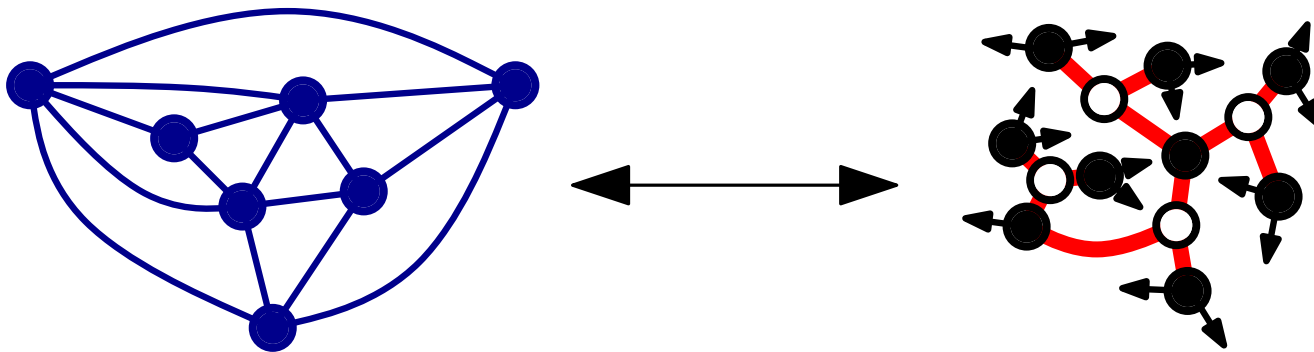


A master bijection for planar maps and its applications

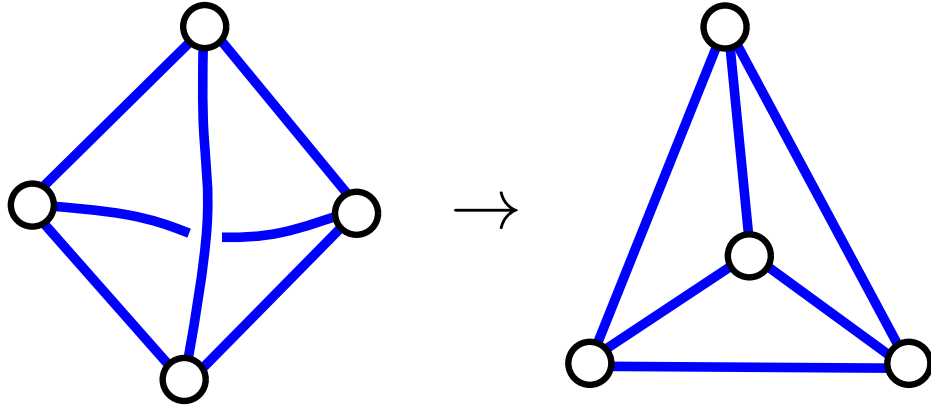
Éric Fusy (CNRS/LIX)
Joint work with Olivier Bernardi (MIT)



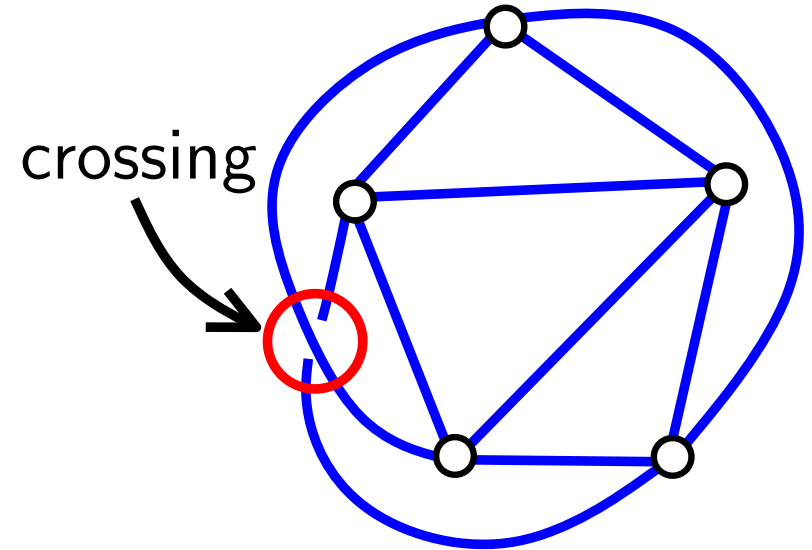
Madrid, June 2011

Planar graphs. Definition

A **planar graph** is a graph that can be drawn in \mathbb{R}^2 without edge-crossing



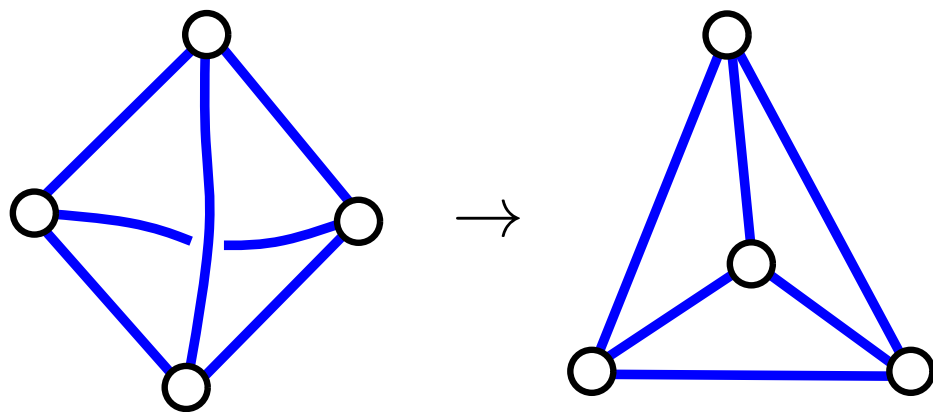
K_4 is planar



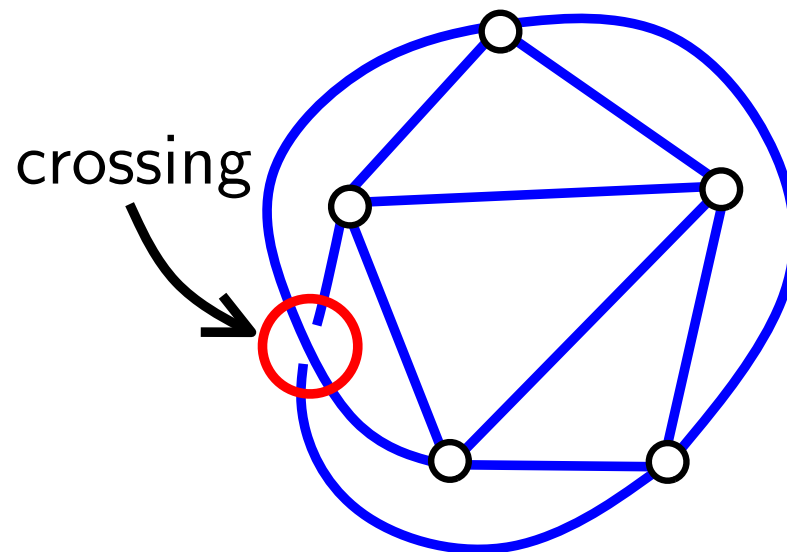
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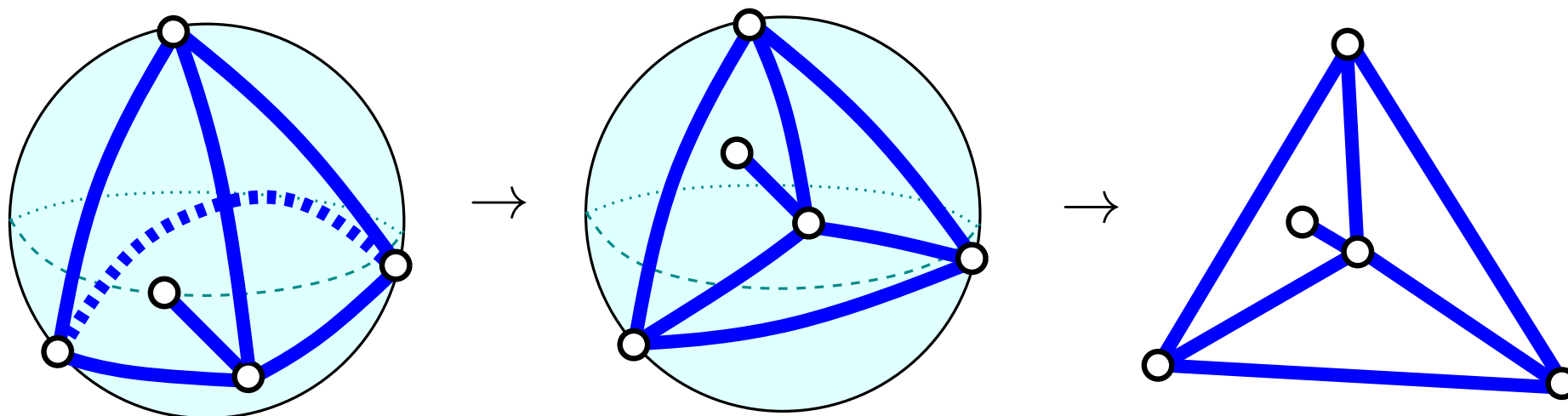


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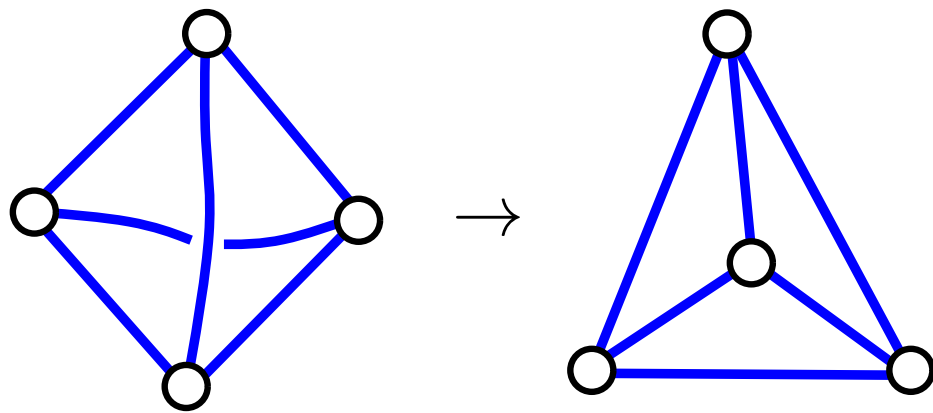
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Rk: Can be drawn in $\mathbb{R}^2 \Leftrightarrow$ can be drawn in S^2

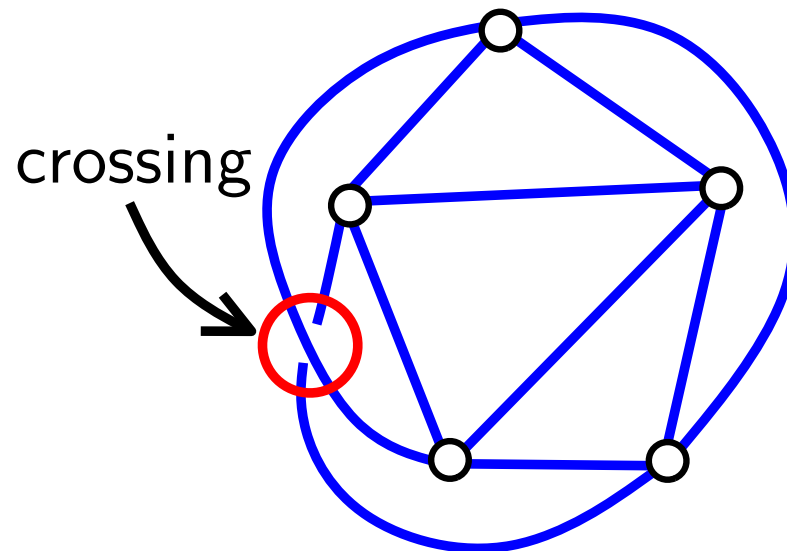


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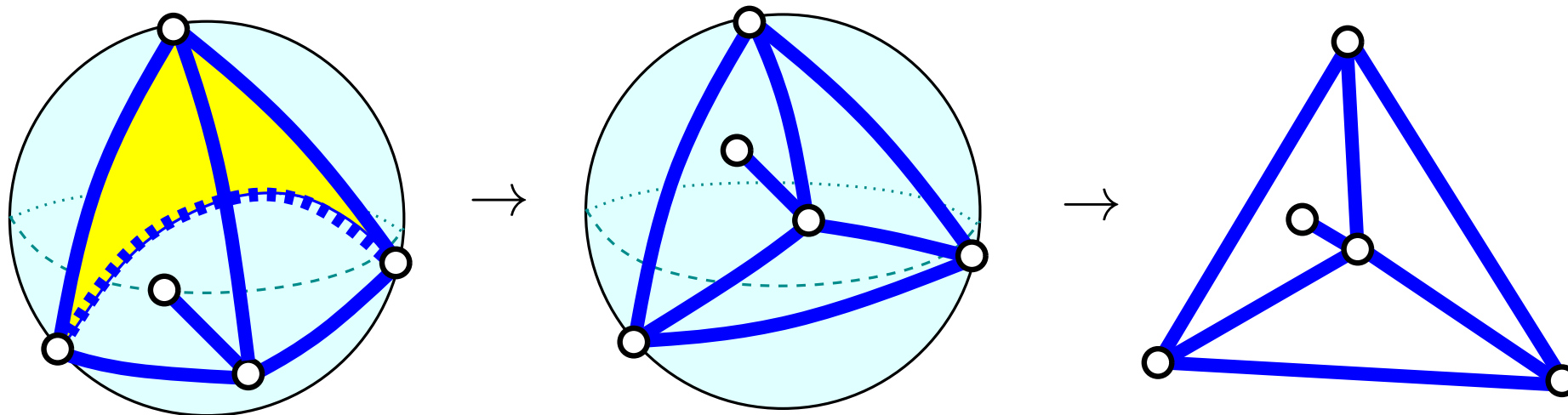


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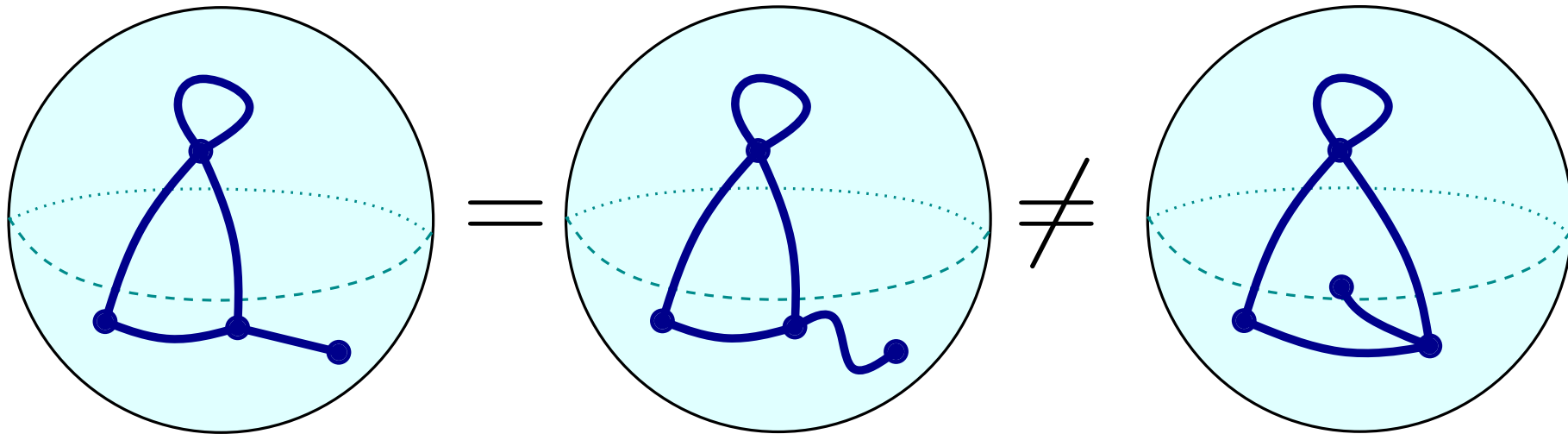
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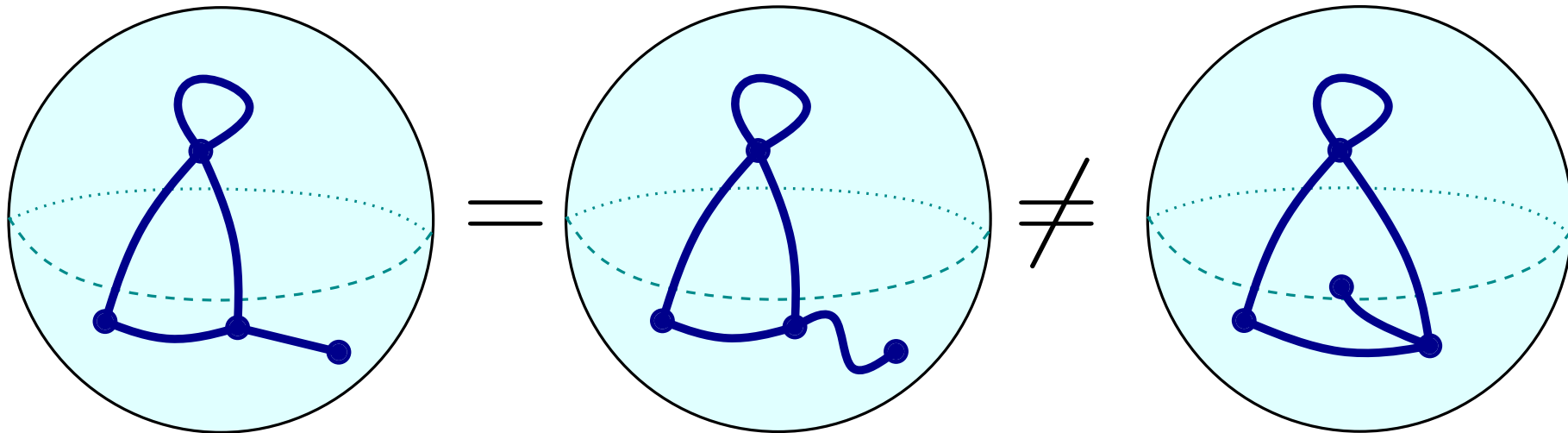
Planar maps and plane maps. **Definition**

- A **planar map** is a connected planar graph drawn in **the sphere** considered up to continuous deformation.



Planar maps and plane maps. **Definition**

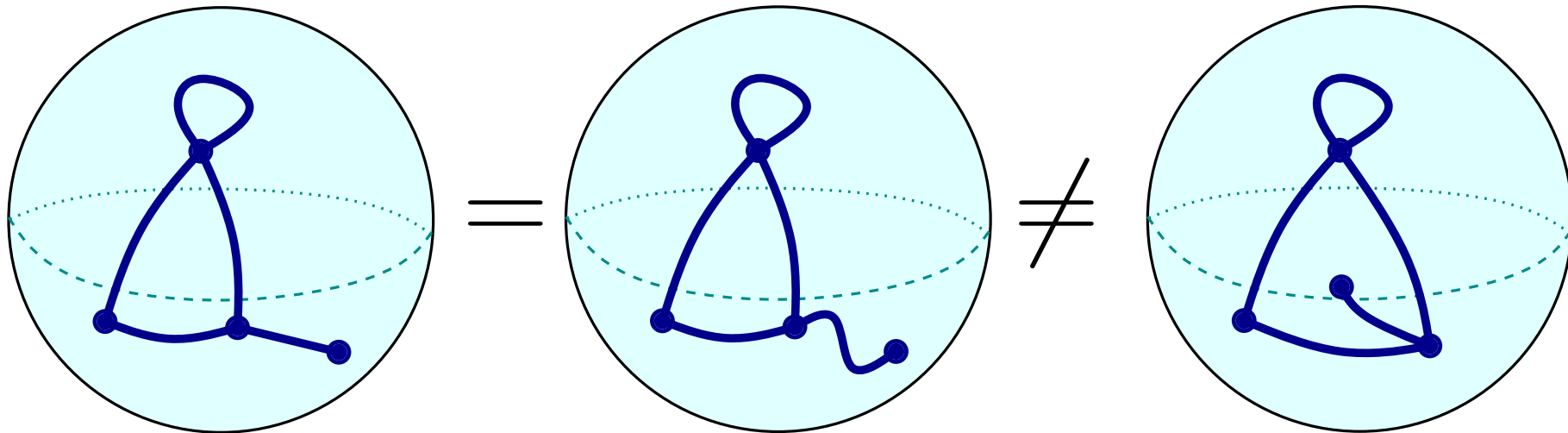
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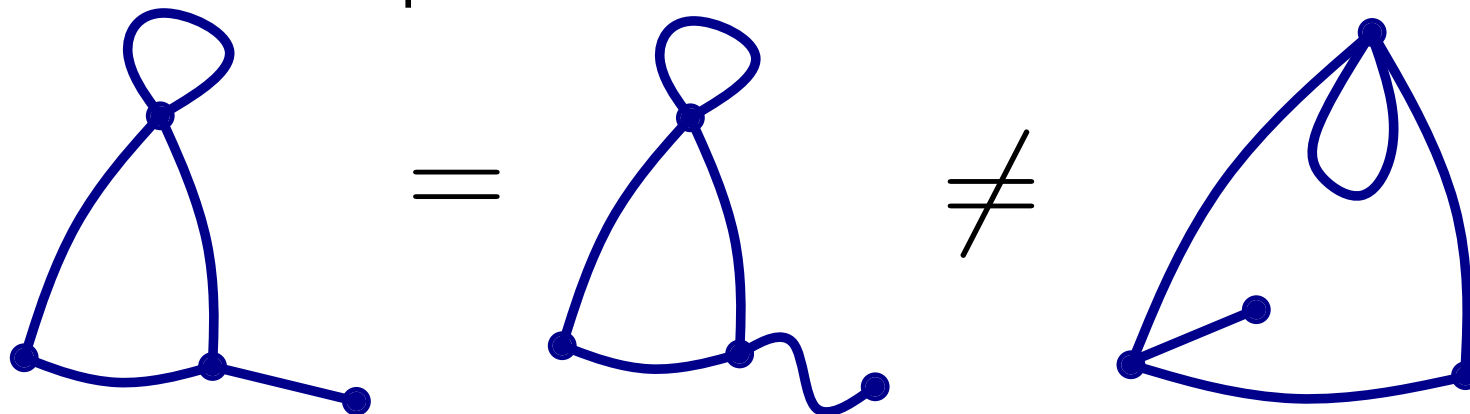
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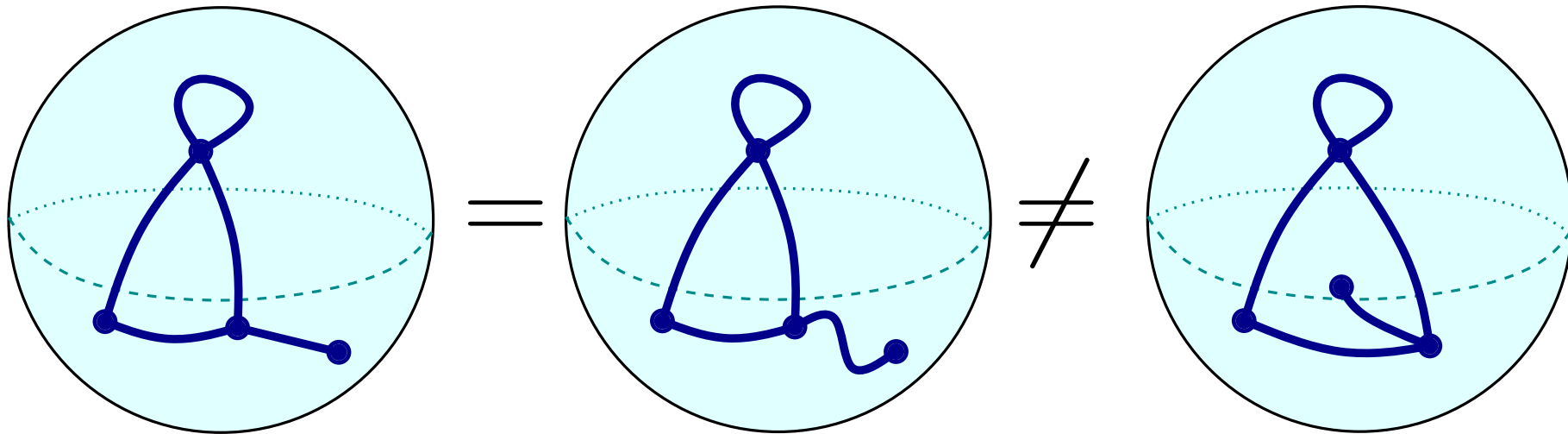
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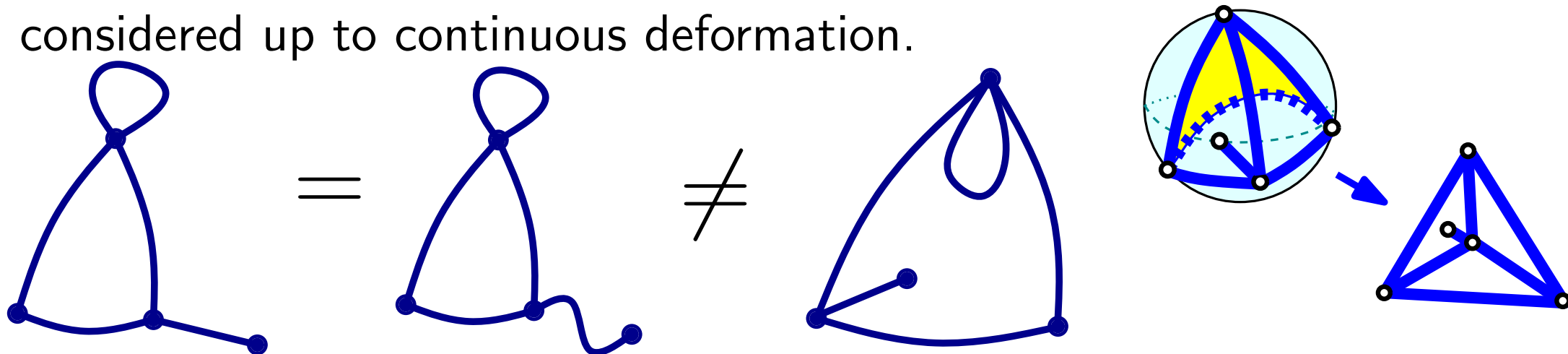
Planar maps and plane maps. **Definition**

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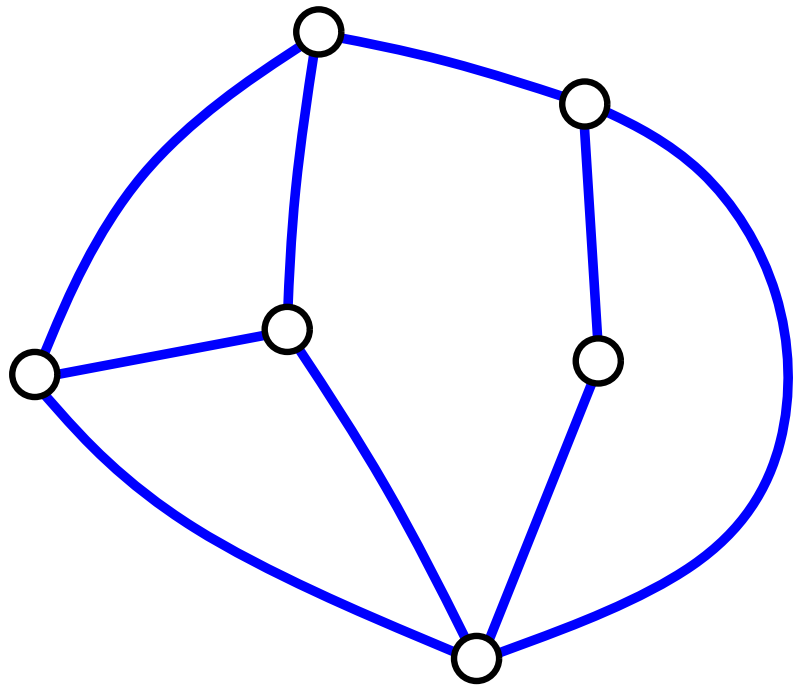


Rk: Plane map = planar map **with a marked face** (the outer face)

The Euler relation

Let $M = (V; E; F)$ be a planar map. Then

$$|V| - |E| + |F| = 2$$



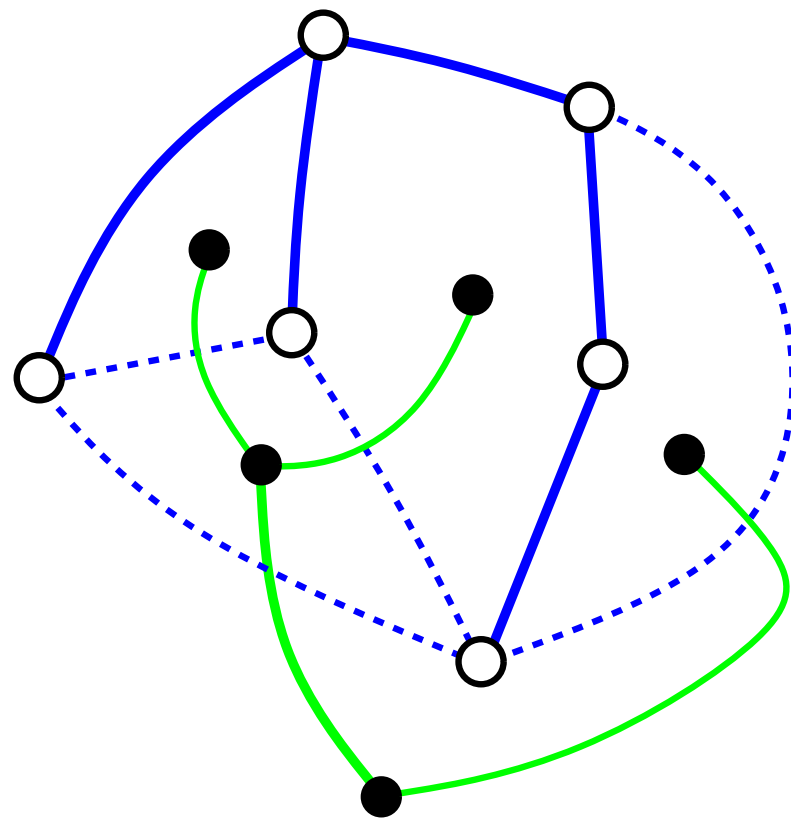
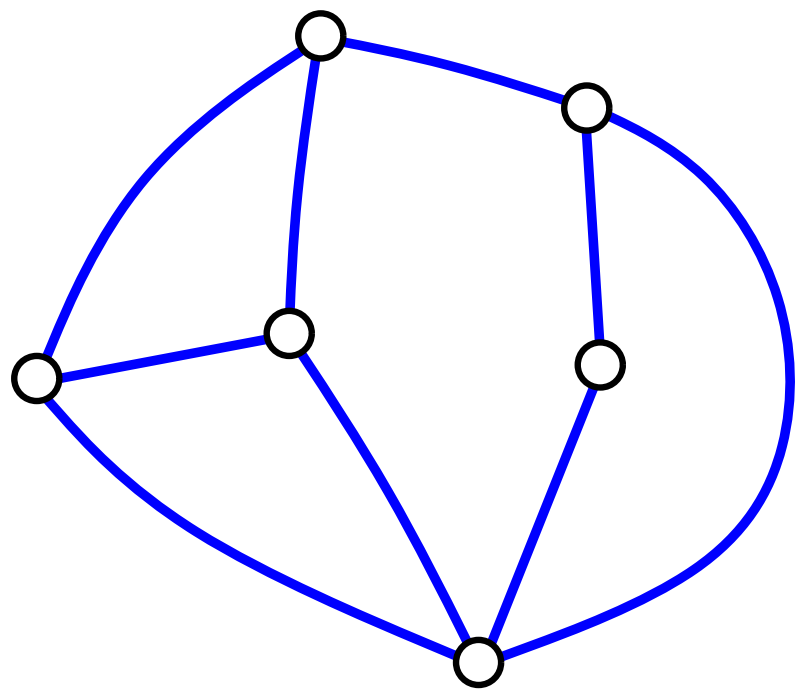
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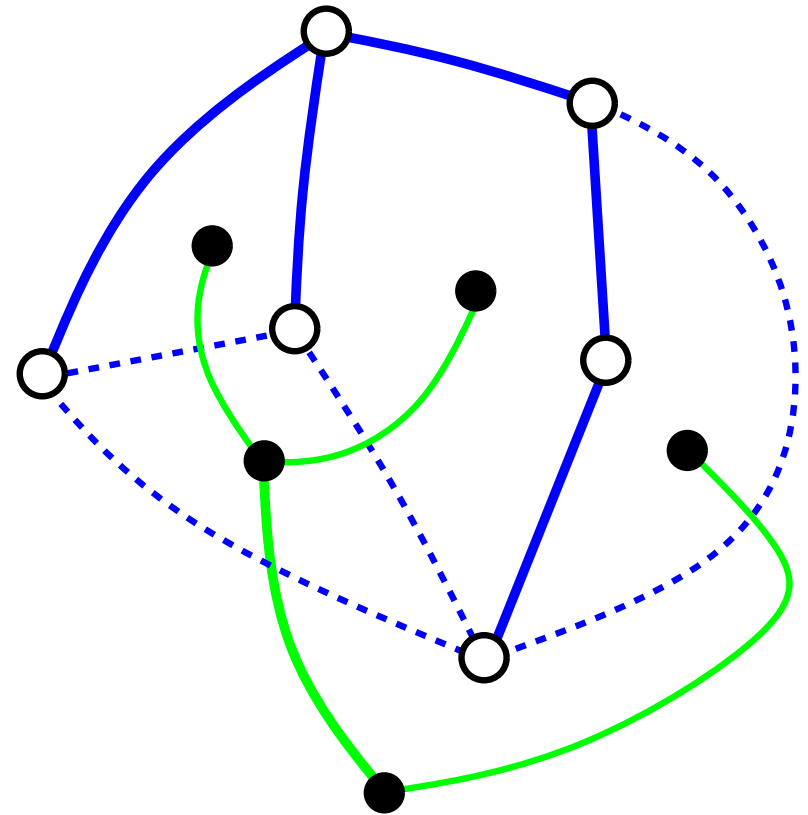
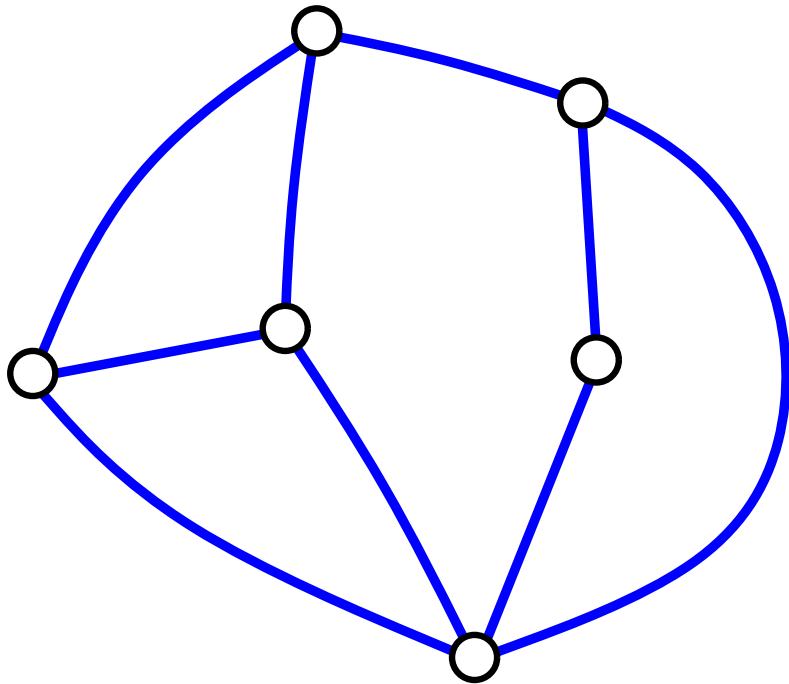
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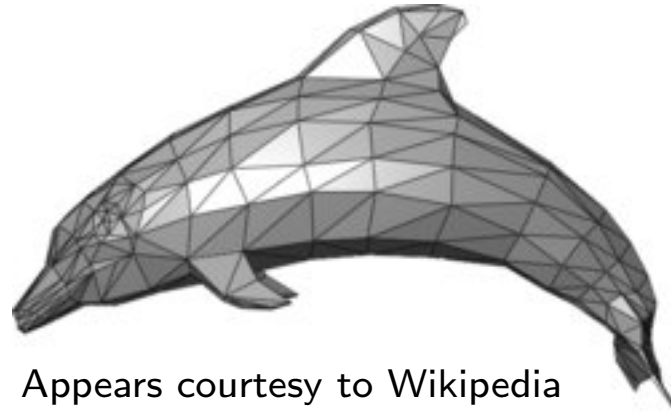
$$|E| = (|V| - 1) + (|F| - 1)$$



\Rightarrow simple planar graph $G = (V; E)$ satisfies $|E| \leq 3|V| - 6$
(hence K_5 has too many edges to be planar)

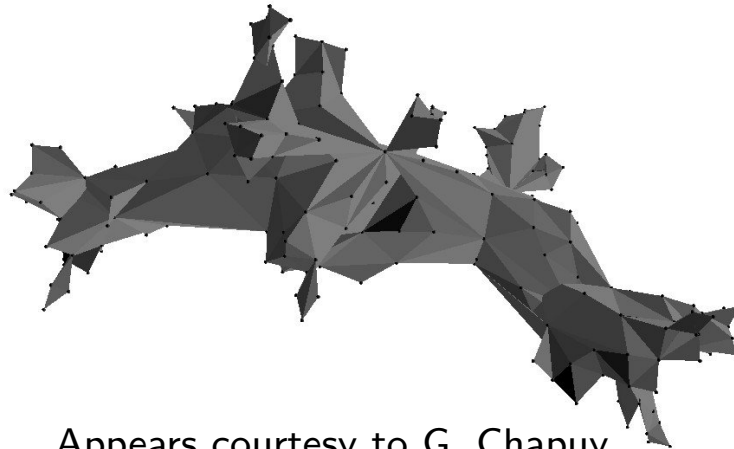
Planar maps. Motivations

- Algorithmic applications: efficient encoding of meshed surfaces.



Appears courtesy to Wikipedia

- Probability and Physics: random lattices, random surfaces.



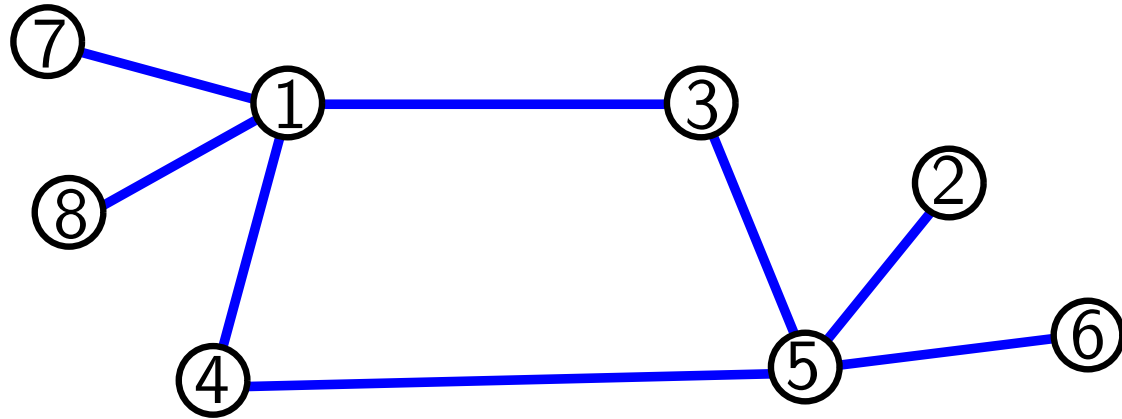
Appears courtesy to G. Chapuy

- Representation Theory: factorization problems.

Symmetry issues.

