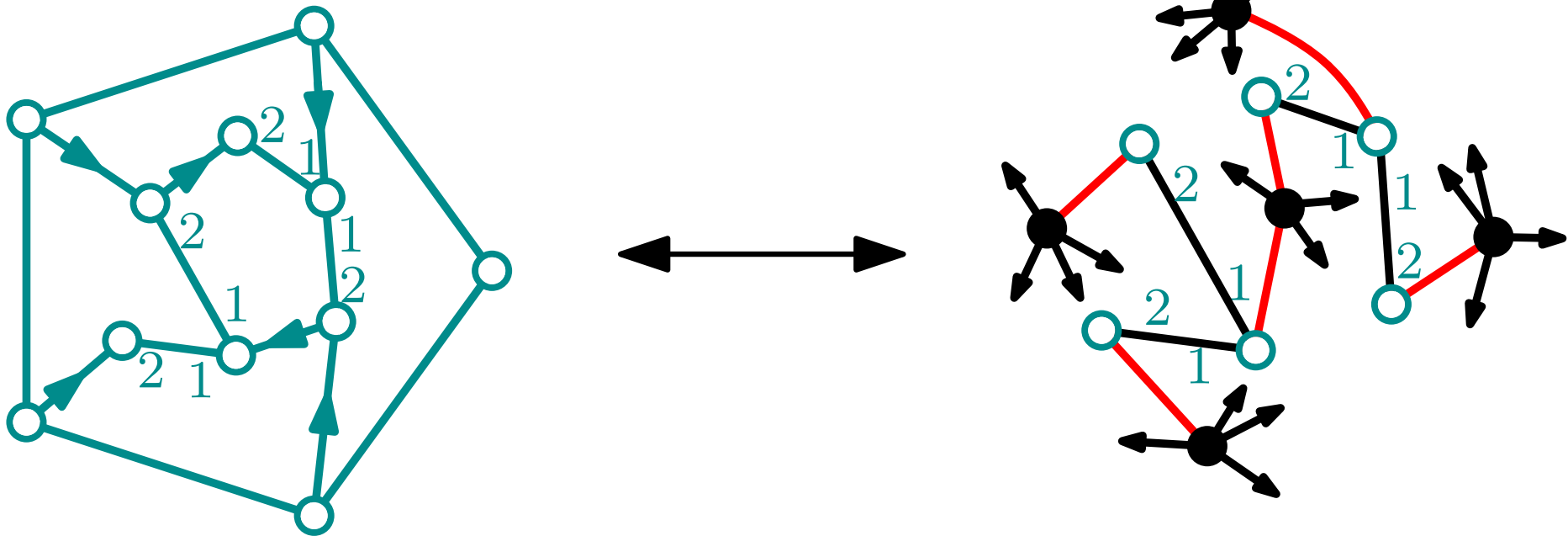


## $d$ -angulations of girth $d$

**Thm [Bernardi-F'10]:** A  $d$ -angulation  $G$  admits a  $d/(d-2)$ -orientation if and only if  $G$  has girth  $d$ .

**Thm [Bernardi-F'10]:** By specializing the master bijection one obtains a bijection between  $d$ -angulations of girth  $d$  and mobiles (with white-white edges having weights summing to  $d-2$ ) such that

- black vertices have degree  $d$
- white vertices have indegree  $d$
- the excess is  $d$  (redundant).



## $d$ -angulations of girth $d$ : counting

**Thm[Bernardi-F'10]:** Let  $W_0, W_1, \dots, W_{d-2}$  be the power series in  $x$  defined by:  $W_{d-2} = x(1 + W_0)^{d-1}$

and  $\forall j < d - 2, \quad W_j = \sum_r \sum_{\substack{i_1, \dots, i_r > 0 \\ i_1 + \dots + i_r = j+2}} W_{i_1} \cdots W_{i_r}.$

The generating function  $F_d$  of rooted  $d$ -angulations of girth  $d$  satisfies

$$F'_d(x) = (1 + W_0)^d.$$

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$$F'_d(x) = (1 + W_0)^d.$$

### Example $d=5$ :

$$W_3 = x(1 + W_0)^4$$

$$W_0 = W_1^2 + W_2$$

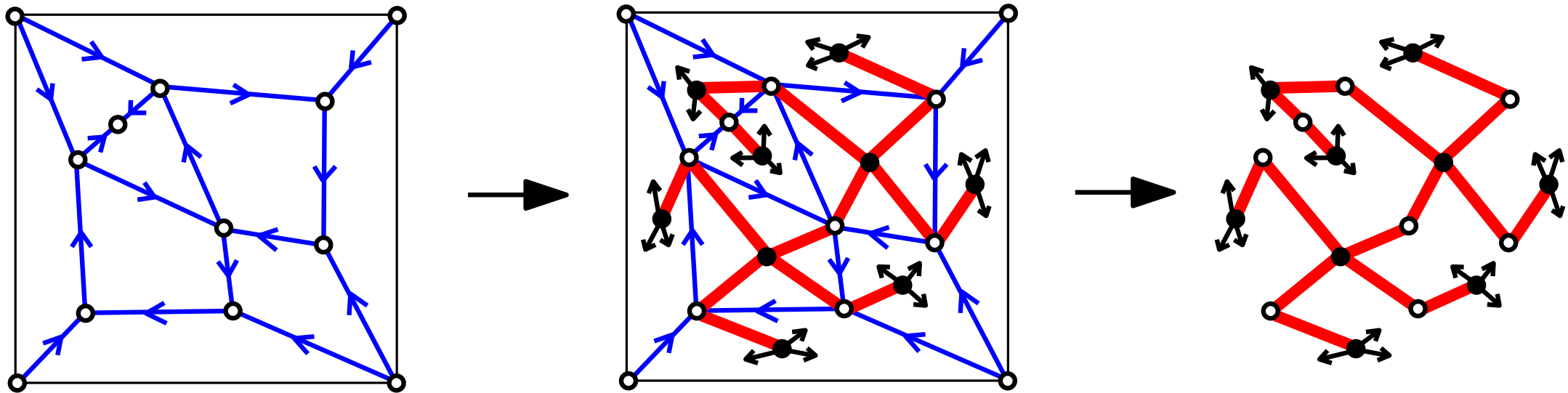
$$W_1 = W_1^3 + 2W_1W_2 + W_3$$

$$W_2 = W_1^4 + 3W_1^2W_2 + 2W_1W_3 + W_2^2$$

## Simplification in the bipartite case

- For  $d$  even,  $d = 2b$ , we have  $\frac{d}{d-2} = \frac{b}{b-1}$
- Can work with  $b/(b-1)$ -orientations:
  - edges have weight  $b-1$
  - vertices have indegree  $b$

