A master bijection for planar maps and its applications

Éric Fusy (CNRS/LIX)
Joint work with Olivier Bernardi (MIT)
Planar graphs. Definition

A **planar graph** is a graph that can be drawn in $\mathbb{R}^2$ without edge-crossing.

$K_4$ is planar

$K_5$ is not planar
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- \( K_4 \) is planar
- \( K_5 \) is not planar

**Rk:** Can be drawn in \( \mathbb{R}^2 \) \( \iff \) can be drawn in \( S^2 \)
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Planar maps and plane maps. **Definition**

- A **planar map** is a connected planar graph drawn in the sphere considered up to continuous deformation.
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(i) A map has vertices and edges (like a graph), **and also faces**

(ii) Encoded by **cyclic order of neighbours** around each vertex
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Planar maps and plane maps. **Definition**

- A **planar map** is a connected planar graph drawn in the sphere considered up to continuous deformation.

(i) A map has vertices and edges (like a graph), and also **faces**
(ii) Encoded by **cyclic order of neighbours** around each vertex

- A **plane map** is a connected planar graph drawn in the plane considered up to continuous deformation.

Rk: Plane map = planar map **with a marked face** (the outer face)
The Euler relation
Let $M = (V, E, F)$ be a planar map. Then

$$|V| - |E| + |F| = 2$$
The Euler relation

Let $M = (V, E, F)$ be a planar map. Then

\[
|V| - |E| + |F| = 2
\]

\(\iff\)

\[
|E| = (|V| - 1) + (|F| - 1)
\]
The Euler relation

Let $M = (V, E, F)$ be a planar map. Then

\[ |V| - |E| + |F| = 2 \]

\[ \iff \quad |E| = (|V| - 1) + (|F| - 1) \]

\[ \Rightarrow \text{ simple planar graph } G = (V, E) \text{ satisfies } |E| \leq 3|V| - 6 \]

(hence $K_5$ has too many edges to be planar)
Planar maps. 

Motivations

- Algorithmic applications: efficient encoding of meshed surfaces.

- Probability and Physics: random lattices, random surfaces.

- Representation Theory: factorization problems.
Symmetry issues.

In order to identify vertices unambiguously (to *avoid symmetry issues*):

- **Planar graphs**: need to **label the vertices**

  ![A labelled planar graph](image)

- **Planar maps**: only need to **mark a corner**

  ![A rooted planar map](image)
Asymptotic behaviour of planar maps/graphs

- **Asymptotic number:**

  
  **Labelled planar graphs** \( n \) vertices:
  \[
  \sim n! \, c \, n^{-7/2} \gamma^n
  \]
  [Giménez, Noy’05]

  **Rooted planar maps** \( n \) edges:
  \[
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  \]
  [Tutte’63]
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**Rk:** In both cases, number of **rooted labelled objects** is \( \sim n! cn^{-5/2} \gamma^n \)
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- **Rooted**

  In both cases, number of **rooted labelled objects** is $\sim n!cn^{-5/2}\gamma^n$

- **Random planar graph/map of size $n$ (for $n$ large):**

  ...
Asymptotic behaviour of planar maps/graphs

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- **Random planar graph/map of size $n$ (for $n$ large):**
  - **Local parameters:** $\mu \cdot n + \sigma \sqrt{n} \cdot X$
  - \( \sim n!cn^{-5/2}\gamma^n \) **gaussian fluctuations**
Asymptotic behaviour of planar maps/graphs

- **Asymptotic number:**
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  - **Local parameters:** $\mu \cdot n + \sigma \sqrt{n} \cdot X$
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    - [Gao Wormald] [Mac Diarmid, Reed] [Drmota et al’2011]
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  - **Diameter:** scale is \( n^{1/4} \)
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    - [Chapuy el al’10]
Asymptotic behaviour of planar maps/graphs

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- **Why do they have same behaviour ?**
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- Why do they have same behaviour?  [Giménez, Noy, Rué’10]

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<tr>
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- **Why do they have same behaviour?** [Giménez, Noy, Rué’10]

Random planar graph $\sim$ random (3-connected) planar map of size $\Theta(n)$ + little pieces attached into it

- **Planar maps:**
  
  - simpler enumeration formulas
  
  - can control distance parameters
  
  - bijections!
The girth parameter

The **girth** of a graph is the length of a shortest cycle within the graph.
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Girth = 3

Rk: If $girth = d$ then all faces have **degree at least** $d$
The girth parameter

The **girth** of a graph is the length of a shortest cycle within the graph.