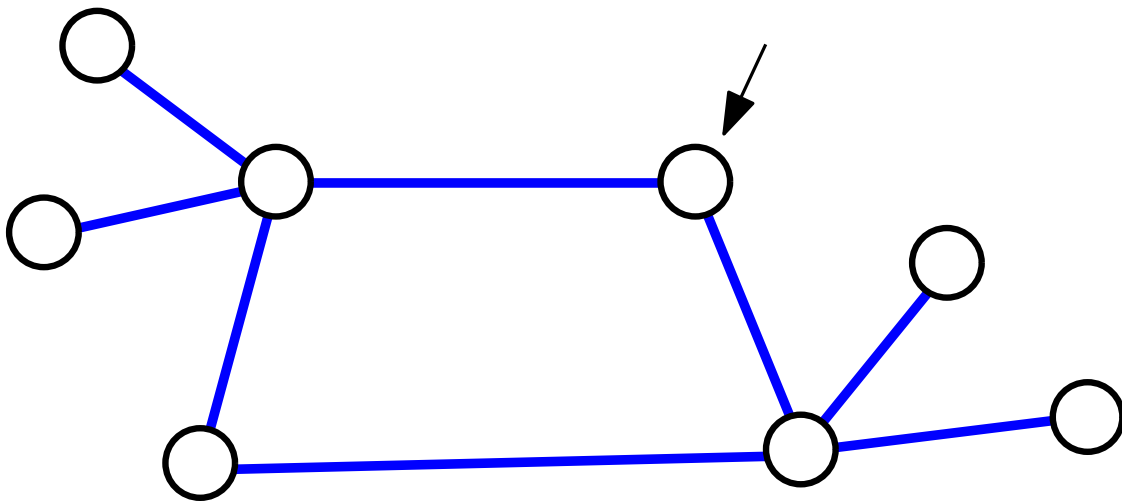
In order to identify vertices unambiguously (to **avoid symmetry issues**):

- Planar graphs: need to **label the vertices**



A labelled planar graph

- Planar maps: only need to **mark a corner**



A rooted planar map

Asymptotic behaviour of planar maps/graphs

- **Asymptotic number:**

Labelled planar graphs n vertices:

$$\sim n! c n^{-7/2} n \quad [\text{Giménez, Noy'05}]$$

Rooted planar maps n edges

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[Chassaing, Schaeffer'04], [Le Gall'11], [Miermont'11] [Chapuy et al'10]

- **Why do they have same behaviour ?**

Asymptotic behaviour of planar maps/graphs

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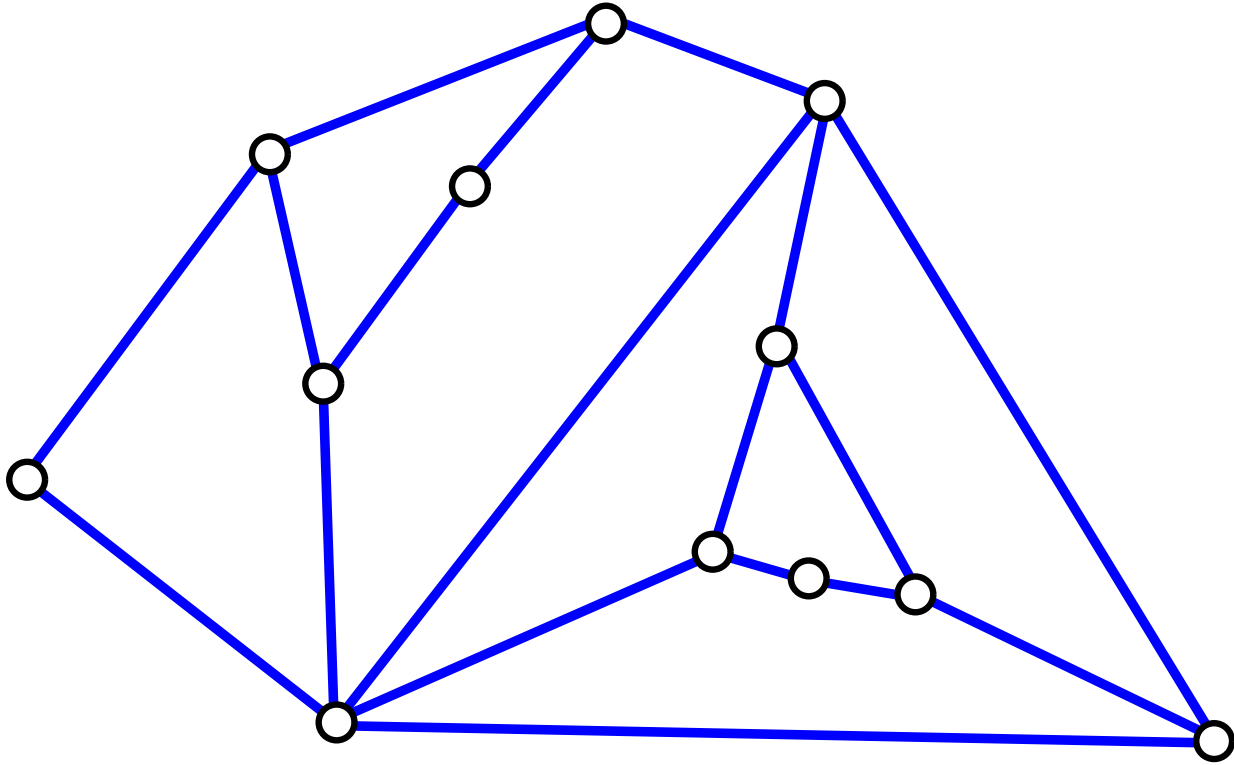
[Chassaing, Schaeffer'04], [Le Gall'11], [Miermont'11] [Chapuy et al'10]

- **Why do they have same behaviour ?** [Giménez, Noy, Rué'10]

Random planar **graph** \sim random (3-connected) planar **map** of size $\Theta(n)$
 + **little pieces** attached into it

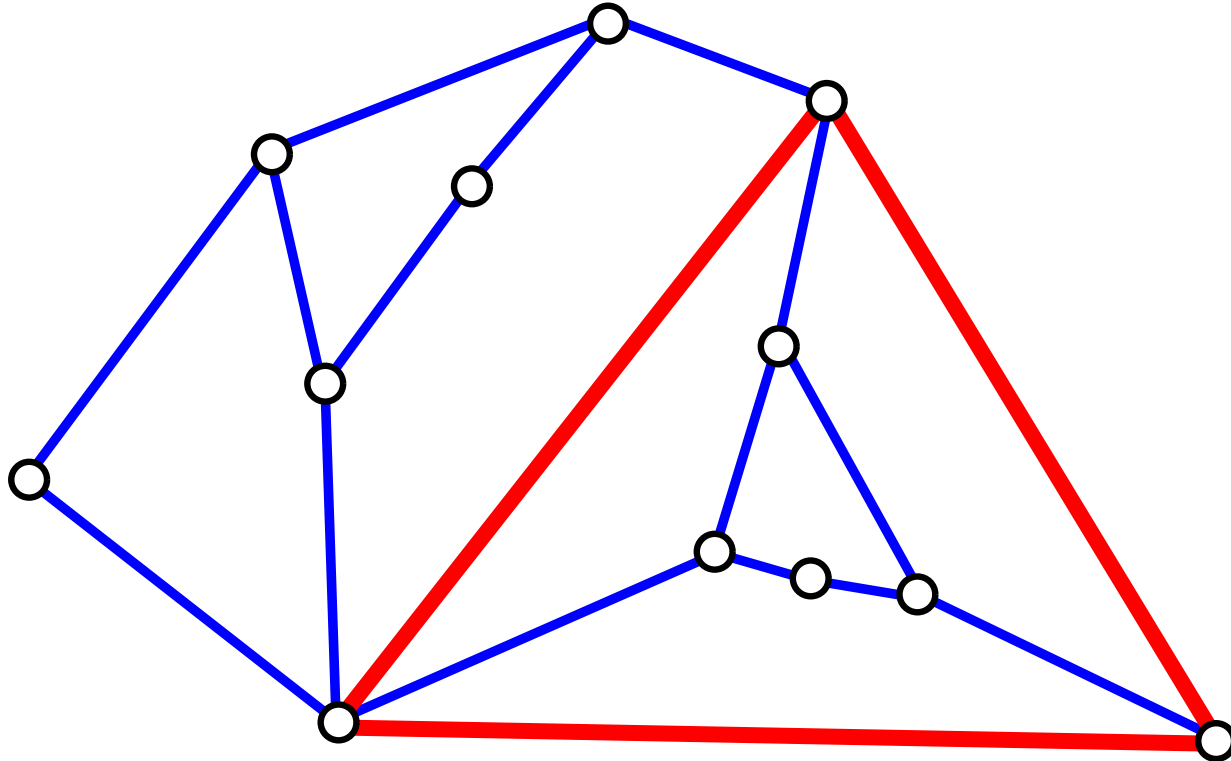
The girth parameter

The **girth** of a graph is the length of a shortest cycle within the graph



The girth parameter

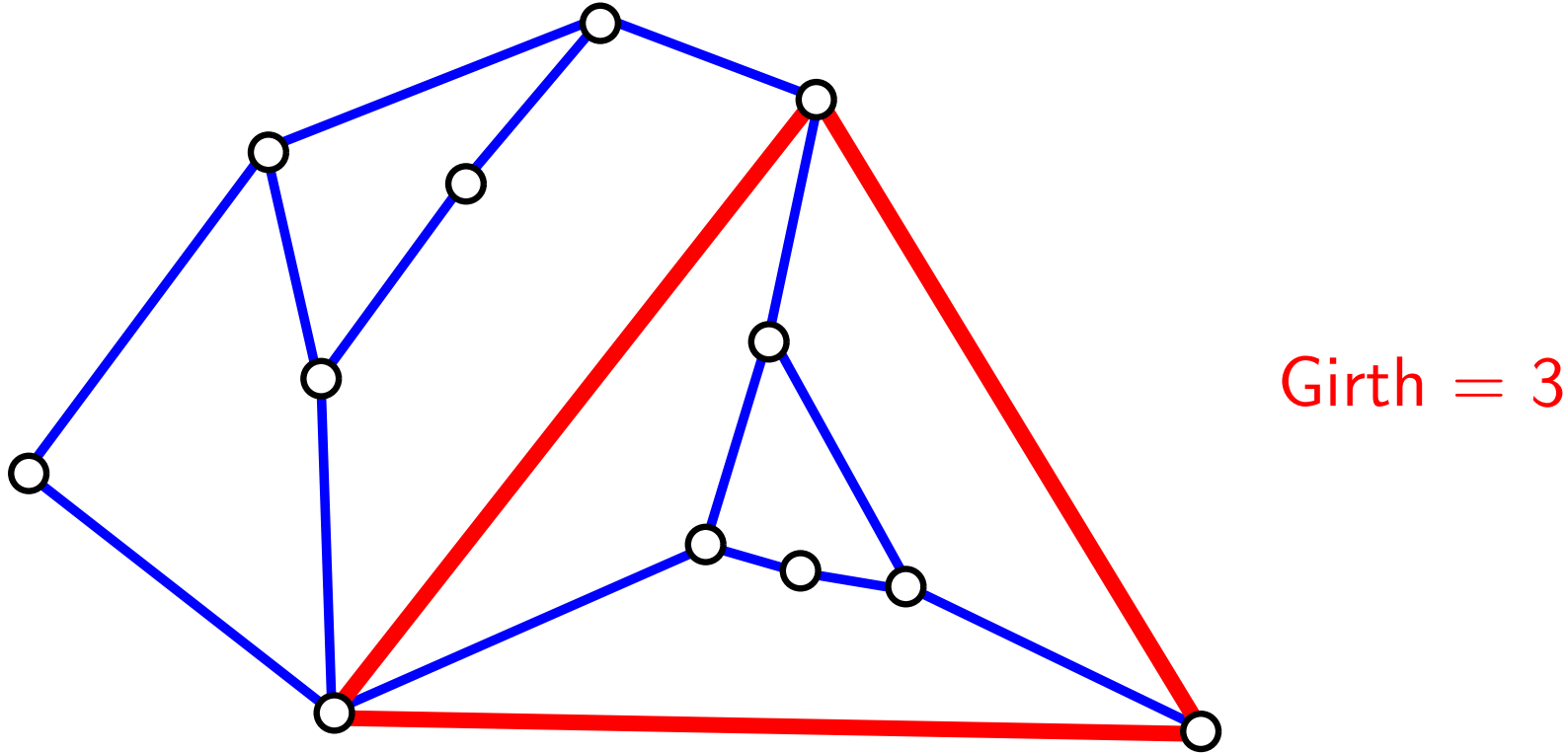
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Girth = 3

The girth parameter

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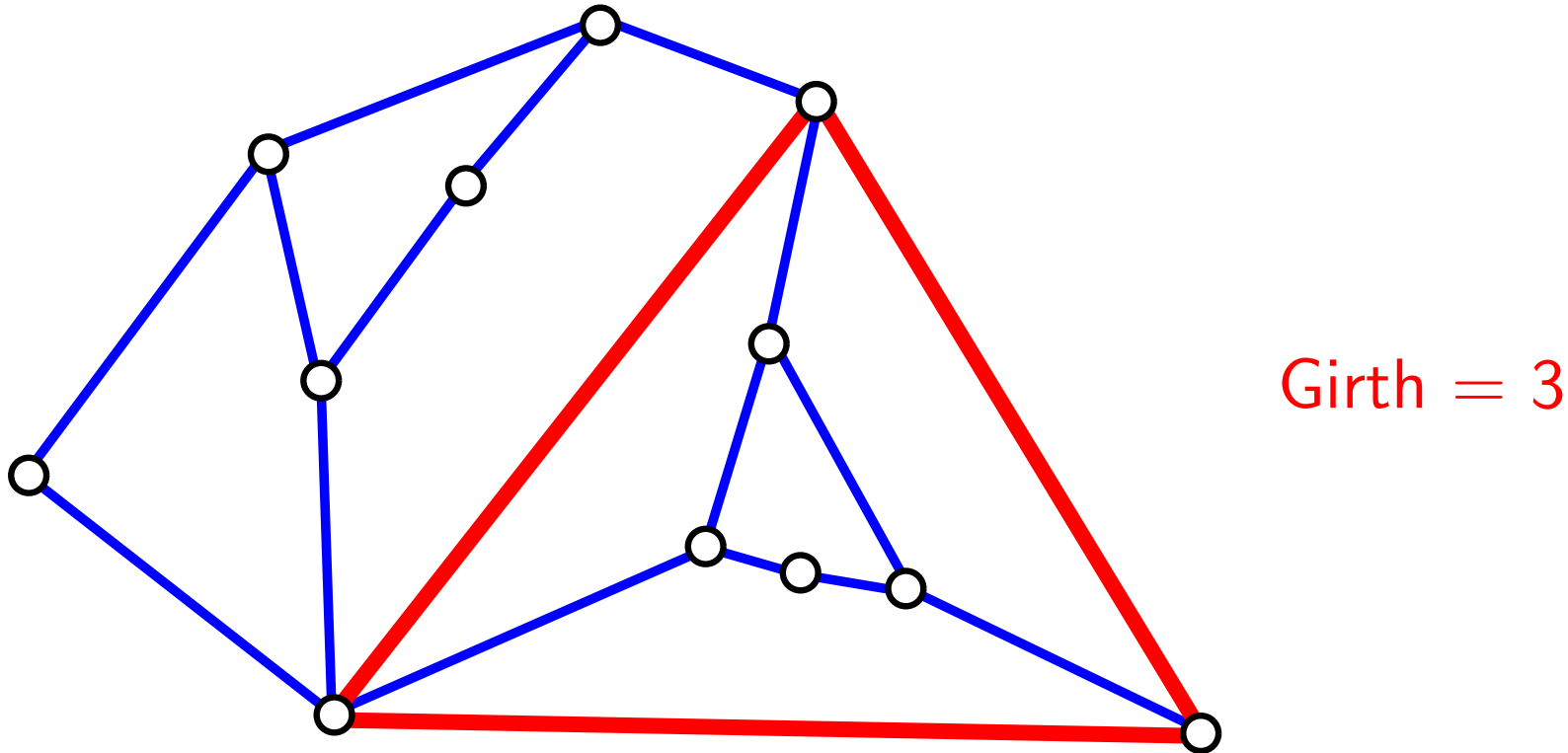


Girth = 3

Rk: If $girth = d$ then all faces have **degree at least d**

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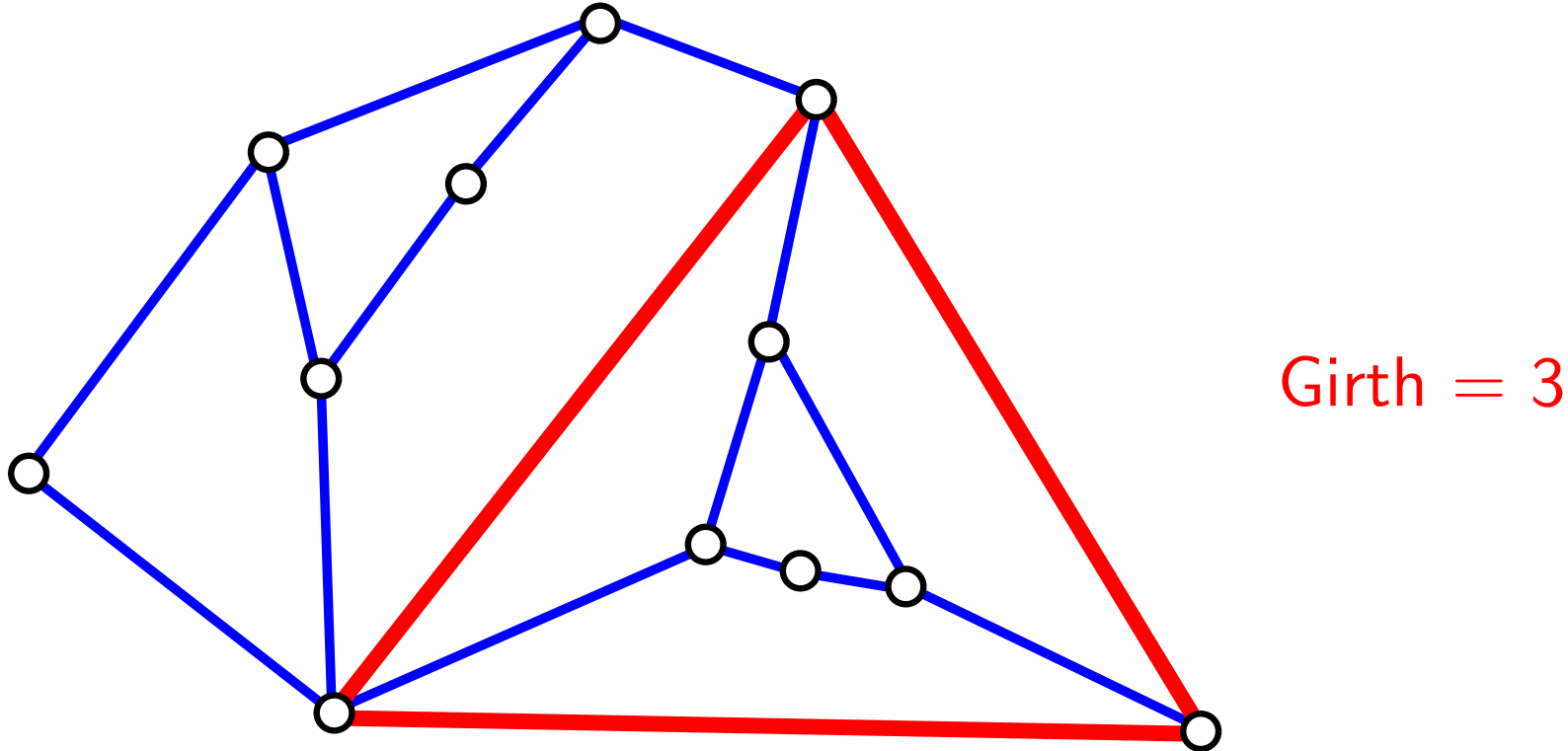
Loopless $\Leftrightarrow girth \geq 2$

Simple $\Leftrightarrow girth \geq 3$

Triangle-free $\Leftrightarrow girth \geq 4$

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Many natural map families are specified by constraints on the **girth** and on the **face-degrees** (loopless triangulations, simple quadrangulations,...)

Planar maps. Exact counting results

- Triangulations ($2n$ faces)

Loopless: $\frac{2^n}{(n+1)(2n+1)} \binom{3n}{n}$

Simple: $\frac{1}{n(2n-1)} \binom{4n-2}{n-1}$

- Quadrangulations (n faces)

General: $\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$

Simple: $\frac{2}{n(n+1)} \binom{3n}{n-1}$

- Bipartite maps (n_i faces of degree $2i$)

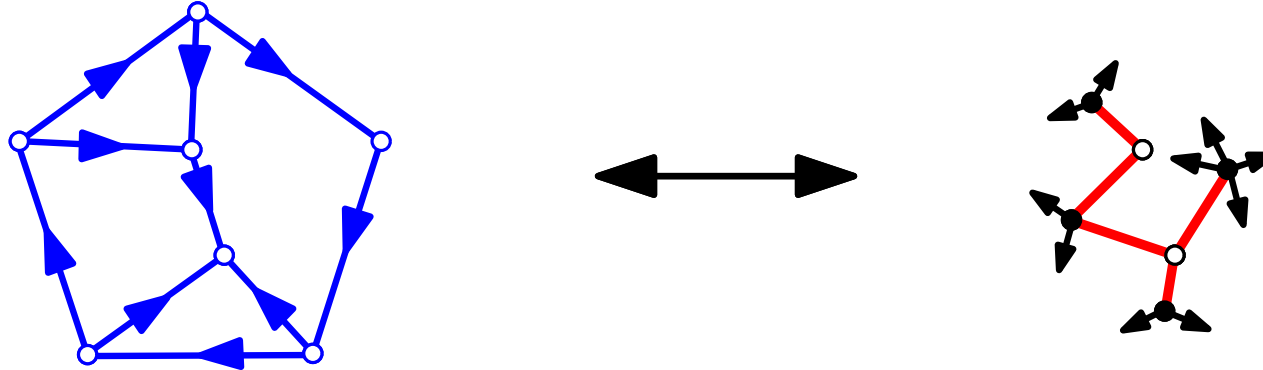
$$\frac{2 \cdot (\sum i n_i)!}{(2 + \sum (i-1)n_i)!} \prod_i \frac{1}{n_i!} \binom{2i-1}{i}^{n_i}$$

Planar maps. **Counting methods**

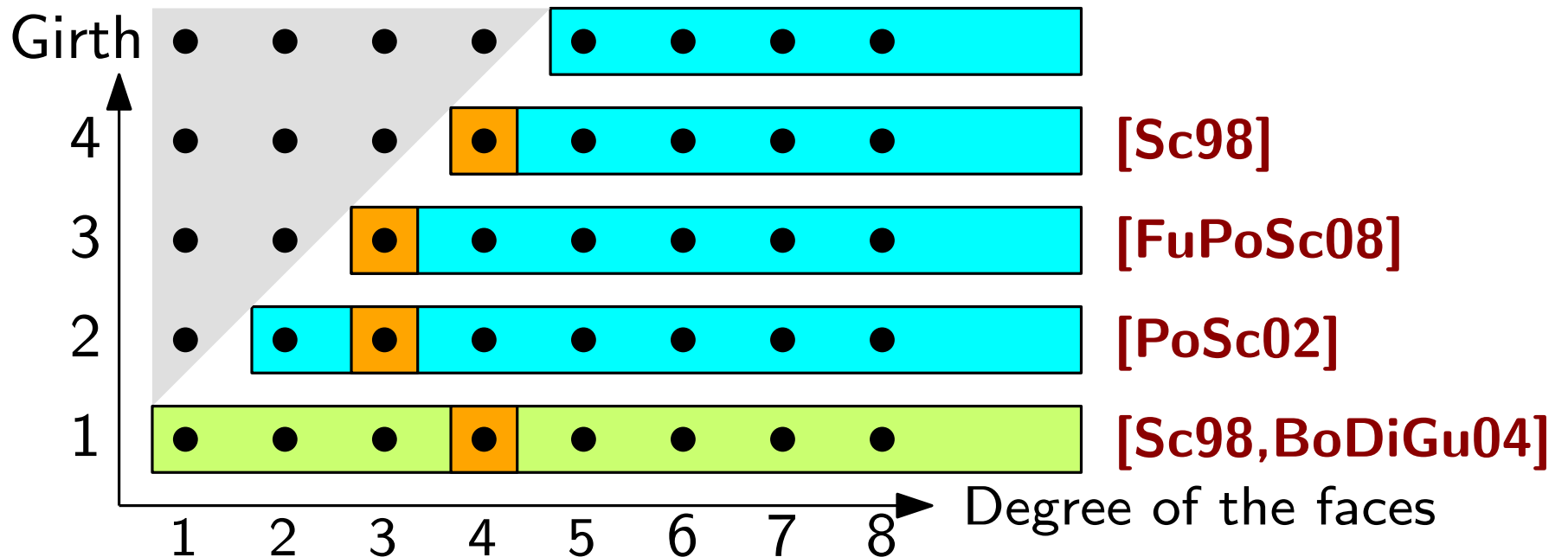
- **Generating functions** [Tutte 63]
Recursive description of maps \rightsquigarrow recurrences.
- **Matrix Integrals** ['t Hooft 74, Brézin et al'78]
Feynmann Diagram \approx maps.
- **Bijections** [Cori-Vauquelin 81, Schaeffer 98]
Maps \rightsquigarrow decorated trees.

Outline

1. **Master bijection** between a class of **oriented maps** and a class of bicolored **decorated trees** (which are called mobiles).

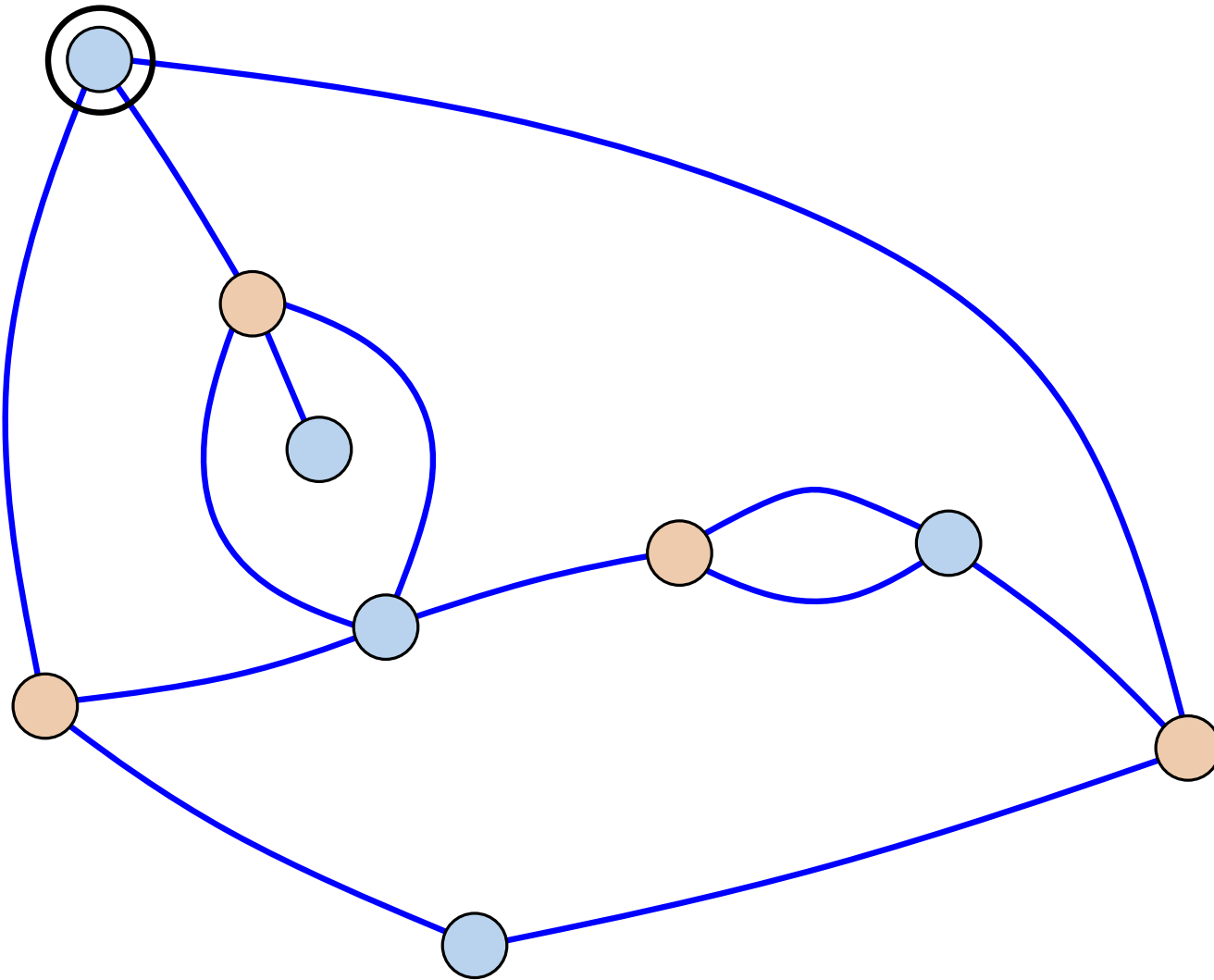


2. **Specializations** to classes of maps (via canonical orientations).

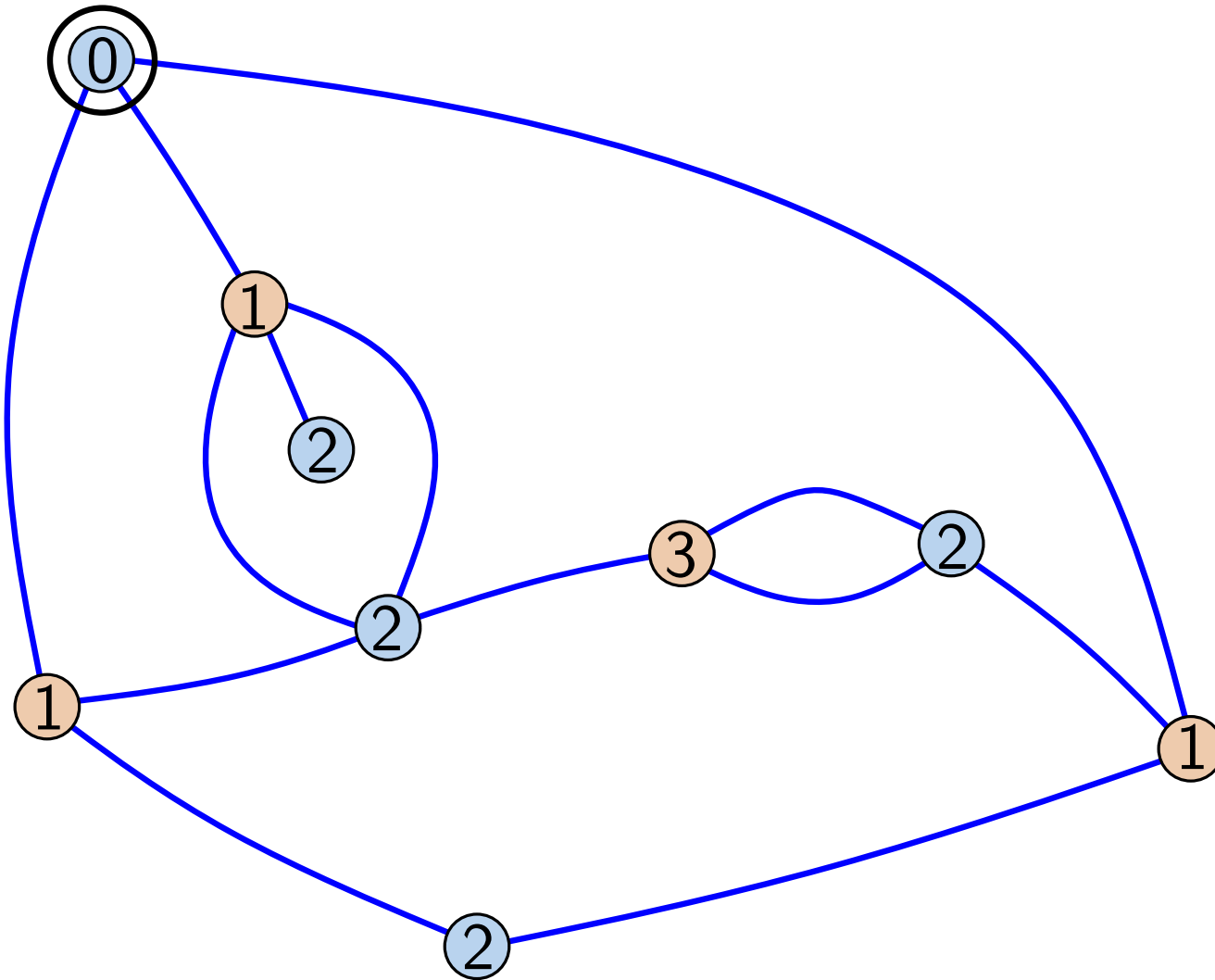


From oriented maps to mobiles

Pointed bipartite map \rightarrow labelled mobile. [Sc98] [BoDiGu04]



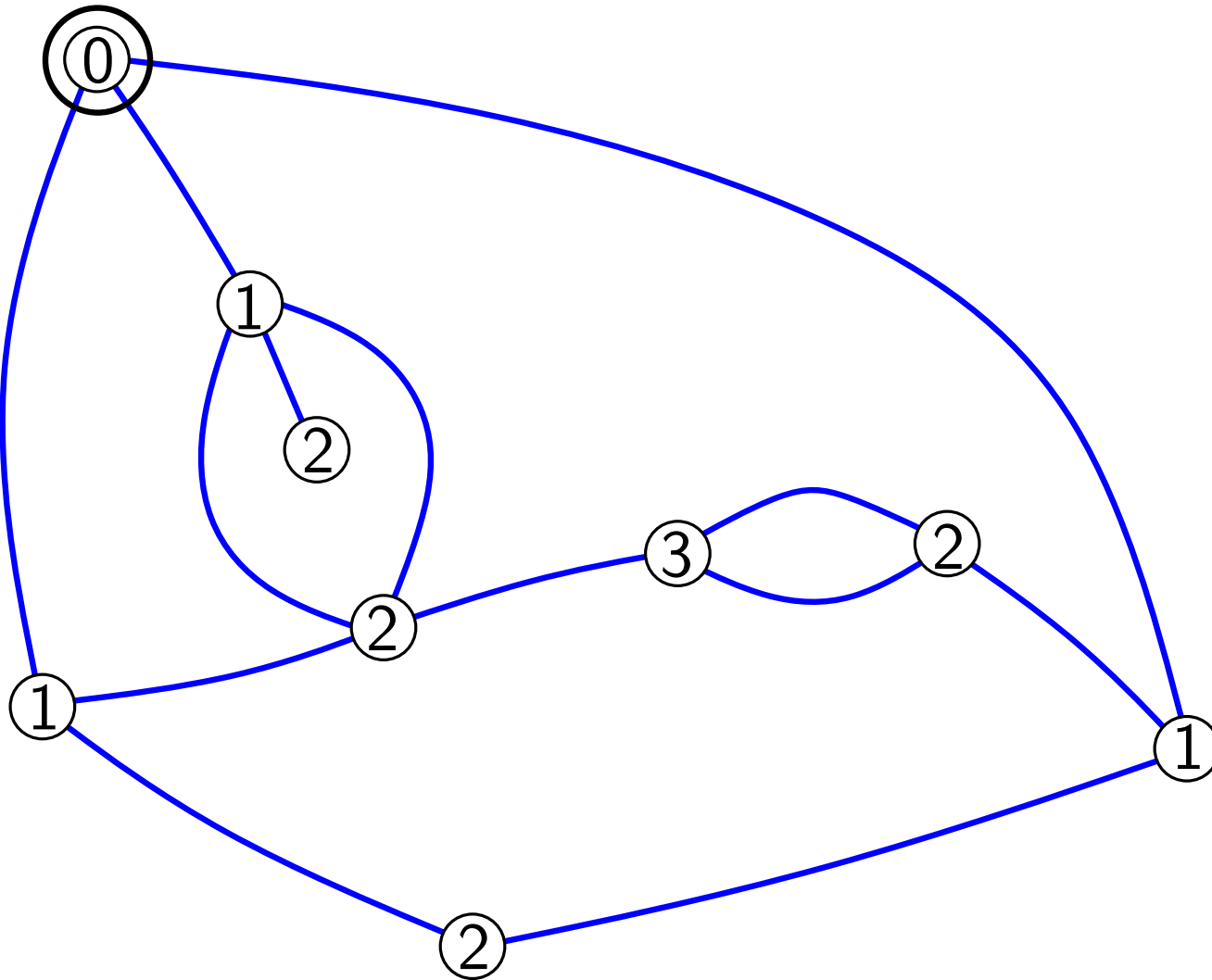
Pointed bipartite map \rightarrow labelled mobile. [Sc98] [BoDiGu04]



Label the vertices
by the distance
from pointed vertex

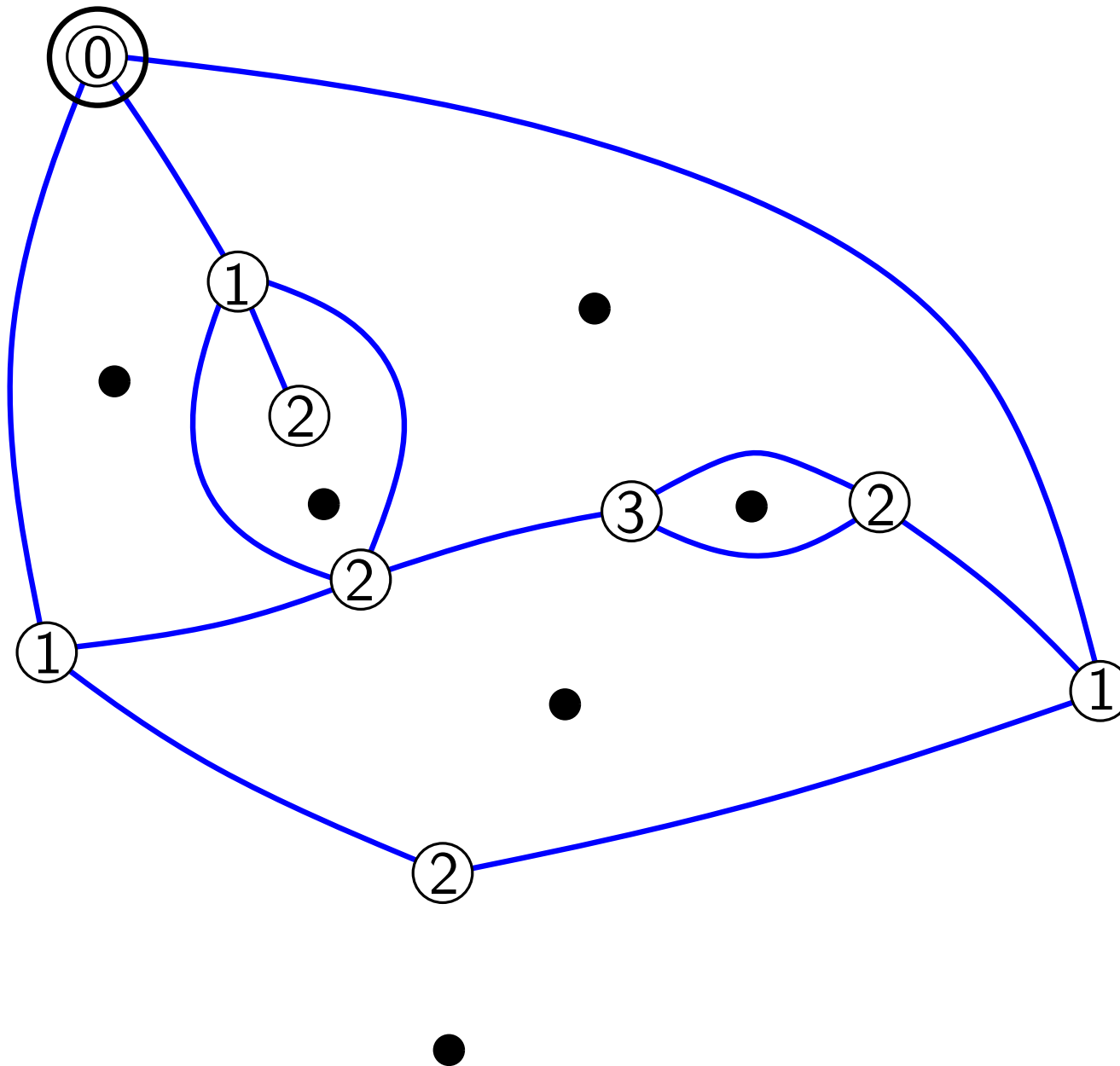
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Construct the
labelled mobile



Pointed bipartite map \rightarrow labelled mobile.

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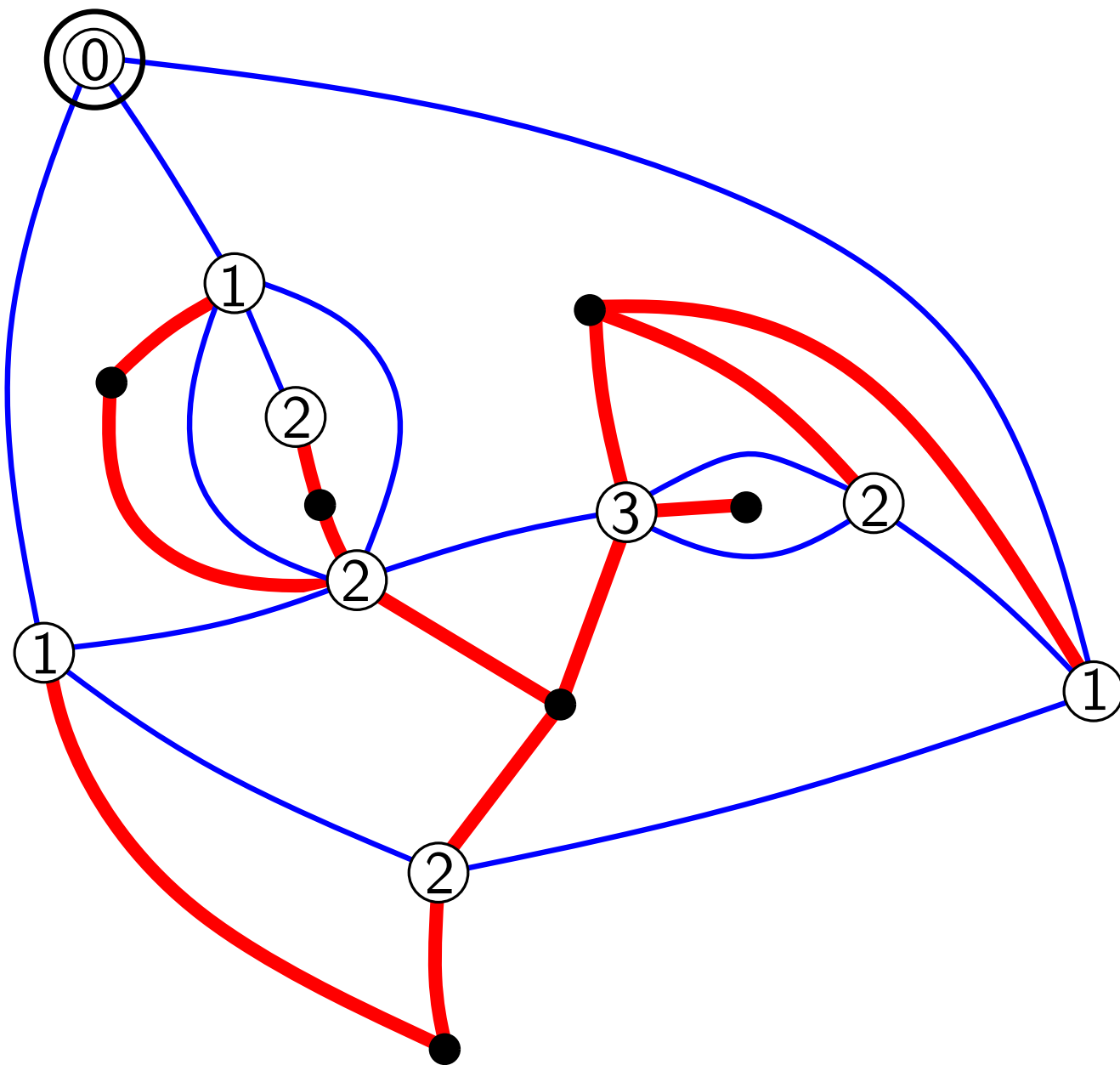


**Construct the
labelled mobile**

(i) put one black
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Pointed bipartite map \rightarrow labelled mobile.

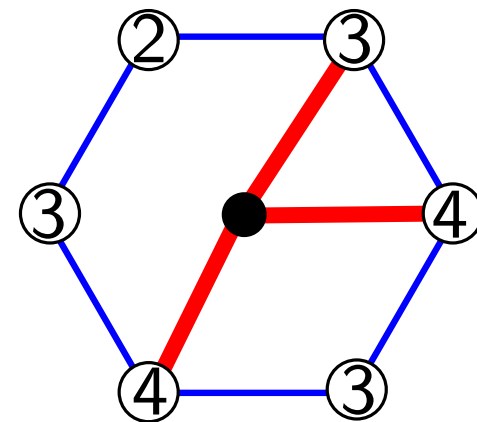
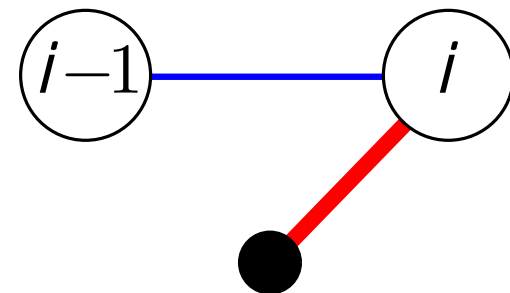
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Construct the labelled mobile

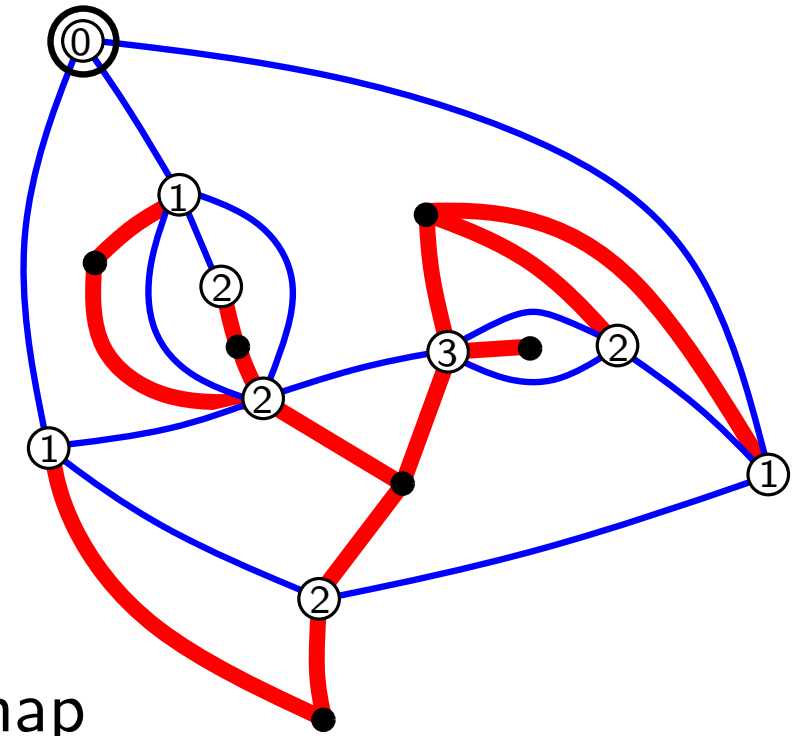
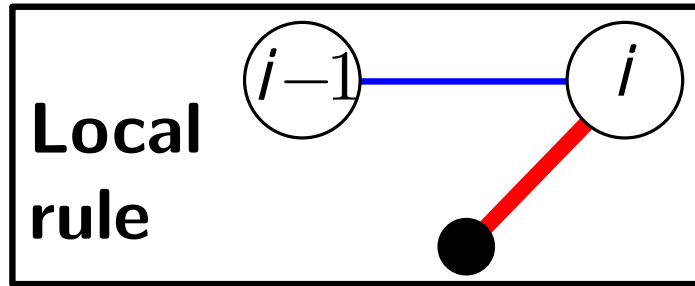
(i) put one black vertex in each face

(ii) each edge of the map gives one edge in the mobile



Pointed bipartite map \rightarrow labelled mobile.

[Sc98] [BoDiGu04]



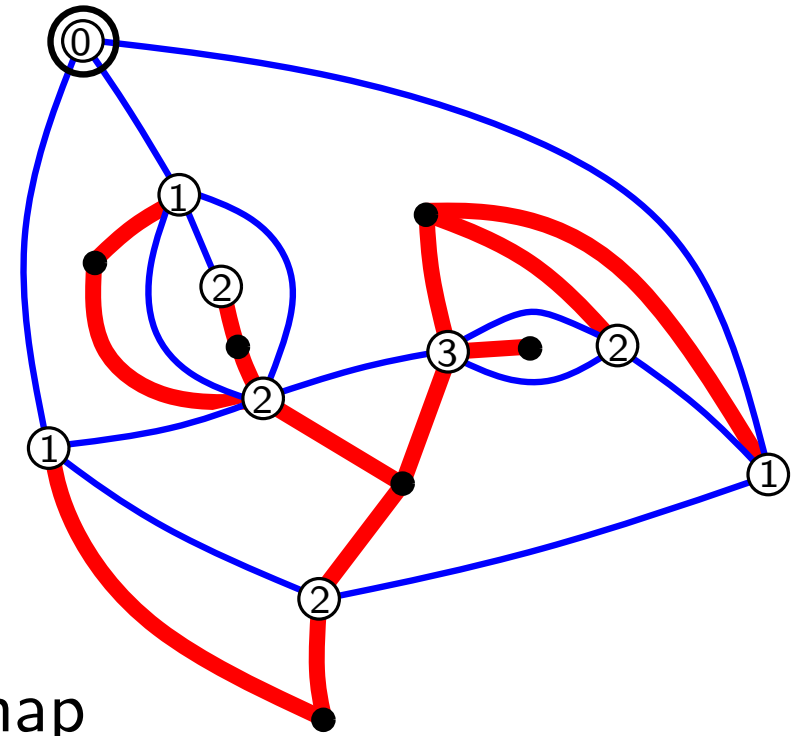
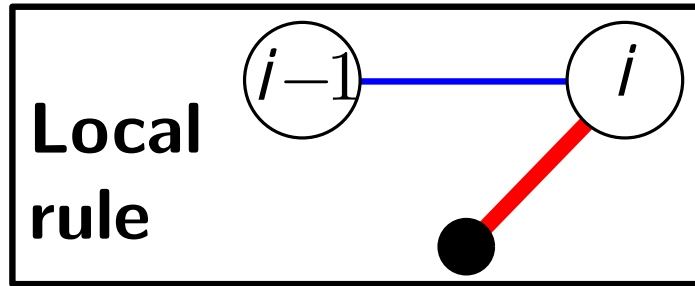
Proof that the mobile is a tree

Let $G = (V; E; F)$ be a pointed bipartite map

Let T be the associated mobile

Pointed bipartite map \rightarrow labelled mobile.