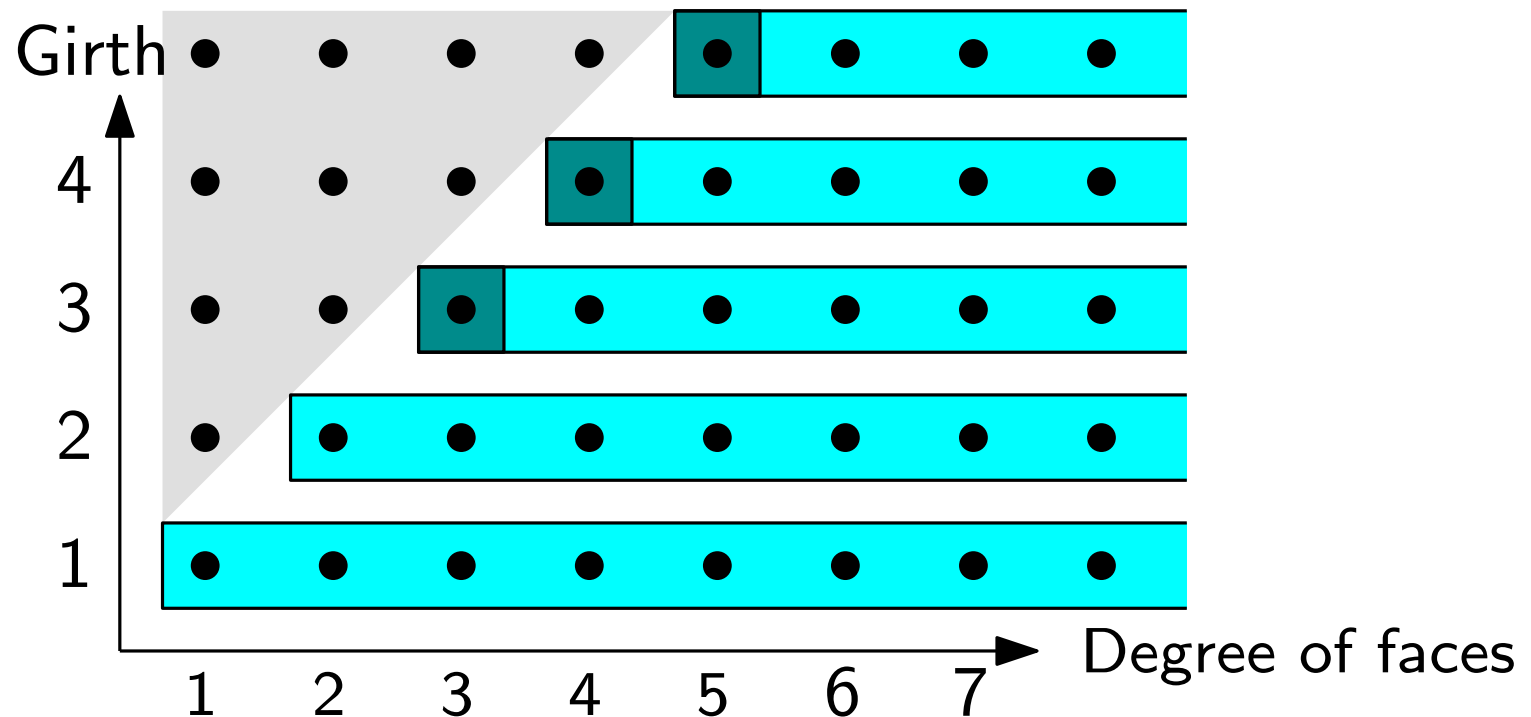
**Example:**  $b = 2$ , simple quadrangulations



recover a bijection of Schaeffer (1999)