**Rk:** If \( girth = d \) then all faces have **degree at least** \( d \)

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<td>Loopless</td>
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\[
Girth = 3
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Many natural map families are specified by constraints on the *girth* and on the **face-degrees** (loopless triangulations, simple quadrangulations, ...).
Planar maps. Exact counting results

- Triangulations ($2n$ faces)
  - Loopless: $\frac{2^n}{(n+1)(2n+1)} \binom{3n}{n}$
  - Simple: $\frac{1}{n(2n-1)} \binom{4n-2}{n-1}$

- Quadrangulations ($n$ faces)
  - General: $\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$
  - Simple: $\frac{2}{n(n+1)} \binom{3n}{n-1}$

- Bipartite maps ($n_i$ faces of degree $2i$)
  $\frac{2 \cdot (\sum i n_i)!}{(2 + \sum (i-1)n_i)!} \prod_i \frac{1}{n_i!} \binom{2i-1}{i}^{n_i}$
Planar maps. Counting methods

- **Generating functions** [Tutte 63]
  Recursive description of maps $\leadsto$ recurrences.

- **Matrix Integrals** [’t Hooft 74, Brézin et al’78]
  Feynmann Diagram $\approx$ maps.

- **Bijections** [Cori-Vauquelin 81, Schaeffer 98]
  Maps $\leadsto$ decorated trees.
Outline

1. **Master bijection** between a class of **oriented maps** and a class of bicolored **decorated trees** (which are called mobiles).

2. **Specializations** to classes of maps (via canonical orientations).

![Diagram showing bijection between oriented maps and decorated trees](image-url)

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</tr>
<tr>
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<td>[PoSc02]</td>
</tr>
<tr>
<td>3</td>
<td>[FuPoSc08]</td>
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<tr>
<td>4</td>
<td>[Sc98]</td>
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References:

- Sc98
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- BoDiGu04
From oriented maps to mobiles
Pointed bipartite map $\rightarrow$ labelled mobile.  [Sc98] [BoDiGu04]
Pointed bipartite map $\rightarrow$ labelled mobile. \cite{Sc98} \cite{BoDiGu04}

Label the vertices by the distance from pointed vertex.
Pointed bipartite map $\rightarrow$ labelled mobile.  \[[\text{Sc98}]\] \[[\text{BoDiGu04}]\]

Construct the labelled mobile.
Pointed bipartite map $\rightarrow$ labelled mobile.  \[\text{[Sc98]} \quad \text{[BoDiGu04]}\]

Construct the labelled mobile

(i) put one black vertex in each face
Pointed bipartite map $\rightarrow$ labelled mobile.  

[Sc98] [BoDiGu04]

Construct the labelled mobile

(i) put one black vertex in each face

(ii) each edge of the map gives one edge in the mobile
Pointed bipartite map $\rightarrow$ labelled mobile. \cite{Sc98,BoDiGu04}

Proof that the mobile is a tree

Let $G = (V, E, F)$ be a pointed bipartite map
Let $T$ be the associated mobile
**Pointed bipartite map $\rightarrow$ labelled mobile.**  

Let $G = (V,E,F)$ be a pointed bipartite map. Let $T$ be the associated mobile.

**Local rule**

$T$ has $|E|$ edges, and has $|V| + |F| - 1 = |E| + 1$ vertices (Euler relation).