[Sc98] [BoDiGu04]



Proof that the mobile is a tree

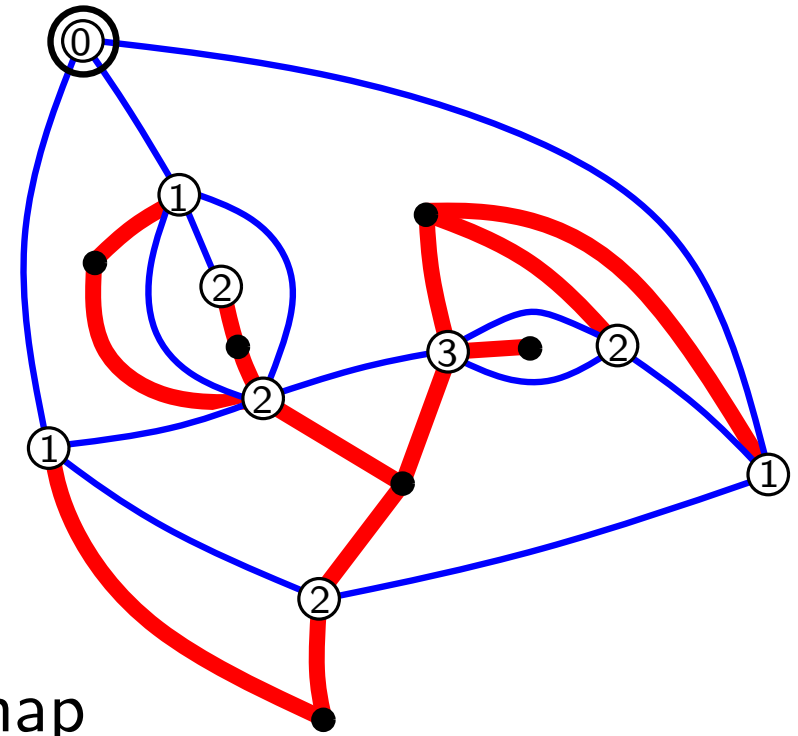
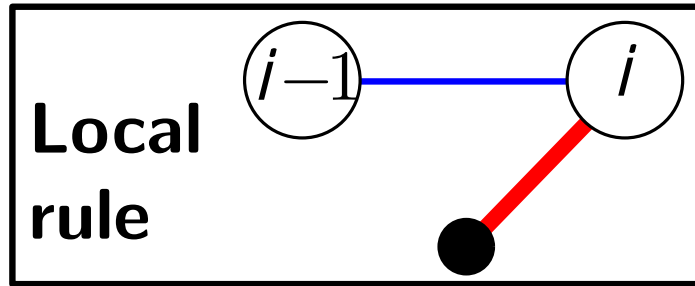
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T has $|E|$ edges, and has $|V| + |F| - 1 = |E| + 1$ vertices (Euler relation)

Pointed bipartite map \rightarrow labelled mobile.

[Sc98] [BoDiGu04]



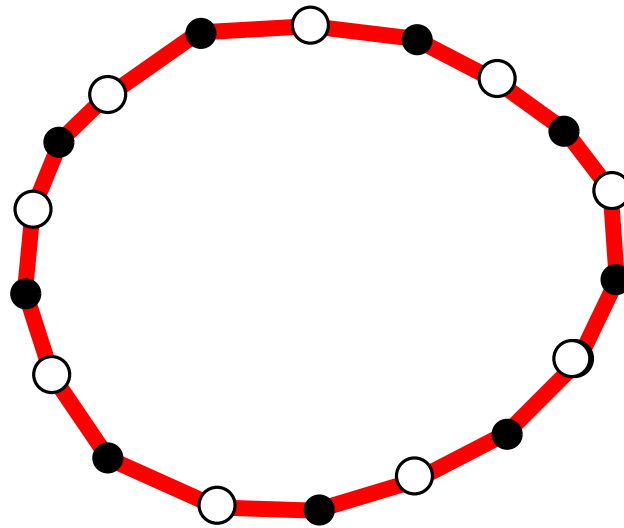
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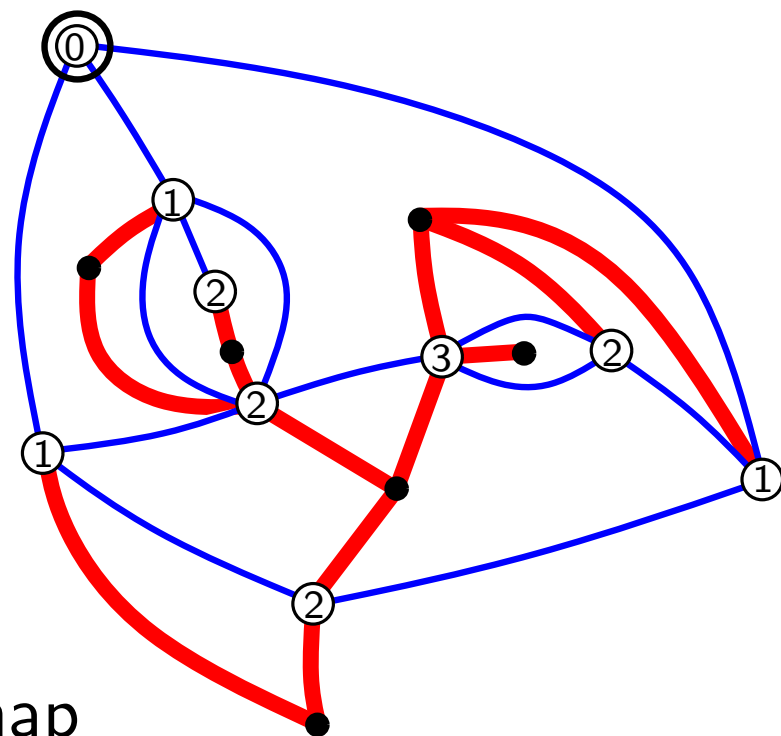
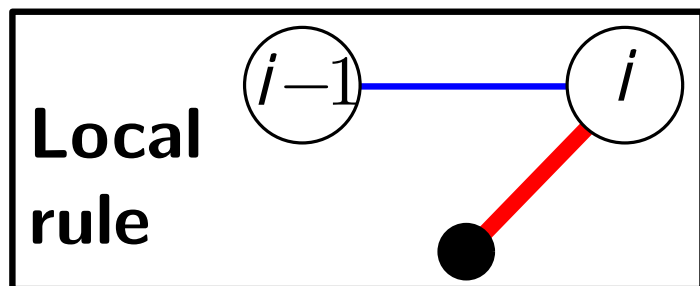
⊙



Assume that T has a cycle C

Pointed bipartite map \rightarrow labelled mobile.

[Sc98] [BoDiGu04]

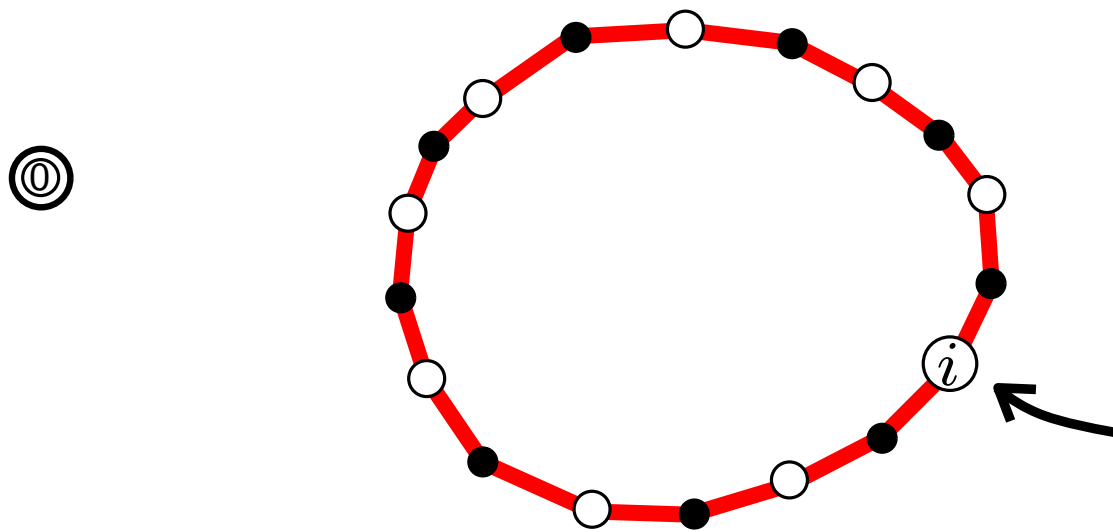


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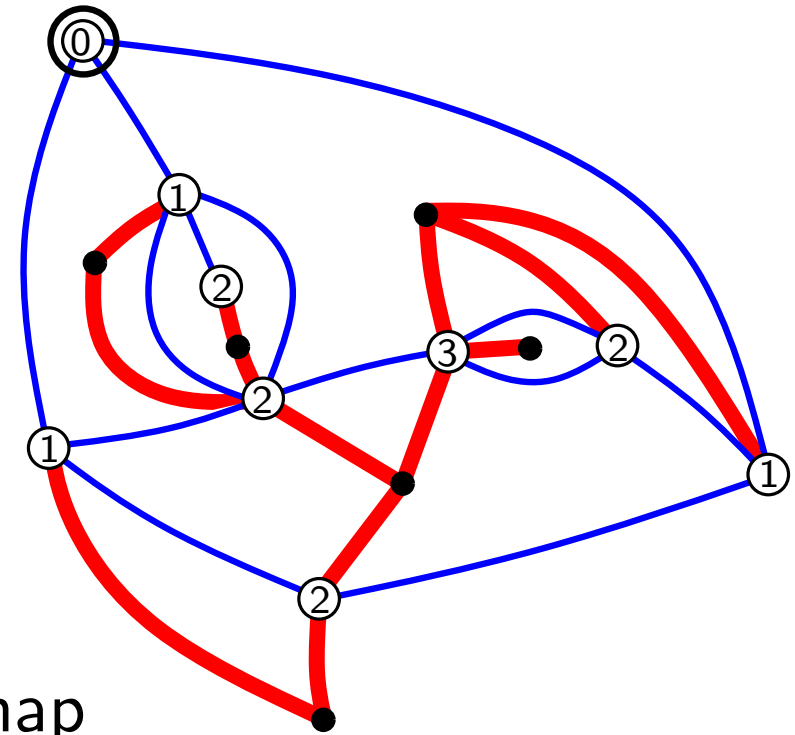
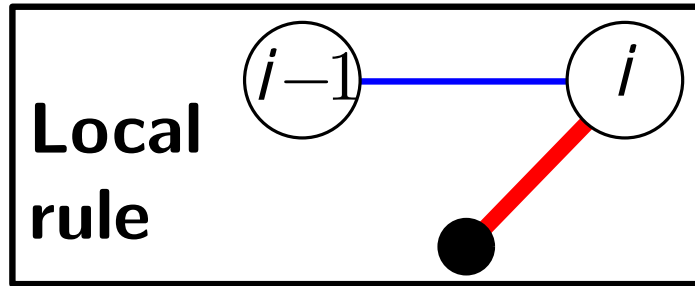


Assume that T has a cycle C

smallest label on C

Pointed bipartite map \rightarrow labelled mobile.

[Sc98] [BoDiGu04]

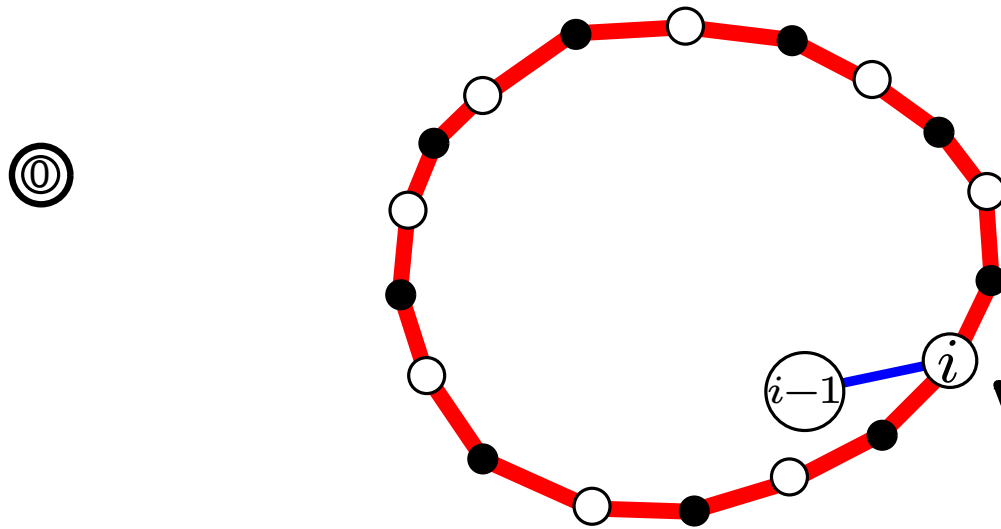


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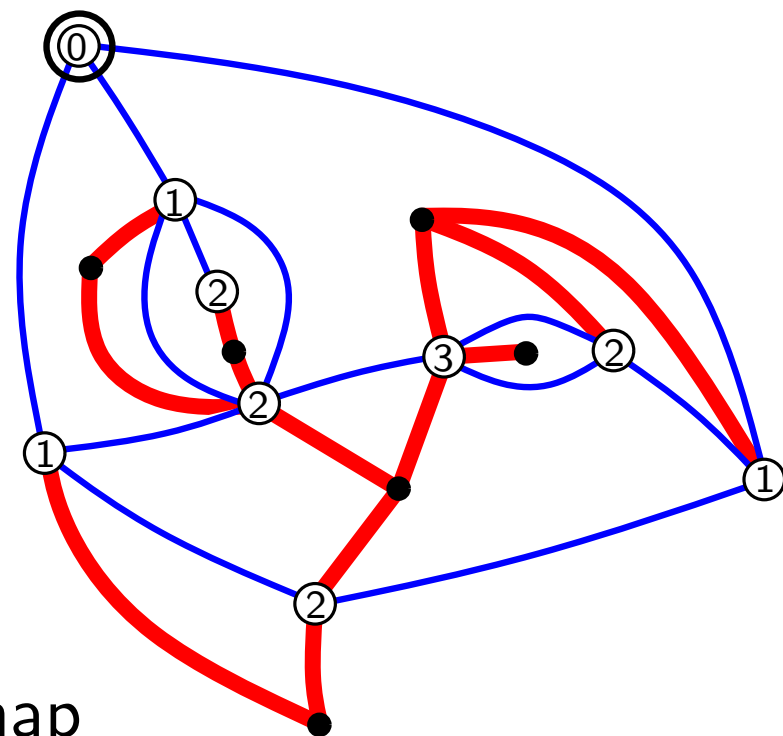
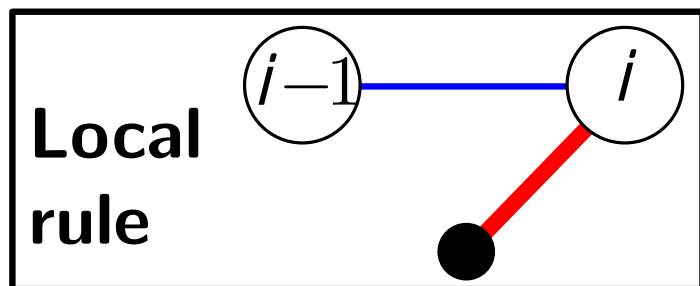


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Pointed bipartite map \rightarrow labelled mobile.

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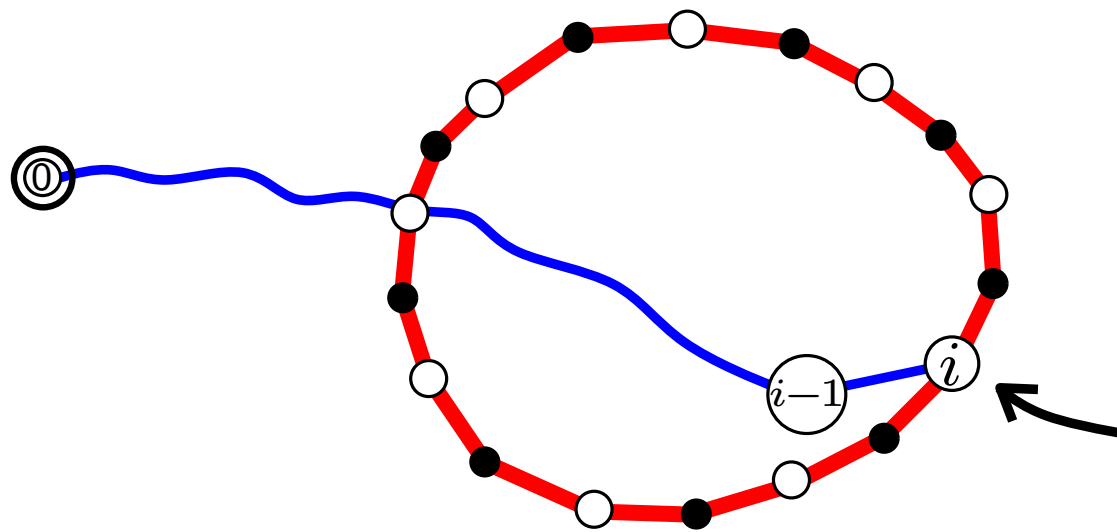


Proof that the mobile is a tree

Let $G = (V; E; F)$ be a pointed bipartite map

Let T be the associated mobile

T has $|E|$ edges, and has $|V| + |F| - 1 = |E| + 1$ vertices (Euler relation)

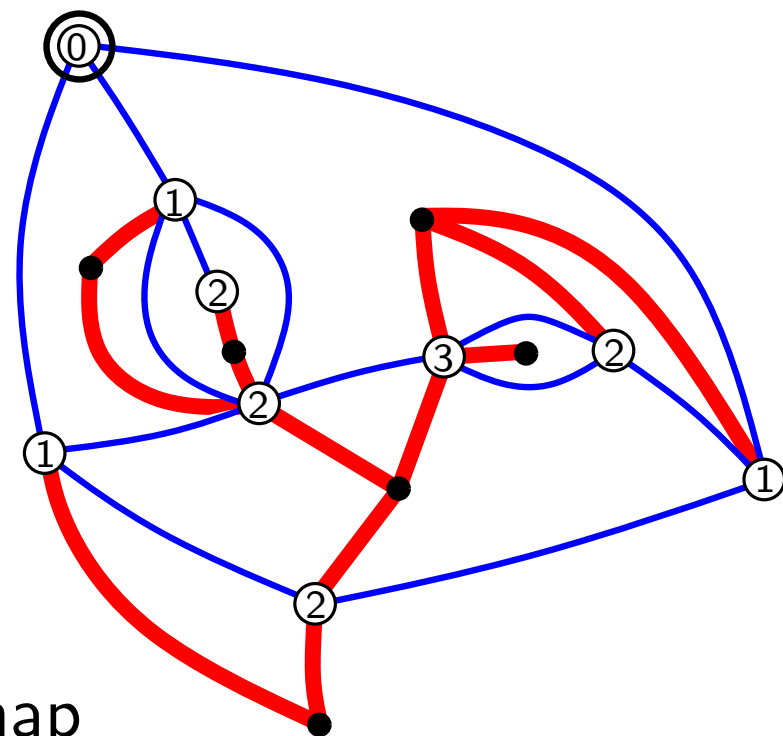
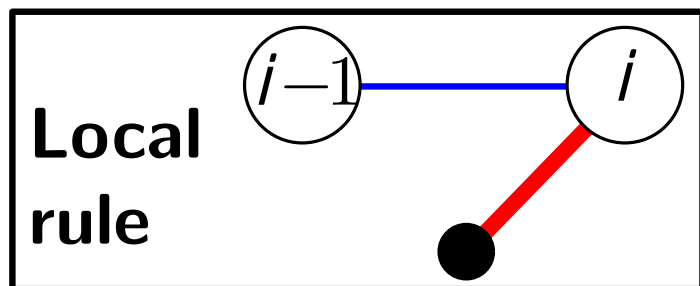


Assume that T has a cycle C

smallest label on C

Pointed bipartite map \rightarrow labelled mobile.

[Sc98] [BoDiGu04]

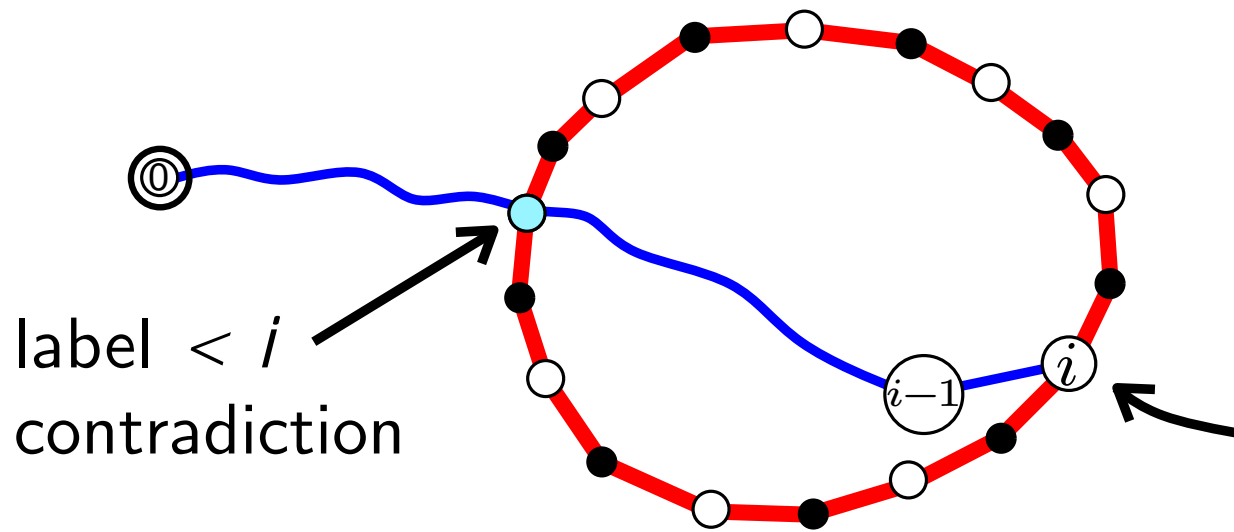


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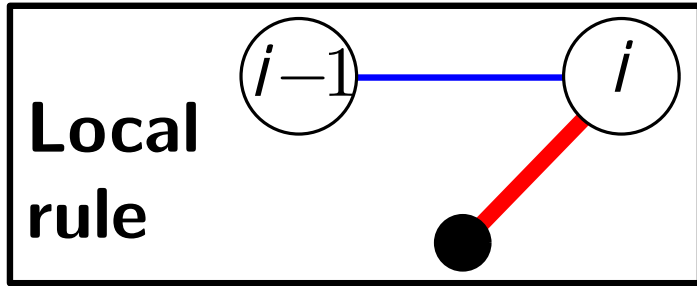
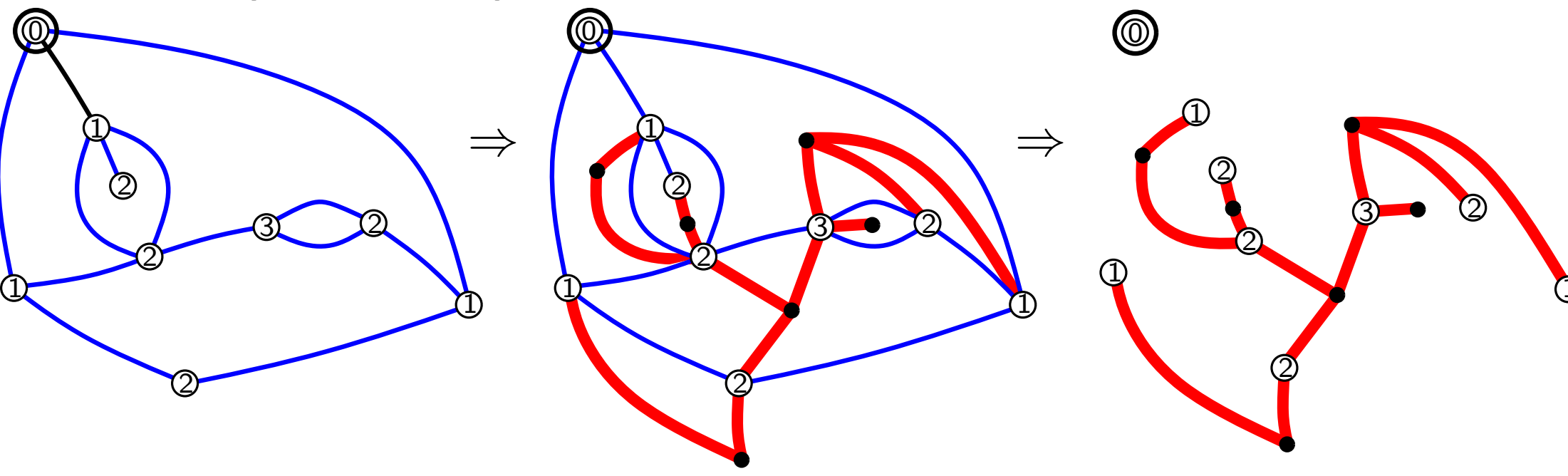
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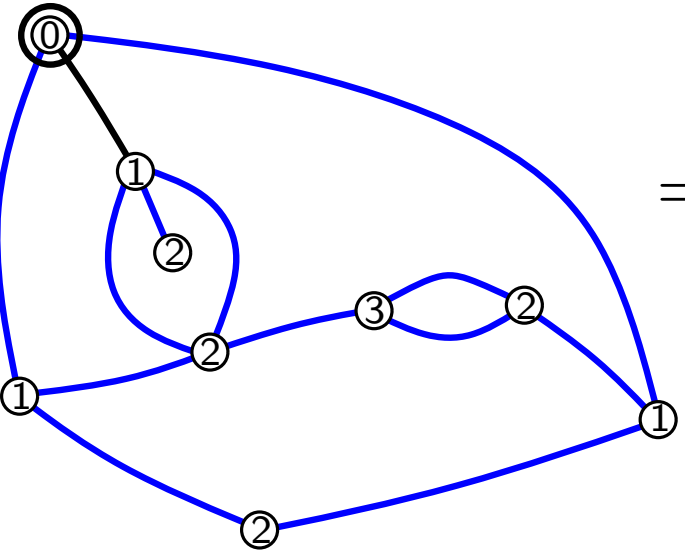
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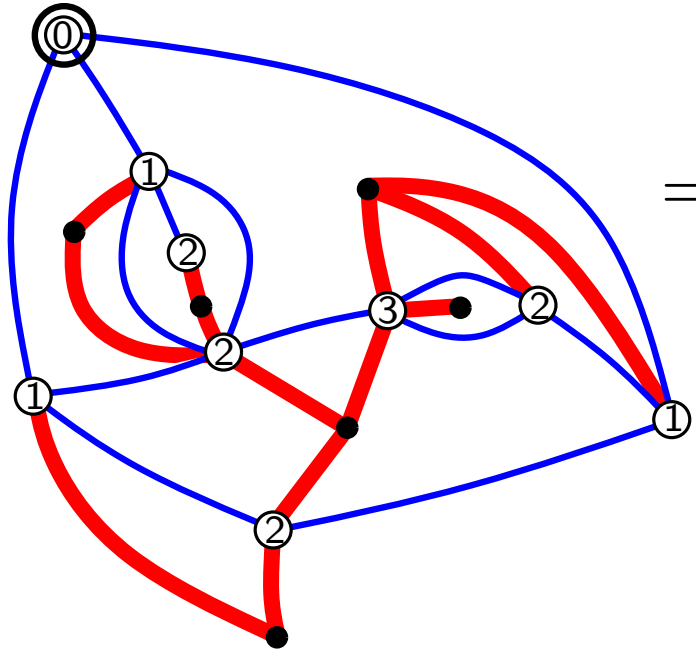
Pointed bipartite map \rightarrow labelled mobile. [Sc98] [BoDiGu04]



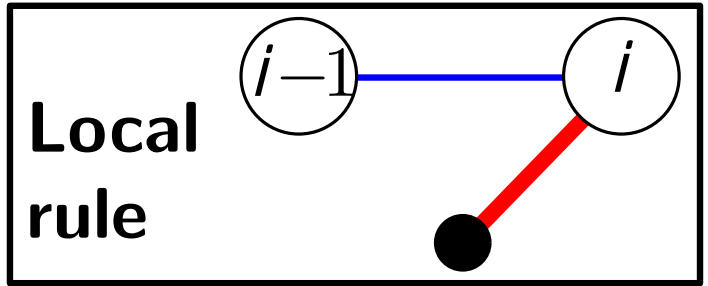
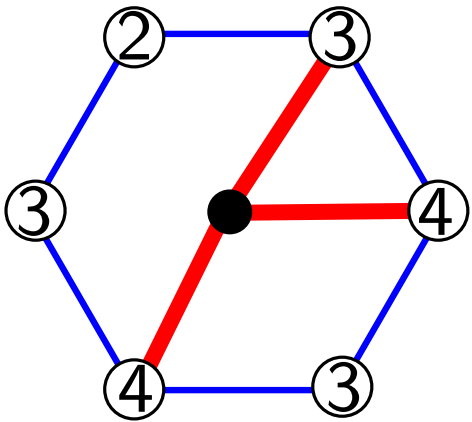
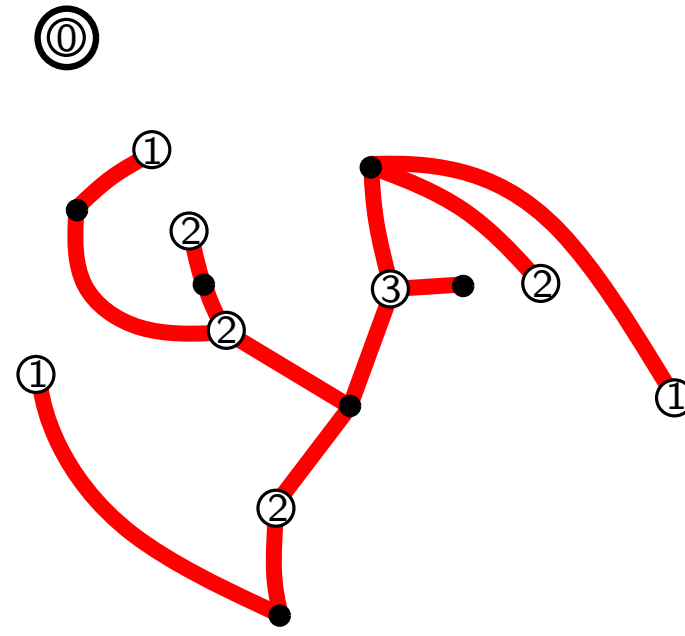
Pointed bipartite map \rightarrow labelled mobile. [Sc98] [BoDiGu04]



\Rightarrow



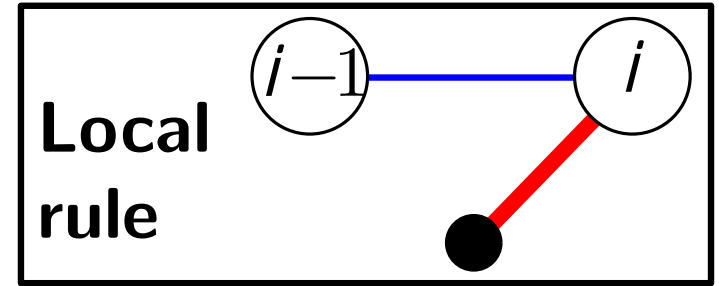
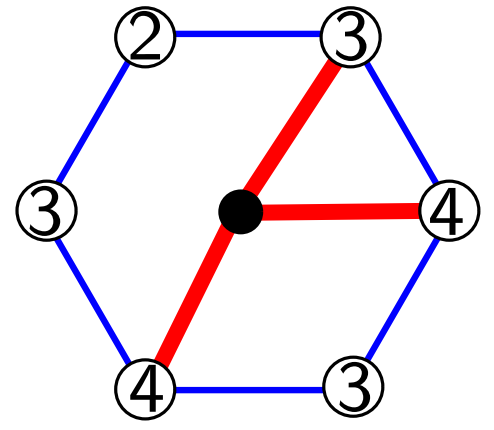
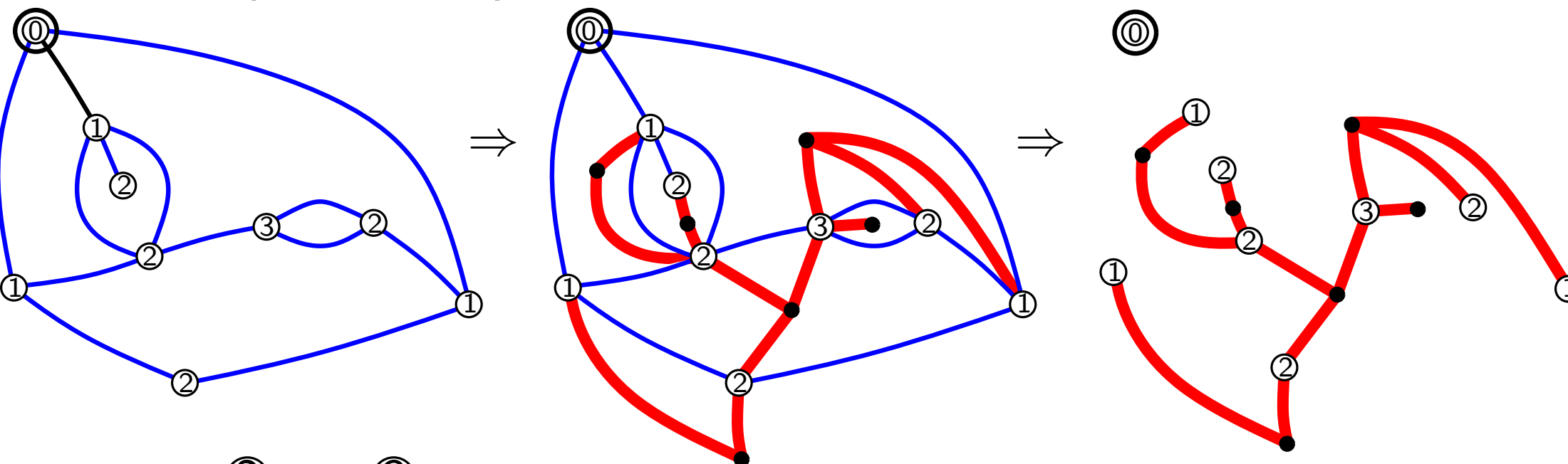
\Rightarrow



Conditions:

- (i) \exists vertex label 1
- (ii) $j \leq i+1$

Pointed bipartite map \rightarrow labelled mobile. [Sc98] [BoDiGu04]

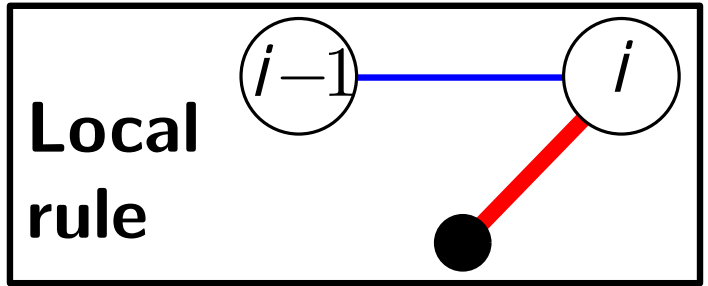
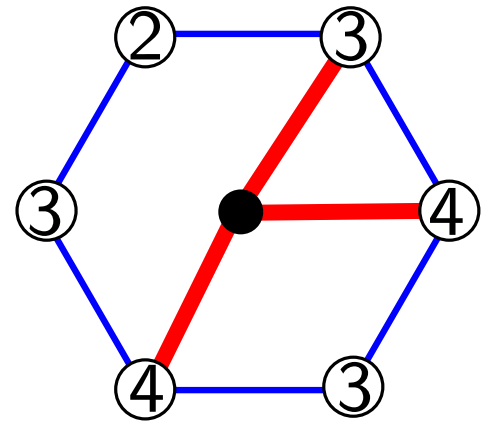
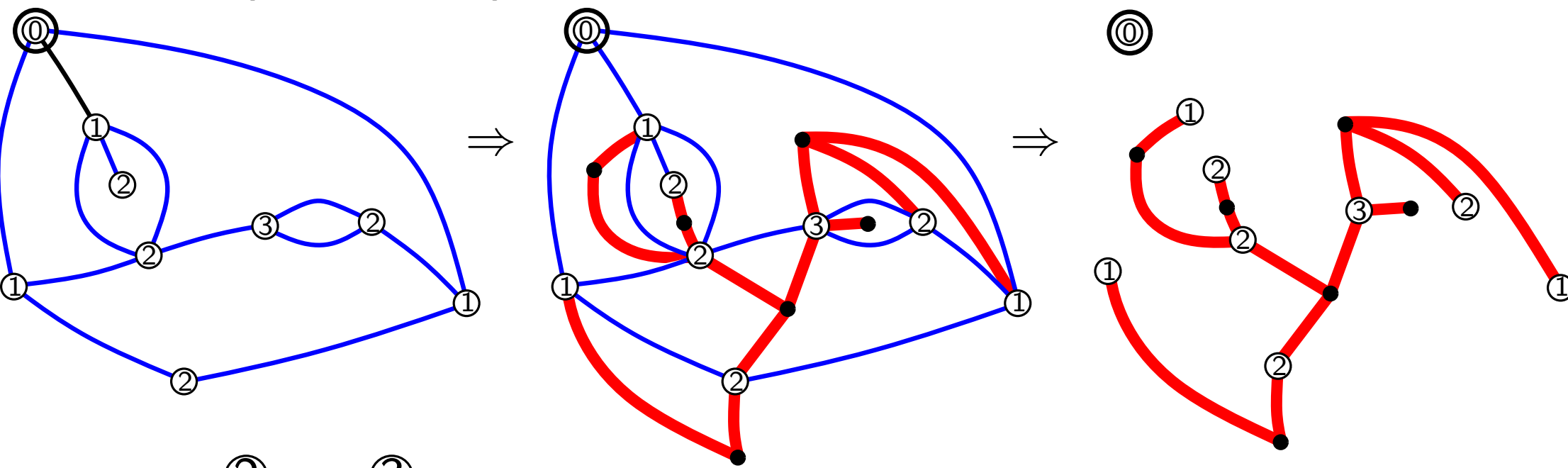


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Theorem: The mapping is a **bijection**. Each **face of degree $2i$** of the bipartite map corresponds to a **black vertex of degree i** in the mobile

Pointed bipartite map \rightarrow labelled mobile. [Sc98] [BoDiGu04]



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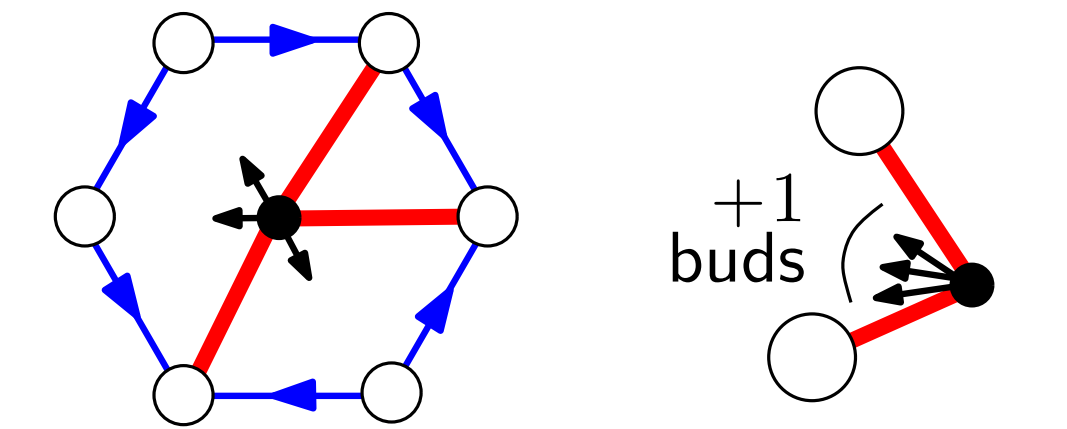
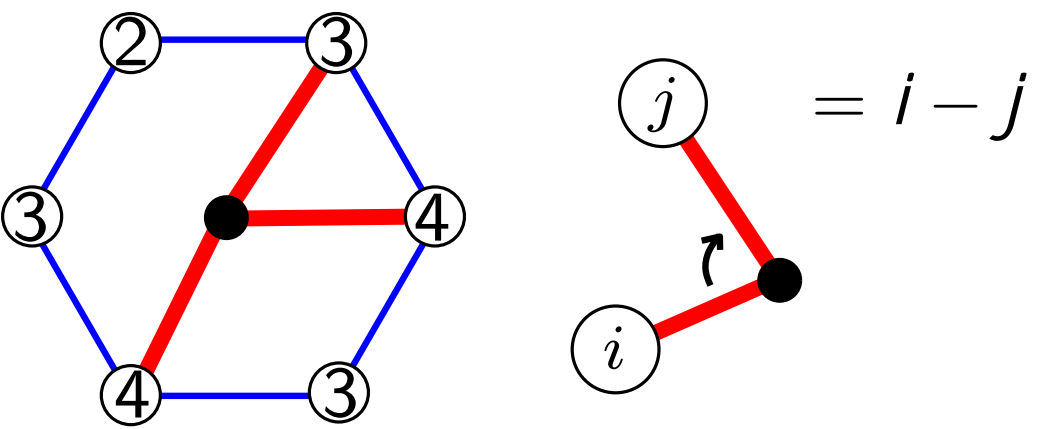
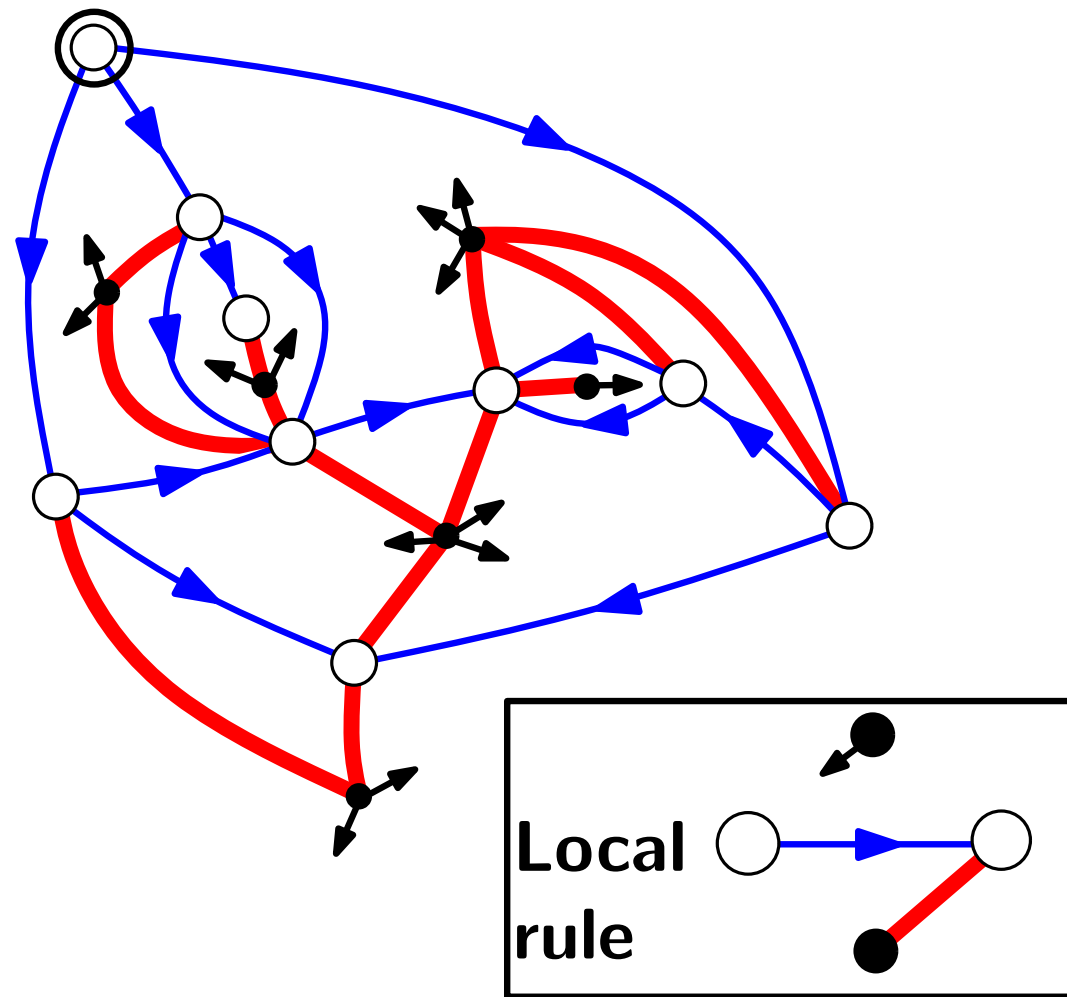
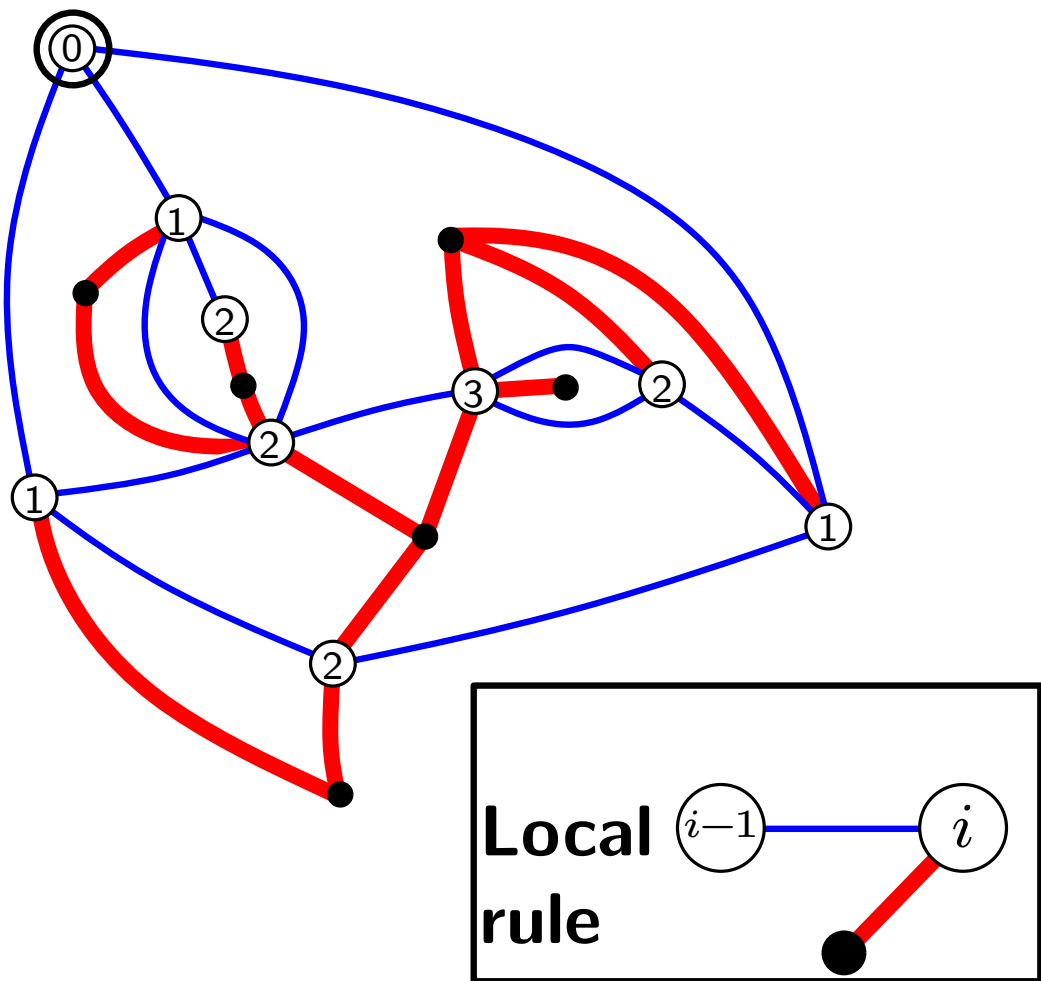
\Rightarrow # rooted bipartite maps with n_i faces of degree $2i$ is

$$\frac{2 \cdot (\sum i n_i)!}{(2 + \sum (i-1)n_i)!} \prod_i \frac{1}{n_i!} \binom{2i-1}{i}^{n_i}$$

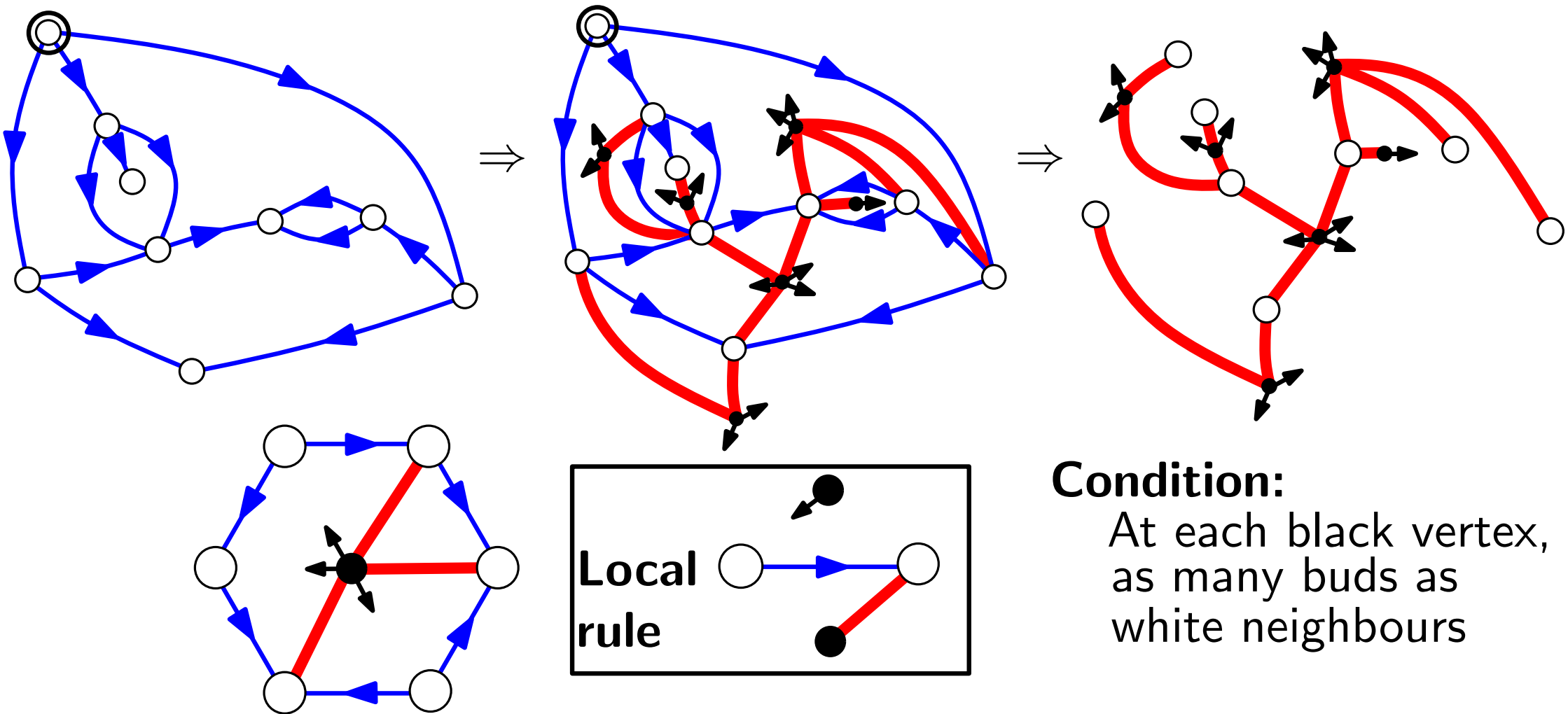
Reformulation with orientations.