# More specializations

## Maps of girth $d$ .

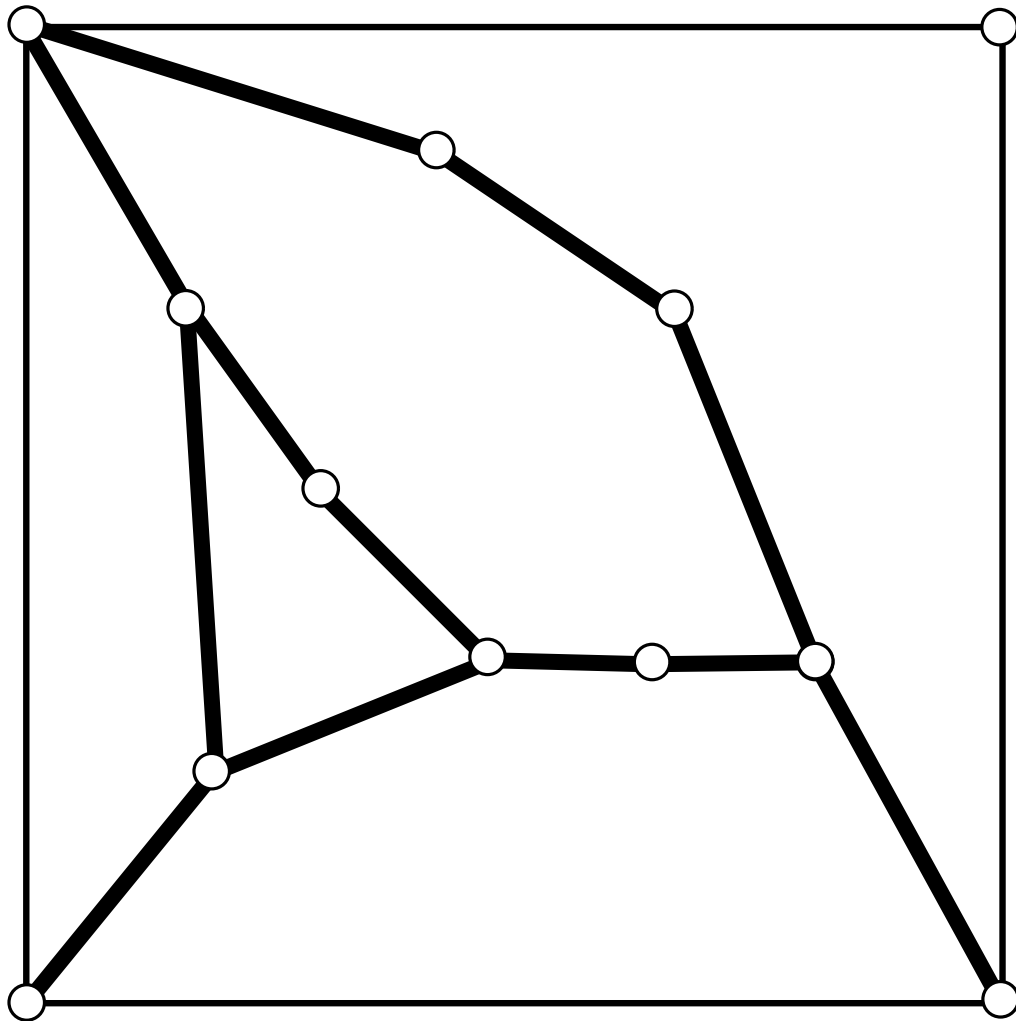


**We show only the bipartite case (simpler)**

Case  $b = 2$  (simple bipartite maps), with quadrangular outer face

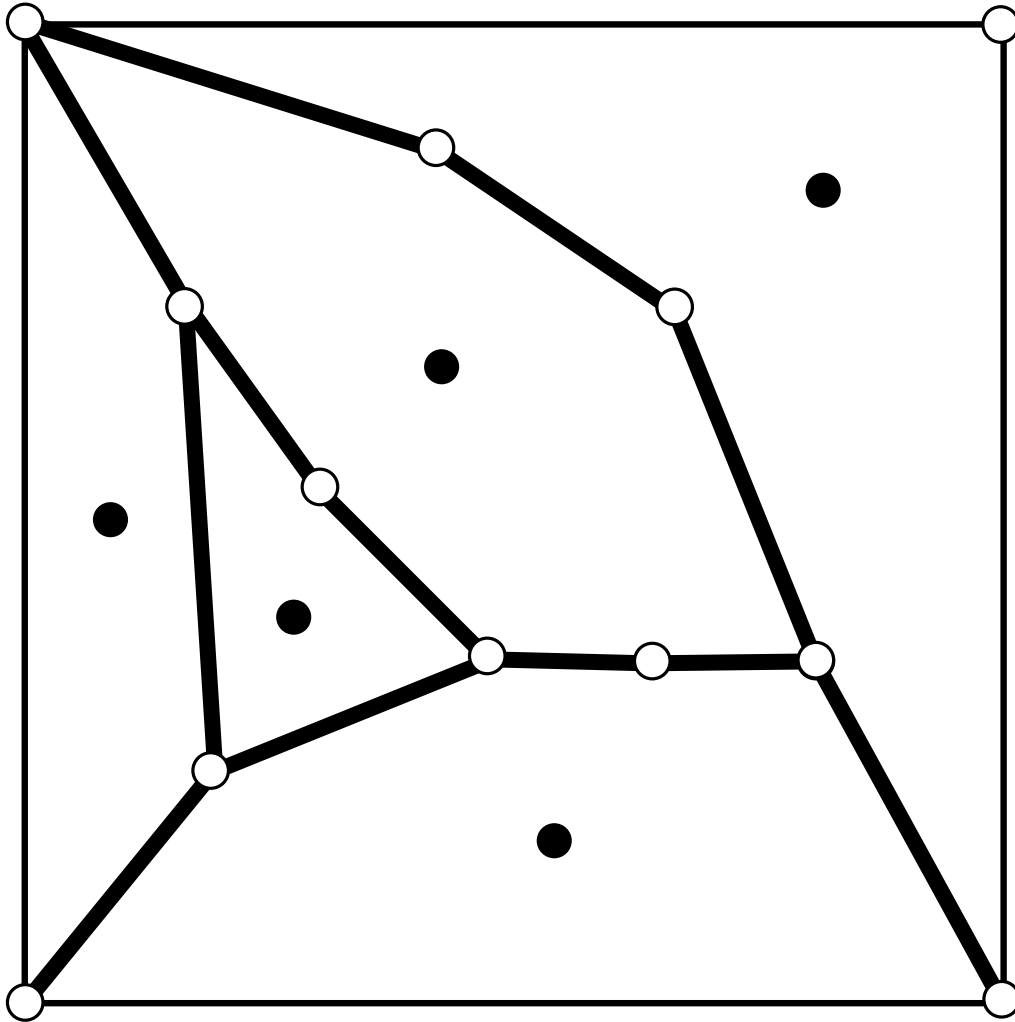
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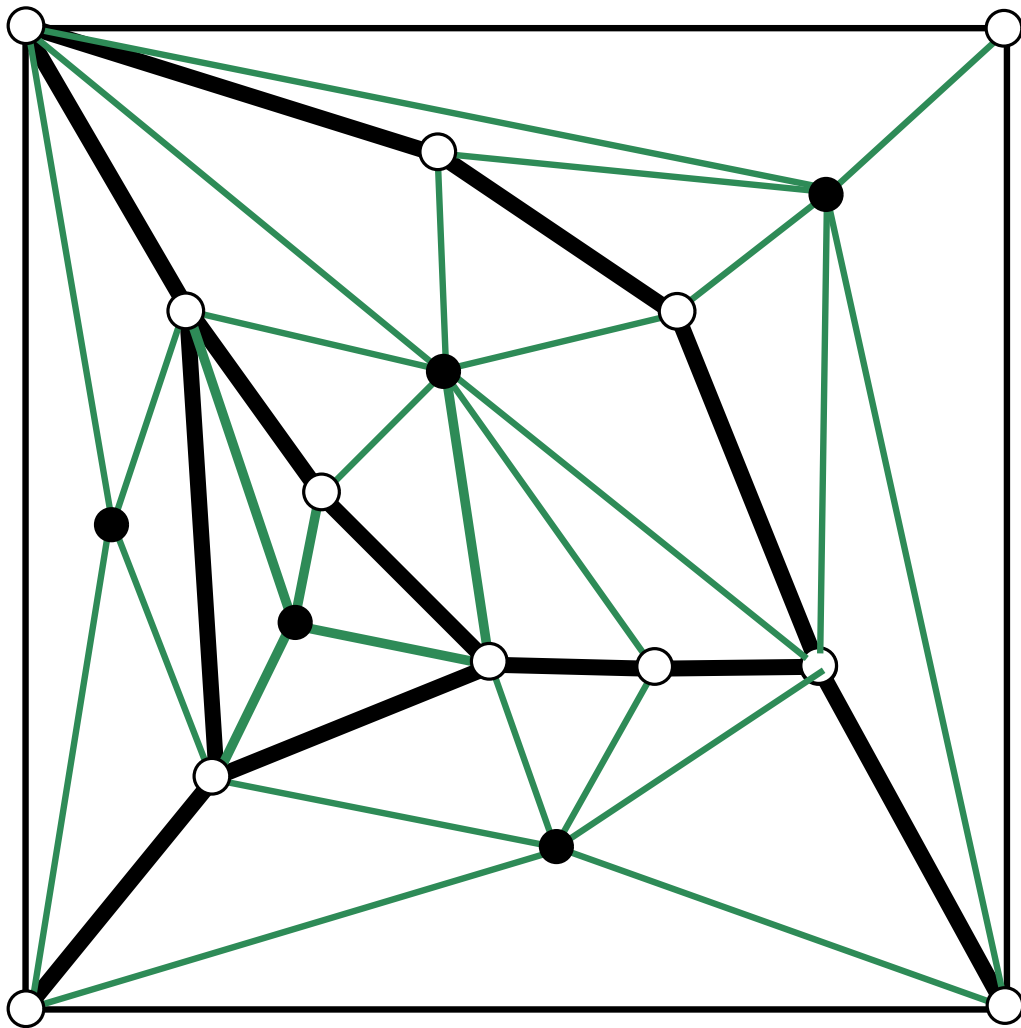
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# We show only the bipartite case (simpler)