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smallest label on $C$
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The smallest label on $C$ is $i$.
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Label $< i$ contradiction.
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Pointed bipartite map $\rightarrow$ labelled mobile. \cite{Sc98, BoDiGu04}

\begin{itemize}
  \item [(i)] $\exists$ vertex label 1
  \item [(ii)] $j \leq i + 1$
\end{itemize}
Pointed bipartite map $\to$ labelled mobile.  

\[ \Rightarrow \]

\begin{itemize}
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\end{itemize}

**Local rule**

**Theorem:** The mapping is a \textbf{bijection}. Each \textbf{face of degree $2i$} of the bipartite map corresponds to a \textbf{black vertex of degree $i$} in the mobile.
Pointed bipartite map \(\rightarrow\) labelled mobile. \([\text{Sc98}]\) \([\text{BoDiGu04}]\)

\[
\begin{align*}
\text{Local rule} & \quad (i) \exists \text{ vertex label } 1 \\
\text{Conditions:} & \quad (\text{i}) \quad j \leq i+1
\end{align*}
\]

**Theorem:** The mapping is a **bijection**. Each face of degree 2\(i\) of the bipartite map corresponds to a **black vertex of degree** \(i\) in the mobile

\[
\# \text{ rooted bipartite maps with } n_i \text{ faces of degree } 2i \text{ is } \frac{2 \cdot (\sum i n_i)!}{(2 + \sum (i - 1)n_i)!} \prod_i \frac{1}{n_i!} \left(\frac{2i - 1}{i}\right)^{n_i}
\]
Reformulation with orientations.

Distance labelling

Geodesic orientation

Local rule

\[ \delta = i - j \]

\[ \delta + 1 \] buds
Reformulation with orientations.

Condition: At each black vertex, as many buds as white neighbours.

Theorem: The mapping is a **bijection**. Each face of degree $2i$ of the bipartite map corresponds to a **black vertex of degree** $2i$ in the mobile.
Reformulation with orientations.

Condition:
At each black vertex, as many buds as white neighbours

Theorem: The mapping is a **bijection**. Each face of degree $2i$ of the bipartite map corresponds to a **black vertex of degree** $2i$ in the mobile
Source-orientations

A source-orientation is an orientation of a pointed map such that

- The pointed vertex (called the source) has only outgoing edges
- Accessibility: Each vertex can be reached from the source
Mobile construction for source-orientations
Mobile construction for source-orientations

Local rule
Mobile construction for source-orientations

Cycle in mobile ⇒ ccw circuit in the source-orientation
Mobile construction for source-orientations

Cycle in mobile $\Rightarrow$ ccw circuit in the source-orientation
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Prisoner cycle lemma
Mobile construction for source-orientations

Cycle in mobile $\Rightarrow$ ccw circuit in the source-orientation

Prisoner cycle lemma
d-gonal source-orientations

We allow the source of the orientation to be a $d$-gon, with $d \geq 0$

Example for $d = 3$
d-gonal source-orientations
We allow the source of the orientation to be a $d$-gon, with $d \geq 0$

Example for $d = 3$

If $d > 0$, can take $d$-gonal source as outer face
**d-gonal source-orientations**

We allow the source of the orientation to be a $d$-gon, with $d \geq 0$

Example for $d = 3$

Let $\mathcal{O}_d$ be the set of $d$-gonal source-orientations with no ccw circuit

Let $\mathcal{O} = \bigcup_{d \geq 0} \mathcal{O}_d$
Mobiles

A mobile is a plane tree with vertices properly colored in black and white, together with buds (half-edges) incident to black vertices.

The excess is the number of buds minus the number of edges.
Mobiles

A **mobile** is a plane tree with vertices properly colored in black and white, together with **buds** (half-edges) incident to black vertices.

![Diagram of a mobile]

The **excess** is the number of buds minus the number of edges.

Let $\mathcal{M}$ be the set of mobiles of nonnegative excess.
Theorem [Bernardi-F’10]: \( \Phi \) is a \textbf{bijection} between \( \mathcal{O} \) and \( \mathcal{M} \).