Distance labelling

Geodesic orientation

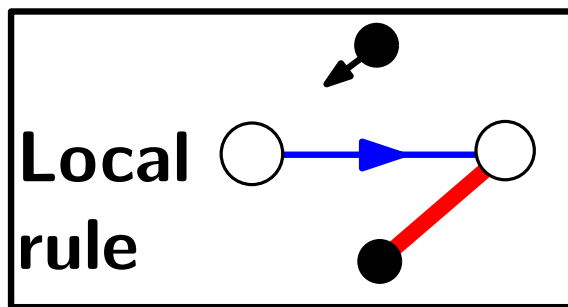
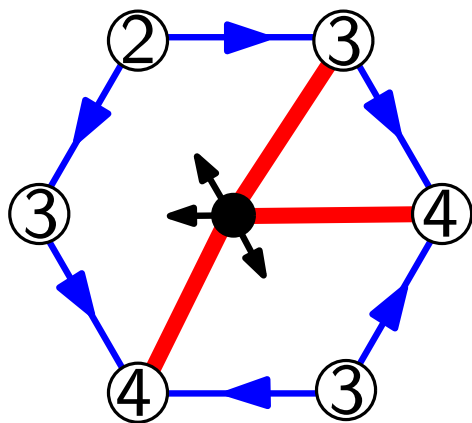
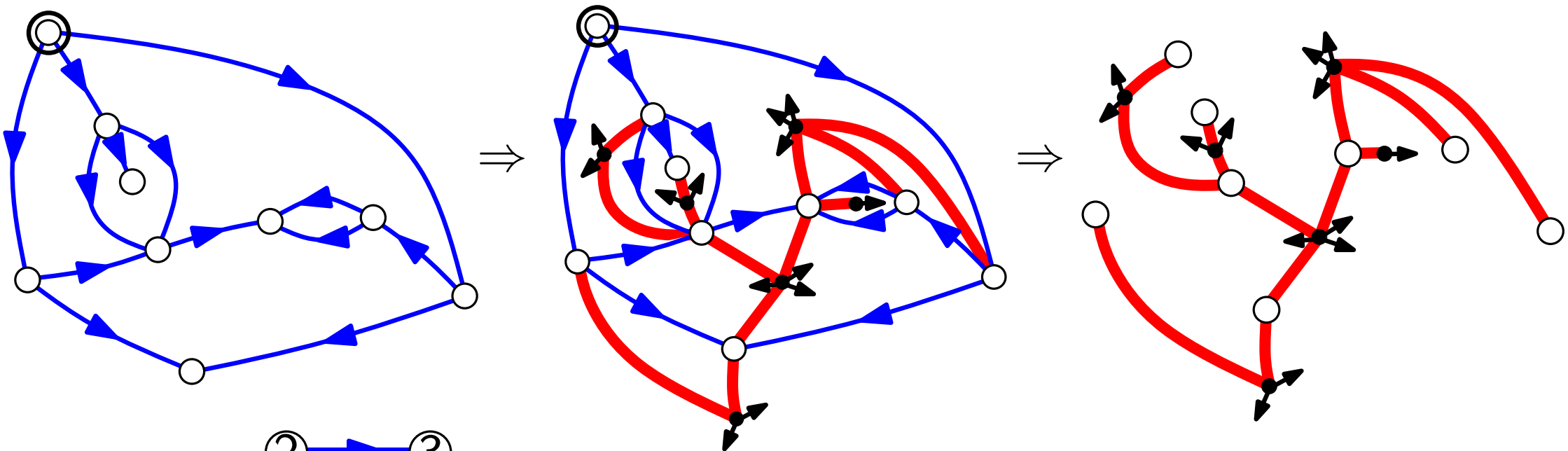


Reformulation with orientations.



Theorem: The mapping is a **bijection**. Each **face of degree $2i$** of the bipartite map corresponds to a **black vertex of degree $2i$** in the mobile

Reformulation with orientations.



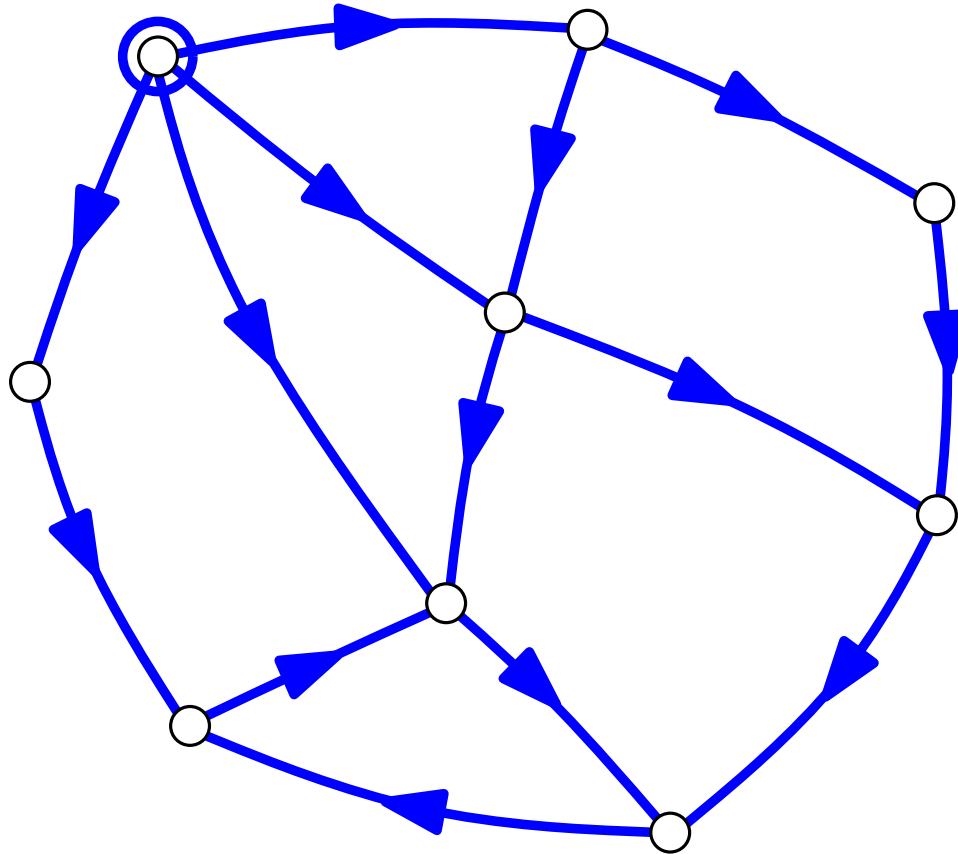
Condition:

At each black vertex, as many buds as white neighbours

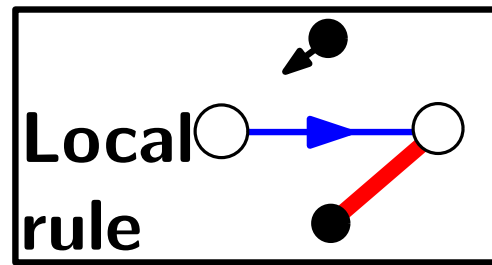
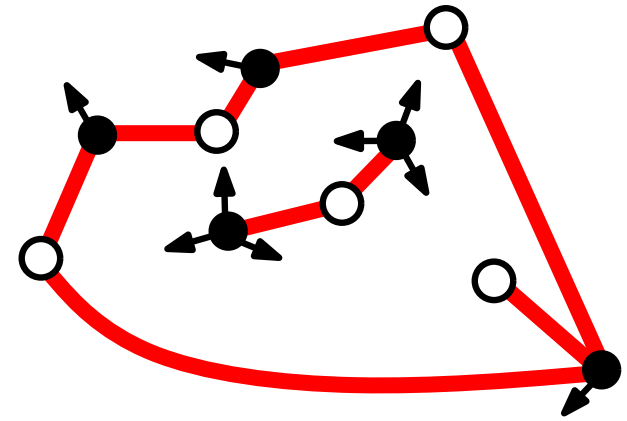
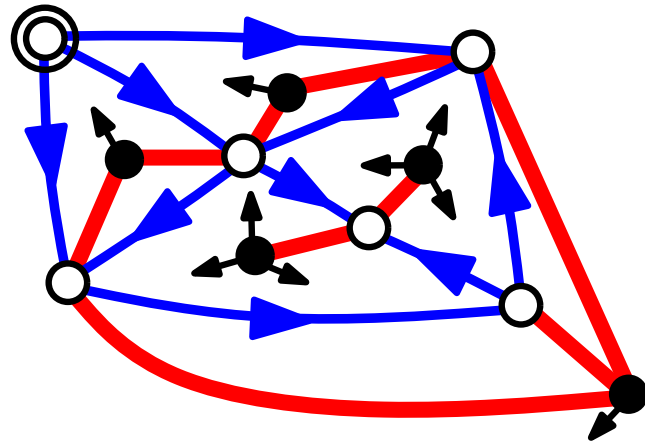
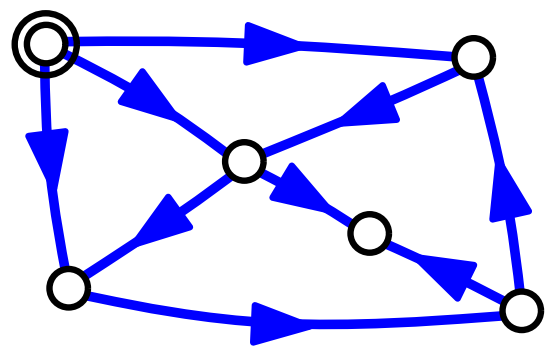
Theorem: The mapping is a **bijection**. Each **face of degree $2i$** of the bipartite map corresponds to a **black vertex of degree $2i$** in the mobile

Source-orientations

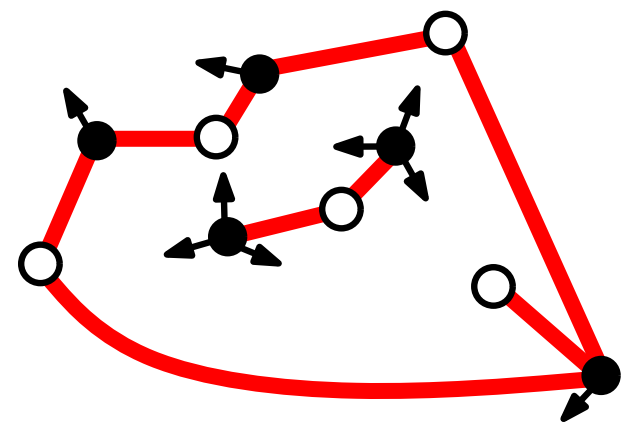
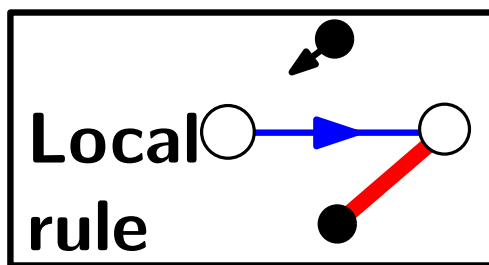
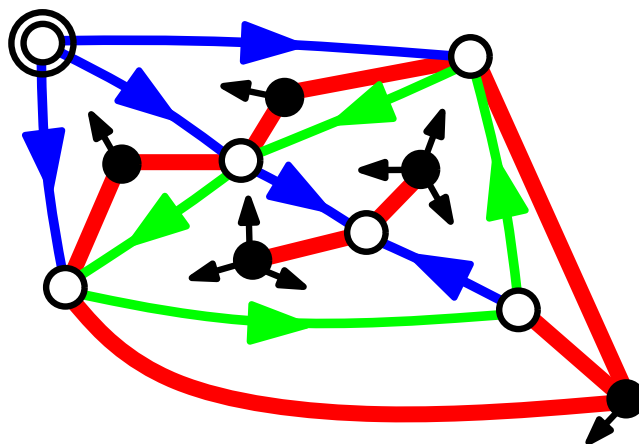
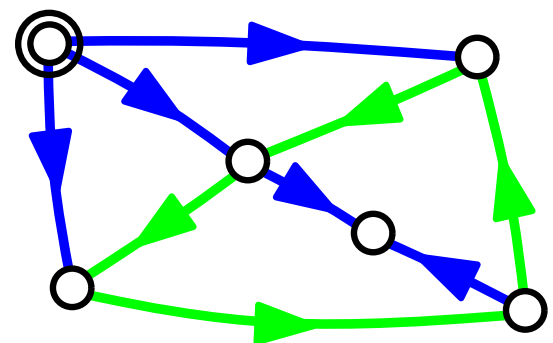
- A **source-orientation** is an orientation of a pointed map such that
- The pointed vertex (called the source) has **only outgoing edges**
 - **Accessibility**: Each vertex can be reached from the source



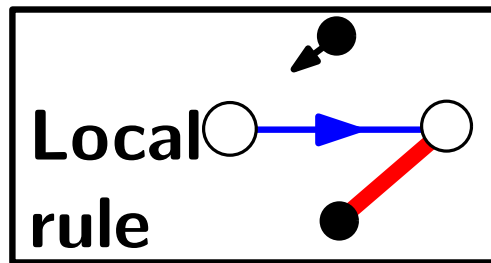
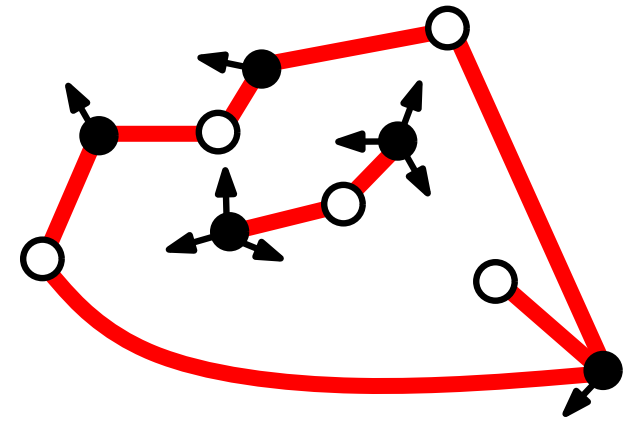
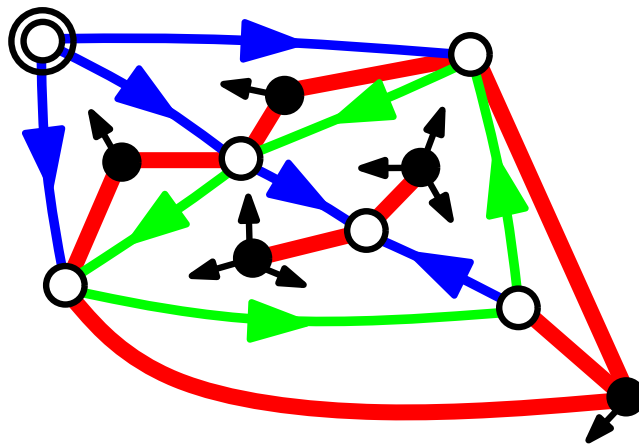
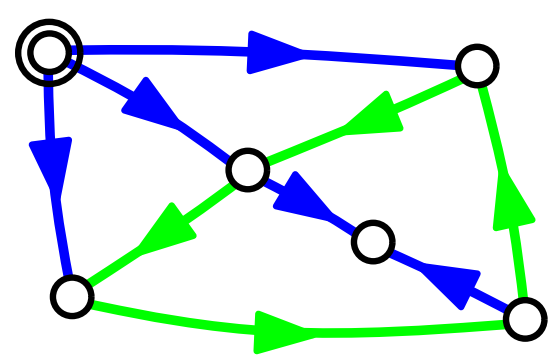
Mobile construction for source-orientations



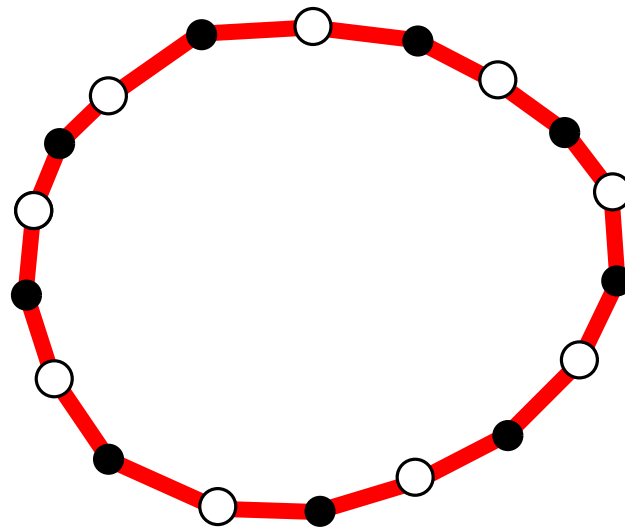
Mobile construction for source-orientations



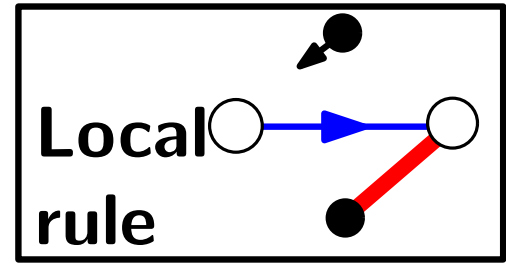
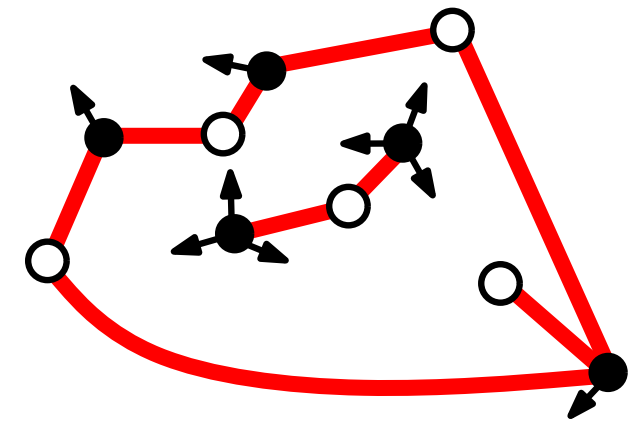
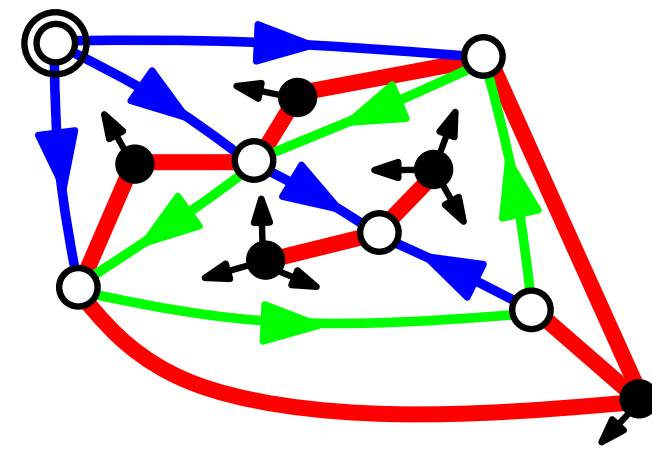
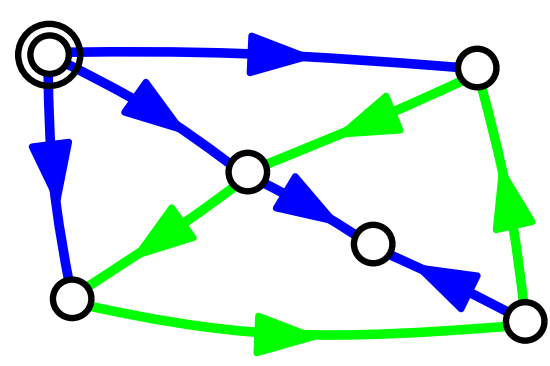
Mobile construction for source-orientations



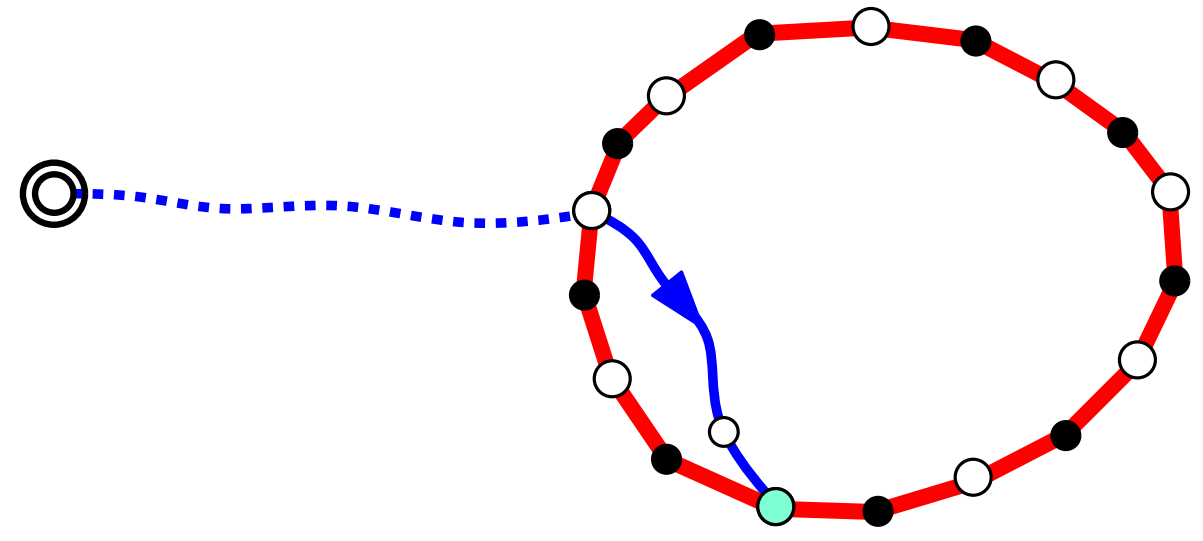
Cycle in mobile \Rightarrow ccw circuit in the source-orientation



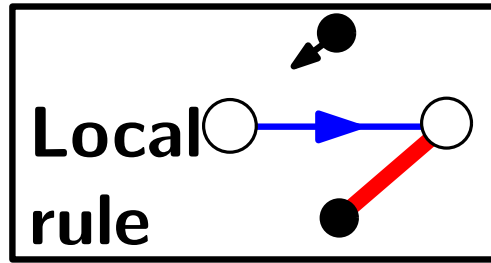
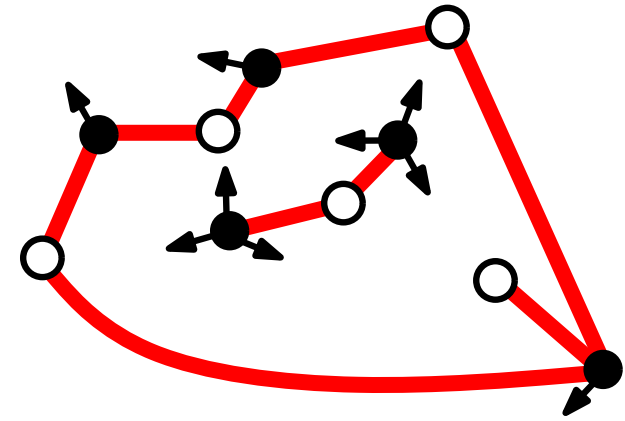
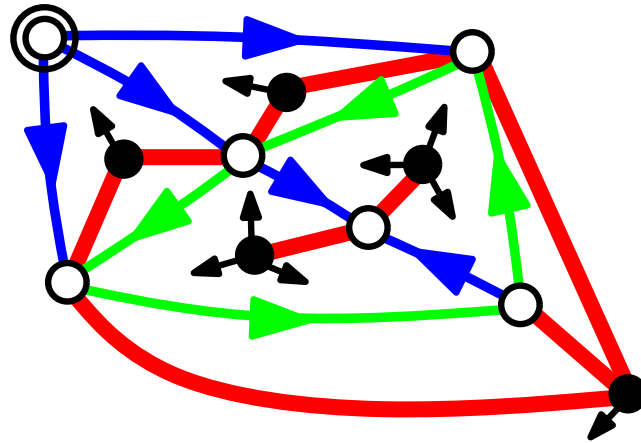
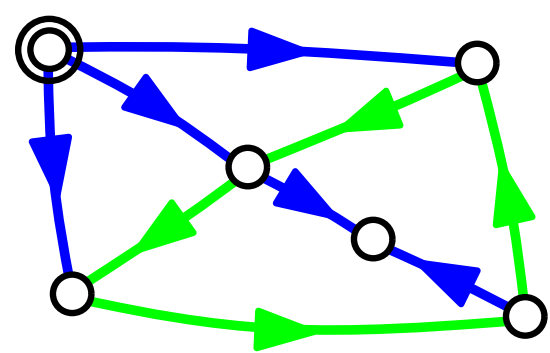
Mobile construction for source-orientations



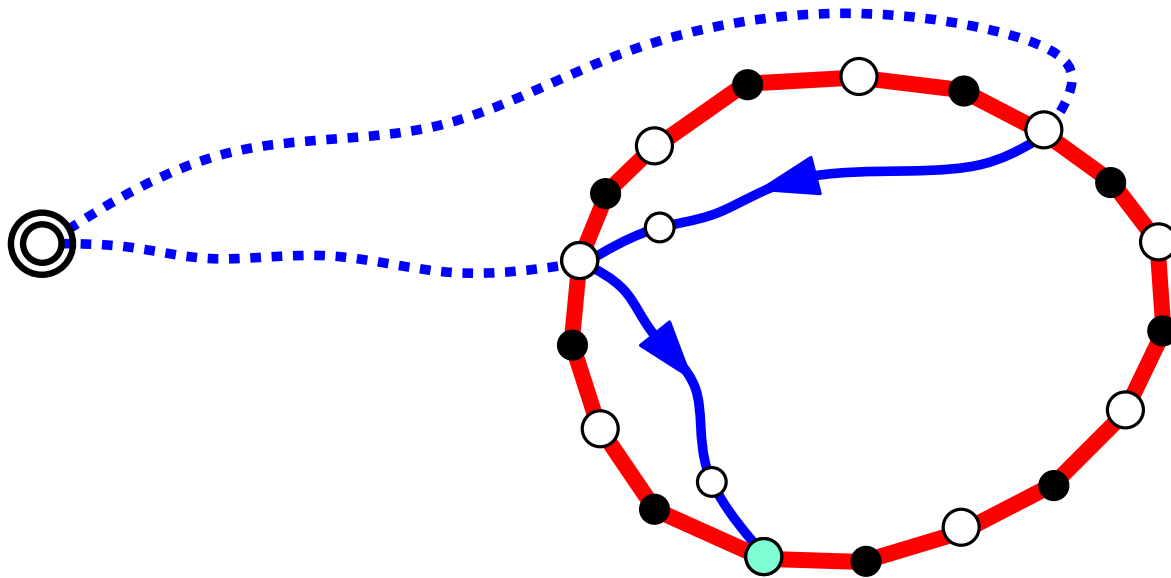
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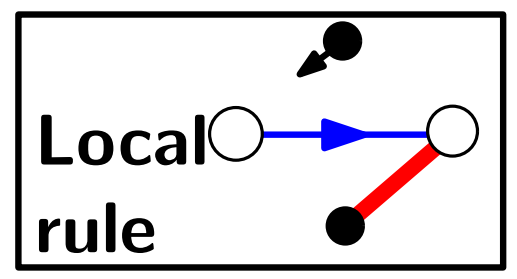
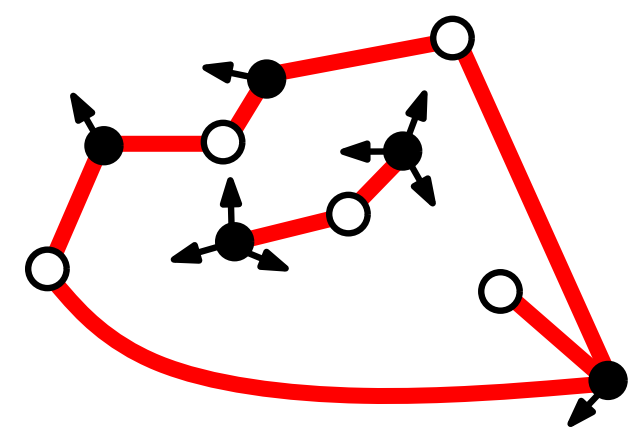
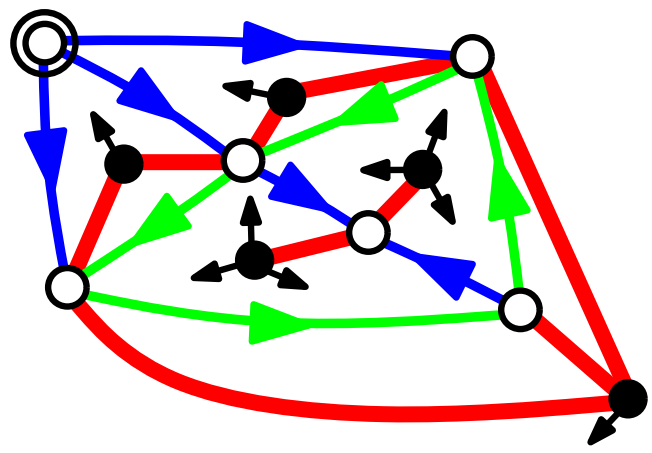
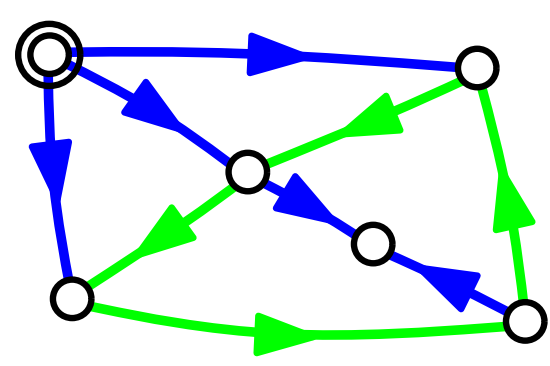
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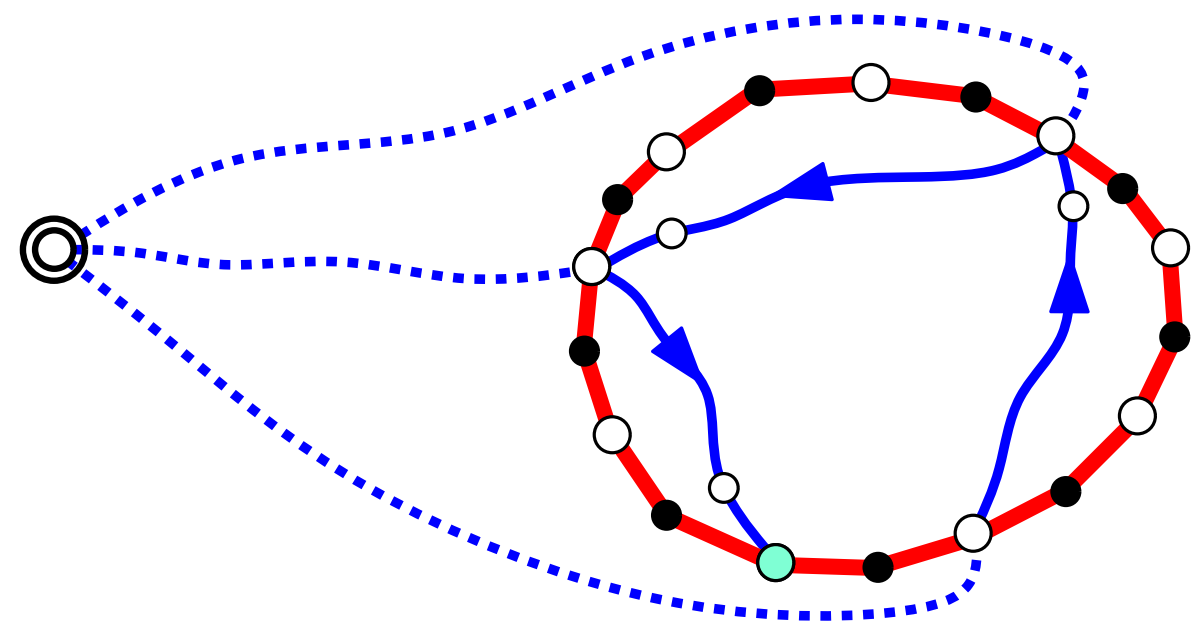
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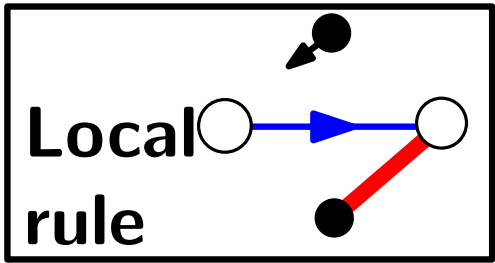
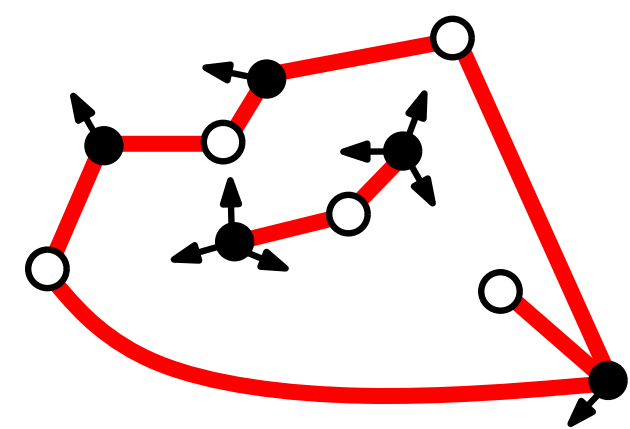
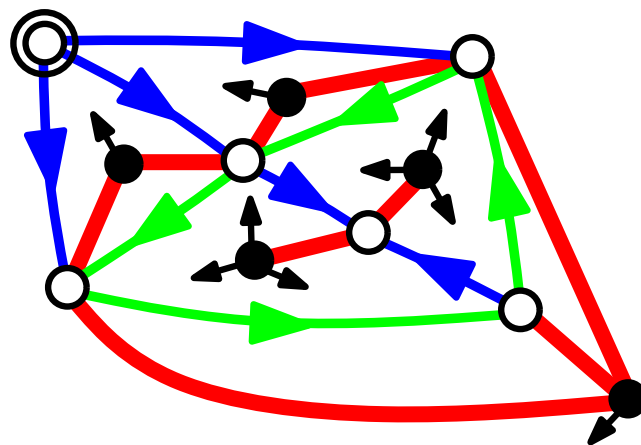
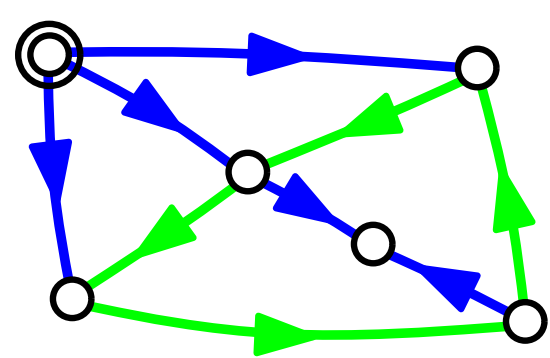
Mobile construction for source-orientations



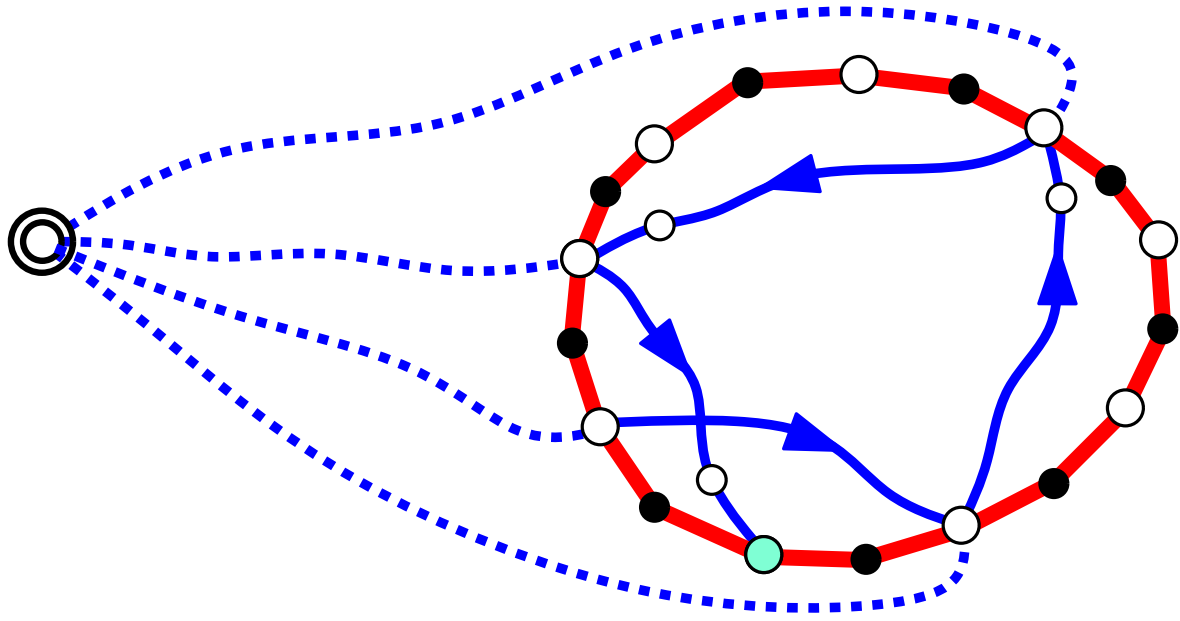
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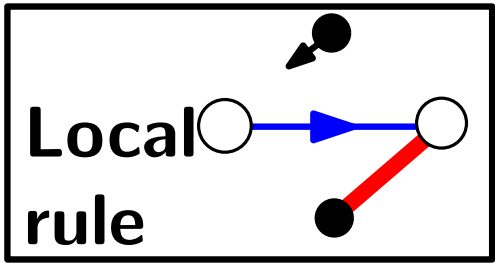
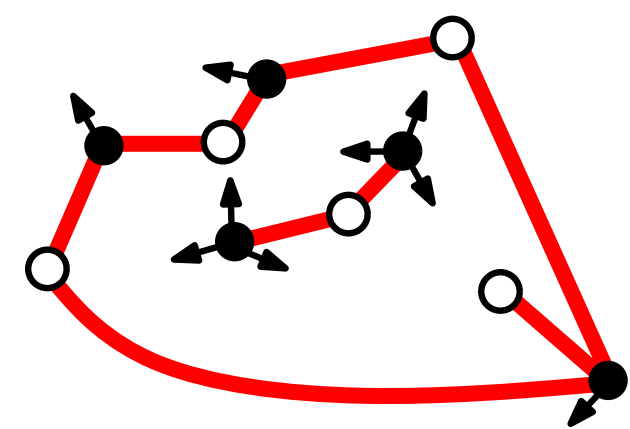
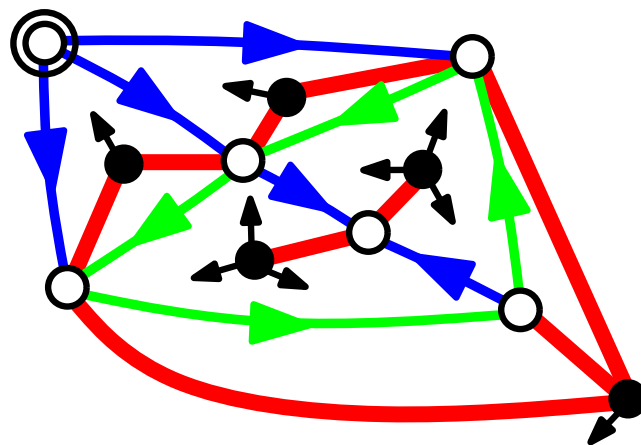
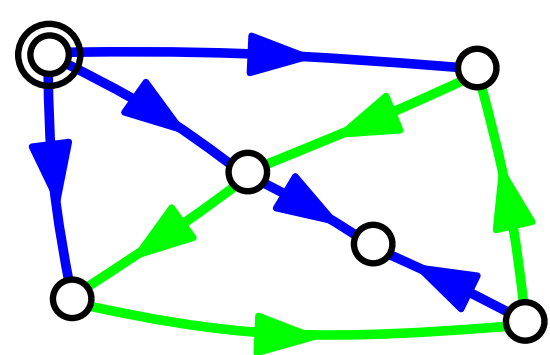
Mobile construction for source-orientations