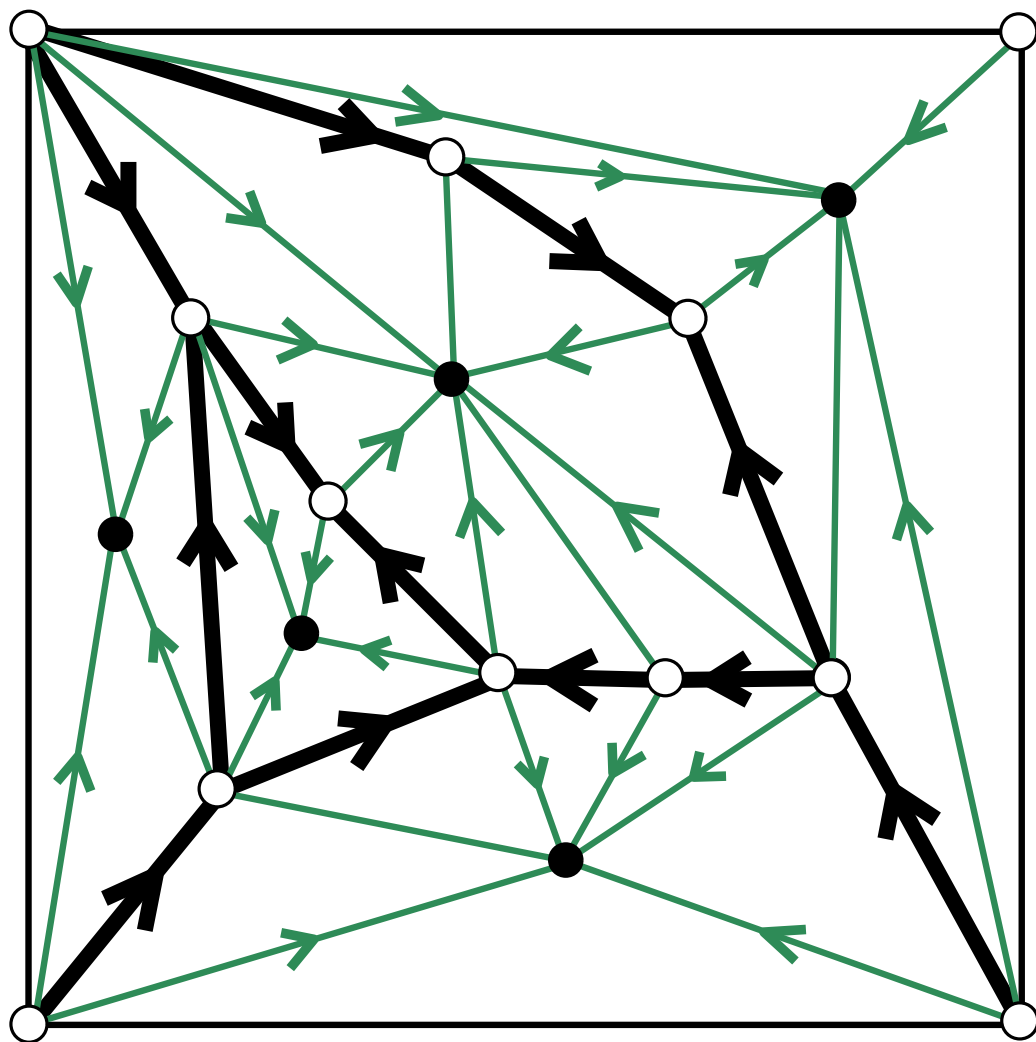
Case  $b = 2$  (simple bipartite maps), with quadrangular outer face



Insert a star  
in each internal face

# We show only the bipartite case (simpler)

Case  $b = 2$  (simple bipartite maps), with quadrangular outer face

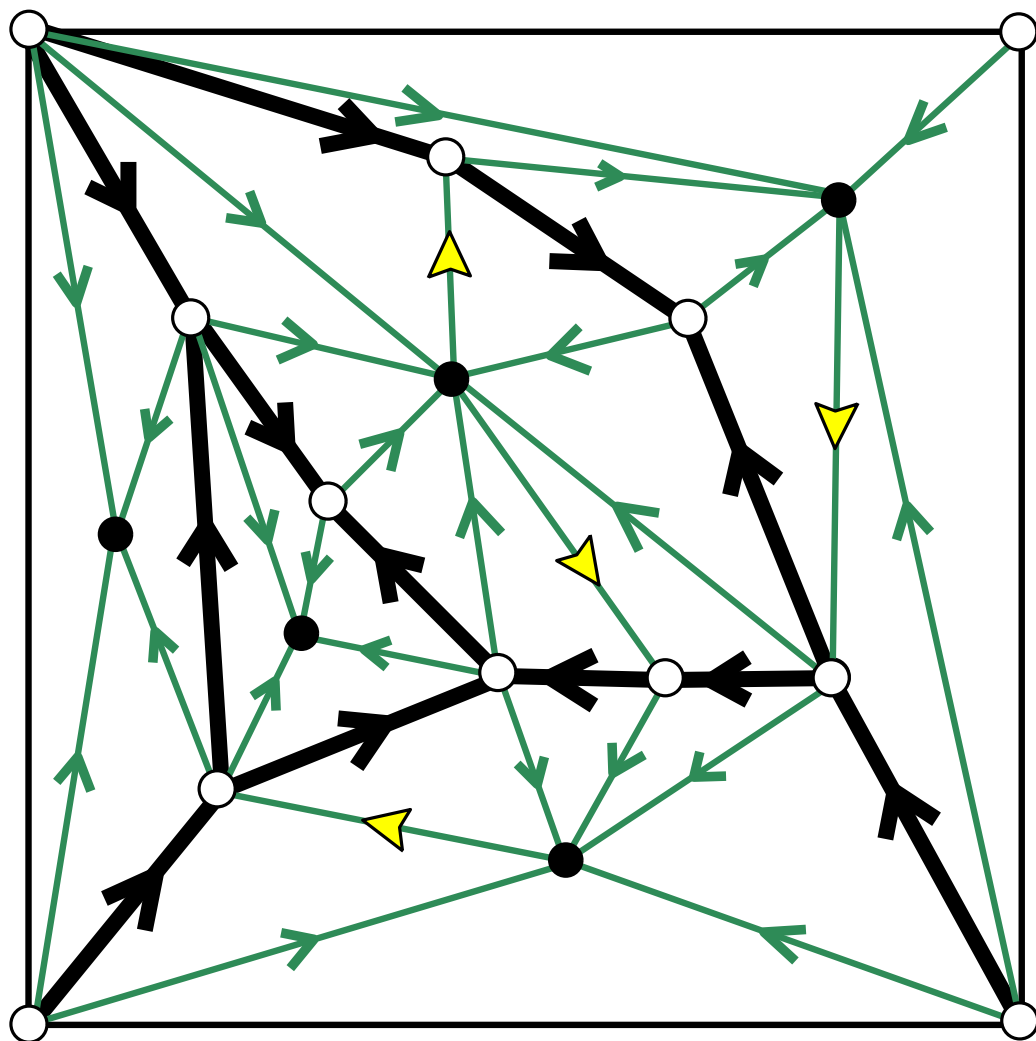


## Generalized 2-orientation

- Each internal white vertex has indegree 2
- Each black vertex of degree  $2i$  has outdegree  $i - 2$