Moreover,

- degree of external face \( \leftrightarrow \) excess
- degree of internal faces \( \leftrightarrow \) degree of black vertices
- indegree of internal vertices \( \leftrightarrow \) degree of white vertices
Master bijection $\Phi$

Theorem [Bernardi-F’10]: $\Phi$ is a bijection between $O$ and $M$.
Moreover,

- degree of external face $\leftrightarrow$ excess
- degree of internal faces $\leftrightarrow$ degree of black vertices
- indegree of internal vertices $\leftrightarrow$ degree of white vertices

cf [Bernardi’07], [Bernardi-Chapuy’10]
Using the master bijection for map enumeration
Main new results

The **Master bijection** between $\mathcal{O}$ (orientations) and $\mathcal{M}$ (mobiles) allows to count maps by girth & face-degrees (via canonical orientations).

<table>
<thead>
<tr>
<th>Girth</th>
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<tbody>
<tr>
<td>2</td>
<td>1, 2, 3, 4, 5, 6, 7, 8</td>
</tr>
<tr>
<td>3</td>
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References:
- [FuPoSc08]
- [Sc98]
- [PoSc02]
Scheme for the strategy

(1) Map family $\mathcal{C}$ identifies with a subfamily $\mathcal{O}_C$ of $\mathcal{O}$ with conditions on:

- Face degrees
- Vertex indegrees
Scheme for the strategy

(1) Map family $\mathcal{C}$ identifies with a **subfamily** $\mathcal{O}_\mathcal{C}$ of $\mathcal{O}$ with conditions on:

- Face degrees
- Vertex indegrees

**Example:** $\mathcal{C} =$ Family of **simple triangulations**

$\mathcal{C} \simeq$ subfamily $\mathcal{O}_\mathcal{C}$ of $\mathcal{O}$ with

- Face-degree $= 3$
- Vertex-indegree $= 3$
Scheme for the strategy

(1) Map family $\mathcal{C}$ identifies with a **subfamily** $\mathcal{O}_C$ of $\mathcal{O}$ with conditions on:
- Face degrees
- Vertex indegrees

**Example:** $\mathcal{C} = \text{Family of simple triangulations}$

\[ \mathcal{C} \simeq \text{subfamily } \mathcal{O}_C \text{ of } \mathcal{O} \text{ with} \]
- Face-degree $= 3$
- Vertex-indegree $= 3$

(2) **Specialize** the master bijection to the subfamily $\mathcal{O}_C$

degree of internal faces $\longleftrightarrow$ degree of black vertices
indegree of internal vertices $\longleftrightarrow$ degree of white vertices
\(\alpha\)-orientations

Let \( G = (V, E) \) be a graph

Let \( \alpha \) be a function from \( V \) to \( \mathbb{N} \)

\[\begin{align*}
\alpha : \quad a &\rightarrow 2 \\
b &\rightarrow 1 \\
c &\rightarrow 2 \\
d &\rightarrow 0 \\
e &\rightarrow 2
\end{align*}\]
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Def: An \( \alpha \)-orientation is an orientation of \( G \) where for each \( v \in V \)

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\text{indegree}(v) = \alpha(v)
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α-orientations
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\[
\text{indegree}(v) = \alpha(v)
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\(\alpha\)-orientations: criteria for existence and accessibility

- If an \(\alpha\)-orientation \textbf{exists}, then

\[
\begin{align*}
(i) & \quad \sum_{v \in V} \alpha(v) = |E| \\
(ii) & \quad \forall S \subseteq V, \sum_{v \in S} \alpha(v) \geq |E_S|
\end{align*}
\]
α-orientations: criteria for existence and accessibility

• If an α-orientation exists, then

\begin{align*}
&\text{(i) } \sum_{v \in V} \alpha(v) = |E| \\
&\text{(ii) } \forall S \subseteq V, \sum_{v \in S} \alpha(v) \geq |E_S| \\
\end{align*}

• If the α-orientation is accessible from a vertex \( u \in V \) then

\[ \sum_{v \in S} \alpha(v) > |E_S| \text{ whenever } u \notin S \text{ and } S \neq \emptyset \]
α-orientations: criteria for existence and accessibility

- If an α-orientation exists, then

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\]

Lemma (folklore): The conditions are necessary and sufficient
\textbf{\(\alpha\)-orientations: criteria for existence and accessibility}