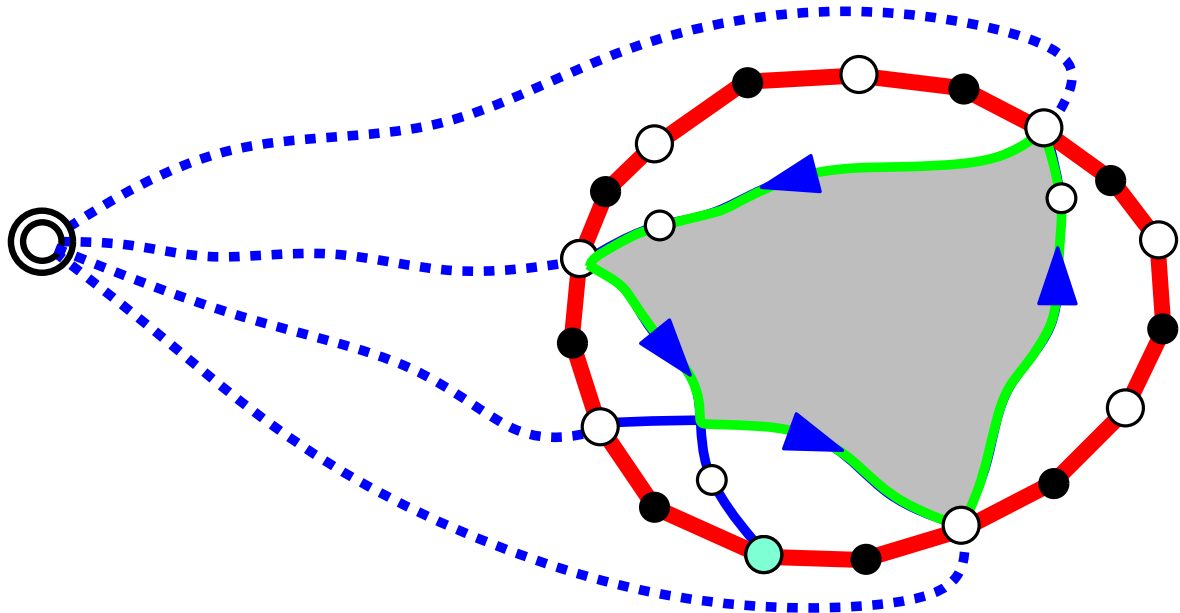
Cycle in mobile \Rightarrow ccw circuit in the source-orientation



Mobile construction for source-orientations

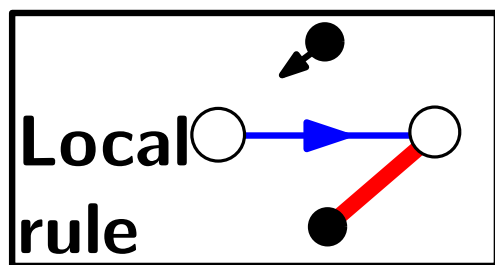
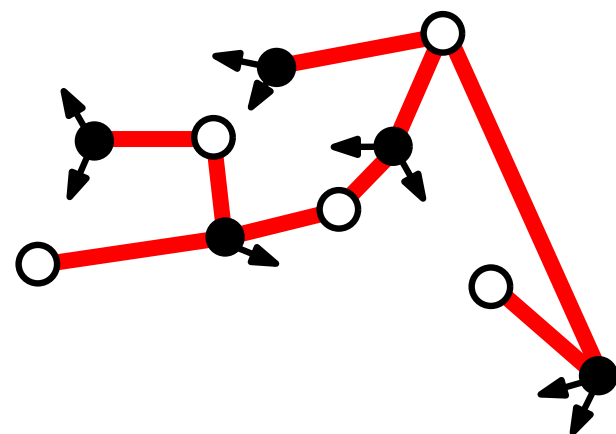
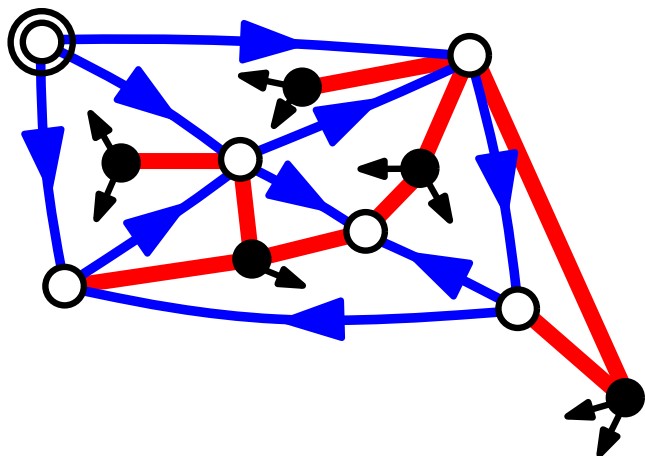
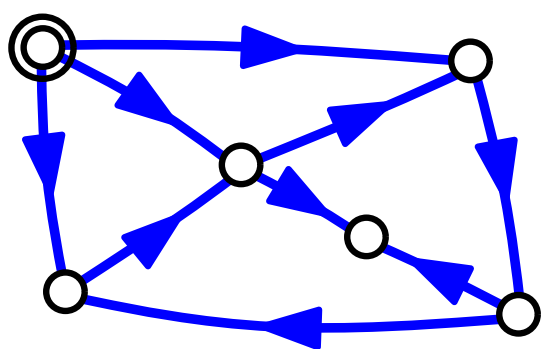


Cycle in mobile \Rightarrow ccw circuit in the source-orientation

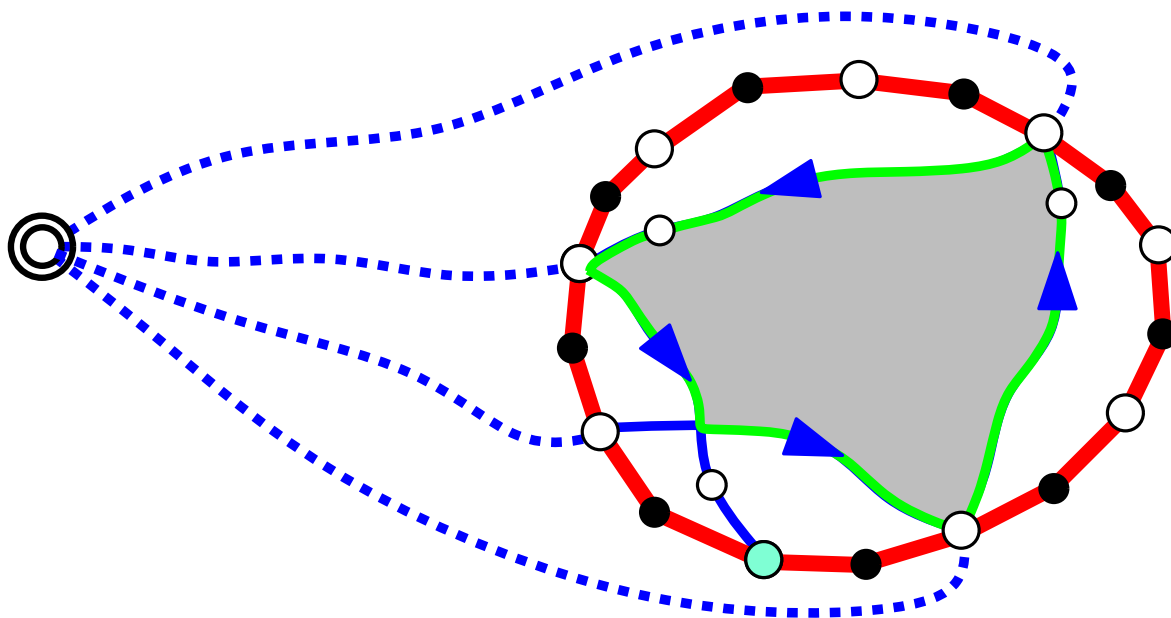


Prisoner cycle lemma

Mobile construction for source-orientations



Cycle in mobile \Rightarrow ccw circuit in the source-orientation

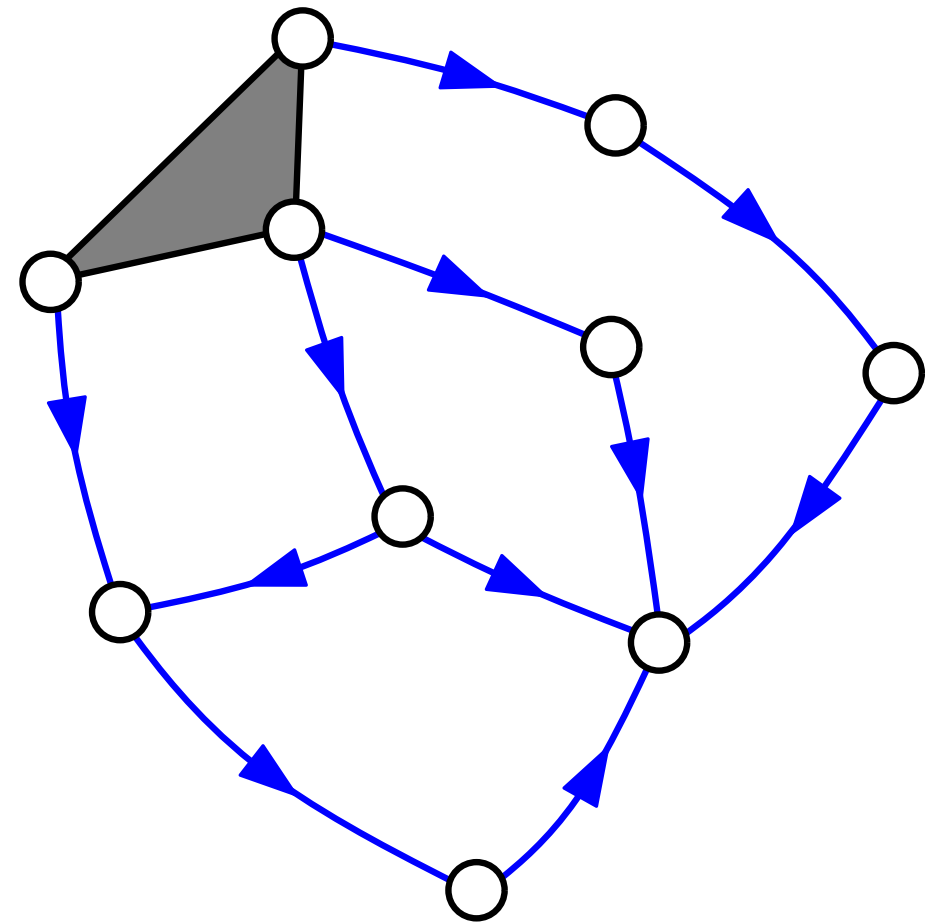


Prisoner
cycle lemma

d-gonal source-orientations

We allow the source of the orientation to be a d -gon, with $d \geq 0$

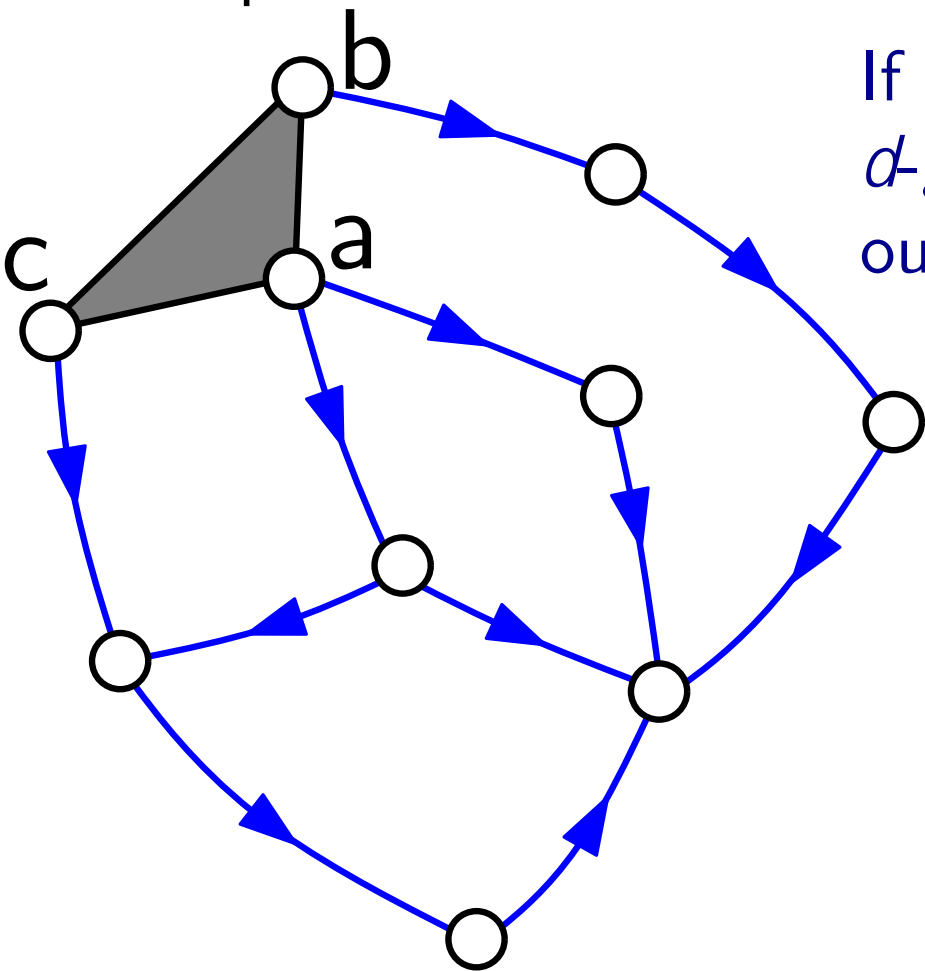
Example for $d = 3$



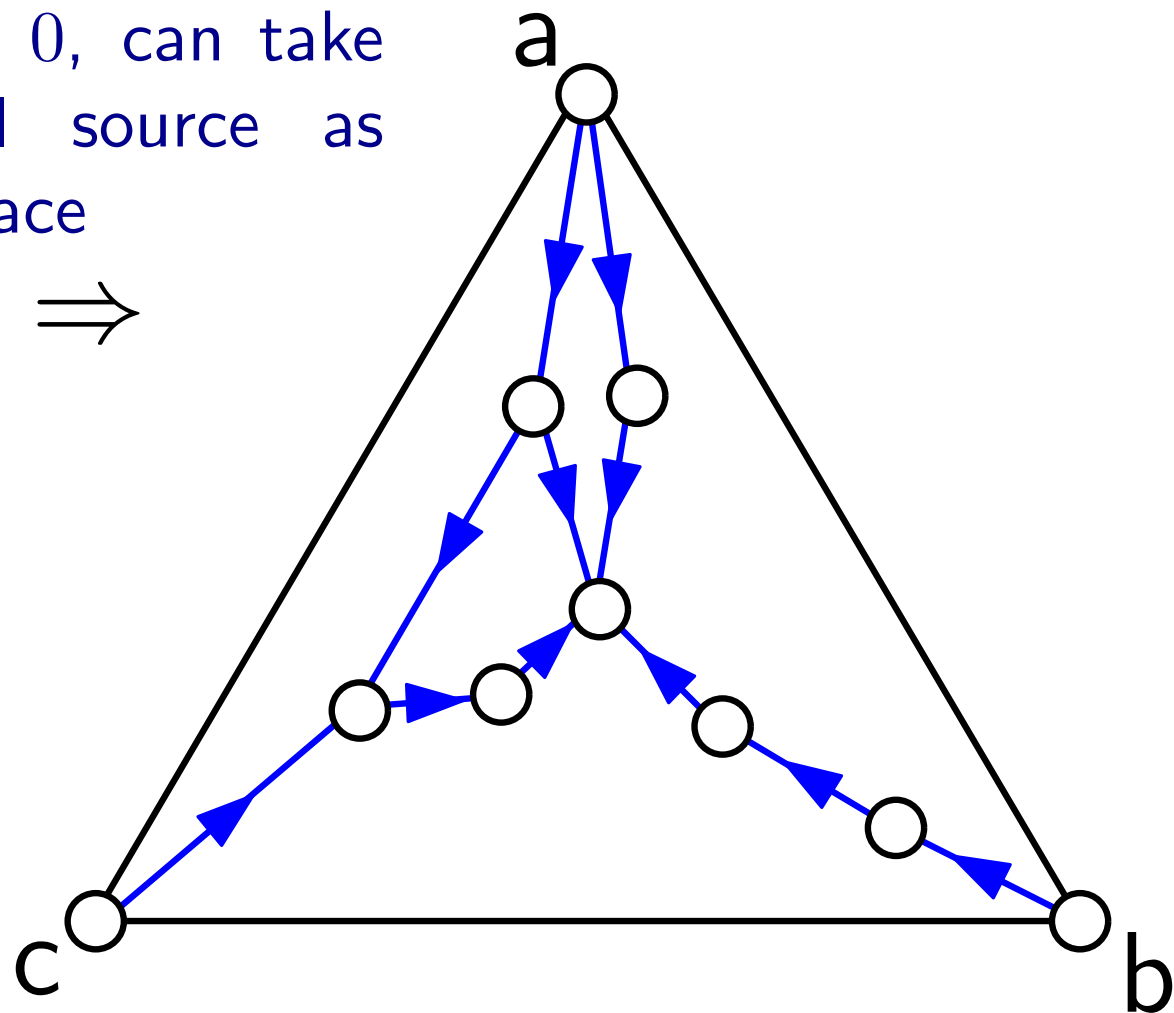
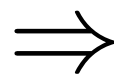
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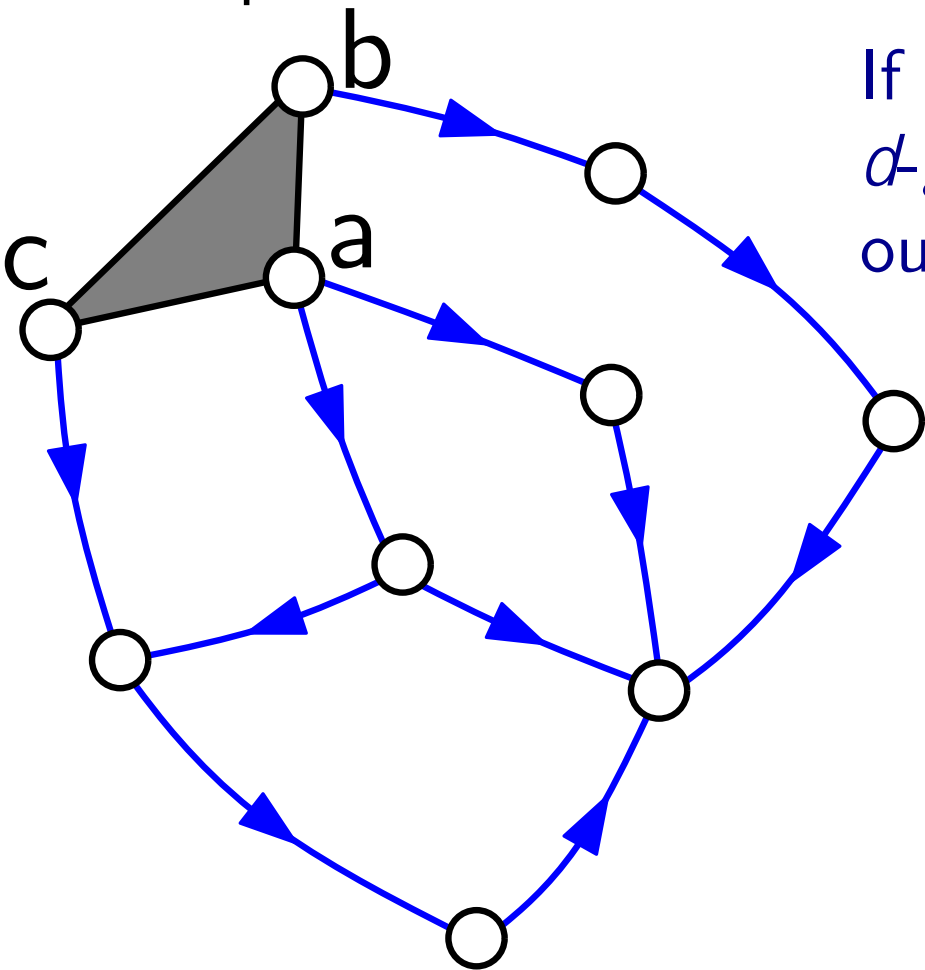
If $d > 0$, can take d -gonal source as outer face



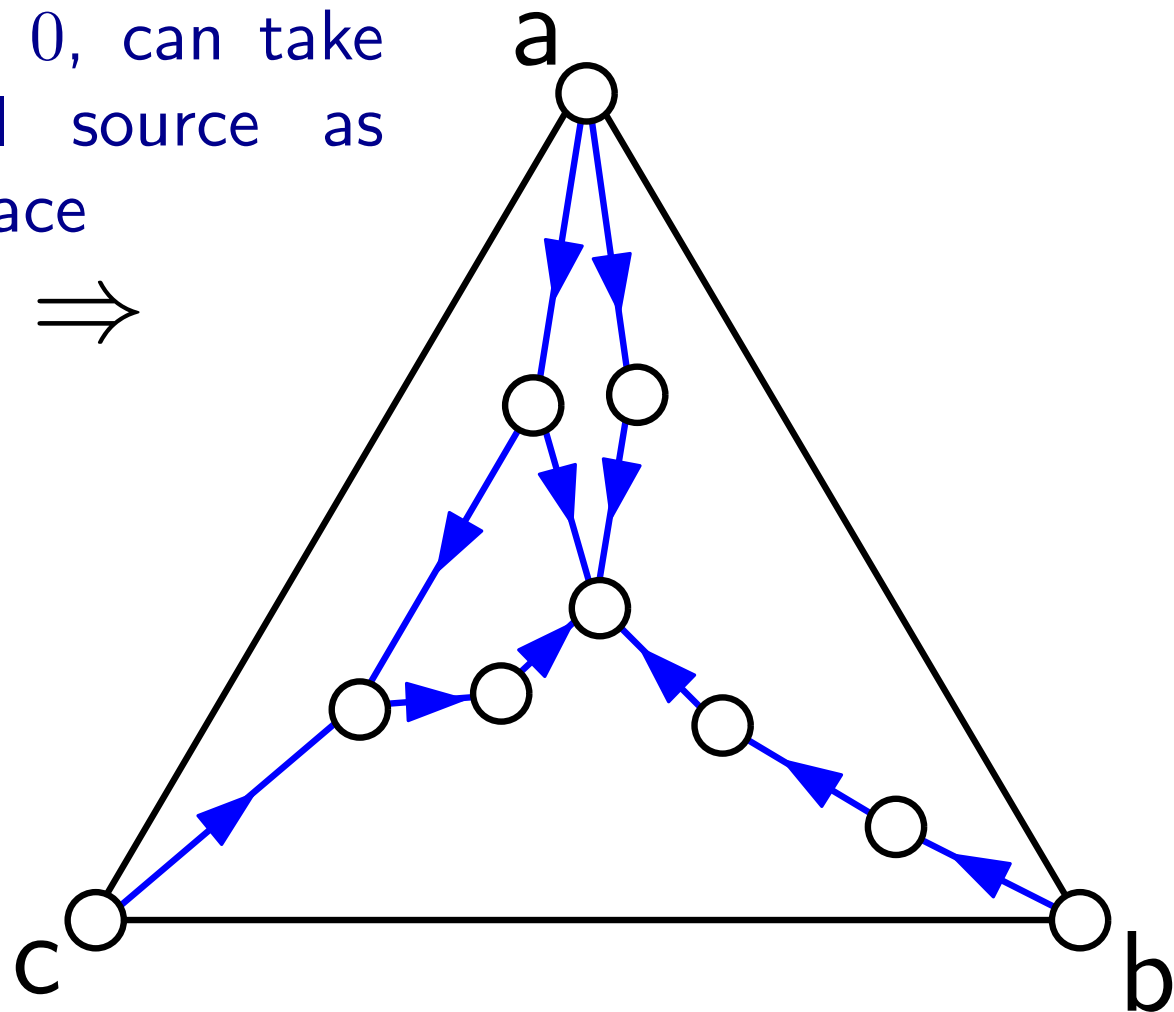
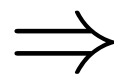
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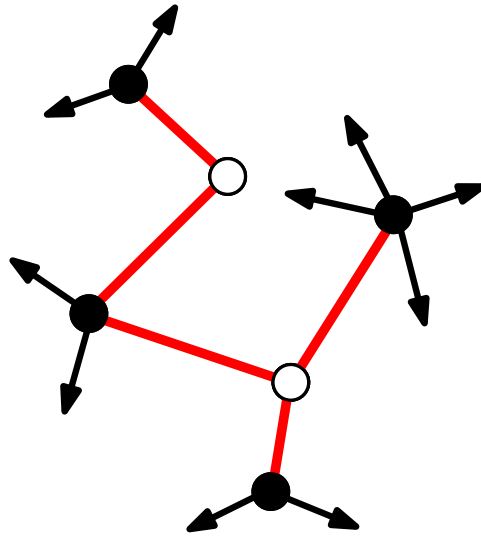


Let \mathcal{O}_d be the set of d -gonal source-orientations with no ccw circuit

Let $\mathcal{O} = \cup_{d \geq 0} \mathcal{O}_d$

Mobiles

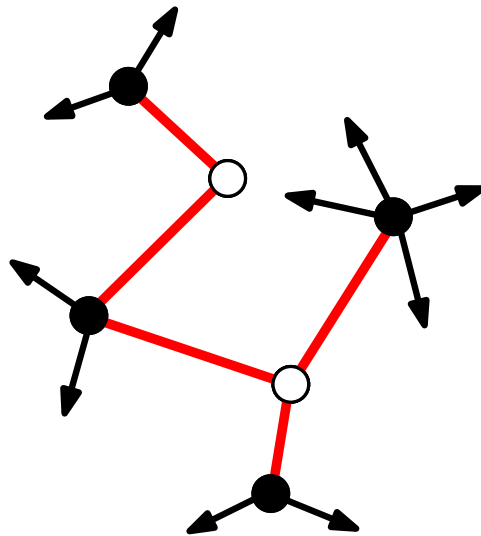
A **mobile** is a plane tree with vertices properly colored in black and white, together with **buds** (half-edges) incident to black vertices.



The **excess** is the number of buds minus the number of edges.

Mobiles

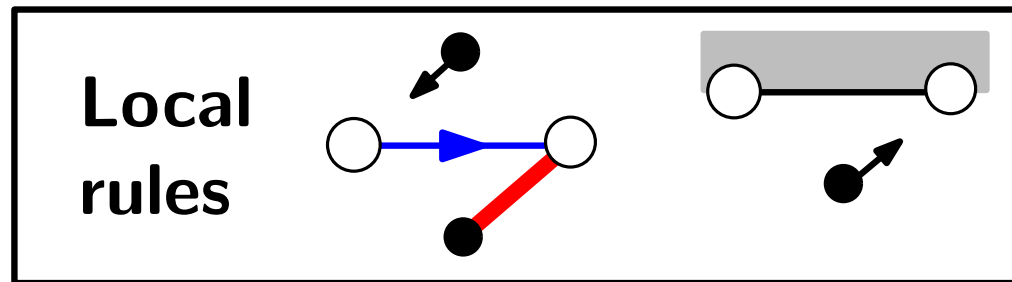
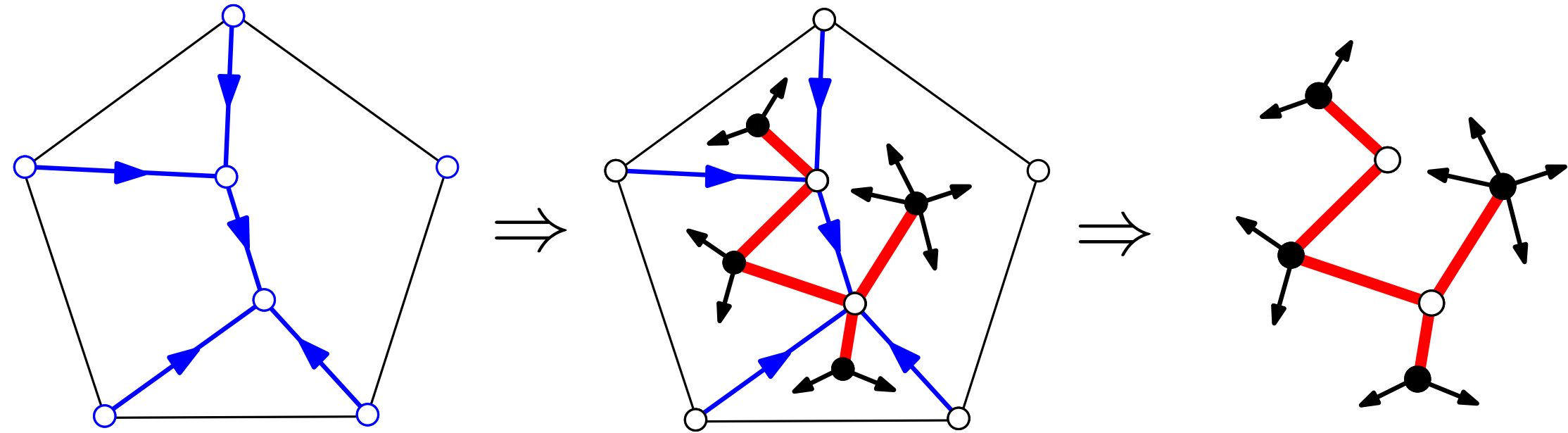
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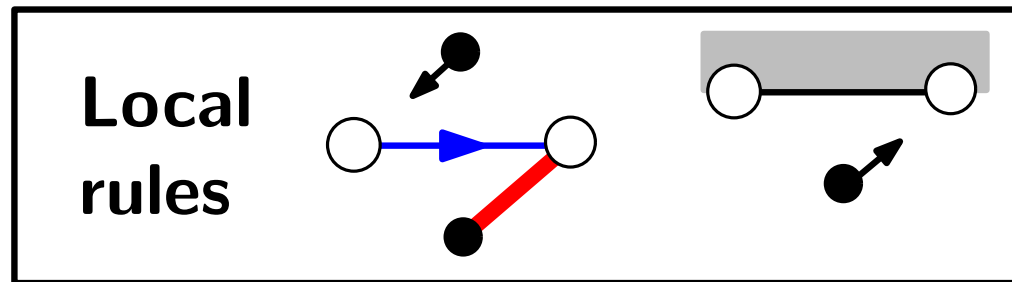
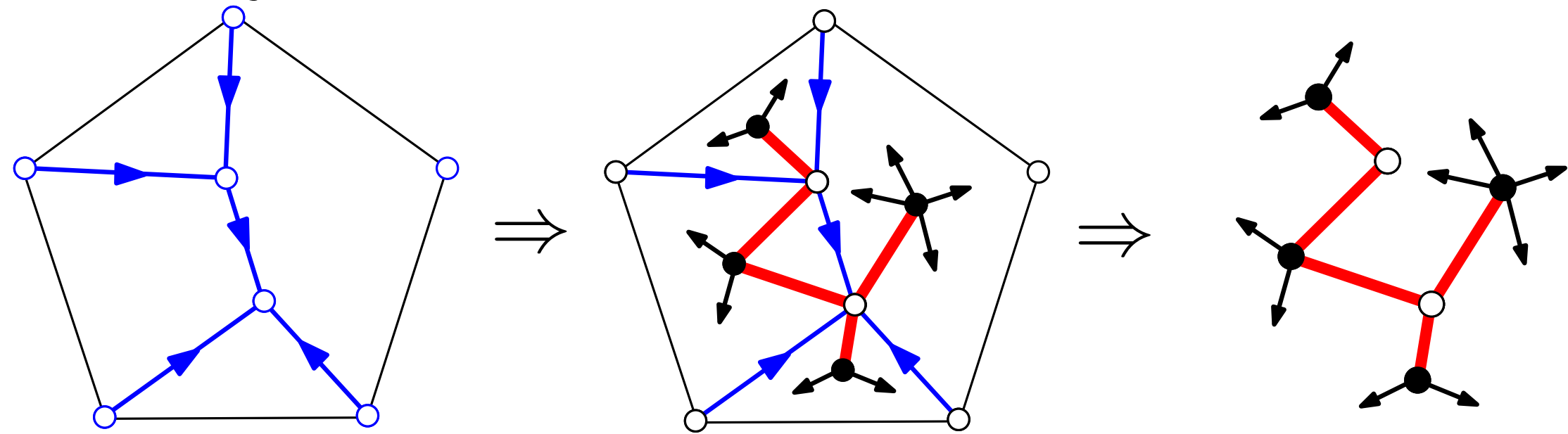
The **excess** is the number of buds minus the number of edges.

Let \mathcal{M} be the set of mobiles of nonnegative excess

Master bijection Φ



Master bijection Φ



Theorem [Bernardi-F'10]: Φ is a **bijection** between \mathcal{O} and \mathcal{M} .

Moreover,

degree of external face

\longleftrightarrow excess

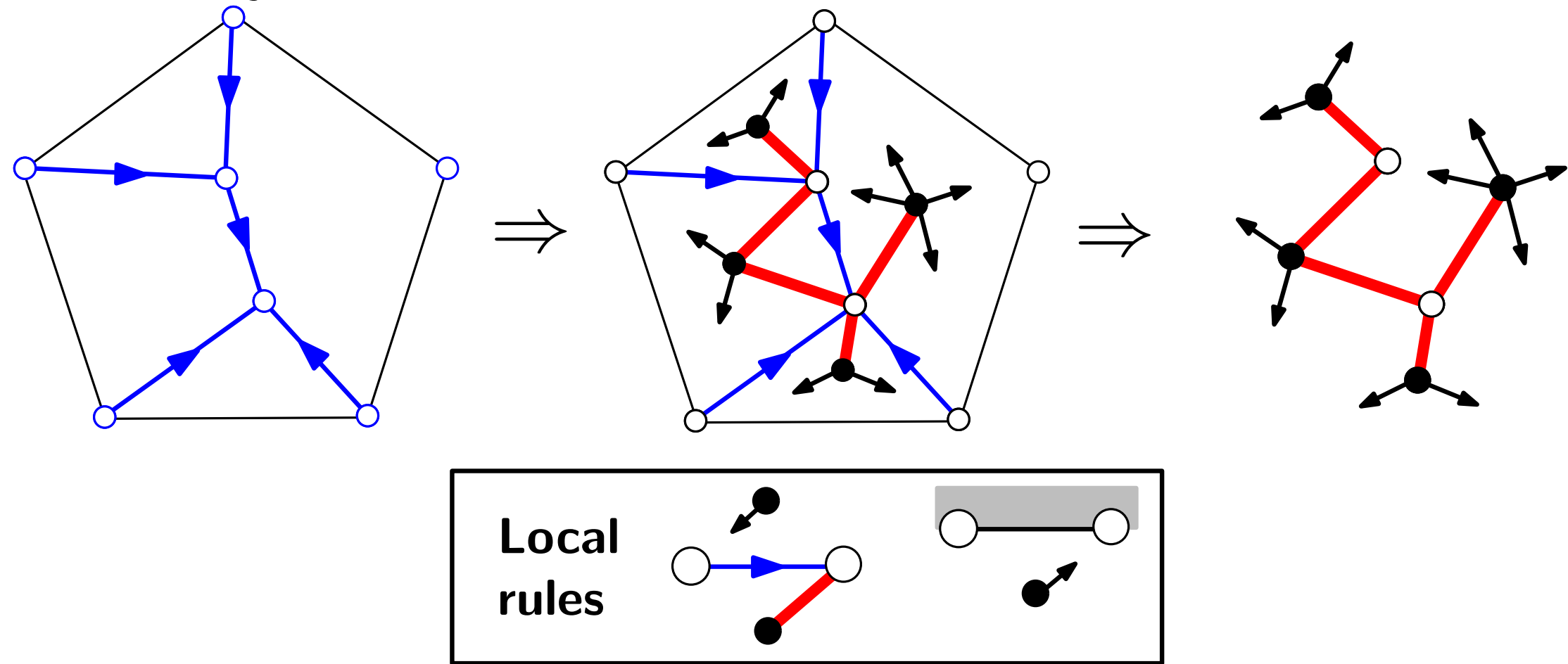
degree of internal faces

\longleftrightarrow degree of black vertices

indegree of internal vertices

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Master bijection Φ



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Moreover,

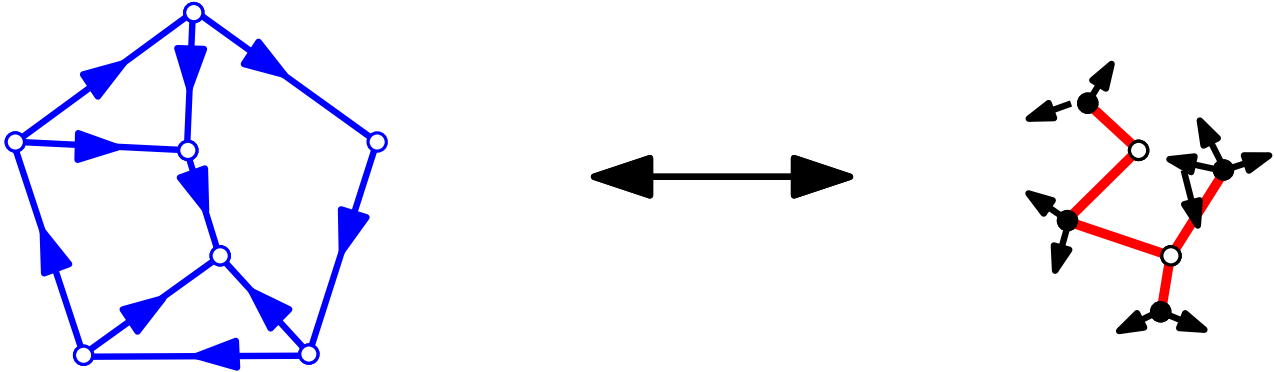
- degree of external face \longleftrightarrow excess
- degree of internal faces \longleftrightarrow degree of black vertices
- indegree of internal vertices \longleftrightarrow degree of white vertices

cf [Bernardi'07], [Bernardi-Chapuy'10]

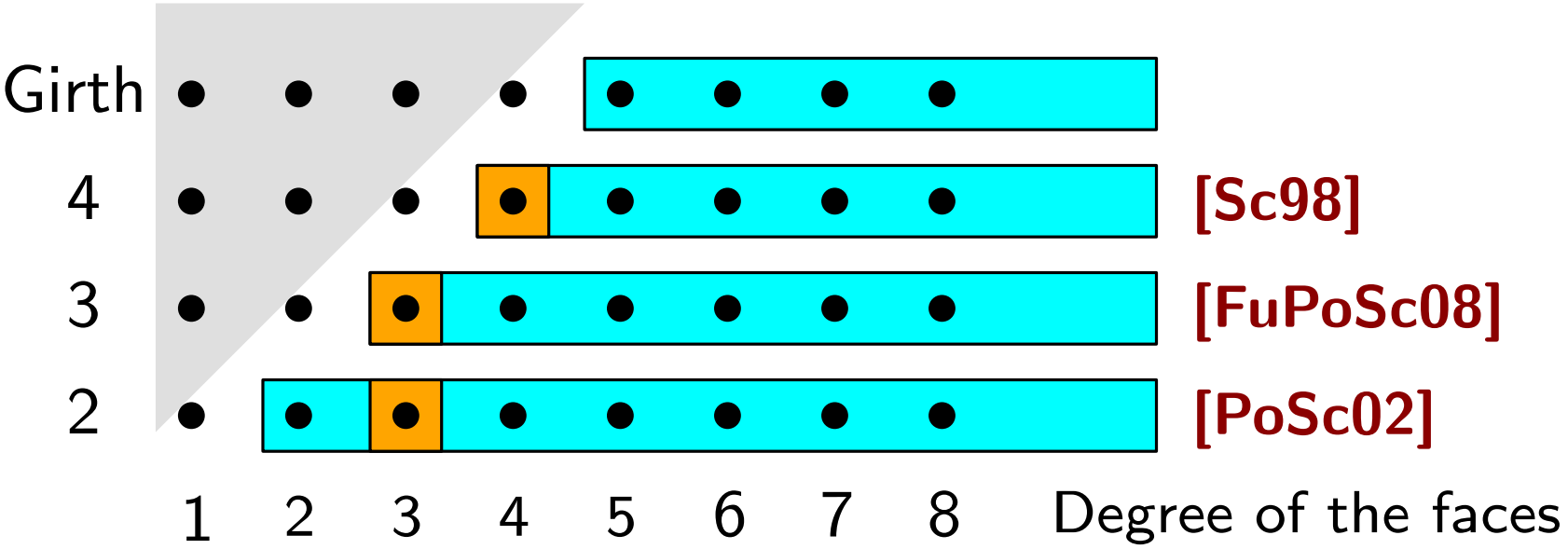
Using the master bijection for map enumeration

Main new results

The **Master bijection** between \mathcal{O} (orientations) and \mathcal{M} (mobiles)



allows to count maps by girth & face-degrees (via canonical orientations).