# We show only the bipartite case (simpler)

Case  $b = 2$  (simple bipartite maps), with quadrangular outer face

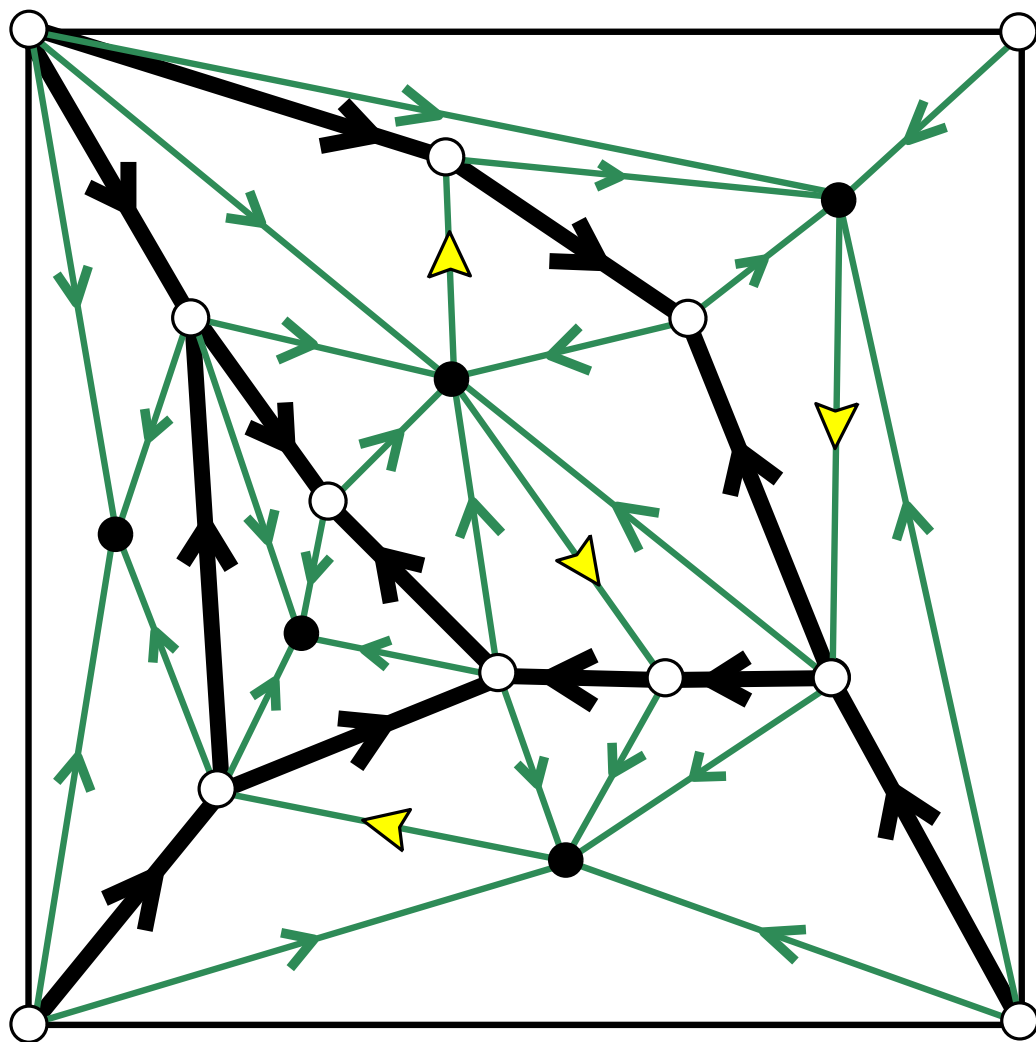


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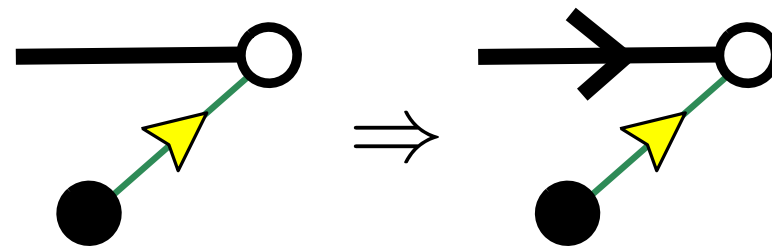
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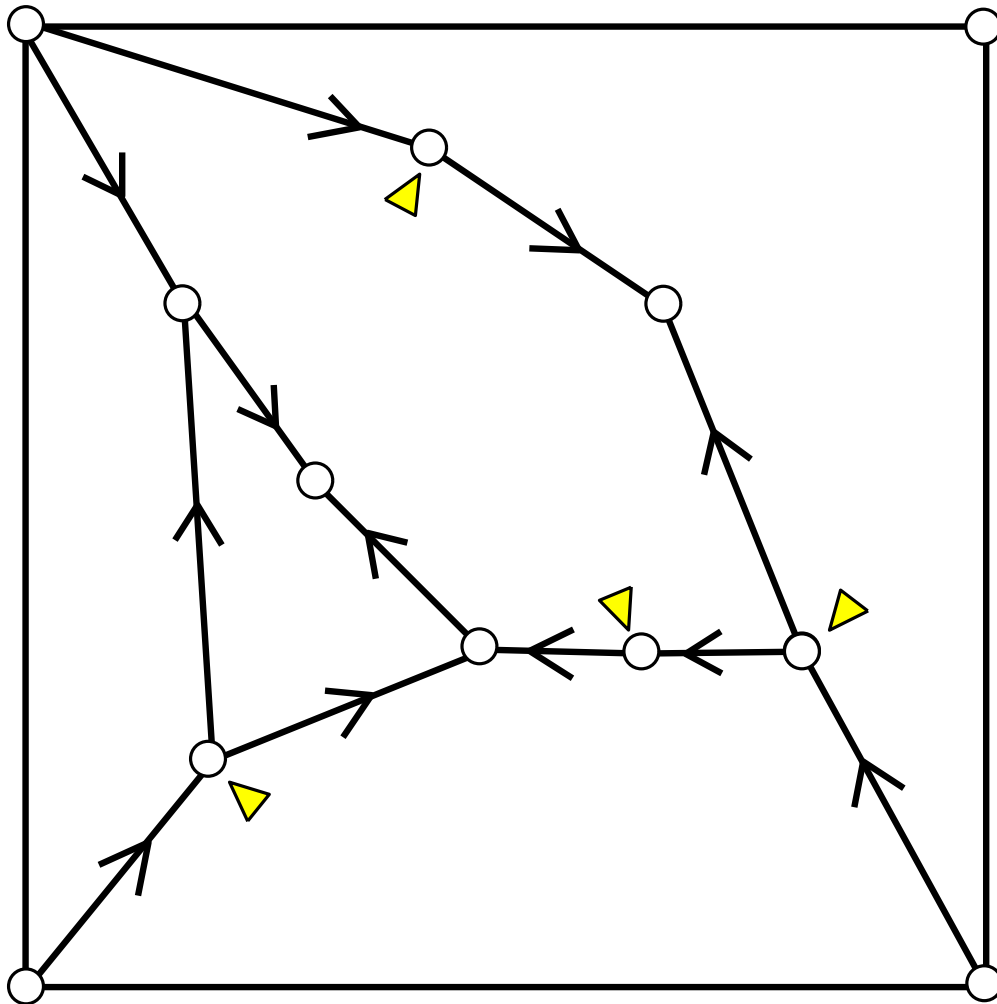
For the minimal one:



& still accessible after deleting the stars

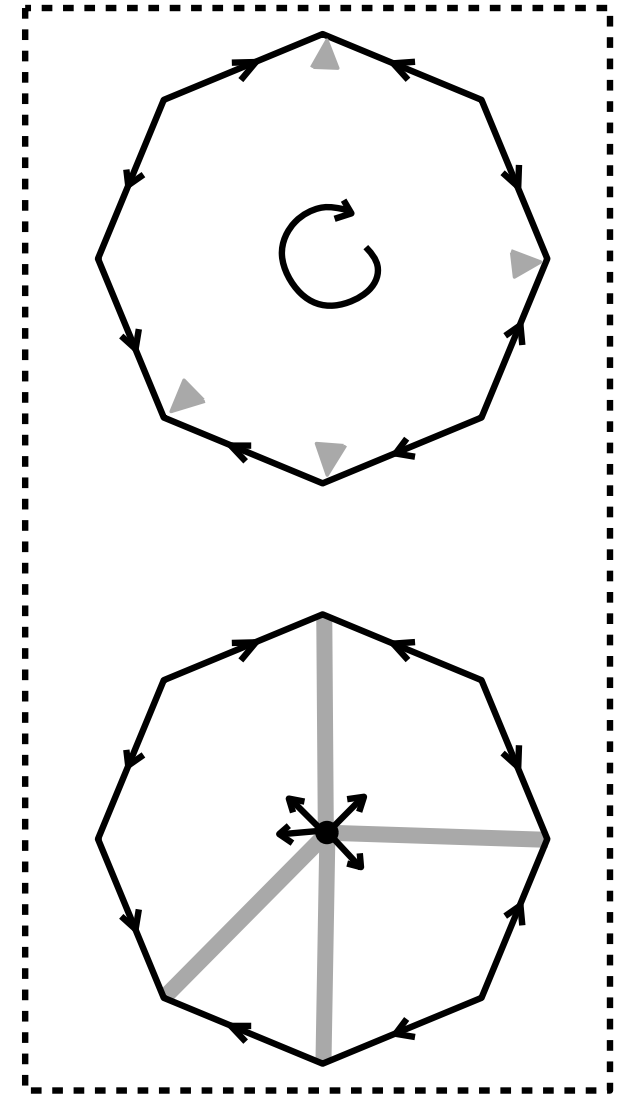
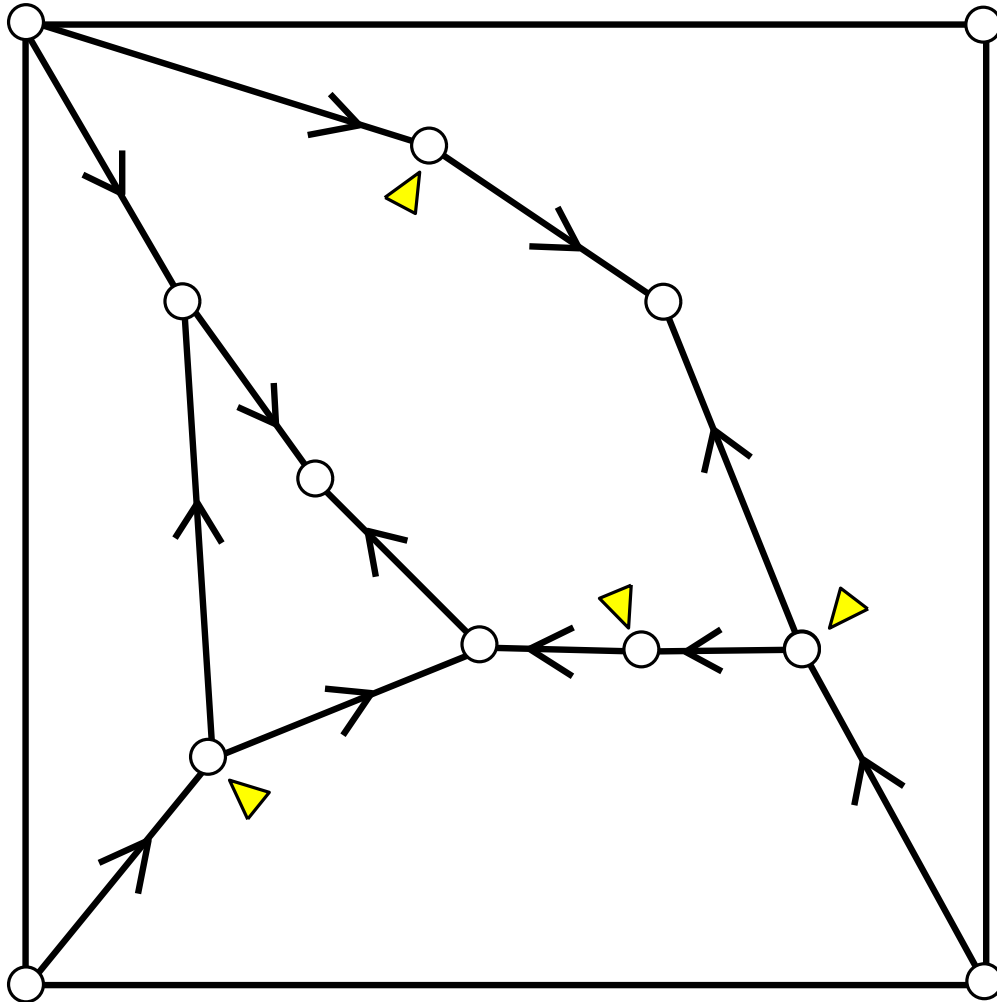
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Case  $b = 2$  (simple bipartite maps), with quadrangular outer face