- If an \(\alpha\)-orientation \textbf{exists}, then

\[ \sum_{v \in V} \alpha(v) = |E| \]

\[ \forall S \subseteq V, \sum_{v \in S} \alpha(v) \geq |E_S| \]

- If the \(\alpha\)-orientation is \textbf{accessible} from a vertex \(u \in V\) then

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\textbf{Lemma (folklore):} The conditions are necessary \textbf{and sufficient} \(\Rightarrow\) accessibility from \(u \in V\) just depends on \(\alpha\) (not on which \(\alpha\)-orientation)
α-orientations for plane maps

**Fundamental lemma:** If a plane map admits an α-orientation, then it admits a **unique α-orientation without ccw circuit**, called **minimal**.
α-orientations for plane maps

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**Uniqueness proof:** if $O_1 \neq O_2$, edges where $O_1$ and $O_2$ **disagree** form an **eulerian suborientation** of $O_1$ ⇒ contains a circuit (ccw in $O_1$ or $O_2$)
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Set of α-orientations = **distributive lattice**

[Khueller et al'93], [Propp'93], [O. de Mendez'94], [Felsner'03]
**α-orientations for plane maps**

**Fundamental lemma:** If a plane map admits an $\alpha$-orientation, then it admits a unique $\alpha$-orientation **without ccw circuit**, called **minimal**

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Set of $\alpha$-orientations = **distributive lattice**

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\(\alpha\)-orientations for plane maps in our setting

- **External polygon** (the source) of the plane map is **unoriented**
- **Indegrees** are only on the **internal vertices**

\[\begin{array}{c}
\alpha : \ a \rightarrow 3 \\
\quad \quad \ b \rightarrow 2 \\
\quad \quad \ c \rightarrow 2 \\
\quad \quad \ d \rightarrow 3
\end{array}\]

An \(\alpha\)-orientation
α-orientations for plane maps in our setting

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- **Indegrees** are only on the **internal vertices**

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    b & \rightarrow 2 \\
    c & \rightarrow 2 \\
    d & \rightarrow 3 
\end{align*} \]

An \(\alpha\)-orientation

Partition \(V\) (vertex-set) as \(V_i \cup V_e\) and \(E\) (edge-set) as \(E_i \cup E_e\)

**Existence:**

(i) \(\sum_{v \in V_i} \alpha(v) = |E_i|\)

(ii) \(\forall S \subseteq V, \sum_{v \in S \cap V_i} \alpha(v) \geq |E_S \cap E_i|\)

**Accessibility from outer face:**

(iii) \(\forall S \subseteq V_i, \sum_{v \in S \cap V_i} \alpha(v) > |E_S \cap E_i|\)

**Distributive lattice** structure
Example: simple triangulations

![Diagram showing simple triangulations with degrees of faces and girths.]
Fact: A triangulation with \( n \) internal vertices has \( 3n \) internal edges.

Proof: The numbers \( v, e, f \) of vertices edges and faces satisfy:
- Incidence relation: \( 3f = 2e \).
- Euler relation: \( v - e + f = 2 \).
Fact: A triangulation with \( n \) internal vertices has \( 3n \) internal edges.

Natural candidate for indegree function:

\[
\alpha : v \mapsto 3 \text{ for each internal vertex } v.
\]

call \textbf{3-orientation} such an \( \alpha \)-orientation
Fact: A triangulation admitting a 3-orientation is simple

$k$ internal vertices
$3k + 1$ internal edges
Thm [Schnyder 89]: A simple triangulation admits a 3-orientation.

New (easier) proof: Any simple planar graph $G = (V, E)$ satisfies

$$\frac{|E| - 3}{|V| - 3} \geq 3$$  

(Euler relation)

hence the existence/accessibility conditions are satisfied. □
The class $\mathcal{T}$ of simple triangulations is identified with the class of plane orientation $\mathcal{O}_T \subset \mathcal{O}$ with faces of degree 3, and internal vertices of indegree 3.