Scheme for the strategy

(1) Map family \mathcal{C} identifies with a **subfamily** $\mathcal{O}_{\mathcal{C}}$ of \mathcal{O} with conditions on:

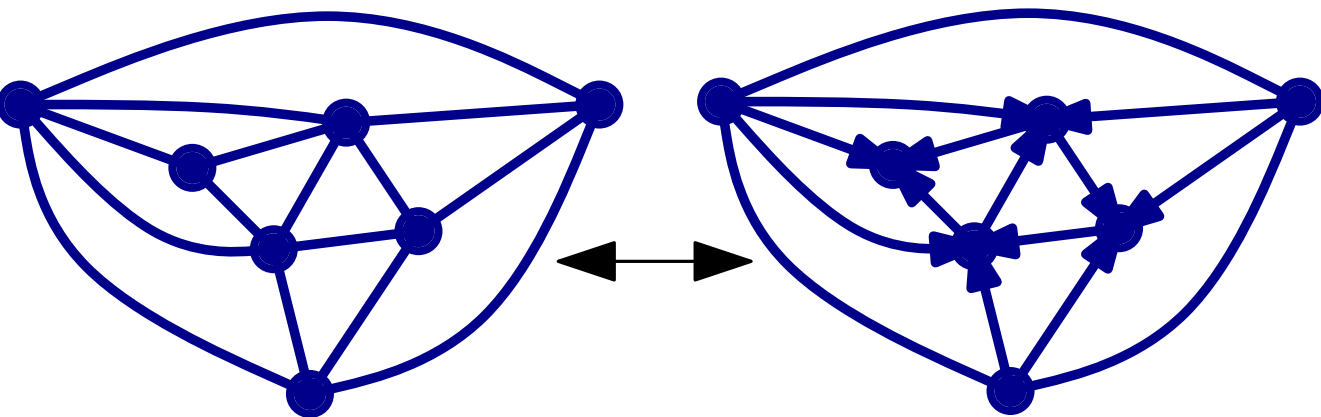
- Face degrees
- Vertex indegrees

Scheme for the strategy

(1) Map family \mathcal{C} identifies with a **subfamily** \mathcal{O}_C of \mathcal{O} with conditions on:

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Example: \mathcal{C} = Family of **simple triangulations**



$\mathcal{C} \simeq$ subfamily \mathcal{O}_C of \mathcal{O} with

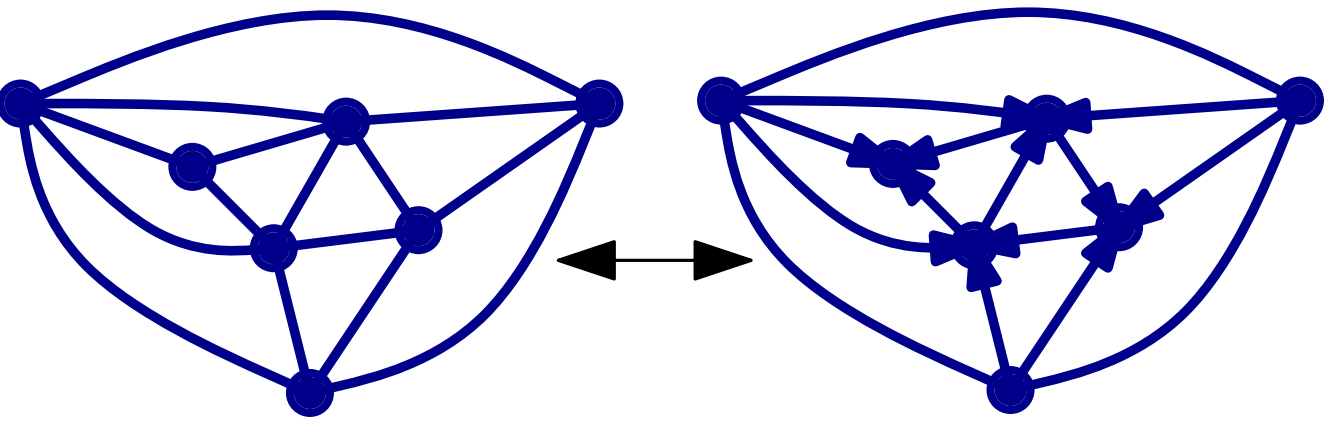
- Face-degree = 3
- Vertex-indegree = 3

Scheme for the strategy

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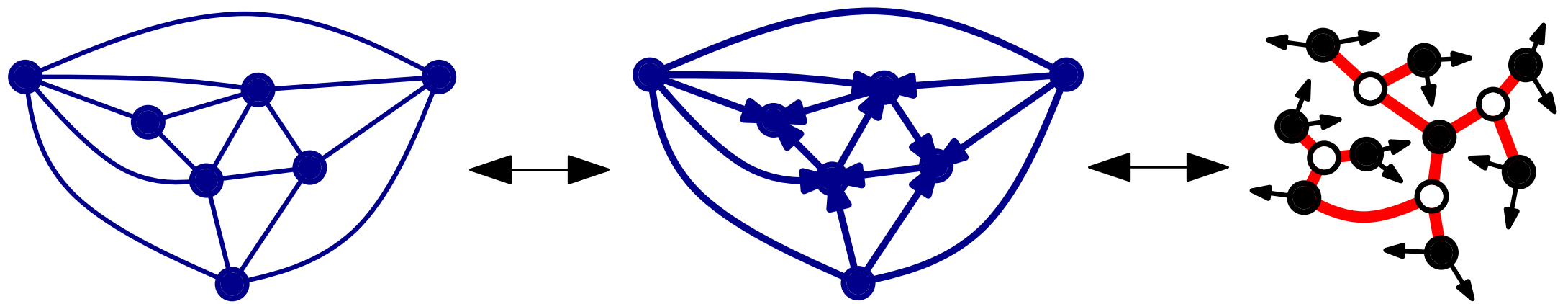
Example: \mathcal{C} = Family of **simple triangulations**



$\mathcal{C} \simeq$ subfamily \mathcal{O}_C of \mathcal{O} with

- Face-degree = 3
- Vertex-indegree = 3

(2) **Specialize** the master bijection to the subfamily \mathcal{O}_C

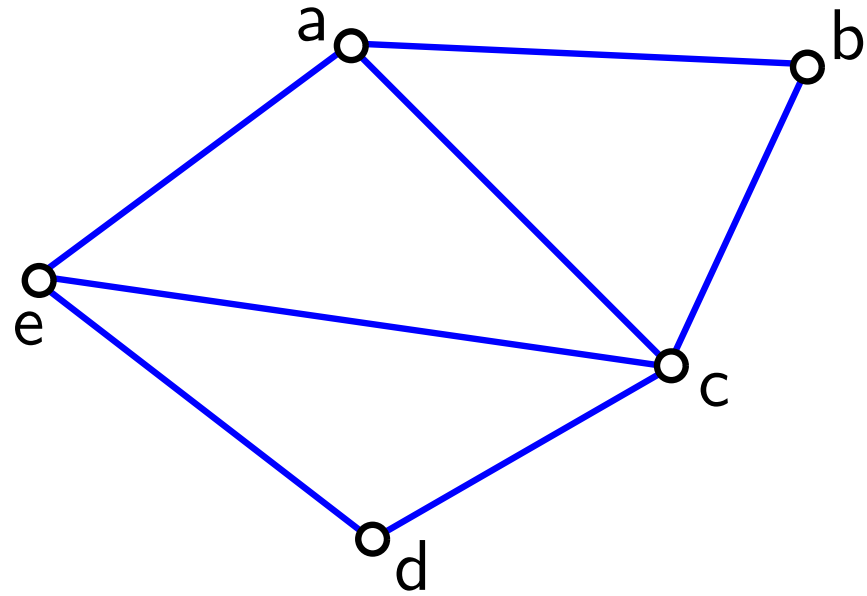


degree of internal faces \longleftrightarrow degree of black vertices
 indegree of internal vertices \longleftrightarrow degree of white vertices

-orientations

Let $G = (V; E)$ be a graph

Let f be a function from V to \mathbb{N}

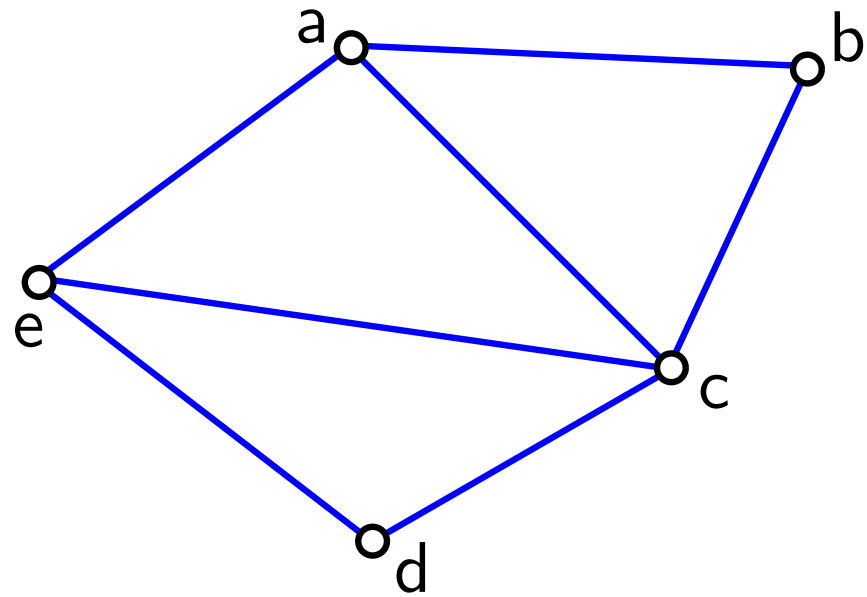


:	a	→	2
:	b	→	1
:	c	→	2
:	d	→	0
:	e	→	2

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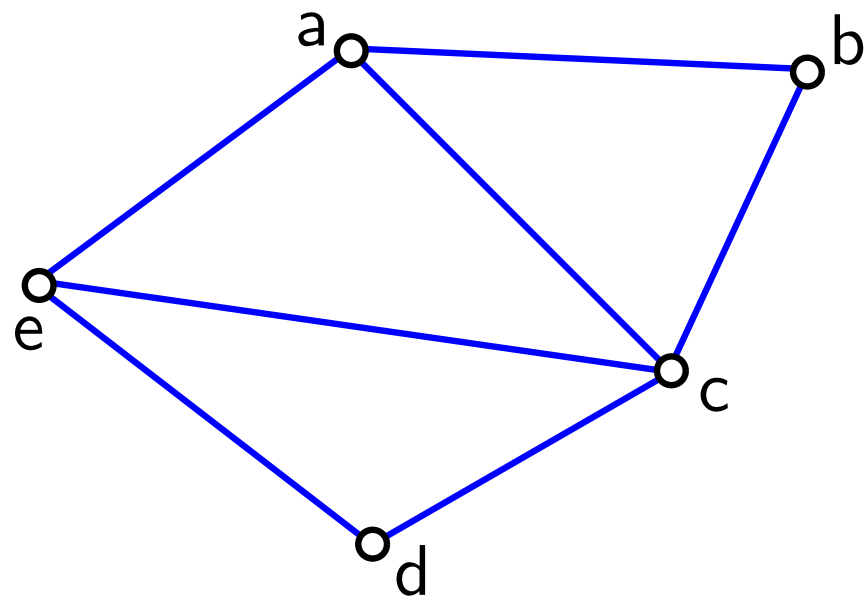
Def: An **-orientation** is an orientation of G where for each $v \in V$

$$\text{indegree}(v) = f(v)$$

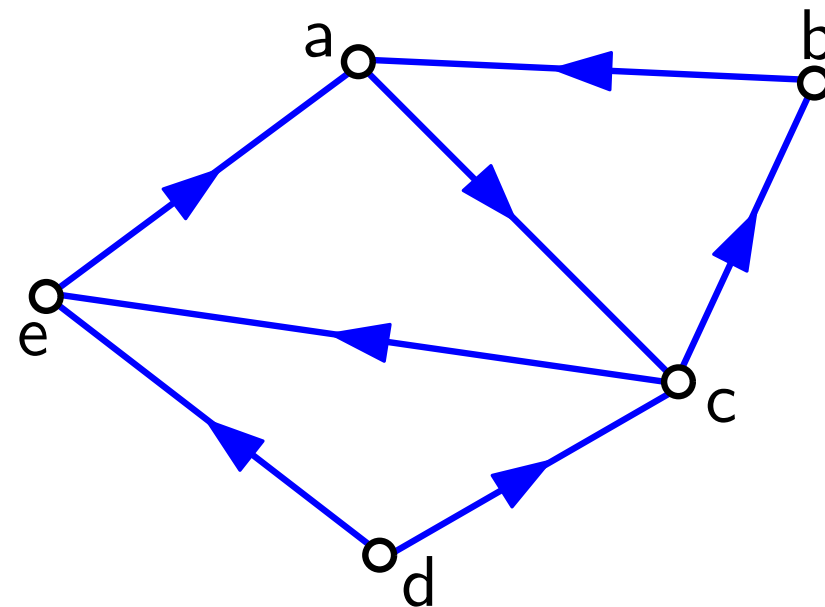
ϕ -orientations

Let $G = (V; E)$ be a graph

Let ϕ be a function from V to \mathbb{N}



ϕ	:	a	\rightarrow	2
		b	\rightarrow	1
		c	\rightarrow	2
		d	\rightarrow	0
		e	\rightarrow	2

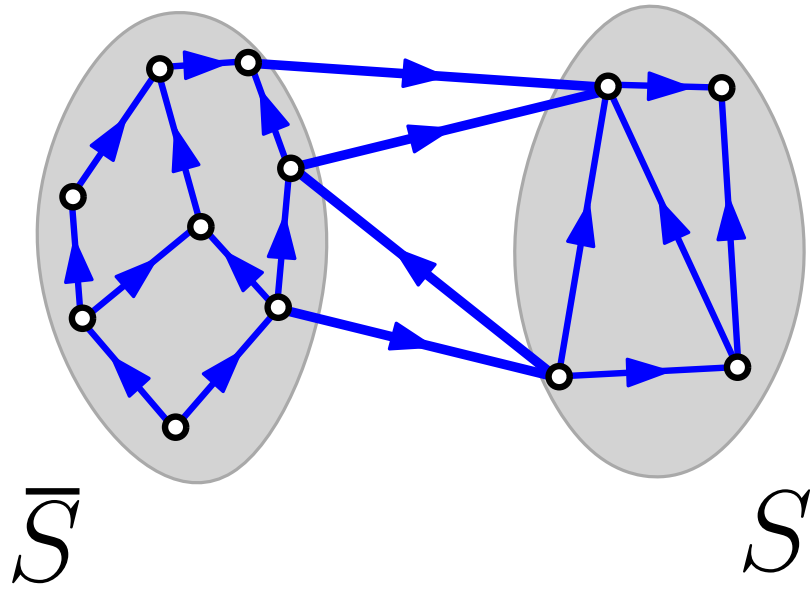


Def: An ϕ -orientation is an orientation of G where for each $v \in V$

$$\text{indegree}(v) = \phi(v)$$

-orientations: criteria for existence and accessibility

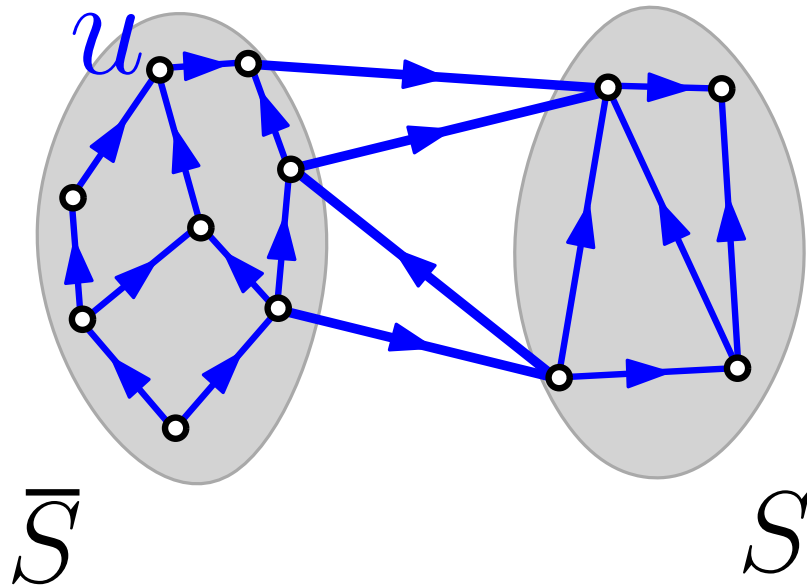
- If an α -orientation **exists**, then



<p>(i) $\sum_{v \in V} \alpha(v) = E$</p> <p>(ii) $\forall S \subseteq V; \sum_{v \in S} \alpha(v) \geq E_S$</p>
--

-orientations: criteria for existence and accessibility

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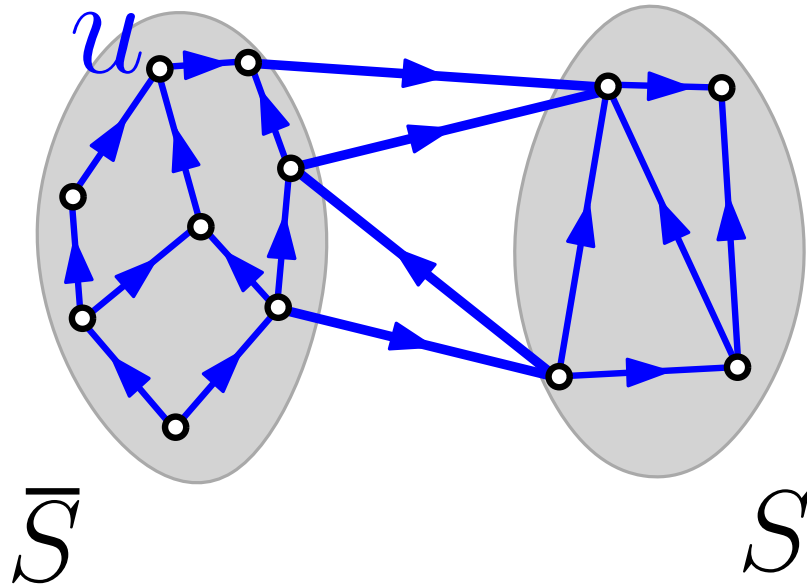
(i) $\sum_{v \in V} \text{deg}^-(v) = E $
(ii) $\forall S \subseteq V; \sum_{v \in S} \text{deg}^-(v) \geq E_S $

- If the -orientation is **accessible** from a vertex $u \in V$ then

(iii) $\sum_{v \in S} \text{deg}^-(v) > E_S $ whenever $u \in S$ and $S \neq \emptyset$
--

-orientations: criteria for existence and accessibility

- If an -orientation **exists**, then



$$\begin{aligned} \text{(i)} \quad & \sum_{v \in V} \text{deg}^-(v) = |E| \\ \text{(ii)} \quad & \forall S \subseteq V; \sum_{v \in S} \text{deg}^-(v) \geq |E_S| \end{aligned}$$

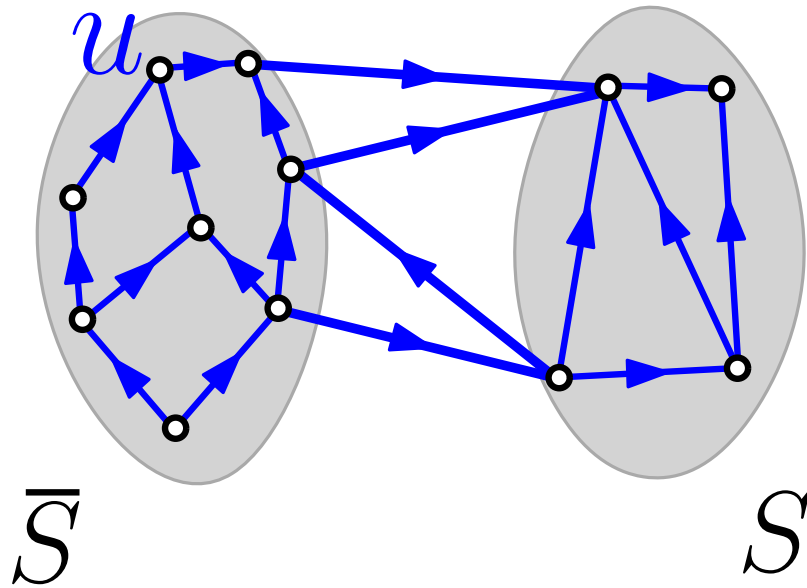
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Lemma (folklore): The conditions are necessary **and sufficient**

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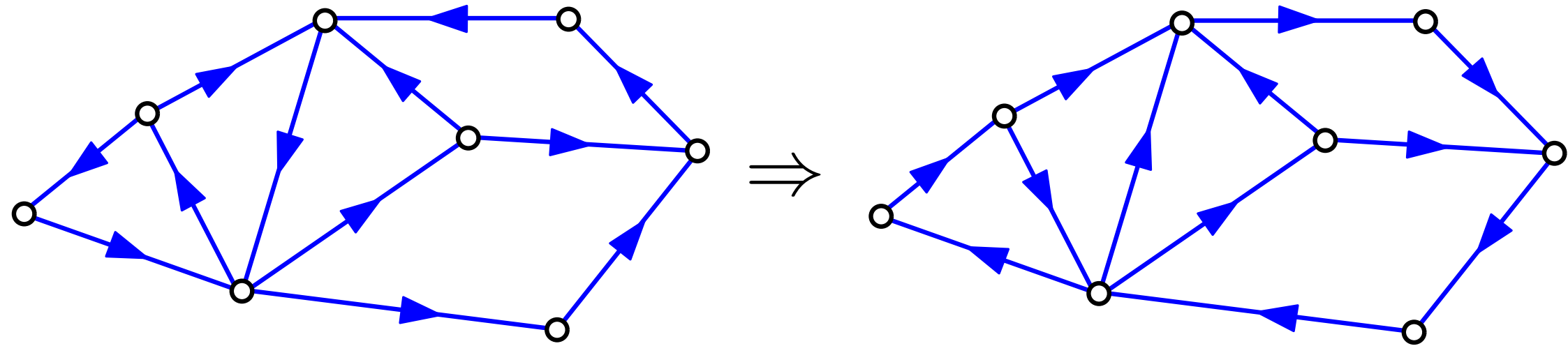
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Lemma (folklore): The conditions are necessary **and sufficient**

\Rightarrow accessibility from $u \in V$ just depends on (not on which -orientation)

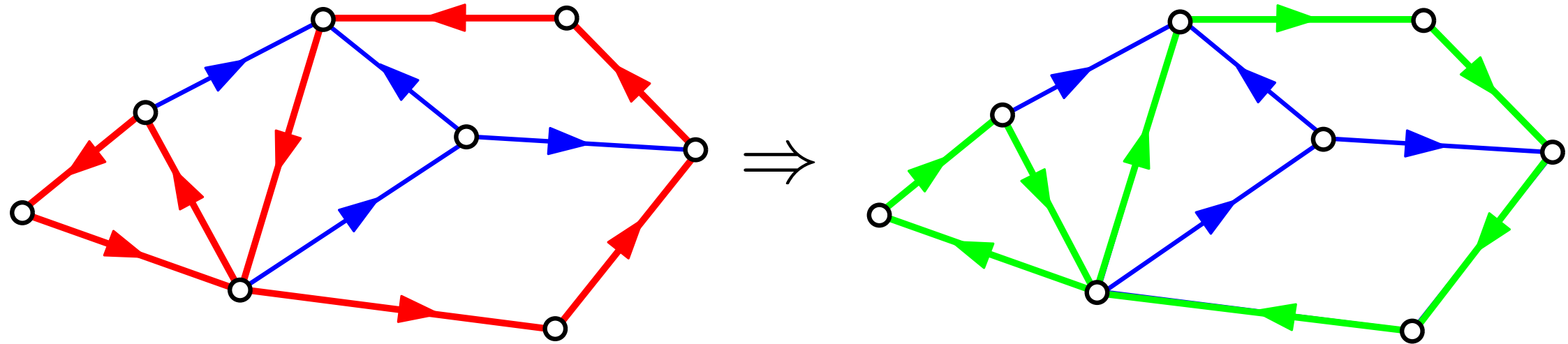
-orientations for plane maps

Fundamental lemma: If a plane map admits an ω -orientation, then it admits a **unique** ω -orientation **without ccw circuit**, called **minimal**



-orientations for plane maps

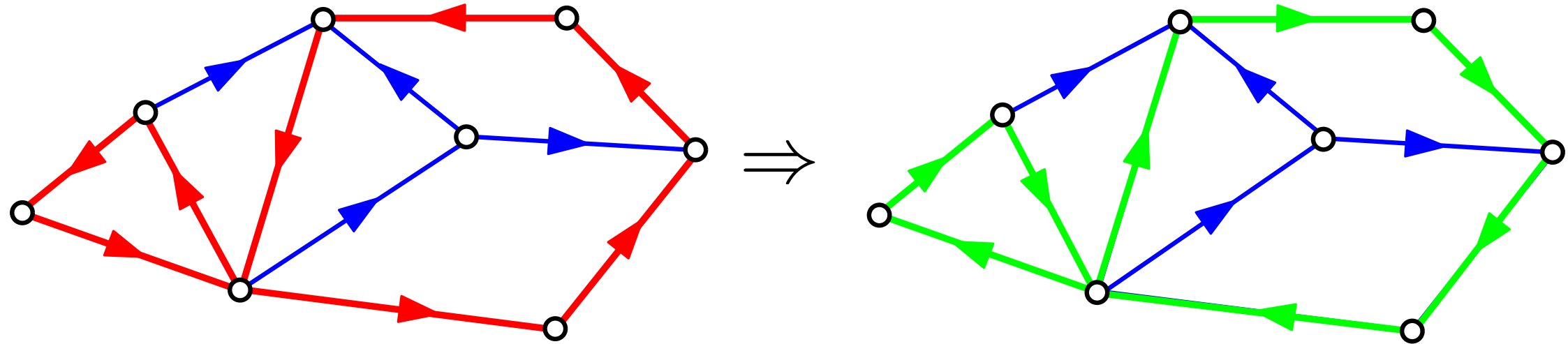
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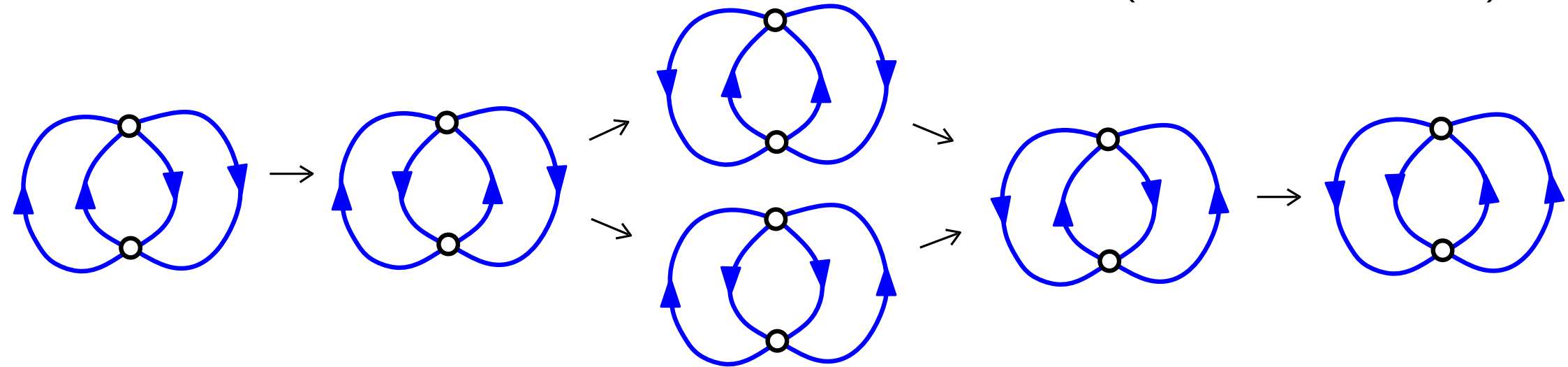
Uniqueness proof: if $O_1 \neq O_2$, edges where O_1 and O_2 **disagree** form an **eulerian suborientation** of $O_1 \Rightarrow$ contains a circuit (ccw in O_1 or O_2)

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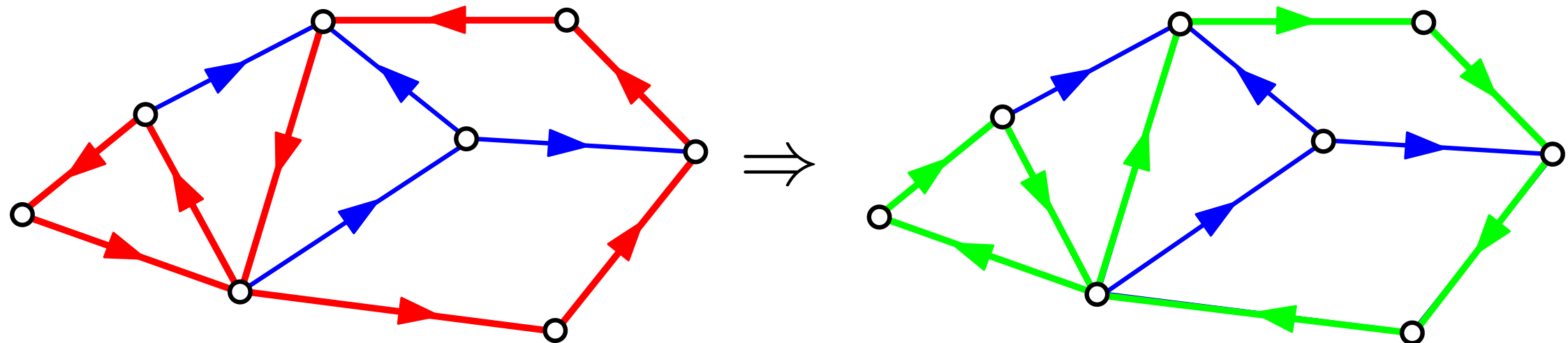
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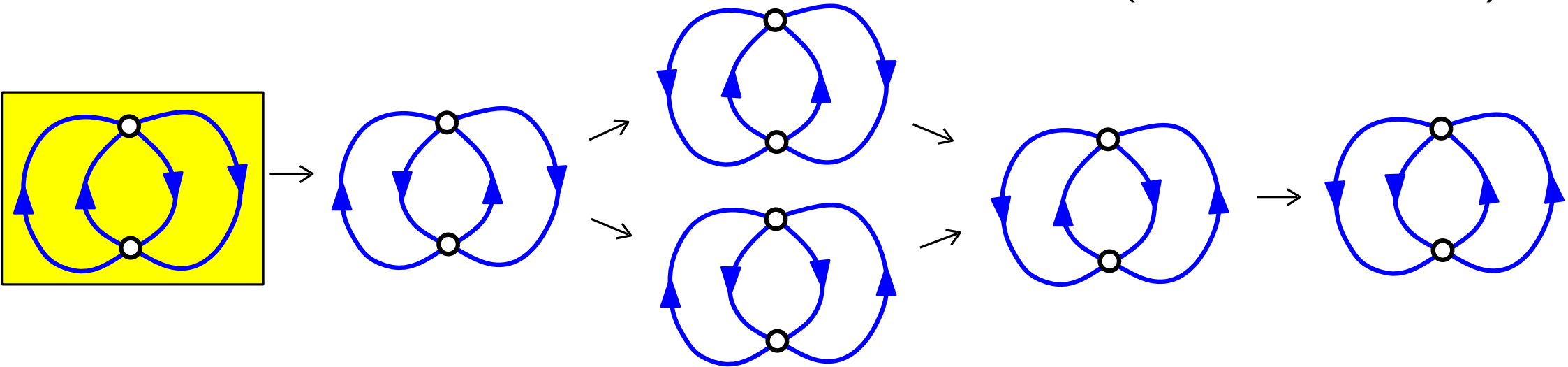
Set of ω -orientations = **distributive lattice**
[Khueler et al'93], [Propp'93], [O. de Mendez'94], [Felsner'03]

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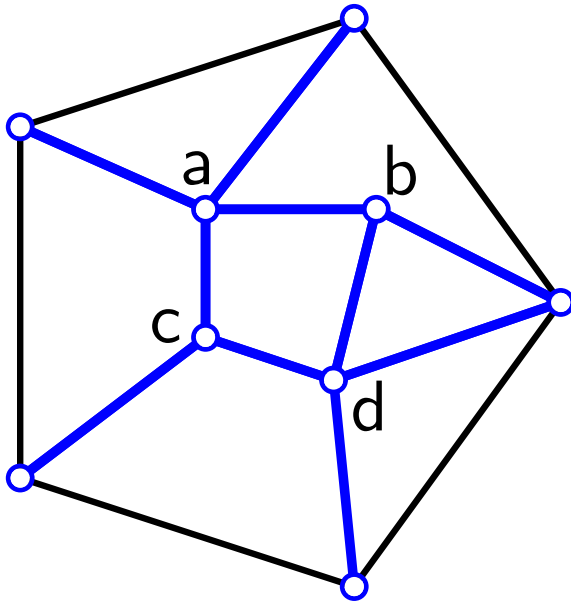
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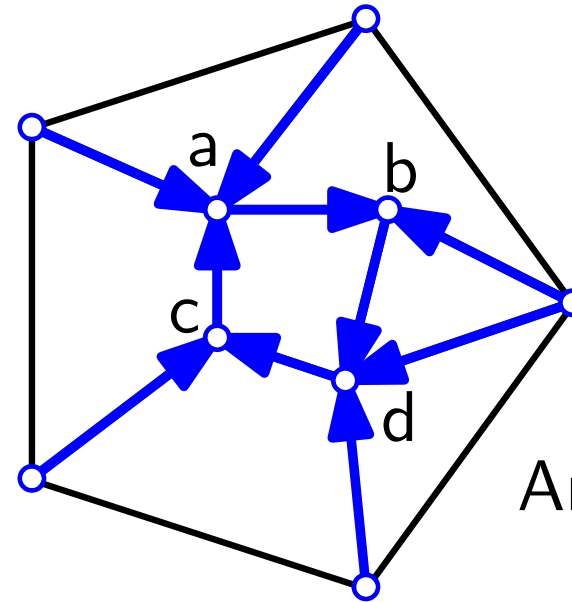
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-orientations for plane maps in our setting

- **External polygon** (the source) of the plane map is **unoriented**
- **Indegrees** are only on the **internal vertices**



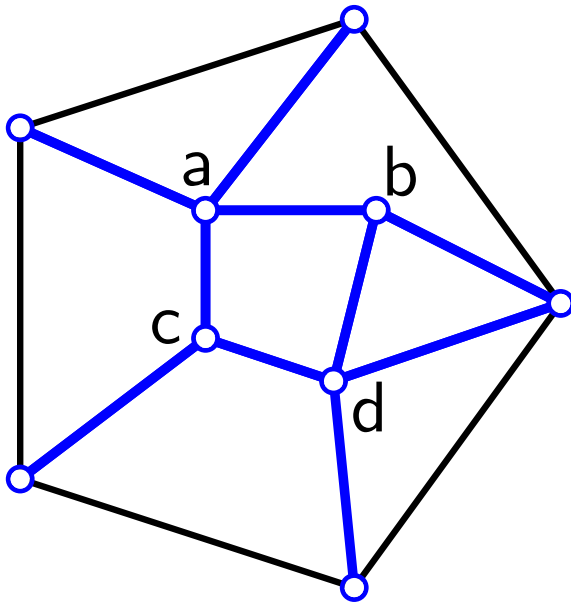
: a \rightarrow 3
b \rightarrow 2
c \rightarrow 2
d \rightarrow 3



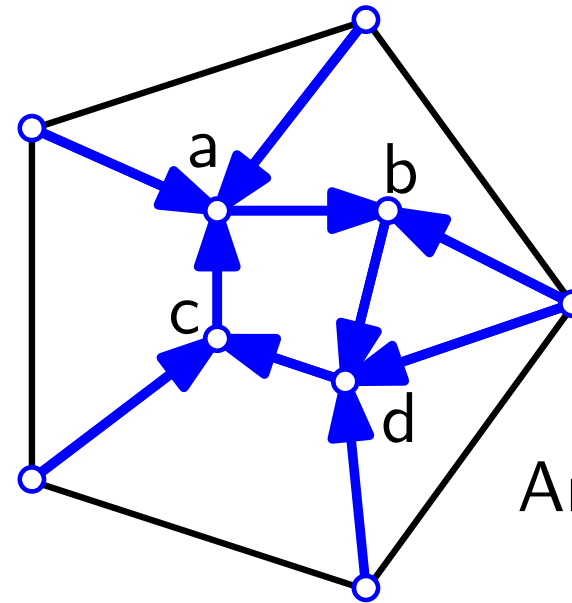
An \rightarrow -orientation

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$$: \begin{aligned} a &\rightarrow 3 \\ b &\rightarrow 2 \\ c &\rightarrow 2 \\ d &\rightarrow 3 \end{aligned}$$



An \rightarrow -orientation

Partition V (vertex-set) as $V_i \cup V_e$ and E (edge-set) as $E_i \cup E_e$

- **Existence:**

$$(i) \sum_{v \in V_i} \text{deg}(v) = |E_i|$$

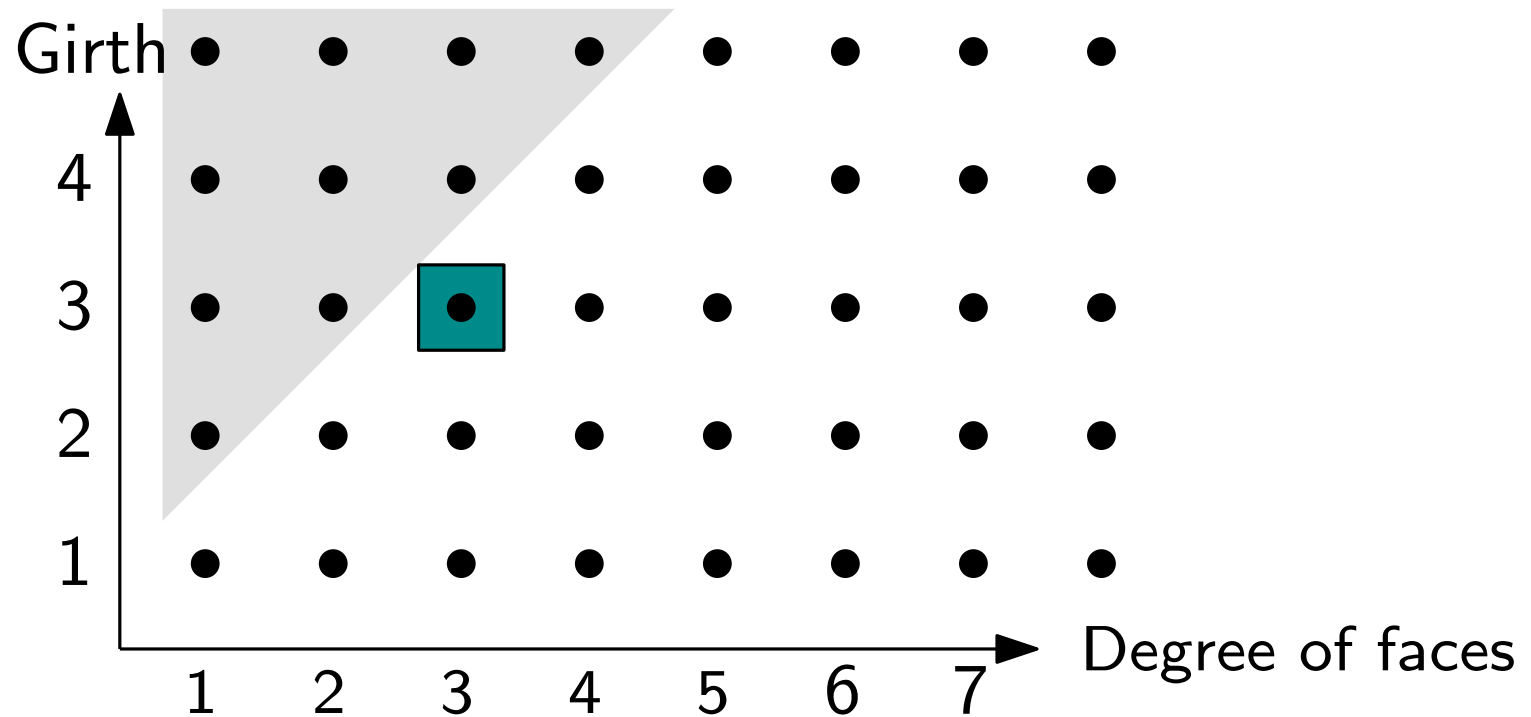
$$(ii) \forall S \subseteq V; \sum_{v \in S \cap V_i} \text{deg}(v) \geq |E_S \cap E_i|$$

- **Accessibility from outer face:**

$$(iii) \forall S \subseteq V_i; \sum_{v \in S \cap V_i} \text{deg}(v) > |E_S \cap E_i|$$

- **Distributive lattice** structure

Example: simple triangulations

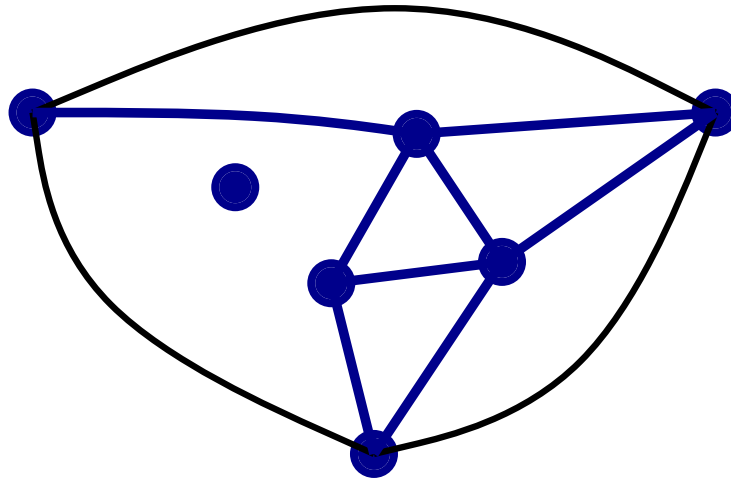


Proof: The numbers v , e , f of vertices edges and faces satisfy:

- Incidence relation: $3f = 2e$.
- Euler relation: $v - e + f = 2$.



f



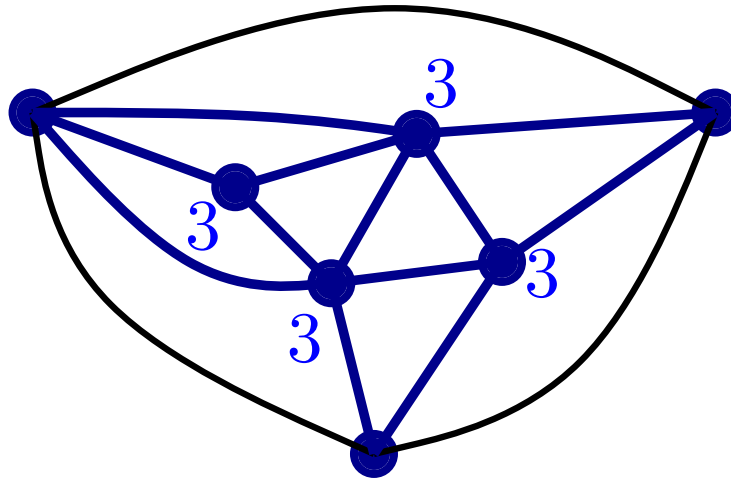
Triangulations

Fact: A triangulation with n internal vertices has $3n$ internal edges.

Natural candidate for indegree function:

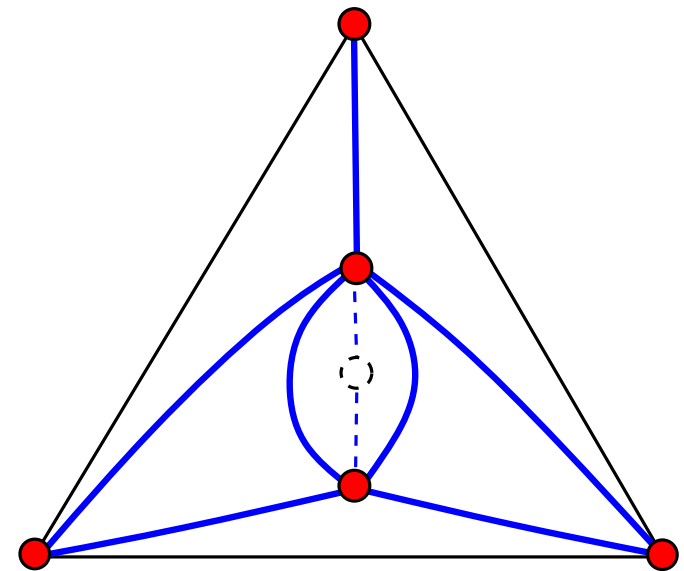
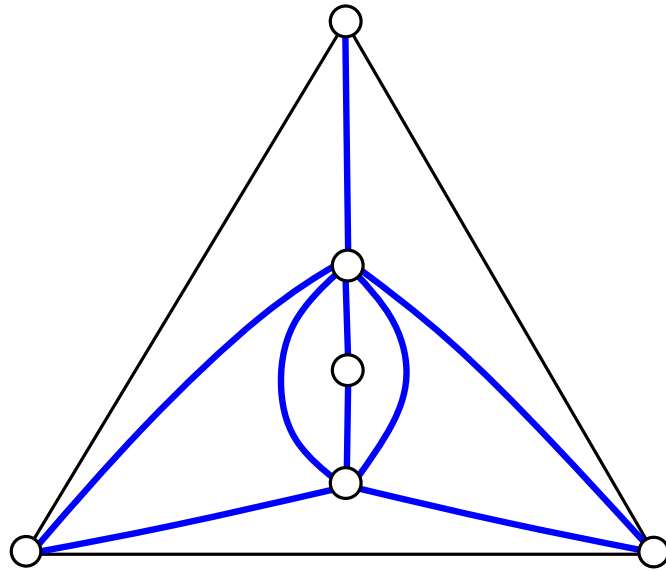
$\deg^{\text{in}} : v \mapsto 3$ for each internal vertex v .

call **3-orientation** such an \deg^{in} -orientation



Triangulations

Fact: A triangulation admitting a 3-orientation is simple



k internal vertices

$3k + 1$ internal edges

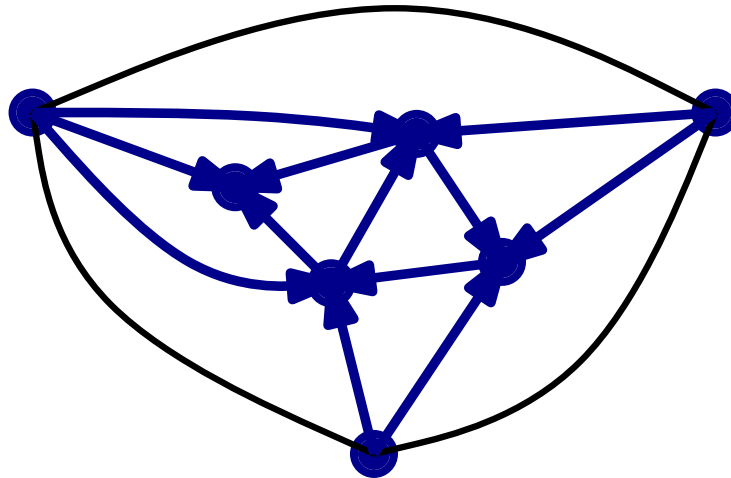
Triangulations

Thm [Schnyder 89]: A simple triangulation admits a 3-orientation.

New (easier) proof: Any simple planar graph $G = (V; E)$ satisfies

$$\frac{|E| - 3}{|V| - 3} \geq 3 \quad (\text{Euler relation})$$

hence the existence/accessibility conditions are satisfied. \square

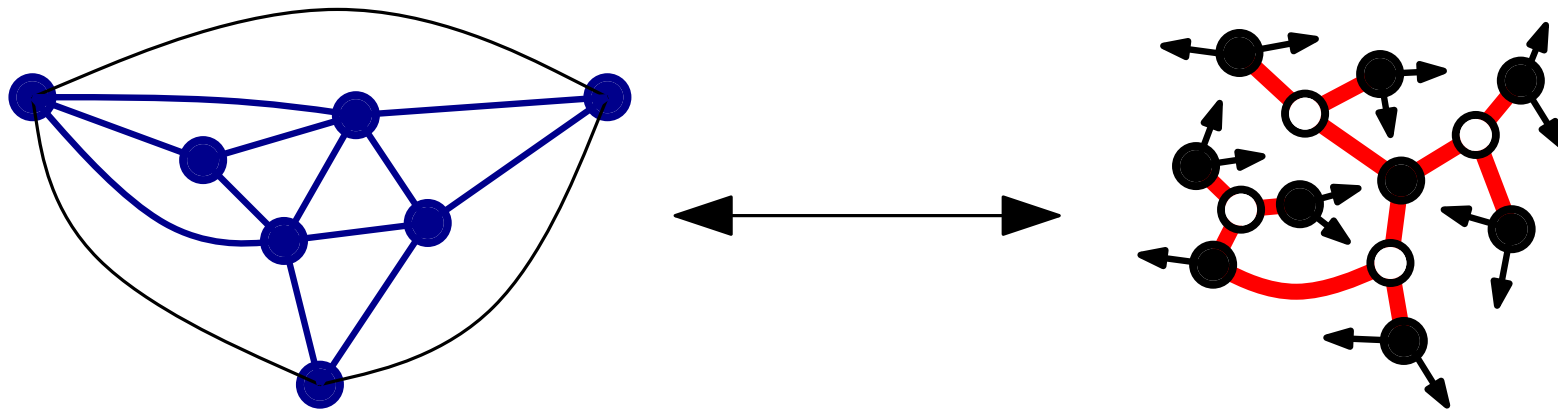


Triangulations

⇒ The class \mathcal{T} of simple triangulations is identified with the class of plane orientation $\mathcal{O}_{\mathcal{T}} \subset \mathcal{O}$ with faces of degree 3, and internal vertices of indegree 3.