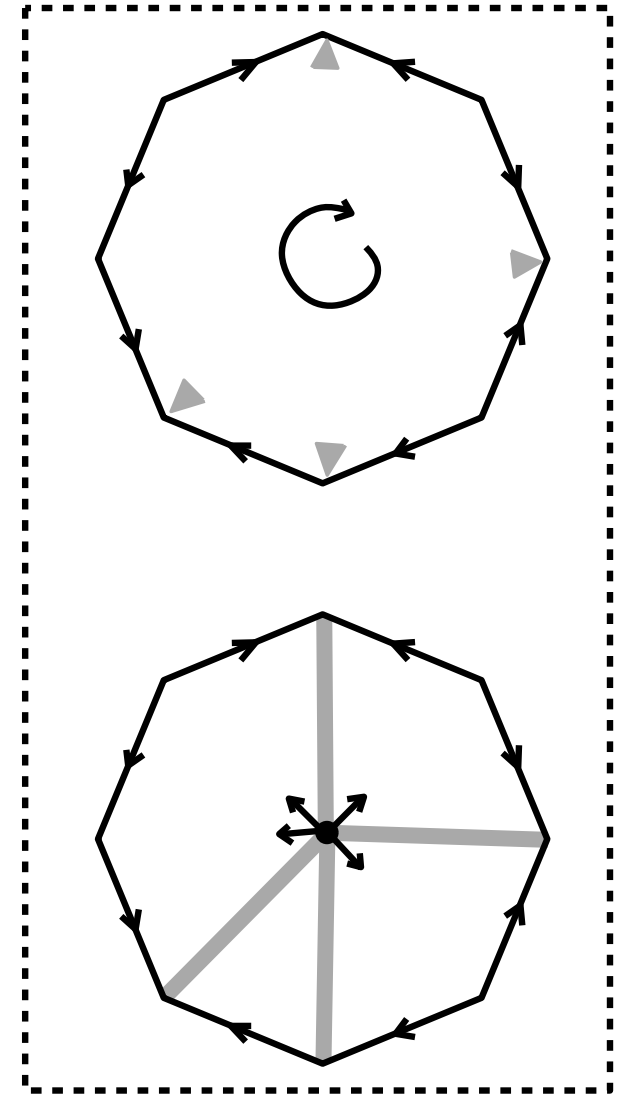
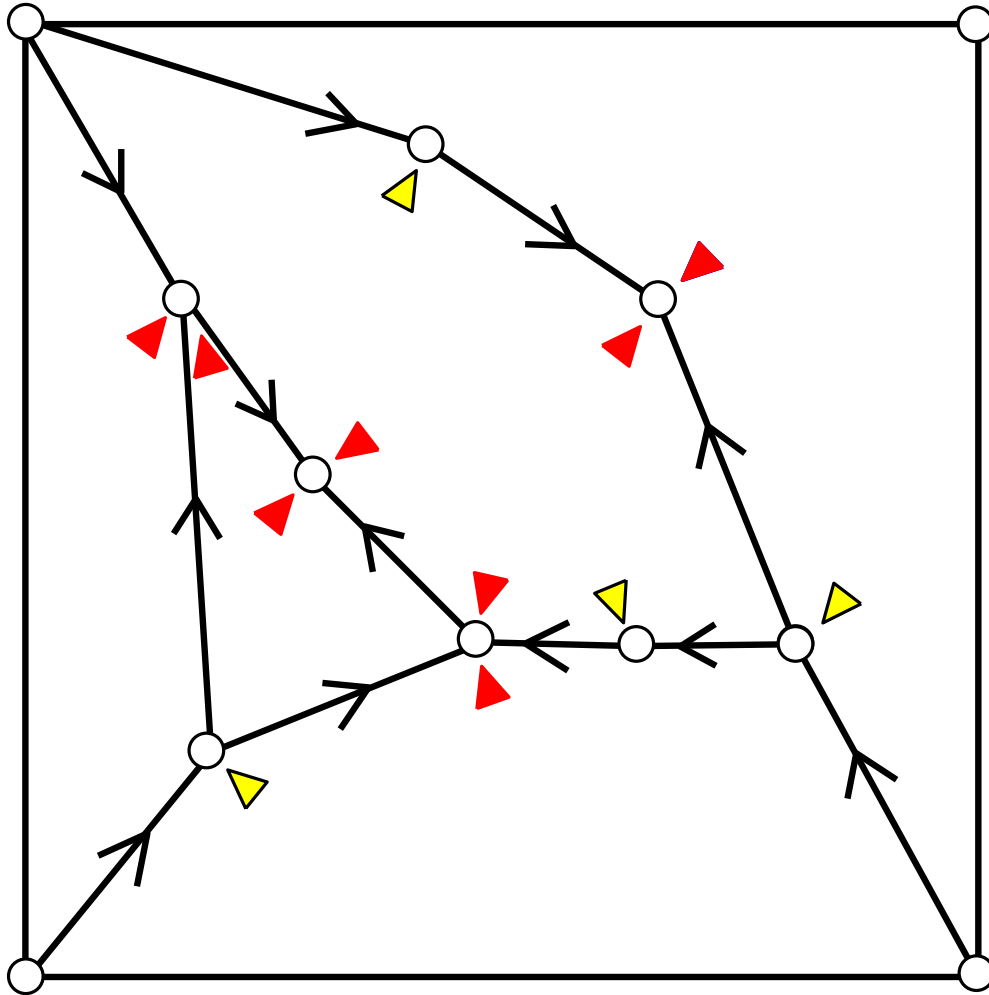
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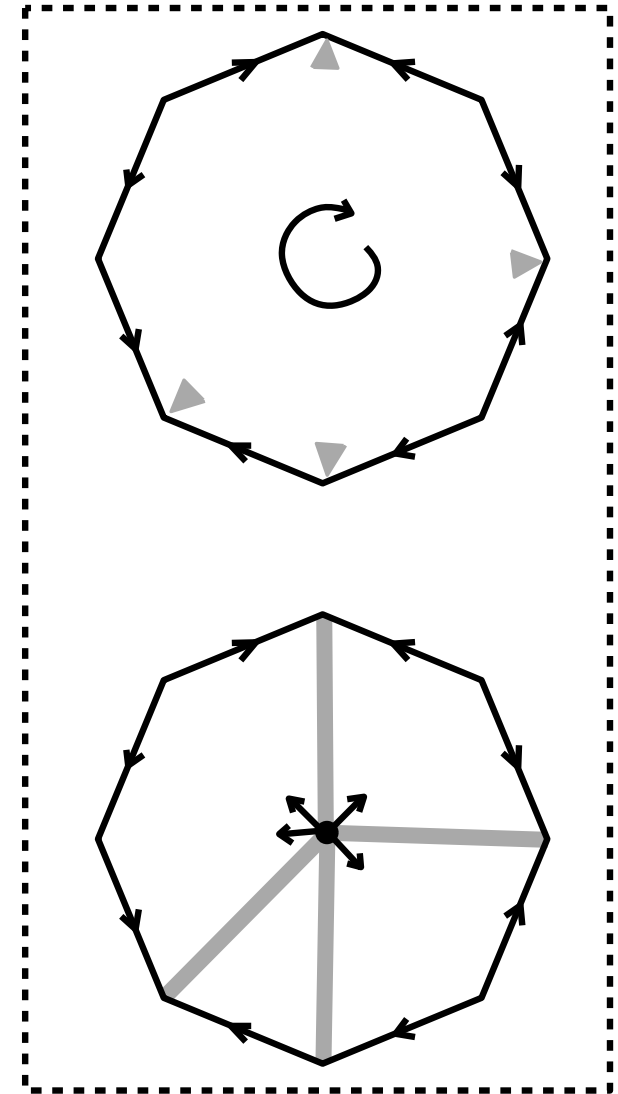
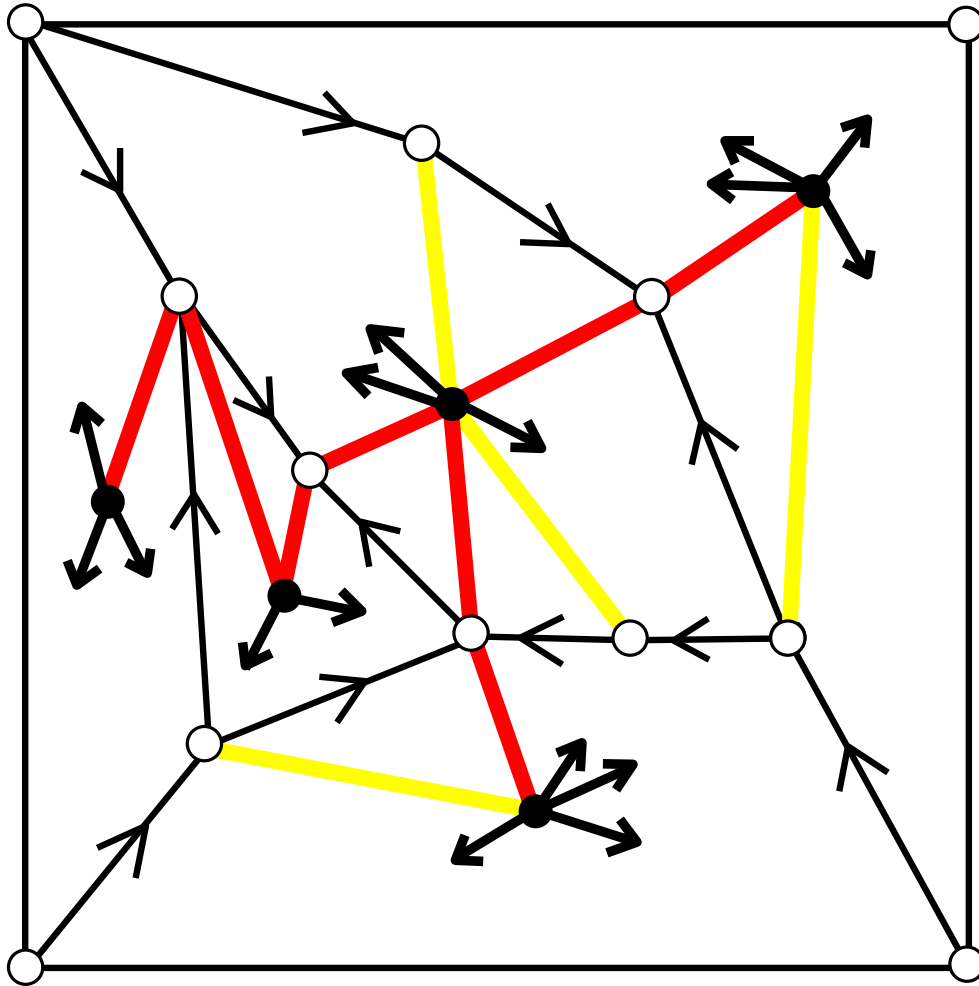
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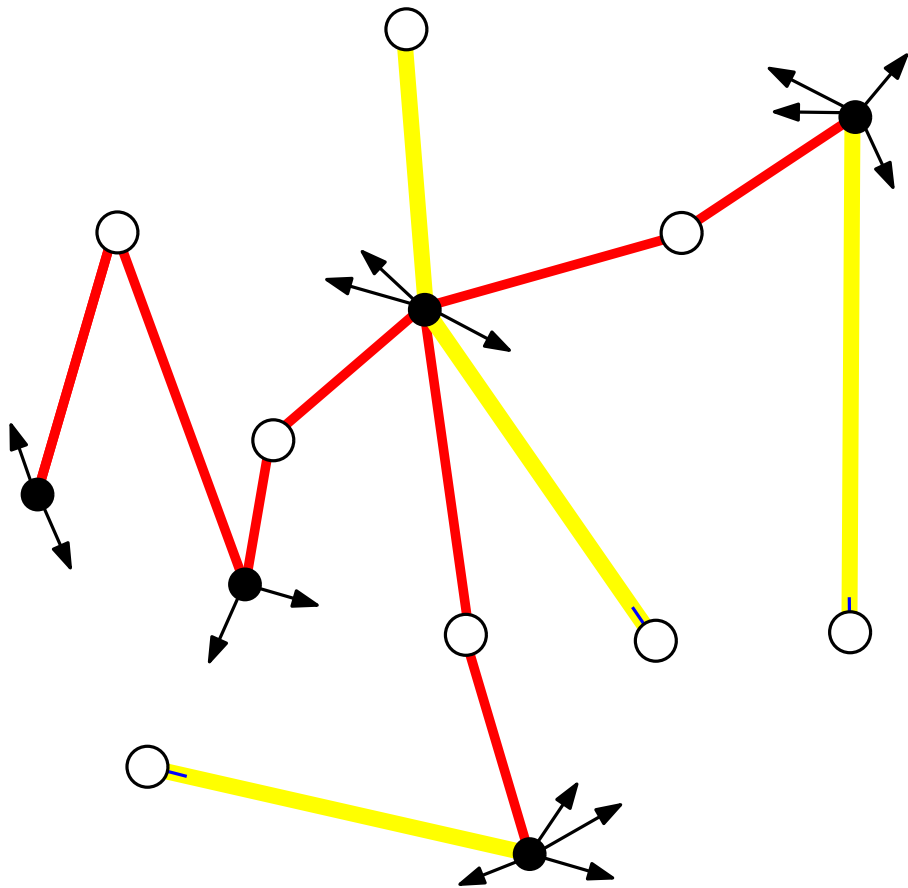
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White vertices either have:

- indegree 2 (middle of red edge)
- indegree 1 (end of leg)

Each black vertex of degree  $2i$  has  $i - 2$  legs

## Closed formulas

**Prop [Bernardi-F'11]:** The number of **rooted simple bipartite maps** with  $n_i$  faces of degree  $2i$  is

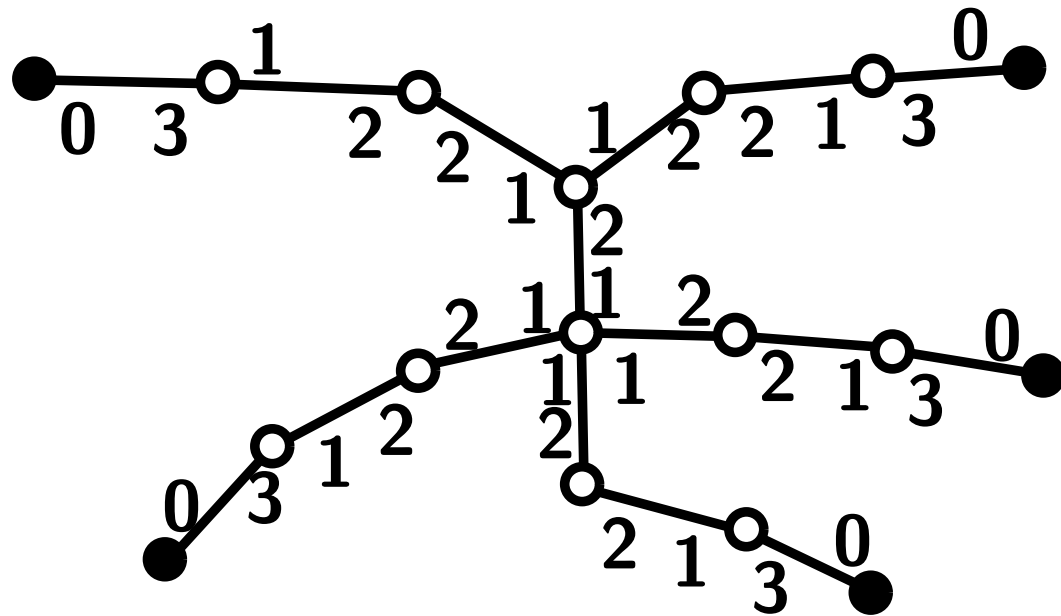
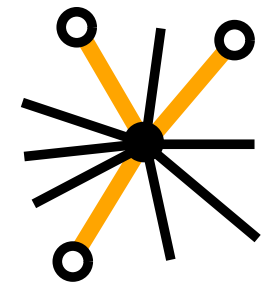
$$2 \frac{(\sum (i+1) n_i - 3)!}{(\sum i n_i - 1)!} \prod_{i \geq 2} \frac{1}{n_i!} \binom{2i-1}{i+1}^{n_i}$$

This can be compared with the formula obtained by Tutte (62) (recovered bijectively by Schaeffer) for **unconstrained rooted bipartite maps**:

$$2 \frac{(\sum i n_i)!}{(\sum (i-1) n_i + 2)!} \prod_{i \geq 1} \frac{1}{n_i!} \binom{2i-1}{i}^{n_i}$$

# Shape of the mobile in higher (bipartite) girth

- Each black vertex of degree  $2i$  has  $i - b$  legs
- There are **connectors** between the black vertices



a connector for  $b = 4$

Connectors, for  $b = 1$ :  $\bullet \overset{0}{\text{---}} \overset{0}{\text{---}} \bullet$   
 $b = 2$ :  $\bullet \overset{0}{\text{---}} \overset{1}{\circ} \overset{1}{\text{---}} \overset{0}{\text{---}} \bullet$   
 $b = 3$ : **binary trees**

# Thanks.

On the ArXiv:

- A bijection for triangulations, quadrangulations, pentagulations, etc.
- Bijective counting of maps by girth and degree.