**Thm [recovering FuPoSc08]:** By specializing the master bijection $\Phi$ to $\mathcal{O}_T$ one obtains a bijection between simple triangulations and mobiles such that

- black vertices have degree 3
- white vertices have degree 3
- the excess is $+3$ (redundant).
Triangulations

**Counting:** The generating function of mobiles with vertices of degree 3 rooted on a white corner is $T(x) = U(x)^3$, where $U(x) = 1 + xU(x)^4$.

Consequently, the number of (rooted) simple triangulations with $2n$ faces is $\frac{1}{n(2n-1)} \binom{4n-2}{n-1}$.
Triangulations: two constructions

- **mobiles**
  - [FuPoSc’08], [Bernardi-F’10]

- **blossoming trees**
  - [PoSc’03], [AlPo’11]
More specializations

$d$-angulations of girth $d$.

<table>
<thead>
<tr>
<th>Girth</th>
<th>Degree of faces</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
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<td>5</td>
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<td>6</td>
<td>6</td>
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<tr>
<td>7</td>
<td>7</td>
</tr>
</tbody>
</table>
$d$-angulations of girth $d$

**Fact:** A $d$-angulation with $(d-2)n$ internal vertices has $dn$ internal edges.
**d-angulations of girth d**

**Fact:** A $d$-angulation with $(d-2)n$ internal vertices has $dn$ internal edges.

**Natural candidate for indegree function:**

$$\alpha : v \mapsto \frac{d}{d-2} \quad \text{for each internal vertex } v...$$
**Fact:** A $d$-angulation with $(d-2)n$ internal vertices has $dn$ internal edges.

**Idea:** We can look for an orientation of $(d-2)G$ with indegree function $\alpha : v \mapsto d$ for each internal vertex $v$. 

---

**$d$-angulations of girth $d$**
$d$-angulations of girth $d$

**Fact:** A $d$-angulation with $(d-2)n$ internal vertices has $dn$ internal edges.

**Idea:** We can look for an orientation of $(d-2)G$ with indegree function $\alpha: v \mapsto d$ for each internal vertex $v$. Call $d/(d-2)$-orientation such an orientation
\(d\)-angulations of girth \(d\)

**Thm [Bernardi-F’10]:** Let \(G\) be a \(d\)-angulation. Then \((d-2)G\) admits a \(d/(d-2)\)-orientation if and only if \(G\) has girth \(d\).
**d-angulations of girth** \( d \)

**Thm [Bernardi-F’10]:** Let \( G \) be a \( d \)-angulation. Then \( (d-2)G \) admits a \( d/(d-2) \)-orientation if and only if \( G \) has girth \( d \).

**Proof:** Similar to \( d = 3 \). Uses the fact that a planar graph \( G = (V, E) \) of girth at least \( d \) satisfies \( \frac{|E| - d}{|V| - d} \geq d \).
Master bijection for weighted orientations

There are now white-white edges in the mobile, with two positive weights summing to $d - 2$. 
Theorem [Bernardi-F’10]: The master bijection can be expressed in the weighted setting:

Moreover,

- degree of internal faces $\leftrightarrow$ degree of black faces
- indegree of internal vertices $\leftrightarrow$ indegree of white vertices
- weights of internal edges $\leftrightarrow$ weights of edges
- degree of external face $\leftrightarrow$ excess
\textbf{d-angulations of girth }d\textbf{ }

\textbf{Thm [Bernardi-F’10]:} A \textit{d}-angulation \(G\) admits a \(d/(d-2)\)-orientation if and only if \(G\) has girth \(d\).

\(\Rightarrow\) The class \(\mathcal{T}_d\) of \textit{d}-angulations of girth \(d\) can be identified with the class of weighted orientations in \(\mathcal{O}\), with faces of degree \(d\), edges of weight \(d-2\), and internal vertices of indegree \(d\).
\textit{d-angulations of girth }d\

\textbf{Thm [Bernardi-F’10]:} A \textit{d-angulation} $G$ admits a $d/(d-2)$-orientation if and only if $G$ has girth $d$.