Thm [recovering FuPoSc08]: By specializing the master bijection Φ to $\mathcal{O}_{\mathcal{T}}$ one obtains a bijection between simple triangulations and mobiles such that

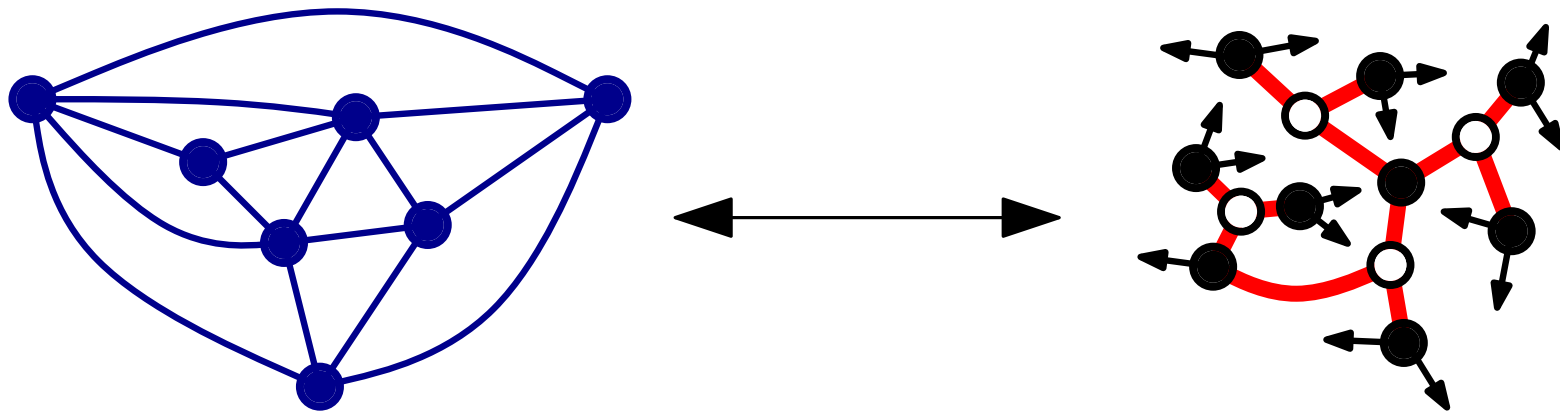
- black vertices have degree 3
- white vertices have degree 3
- the excess is +3 (redundant).



Triangulations

Counting: The generating function of mobiles with vertices of degree 3 rooted on a white corner is $T(x) = U(x)^3$, where $U(x) = 1 + xU(x)^4$.

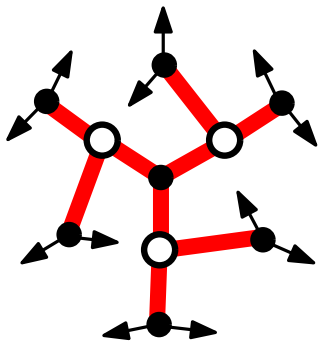
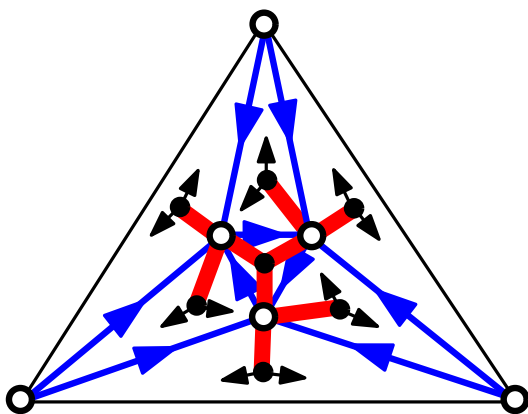
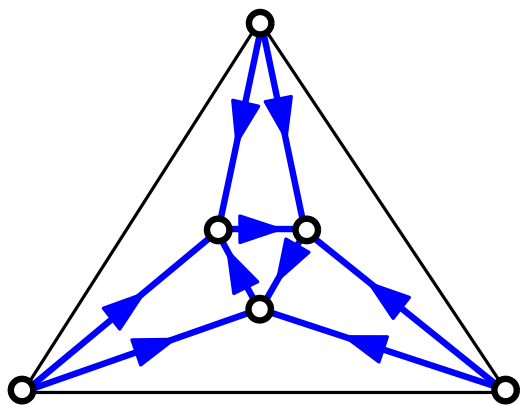
Consequently, the number of (rooted) simple triangulations with $2n$ faces is $\frac{1}{n(2n-1)} \binom{4n-2}{n-1}$.



Triangulations: two constructions

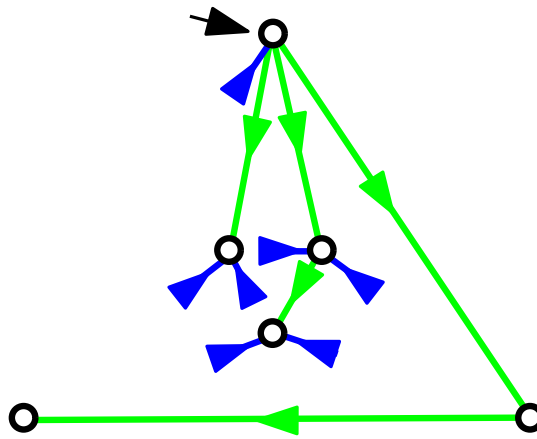
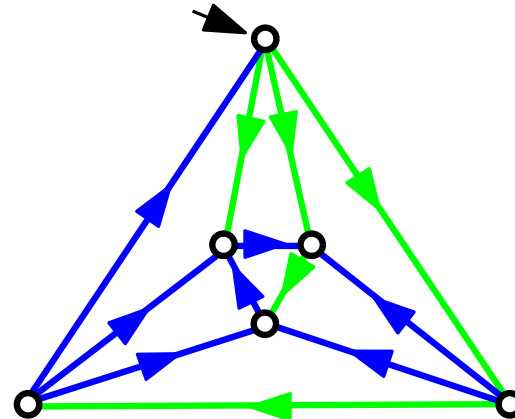
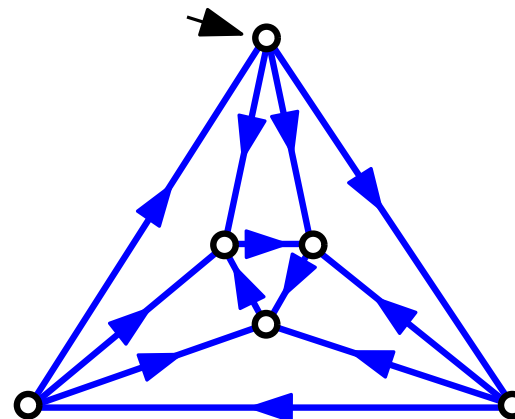
mobiles

[FuPoSc'08], [Bernardi-F'10]



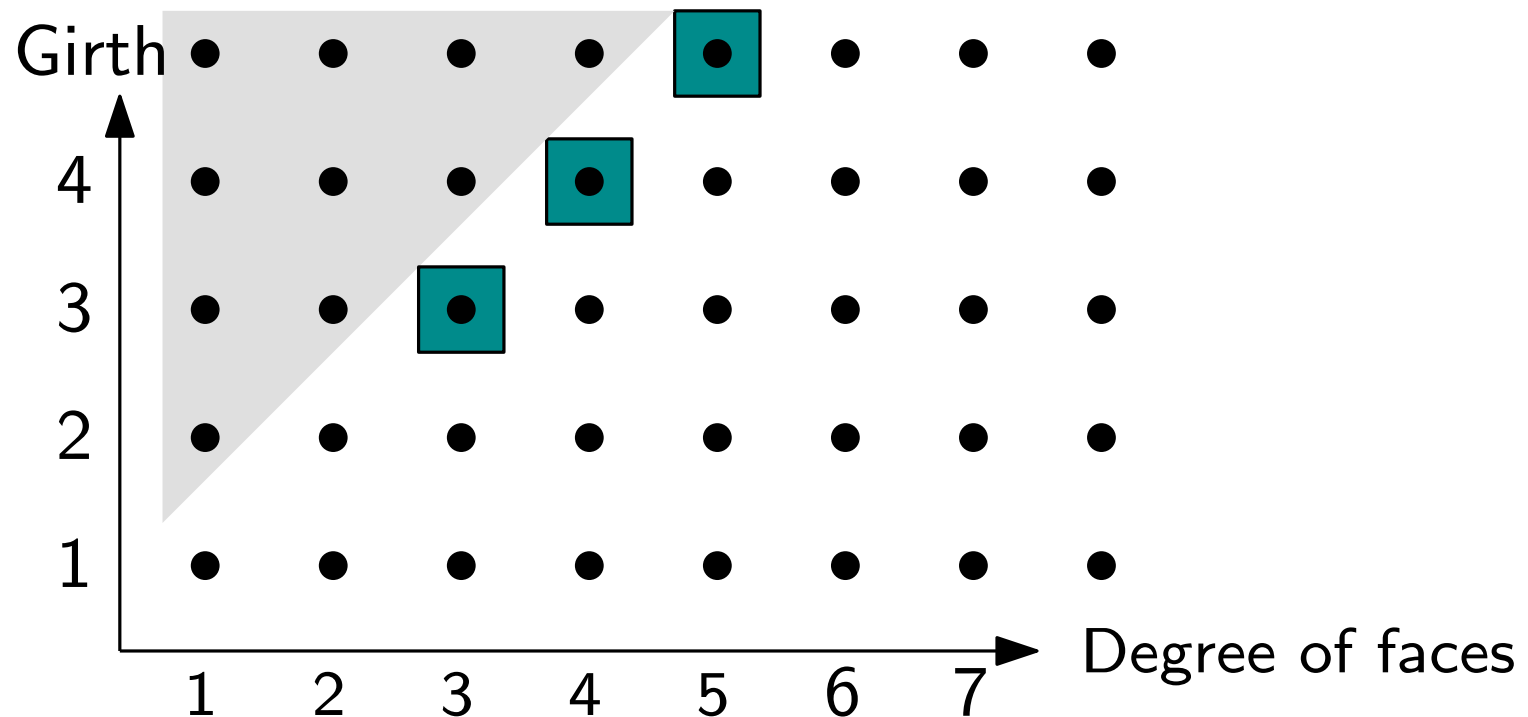
blossoming trees

[PoSc'03], [AlPo'11]



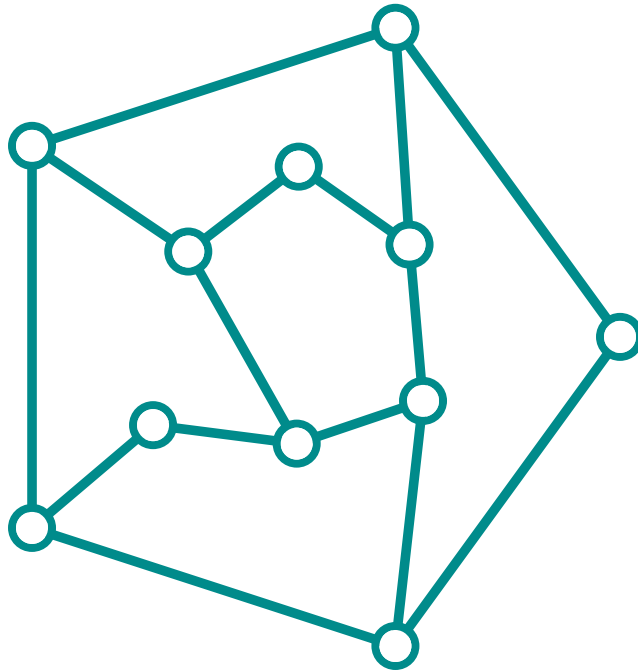
More specializations

d -angulations of girth d .



d -angulations of girth d

Fact: A d -angulation with $(d-2)n$ internal vertices has dn internal edges.

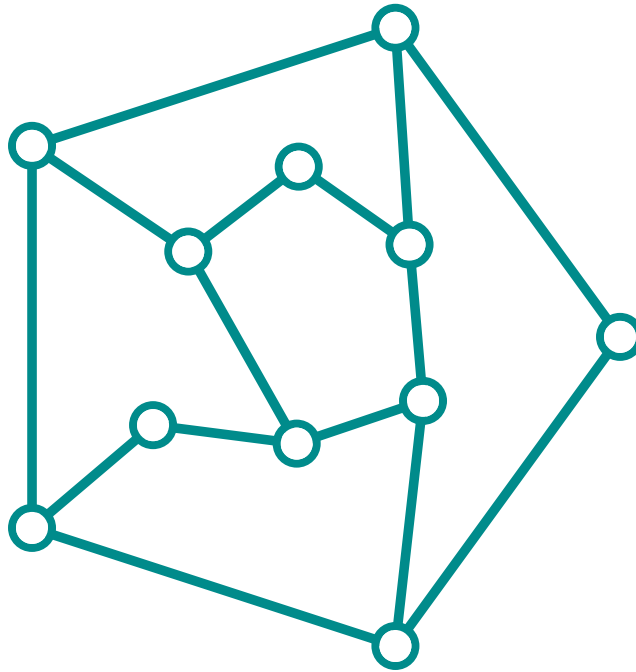


d -angulations of girth d

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Natural candidate for indegree function:

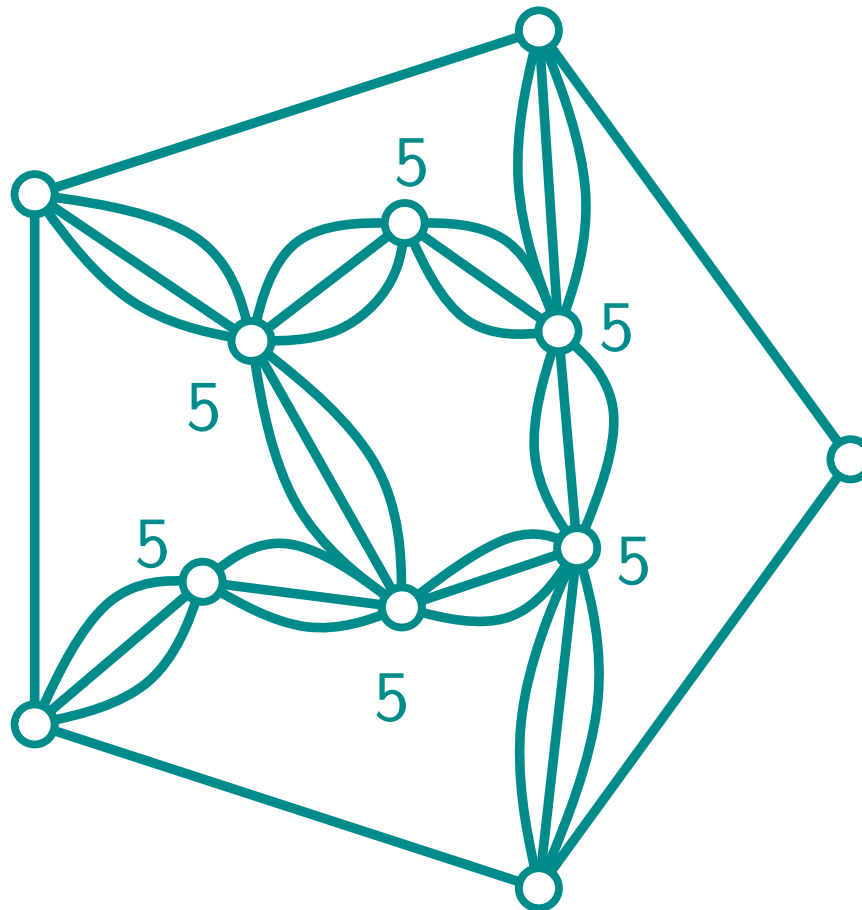
$$: v \mapsto \frac{d}{d-2} \text{ for each internal vertex } v \dots$$



d -angulations of girth d

Fact: A d -angulation with $(d-2)n$ internal vertices has dn internal edges.

Idea: We can look for an orientation of $(d-2)G$ with indegree function
: $v \mapsto d$ for each internal vertex v .

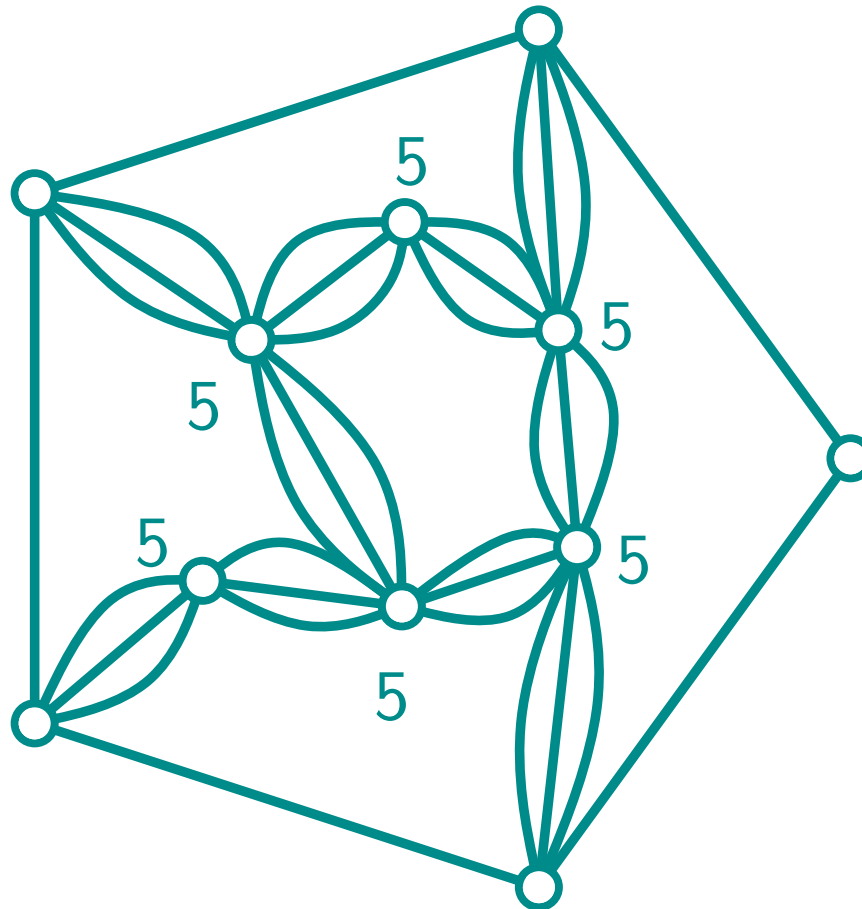


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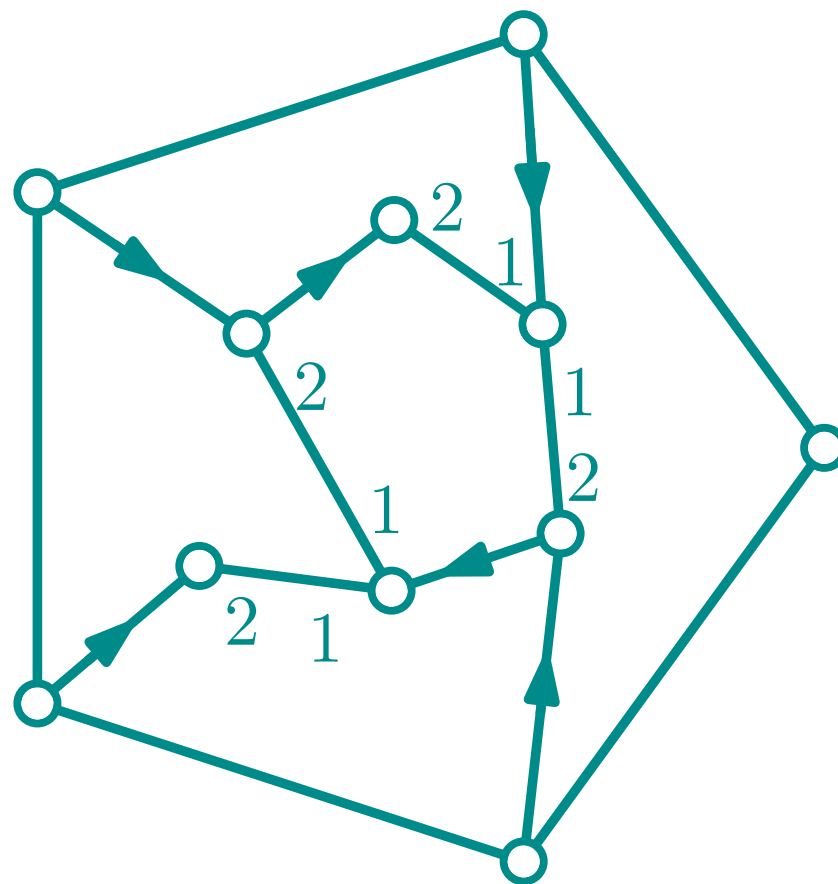
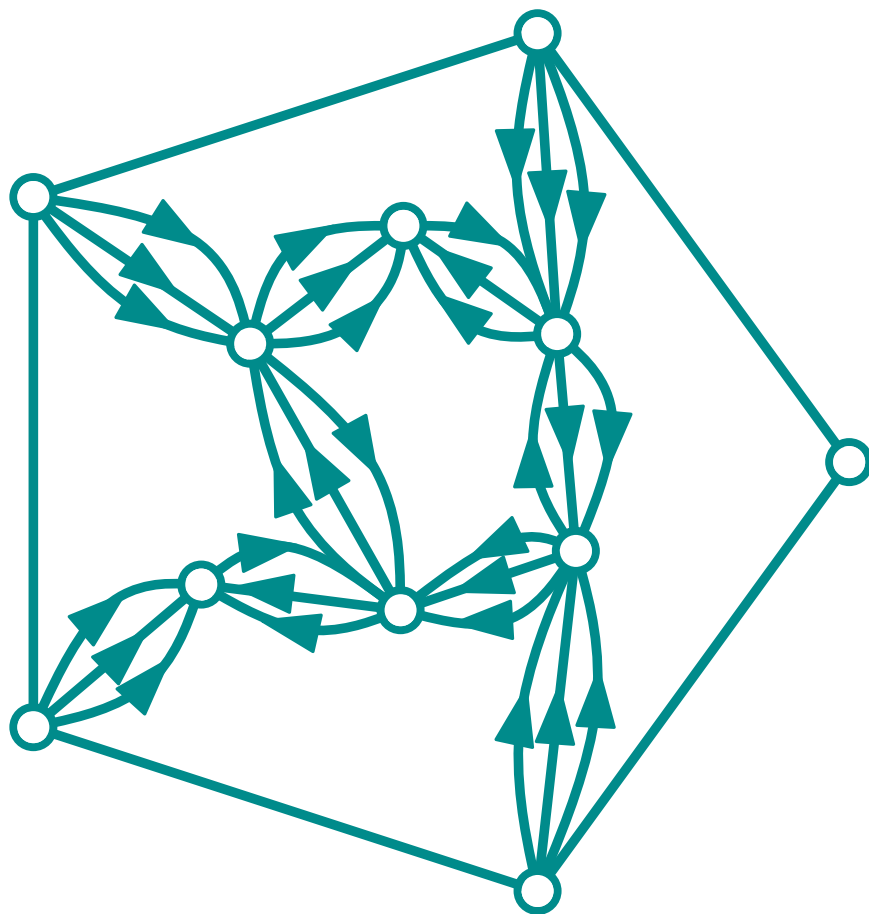
call $d=(d-2)$ -orientation such an orientation



d -angulations of girth d

Thm [Bernardi-F'10]: Let G be a d -angulation. Then $(d-2)G$ admits a $d=(d-2)$ -orientation if and only if G has girth d .

$d = 5$

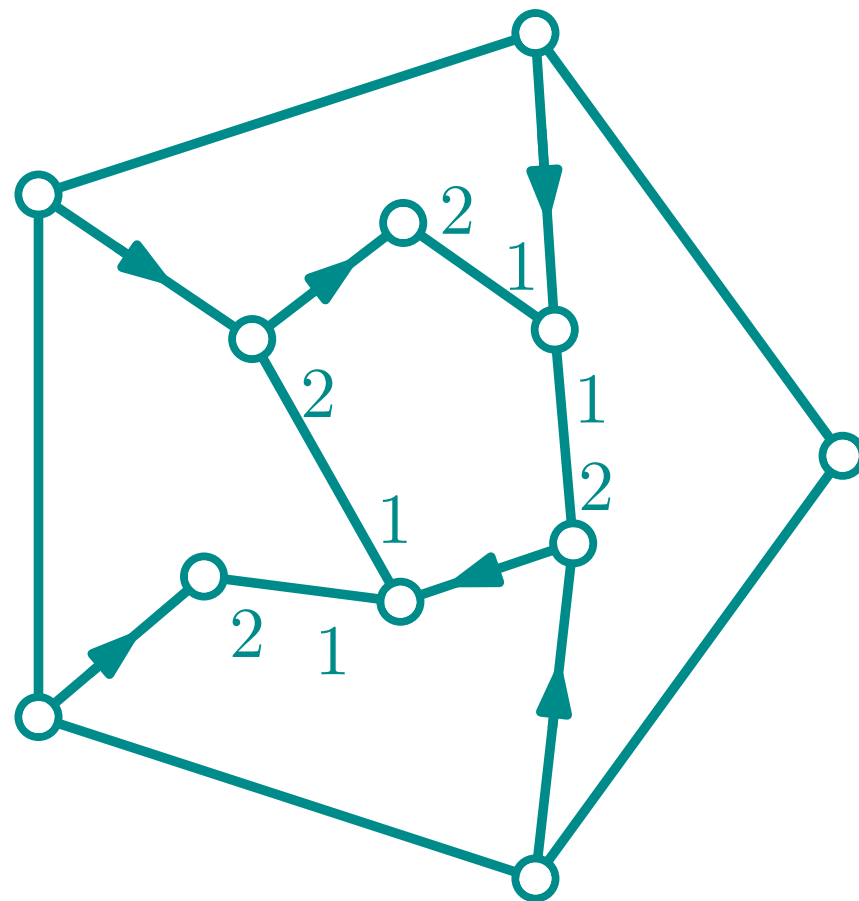
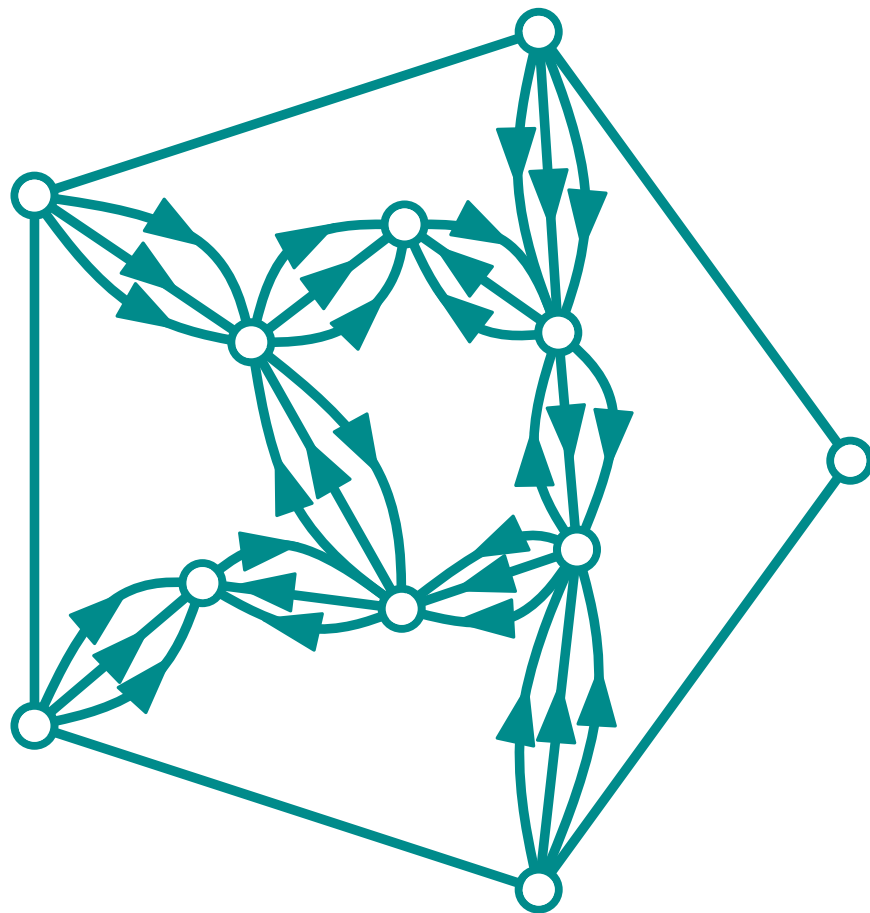


d -angulations of girth d

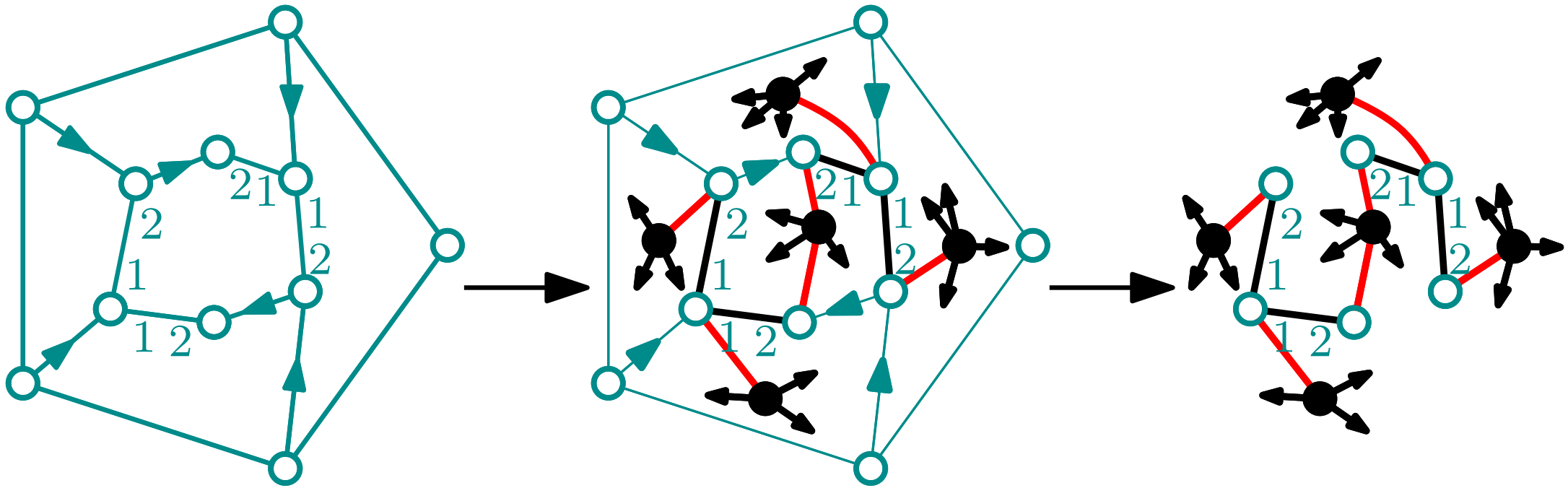
Thm [Bernardi-F'10]: Let G be a d -angulation. Then $(d-2)G$ admits a $d=(d-2)$ -orientation if and only if G has girth d .

Proof: Similar to $d = 3$. Uses the fact that a planar graph $G = (V; E)$ of girth at least d satisfies $\frac{|E| - d}{|V| - d} \geq d$

$d = 5$

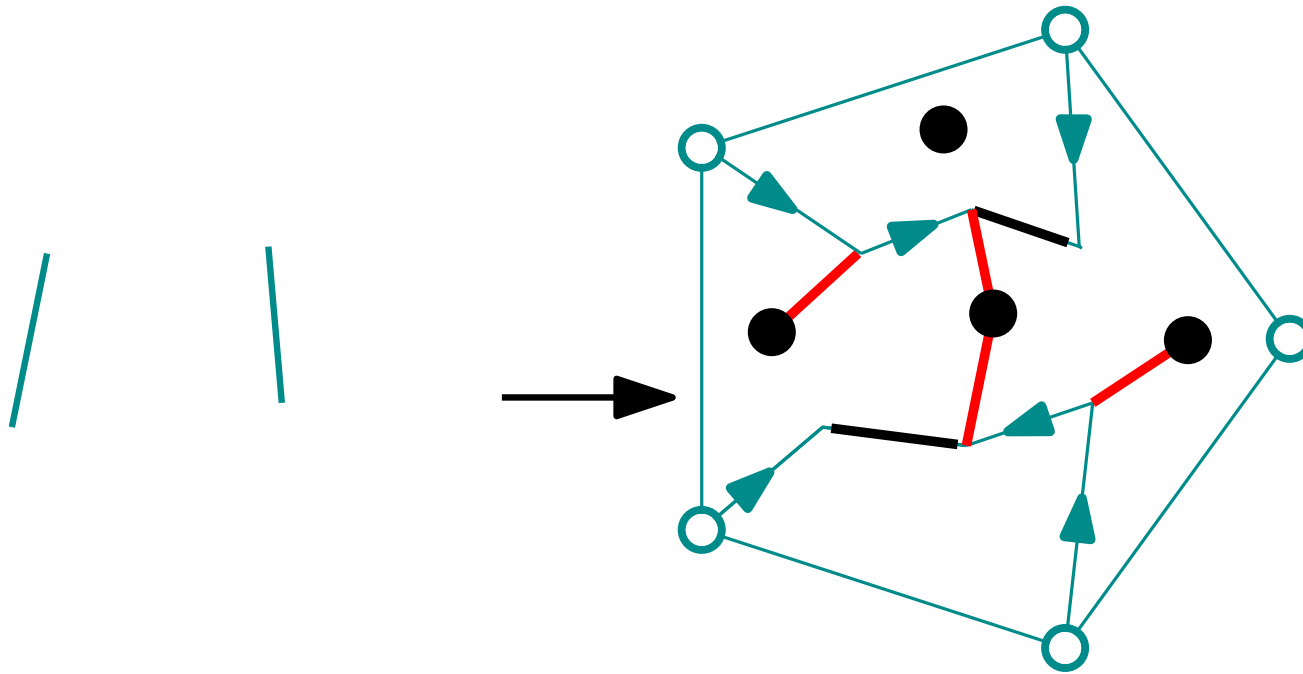


Master bijection for weighted orientations



There are now white-white edges in the mobile, with two positive weights summing to $d - 2$.

Master bijection for weighted orientations



Theorem [Bernardi-F'10]: The master bijection can be expressed in the weighted setting:

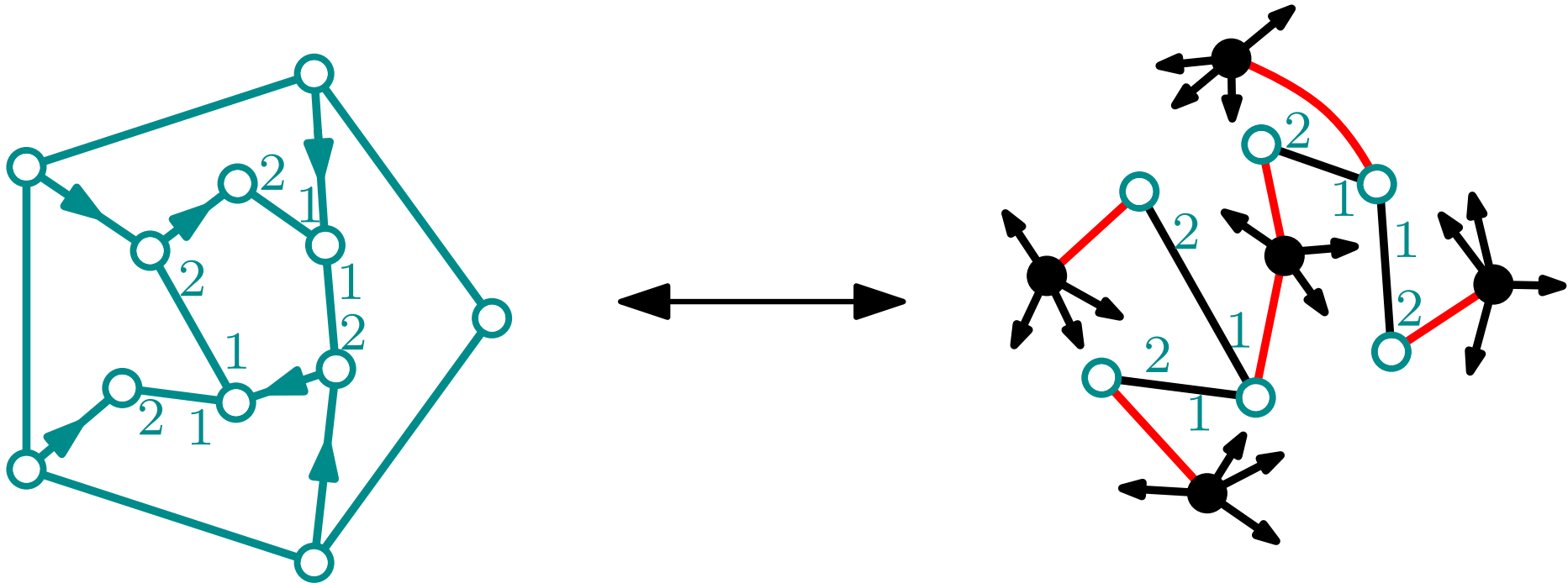
Moreover,

degree of internal faces	\longleftrightarrow	degree of black faces
indegree of internal vertices	\longleftrightarrow	indegree of white vertices
weights of internal edges	\longleftrightarrow	weights of edges
degree of external face	\longleftrightarrow	excess

d -angulations of girth d

Thm [Bernardi-F'10]: A d -angulation G admits a $d=(d-2)$ -orientation if and only if G has girth d .

\Rightarrow The class \mathcal{T}_d of d -angulations of girth d can be identified with the class of weighted orientations in \mathcal{O} , with faces of degree d , edges of weight $d-2$, and internal vertices of indegree d .

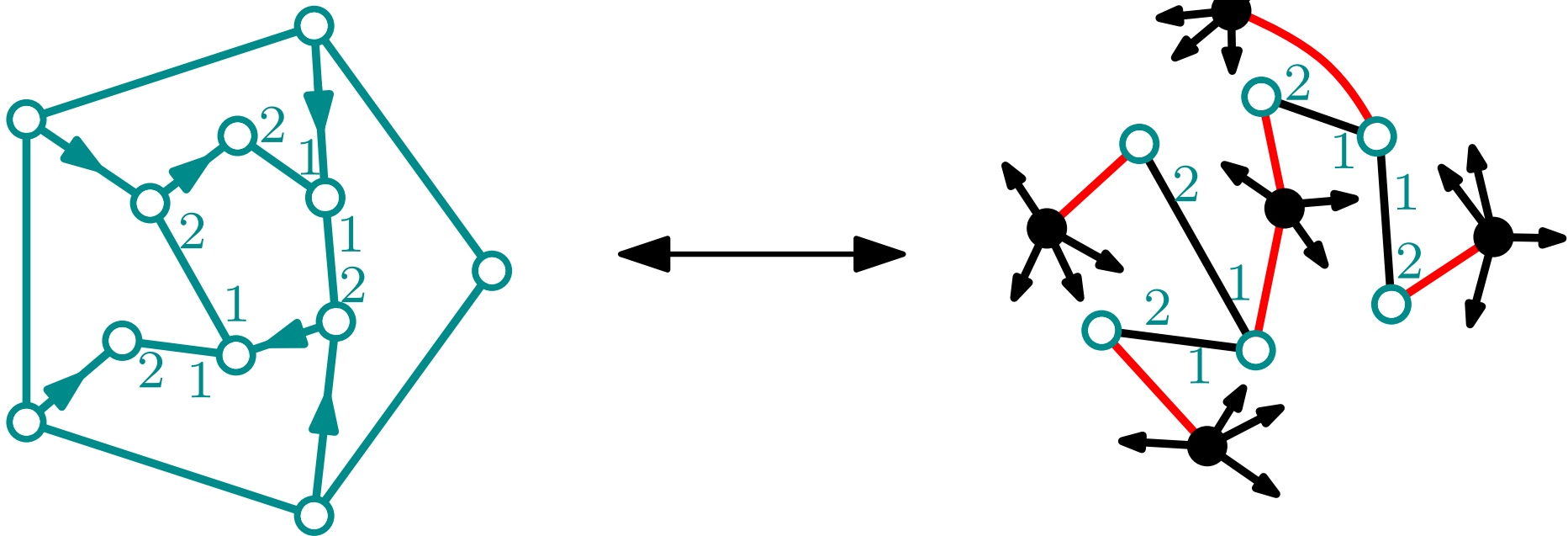


d -angulations of girth d

Thm [Bernardi-F'10]: A d -angulation G admits a $d=(d-2)$ -orientation if and only if G has girth d .

Thm [Bernardi-F'10]: By specializing the master bijection one obtains a bijection between d -angulations of girth d and mobiles (with white-white edges having weights summing to $d-2$) such that

- black vertices have degree d
- white vertices have indegree d
- the excess is d (redundant).



d -angulations of girth d : counting

Thm[Bernardi-F'10]: Let $W_0; W_1; \dots; W_{d-2}$ be the power series in x defined by: $W_{d-2} = x(1 + W_0)^{d-1}$

and $\forall j < d - 2; \quad W_j = \sum_r \sum_{\substack{i_1, \dots, i_r > 0 \\ i_1 + \dots + i_r = j+2}} W_{i_1} \cdots W_{i_r}.$

The generating function F_d of rooted d -angulations of girth d satisfies

$$F'_d(x) = (1 + W_0)^d.$$

d -angulations of girth d : counting

Thm[Bernardi-F'10]: Let $W_0; W_1; \dots; W_{d-2}$ be the power series in x defined by: $W_{d-2} = x(1 + W_0)^{d-1}$

$$\text{and } \forall j < d-2; \quad W_j = \sum_r \sum_{\substack{i_1, \dots, i_r > 0 \\ i_1 + \dots + i_r = j+2}} W_{i_1} \cdots W_{i_r}.$$

The generating function F_d of rooted d -angulations of girth d satisfies

$$F'_d(x) = (1 + W_0)^d.$$

Example $d=5$:

$$W_3 = x(1 + W_0)^4$$

$$W_0 = W_1^2 + W_2$$

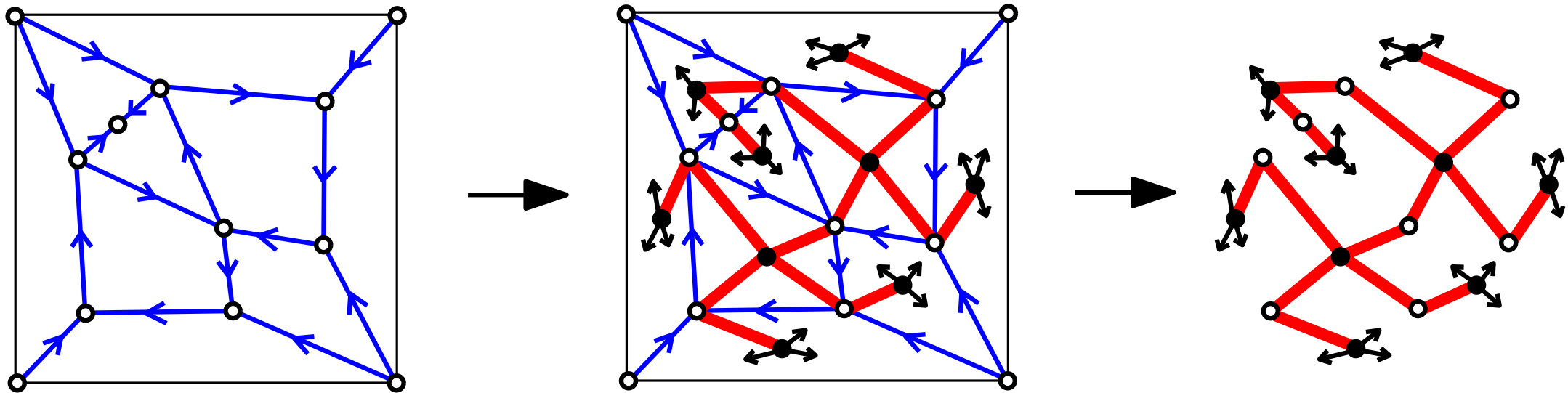
$$W_1 = W_1^3 + 2W_1W_2 + W_3$$

$$W_2 = W_1^4 + 3W_1^2W_2 + 2W_1W_3 + W_2^2$$

Simplification in the bipartite case

- For d even, $d = 2b$, we have $\frac{d}{d-2} = \frac{b}{b-1}$
- Can work with $b=(b-1)$ -orientations:
 - edges have weight $b-1$
 - vertices have indegree b

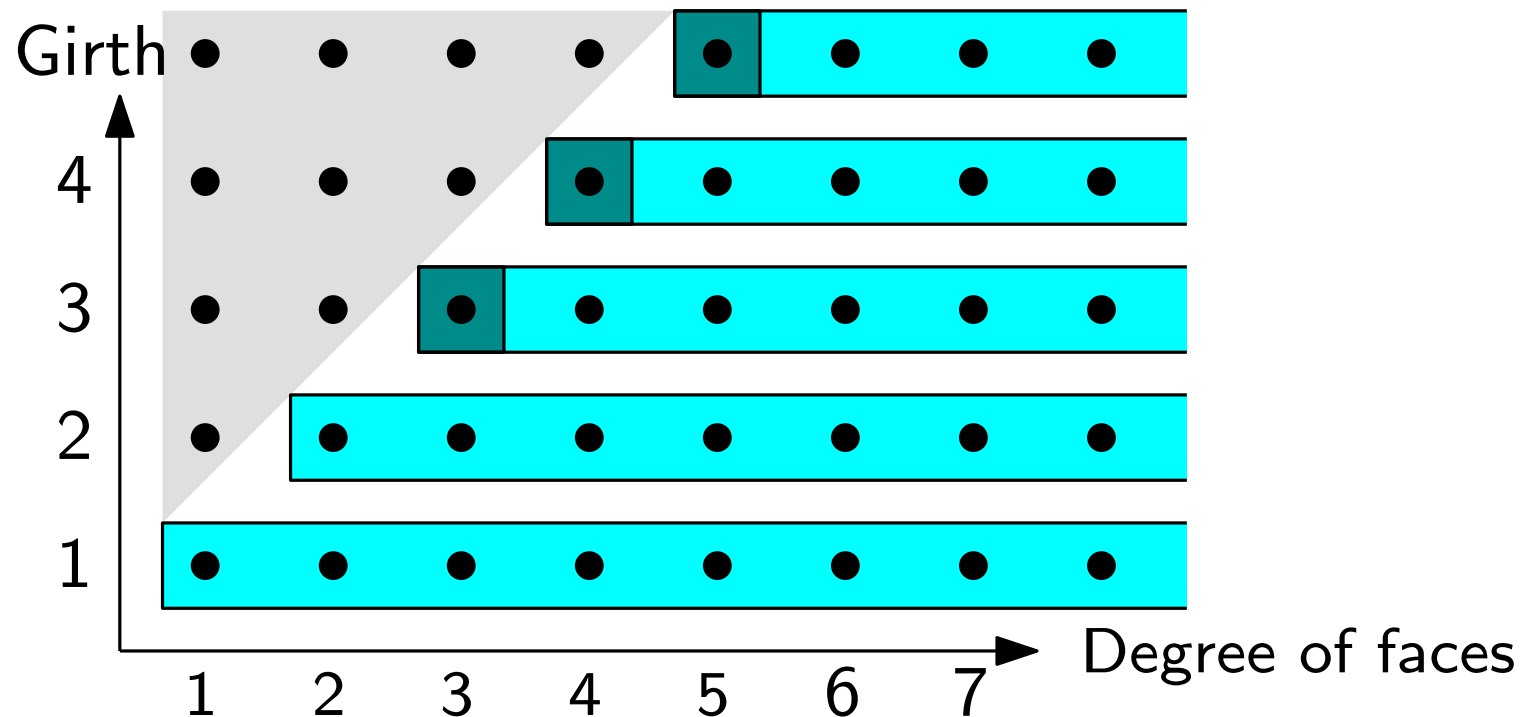
Example: $b = 2$, simple quadrangulations



recover a bijection of Schaeffer (1999)

More specializations

Maps of girth d .

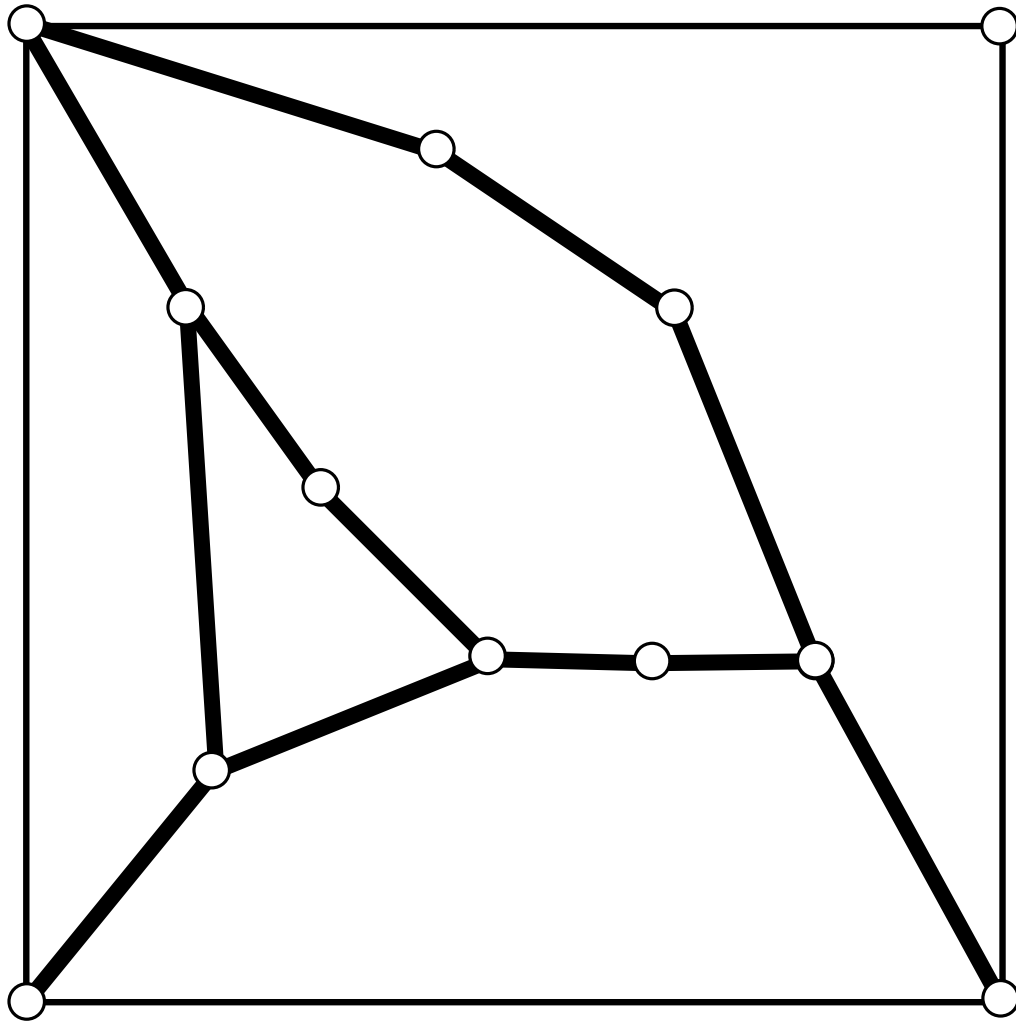


We show only the bipartite case (simpler)

Case $b = 2$ (simple bipartite maps), with quadrangular outer face

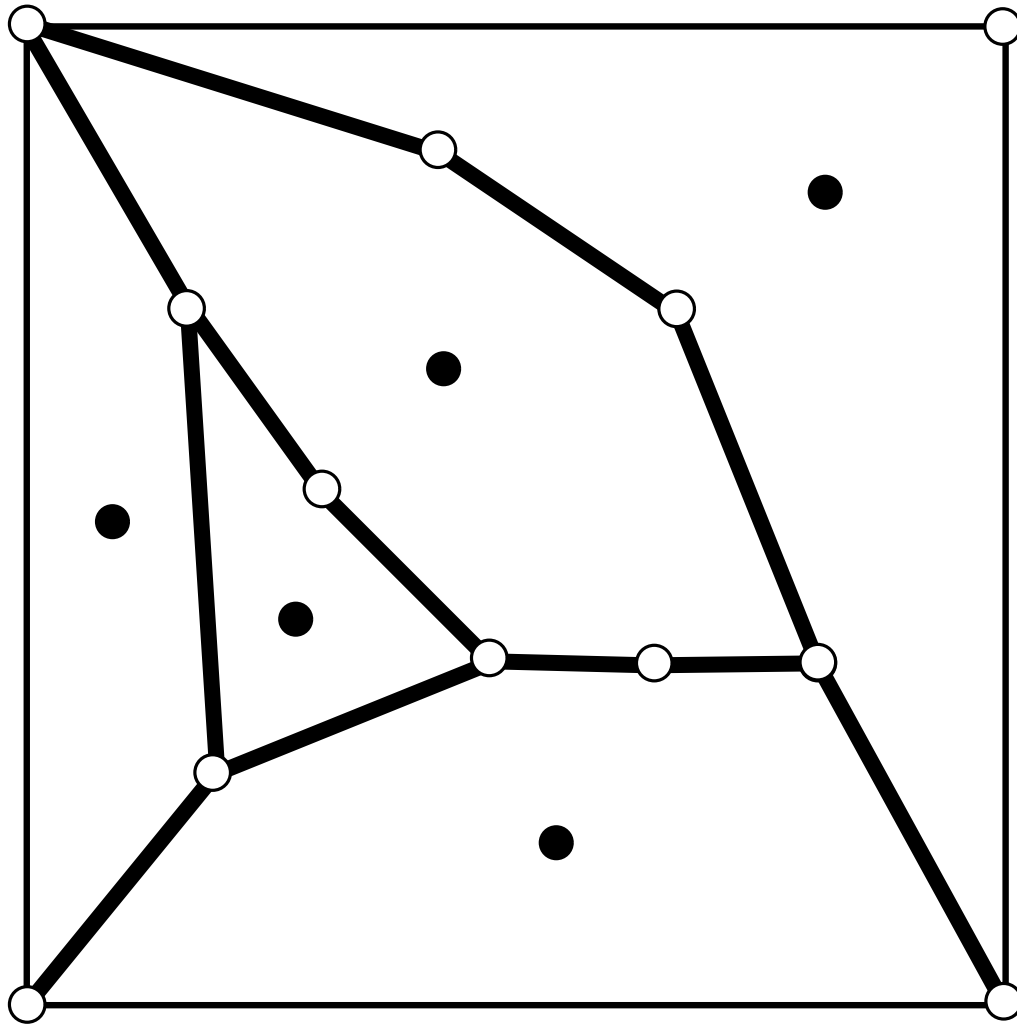
We show only the bipartite case (simpler)

Case $b = 2$ (simple bipartite maps), with quadrangular outer face



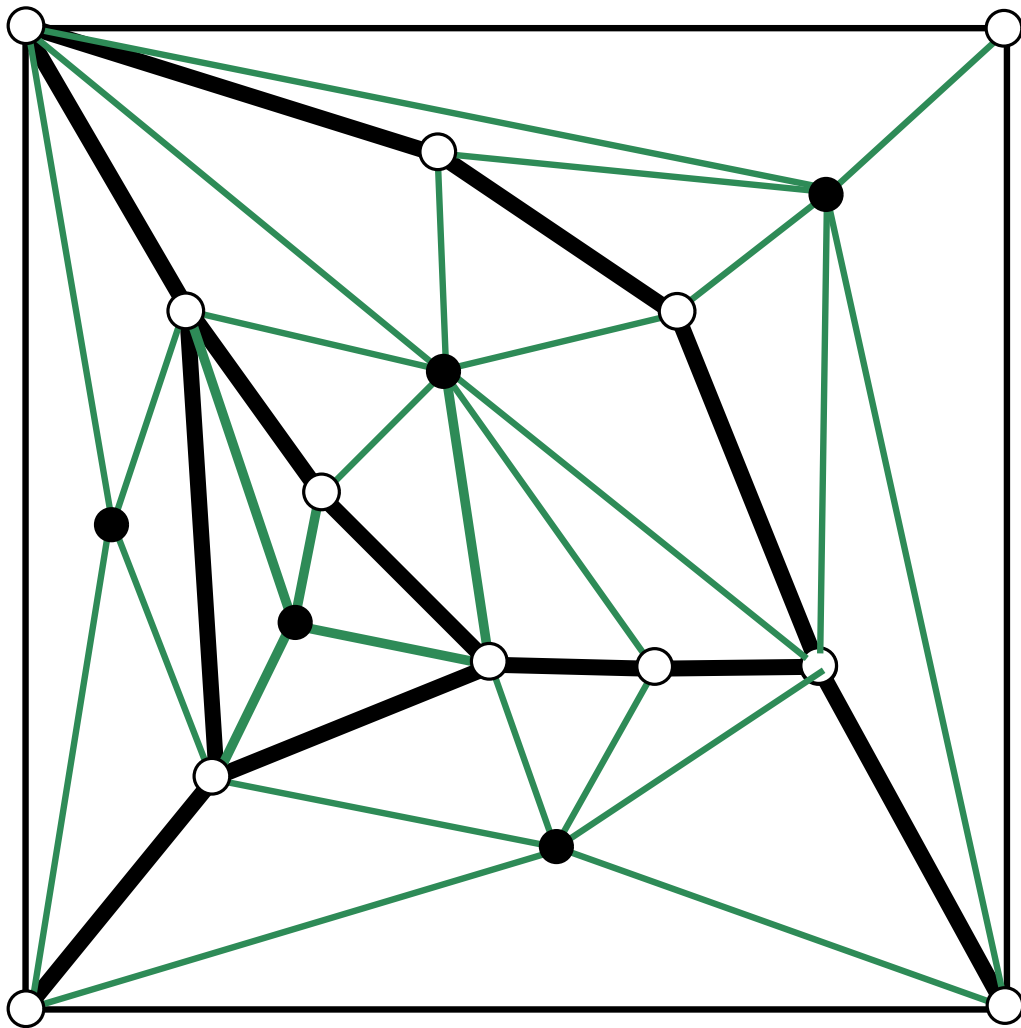
We show only the bipartite case (simpler)

Case $b = 2$ (simple bipartite maps), with quadrangular outer face



We show only the bipartite case (simpler)

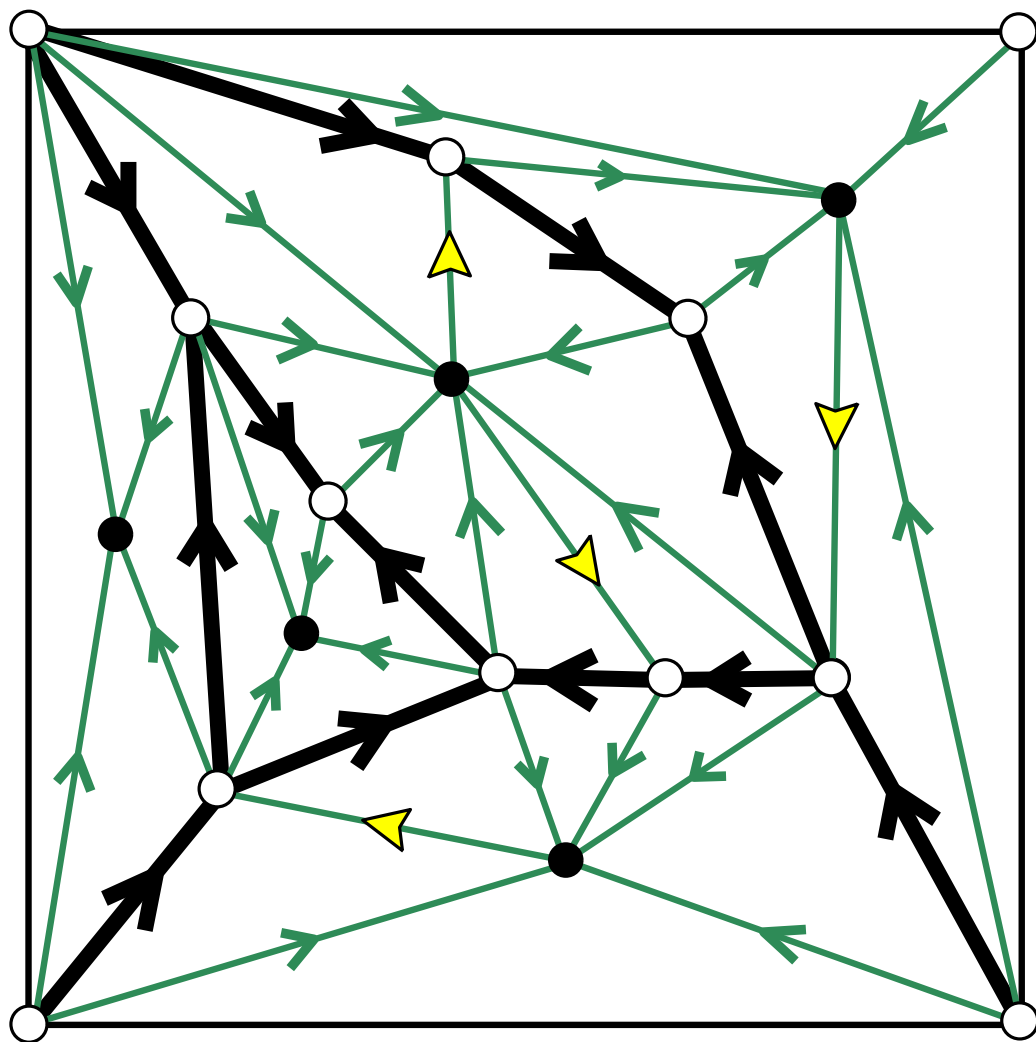
Case $b = 2$ (simple bipartite maps), with quadrangular outer face



Insert a star
in each internal face

We show only the bipartite case (simpler)

Case $b = 2$ (simple bipartite maps), with quadrangular outer face

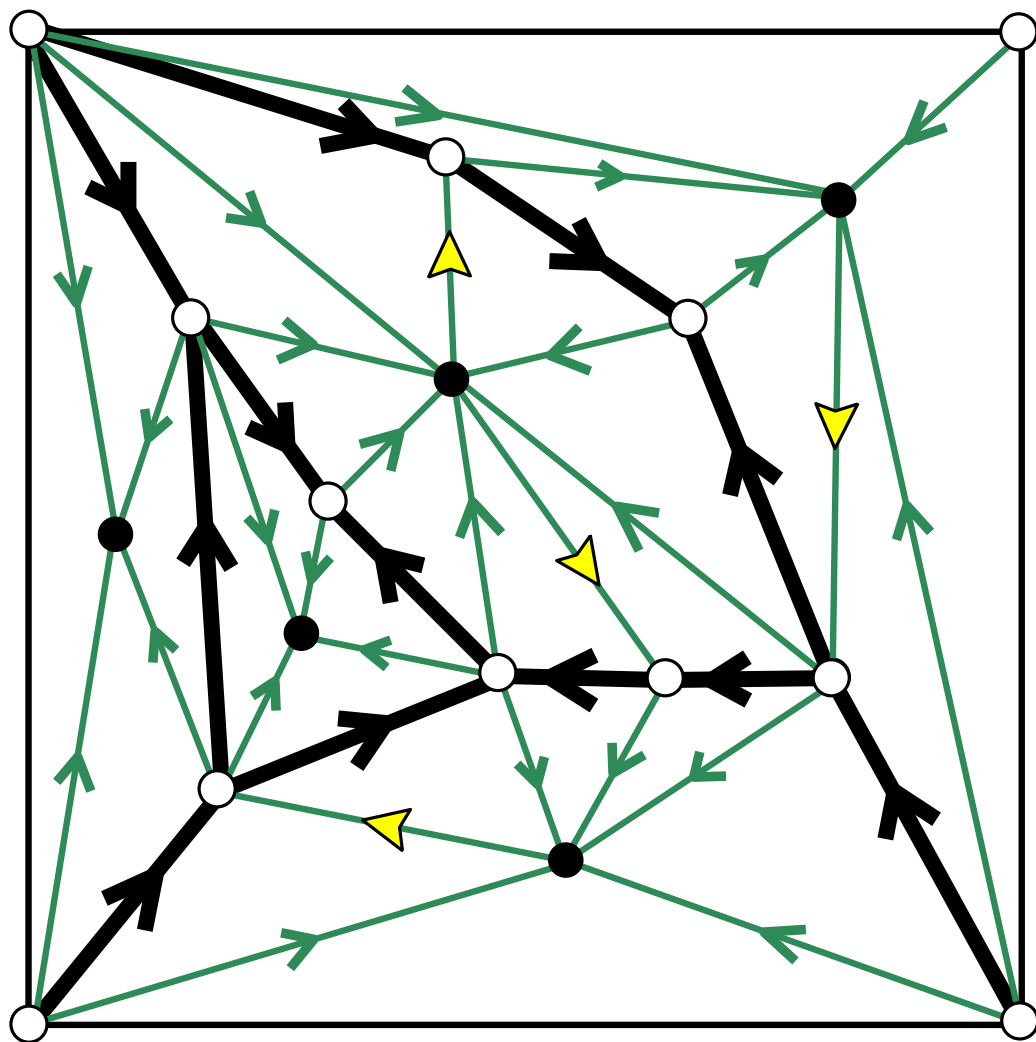


Generalized 2-orientation

- Each internal white vertex has indegree 2
- Each black vertex of degree $2i$ has outdegree $i - 2$

We show only the bipartite case (simpler)

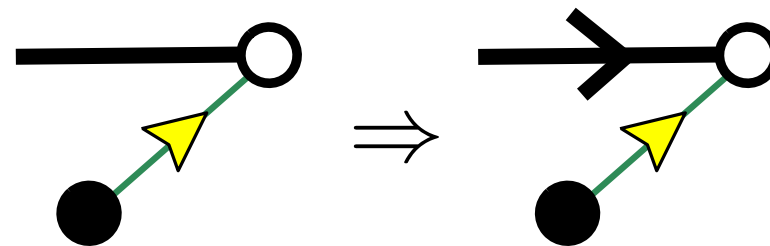
Case $b = 2$ (simple bipartite maps), with quadrangular outer face



Generalized 2-orientation

- Each internal white vertex has indegree 2
- Each black vertex of degree $2i$ has outdegree $i - 2$

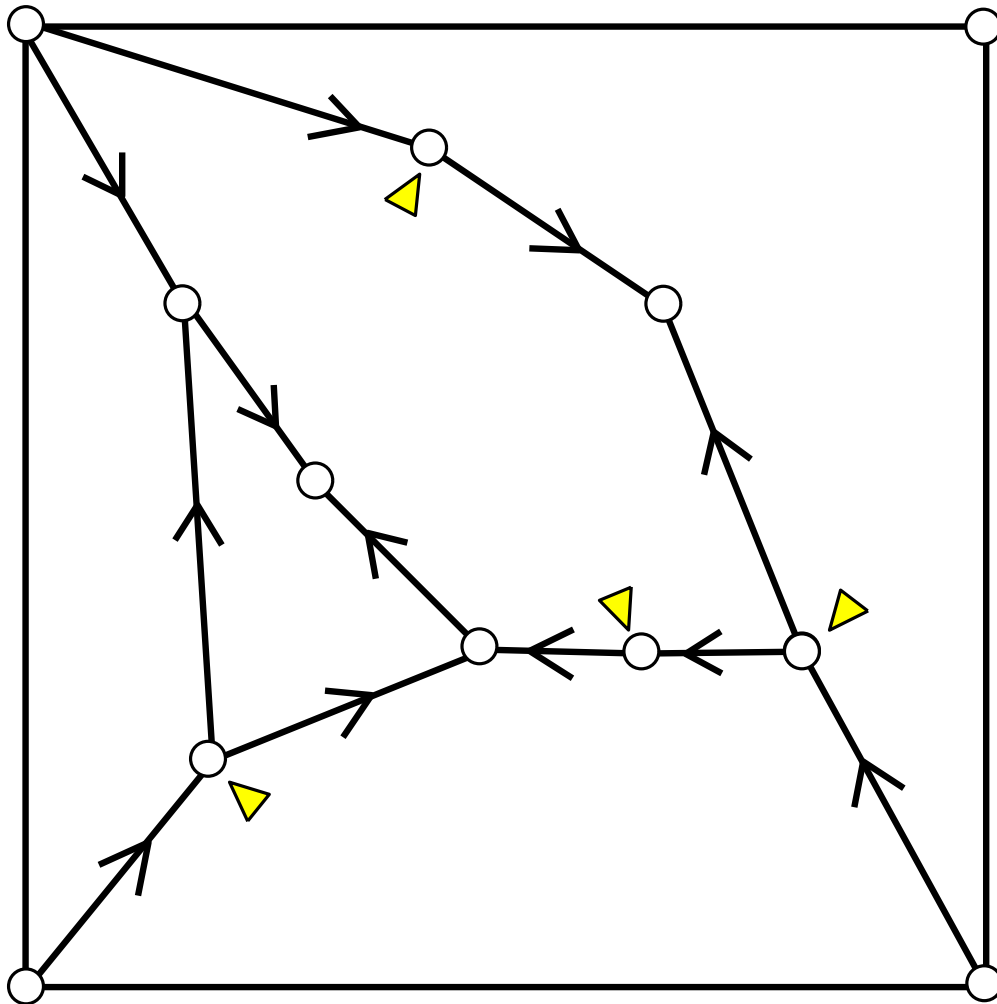
For the minimal one:



& still accessible after deleting the stars

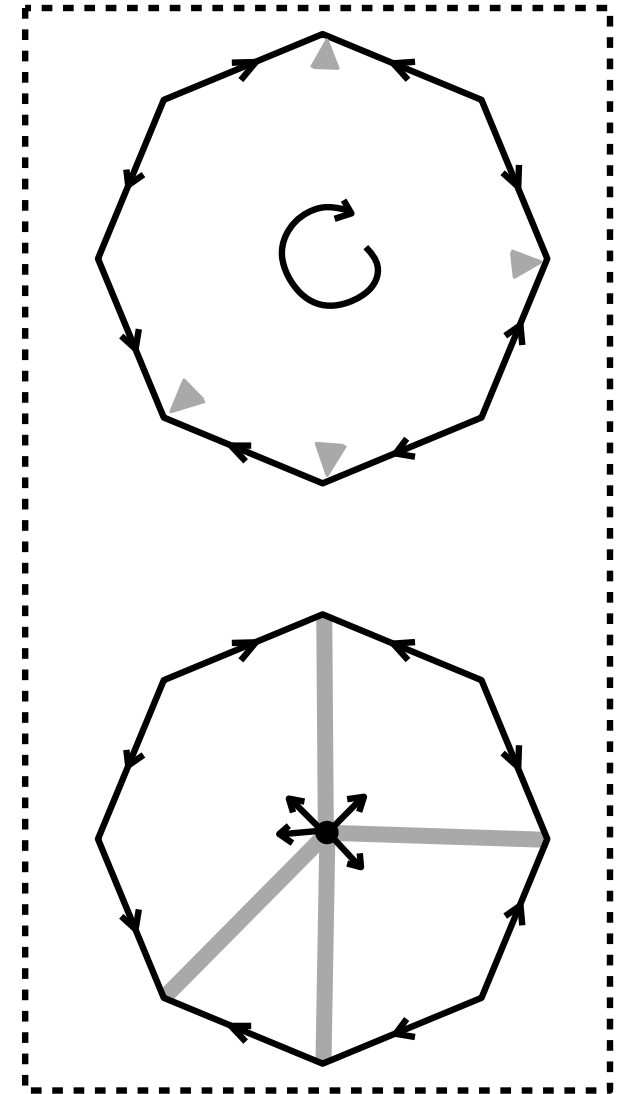
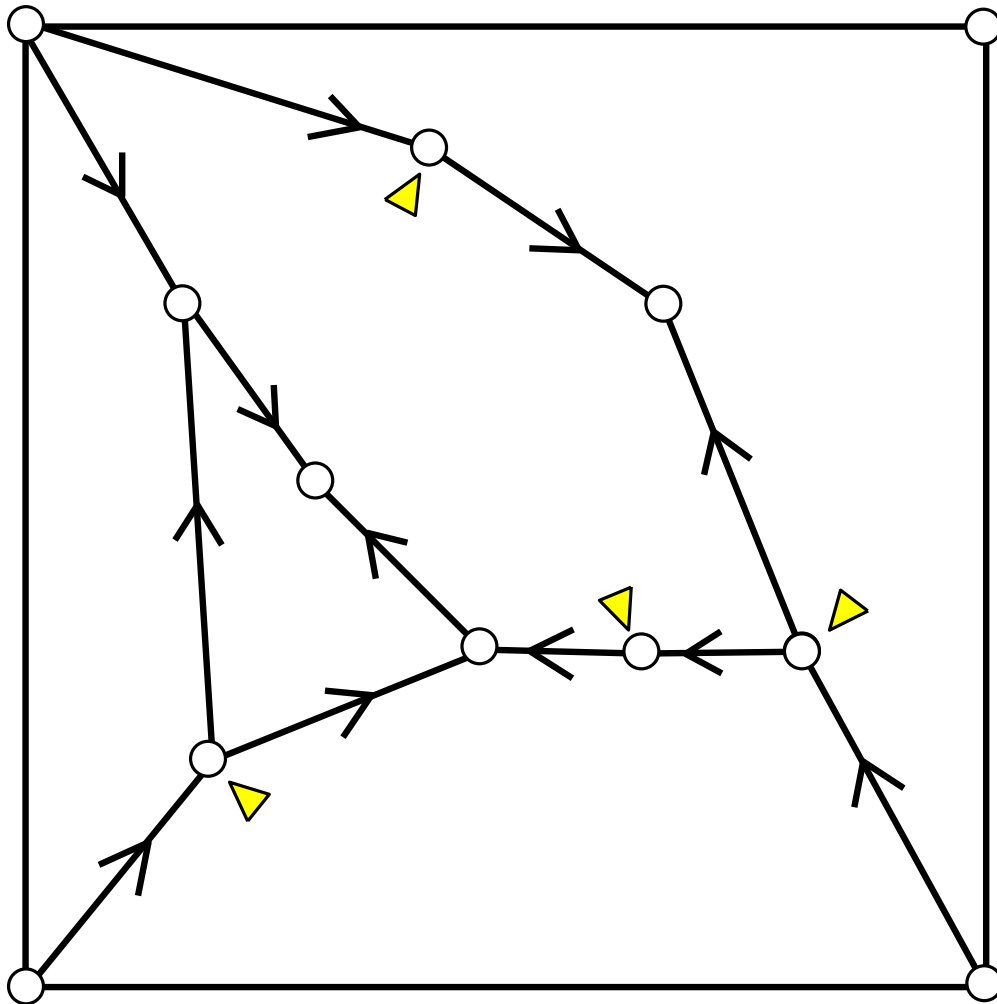
We show only the bipartite case (simpler)

Case $b = 2$ (simple bipartite maps), with quadrangular outer face



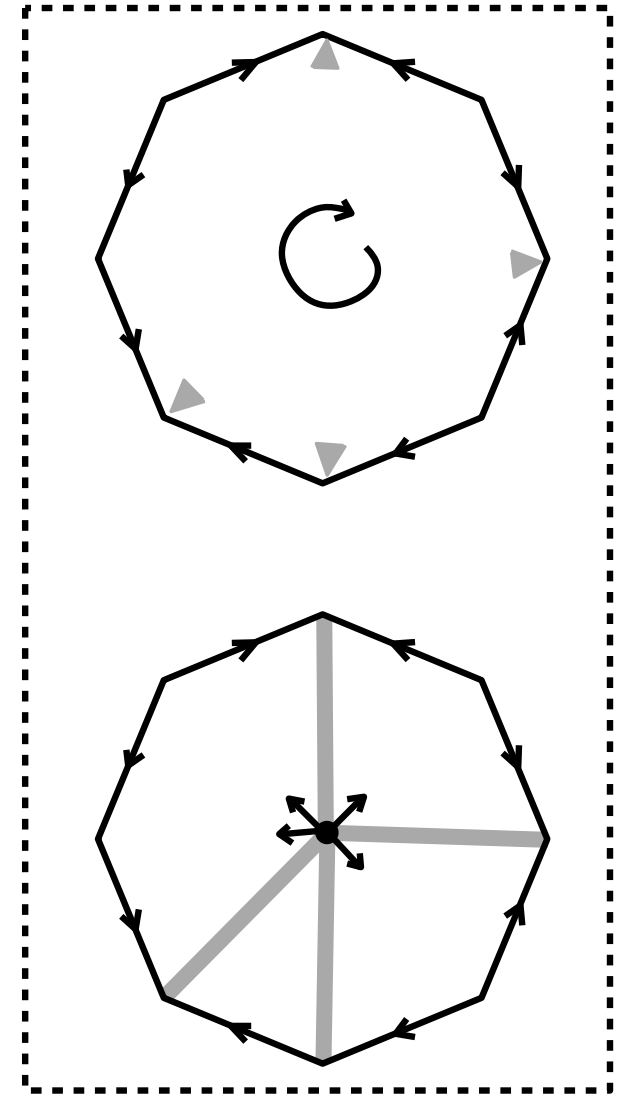
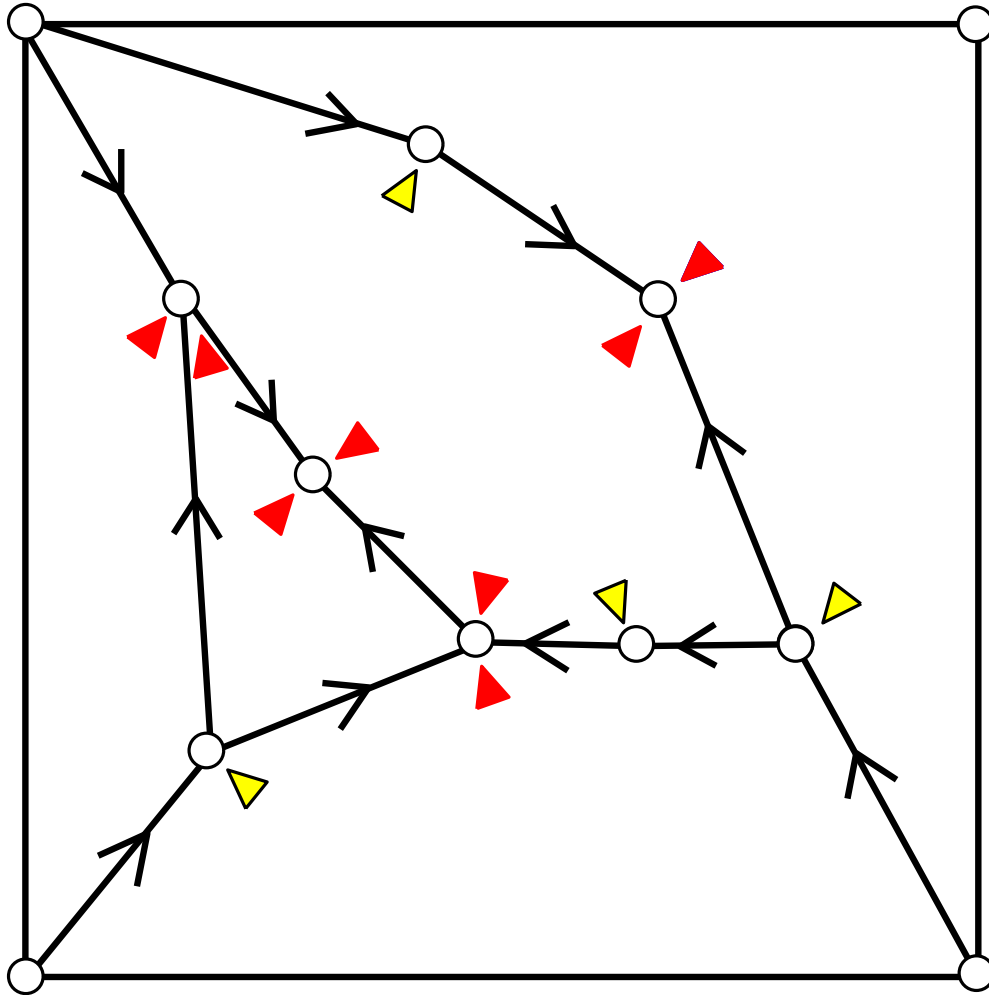
We show only the bipartite case (simpler)

Case $b = 2$ (simple bipartite maps), with quadrangular outer face



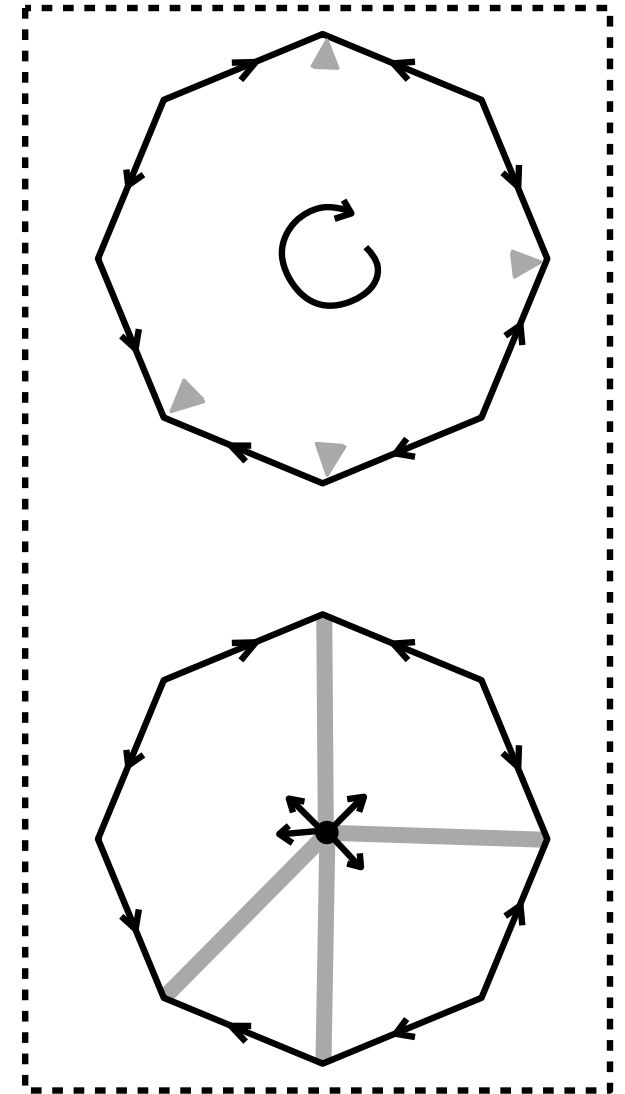
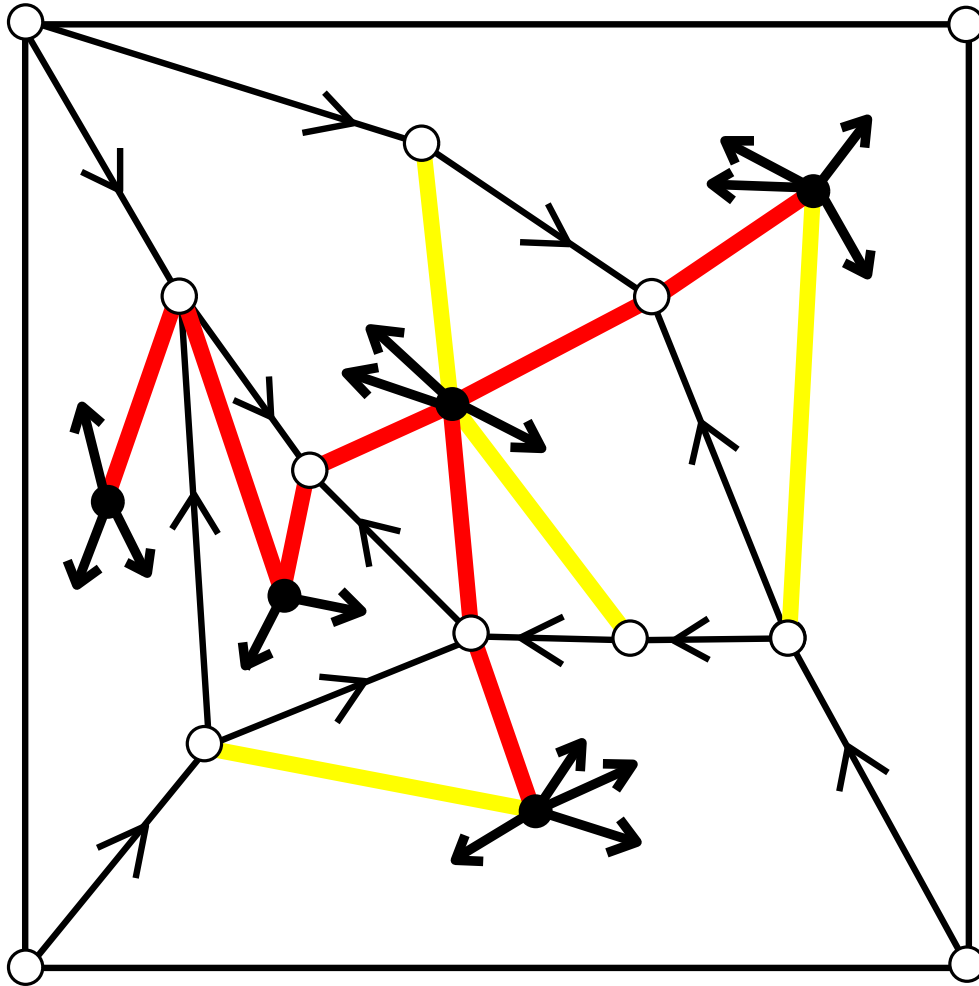
We show only the bipartite case (simpler)

Case $b = 2$ (simple bipartite maps), with quadrangular outer face



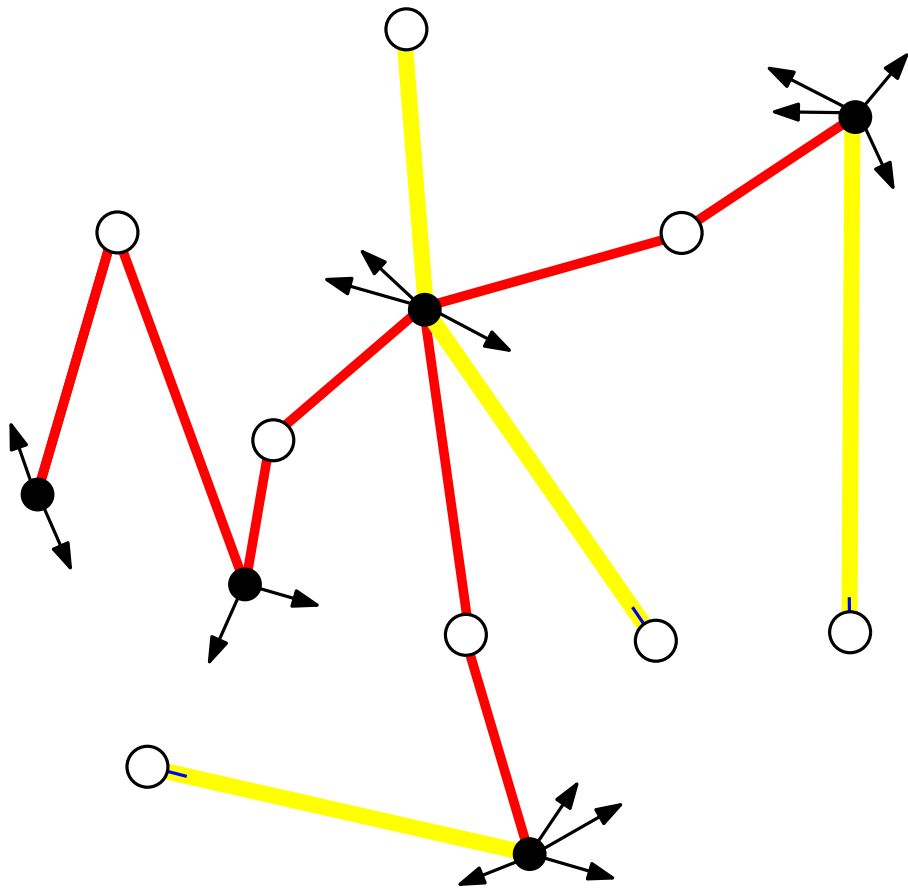
We show only the bipartite case (simpler)

Case $b = 2$ (simple bipartite maps), with quadrangular outer face



We show only the bipartite case (simpler)

Case $b = 2$ (simple bipartite maps), with quadrangular outer face



White vertices either have:

- indegree 2 (middle of red edge)
- indegree 1 (end of leg)

Each black vertex of degree $2i$ has $i - 2$ legs

Closed formulas

Prop [Bernardi-F'11]: The number of **rooted simple bipartite maps** with n_i faces of degree $2i$ is

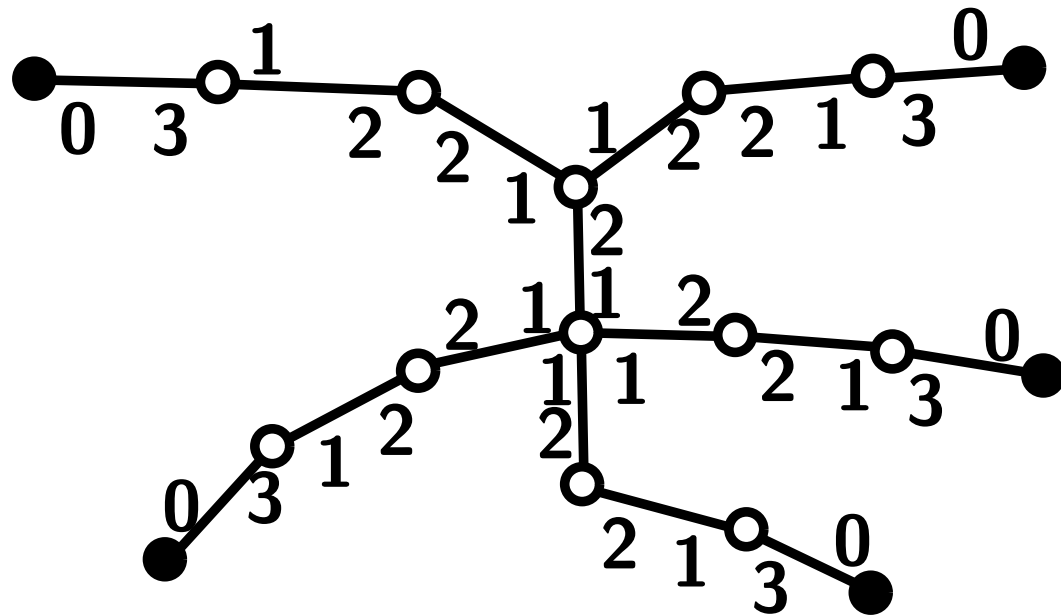
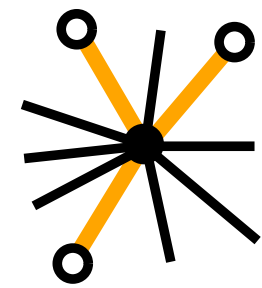
$$2 \frac{(\sum (i+1) n_i - 3)!}{(\sum i n_i - 1)!} \prod_{i \geq 2} \frac{1}{n_i!} \binom{2i-1}{i+1}^{n_i}$$

This can be compared with the formula obtained by Tutte (62) (recovered bijectively by Schaeffer) for **unconstrained rooted bipartite maps**:

$$2 \frac{(\sum i n_i)!}{(\sum (i-1) n_i + 2)!} \prod_{i \geq 1} \frac{1}{n_i!} \binom{2i-1}{i}^{n_i}$$

Shape of the mobile in higher (bipartite) girth

- Each black vertex of degree $2i$ has $i - b$ legs
- There are **connectors** between the black vertices



a connector for $b = 4$

Connectors, for $b = 1$: $\bullet \overset{0}{\text{---}} \overset{0}{\text{---}} \bullet$
 $b = 2$: $\bullet \overset{0}{\text{---}} \overset{1}{\circ} \overset{1}{\text{---}} \overset{0}{\text{---}} \bullet$
 $b = 3$: **binary trees**

Thanks.

On the ArXiv:

- A bijection for triangulations, quadrangulations, pentagulations, etc.
- Bijective counting of maps by girth and degree.