\textbf{Thm [Bernardi-F’10]:} By specializing the master bijection one obtains a bijection between \textit{d-angulations of girth} $d$ and mobiles (with white-white edges having weights summing to $d-2$) such that

- black vertices have degree $d$
- white vertices have indegree $d$
- the excess is $d$ (redundant).
$d$-angulations of girth $d$: counting

**Thm[Bernardi-F’10]:** Let $W_0, W_1, \ldots, W_{d-2}$ be the power series in $x$ defined by: $W_{d-2} = x(1 + W_0)^{d-1}$ and $\forall j < d - 2$, $W_j = \sum_{r} \sum_{i_1, \ldots, i_r > 0 \text{ s.t. } i_1 + \cdots + i_r = j + 2} W_{i_1} \cdots W_{i_r}$.

The generating function $F_d$ of rooted $d$-angulations of girth $d$ satisfies

\[
F'_d(x) = (1 + W_0)^d.
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The generating function $F_d$ of rooted $d$-angulations of girth $d$ satisfies

$$F'_d(x) = (1 + W_0)^d.$$ 

\textbf{Example $d=5$:}

$W_3 = x(1 + W_0)^4$
$W_0 = W_1^2 + W_2$
$W_1 = W_1^3 + 2W_1W_2 + W_3$
$W_2 = W_1^4 + 3W_1^2W_2 + 2W_1W_3 + W_2^2$
Simplification in the bipartite case

- For $d$ even, $d = 2b$, we have $\frac{d}{d-2} = \frac{b}{b-1}$

- Can work with $b/(b-1)$-orientations:
  - edges have weight $b - 1$
  - vertices have indegree $b$

Example: $b = 2$, simple quadrangulations

recover a bijection of Schaeffer (1999)
More specializations
Maps of girth \( d \).
We show only the bipartite case (simpler)
Case \( b = 2 \) (simple bipartite maps), with quadrangular outer face
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Case $b = 2$ (simple bipartite maps), with quadrangular outer face

Insert a star in each internal face
We show only the bipartite case (simpler)
Case $b = 2$ (simple bipartite maps), with quadrangular outer face

Generalized 2-orientation
- Each internal white vertex has indegree 2
- Each black vertex of degree $2i$ has outdegree $i - 2$
We show only the bipartite case (simpler)
Case $b = 2$ (simple bipartite maps), with quadrangular outer face

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Case $b = 2$ (simple bipartite maps), with quadrangular outer face

- Each internal white vertex has indegree 2
- Each black vertex of degree $2i$ has outdegree $i - 2$

For the minimal one:

& still accessible after deleting the stars
We show only the bipartite case (simpler)
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Case $b = 2$ (simple bipartite maps), with quadrangular outer face

White vertices either have:
- indegree 2 (middle of red edge)
- indegree 1 (end of leg)

Each black vertex of degree $2i$ has $i - 2$ legs
Closed formulas

**Prop [Bernardi-F’11]:** The number of **rooted simple bipartite maps** with $n_i$ faces of degree $2i$ is

$$2 \frac{\left( \sum (i+1) n_i - 3 \right)!}{\left( \sum i n_i - 1 \right)!} \prod_{i \geq 2} \frac{1}{n_i!} \binom{2i - 1}{i+1}^{n_i}$$

This can be compared with the formula obtained by Tutte (62) (recovered bijectively by Schaeffer) for **unconstrained rooted bipartite maps**:

$$2 \frac{\left( \sum i n_i \right)!}{\left( \sum (i-1)n_i + 2 \right)!} \prod_{i \geq 1} \frac{1}{n_i!} \binom{2i - 1}{i}^{n_i}$$
Shape of the mobile in higher (bipartite) girth

- Each black vertex of degree $2i$ has $i - b$ legs
- There are connectors between the black vertices

Connectors, for $b = 1$: $\bullet 0 0 \bullet$

$b = 2$: $\bullet 0 1 1 0 \bullet$

$b = 3$: binary trees
Thanks.

On the ArXiv:
- A bijection for triangulations, quadrangulations, pentagulations, etc.
- Bijective counting of maps by girth and degree.