Schnyder woods generalized to higher genus

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joint work with Luca Castelli Aleardi and Thomas Lewiner
Combinatorics of maps
Surfaces

- All surfaces here are closed and orientable
- Classification: one surface in each genus $g$

\[\begin{align*}
g=0 \text{ (sphere)} & \quad | \quad g=1 \text{ (torus)} & \quad | \quad g=2 \\
\end{align*}\]
Graphs on surfaces, maps

- **Graph on surface** = graph $G$ *embedded* on a surface $S_g$ (no edge-crossings)

- $G$ is a map if the components of $G \setminus S_g$ are topological disks

  - **Not a map** (cylindric component)
  - **A map** (3 faces)
How to display a map?

- $g=0$: project on the plane

- $g=1$: $S_g$ like a square with identified opposite sides

- $g>1$: $S_g$ like a $4g$-polygon + identifications of sides
Enumeration of planar maps

• **Strikingly simple counting formulas**
  - Triangulations: $|T_n| = \frac{2(4n-3)!}{n!(3n-2)!}$
  - Quadrangulations: $|Q_n| = \frac{2(3n-3)!}{n!(2n-2)!}$
  - Tetravalent: $|E_n| = \frac{2 \cdot 3^n (2n)!}{n!(n+2)!}$

• **Recursive method**: [Tutte 60’s]
• **Bijective method**: [Cori-Vauquelin’84], [Schaeffer’97]
  (bijections rely on combinatorial structures: orientations, …)
Counting maps in higher genus

- No exact counting formula known, but
  - Can write recurrences [Bender-Canfield’84]
  - Some bijections work [Chapuy-Marcus-Schaeffer’98]
  - Simple asymptotic pattern [Bender-Canfield’86, Gao’93]]

\[ \mathcal{M} = \bigcup_{g,n} \mathcal{M}_g[n] \] a map family (e.g. triangulations)

Then \[ |\mathcal{M}_g[n]| \sim c_g \gamma^n n^{5(g-1)/2}. \]
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\[ \Rightarrow |\mathcal{M}_g[n]| \xrightarrow{n \to \infty} |\mathcal{M}_0[n]| \cdot n^{\Theta(g)}. \]
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A map in genus \( g \) ‘is like’ a planar map with \( \Theta(g) \) marked edges
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Then
$$|\mathcal{M}_g[n]| \sim c_g \gamma^n n^{5(g-1)/2}.$$ 

$$\Rightarrow \quad |\mathcal{M}_g[n]| \sim |\mathcal{M}_0[n]| \cdot n^{\Theta(g)}$$

A map in genus $g$ ‘is like’ a planar map with $\Theta(g)$ marked edges

- ‘is like’ can be made rigorous in some cases:
  - Counting maps with a unique face [Chapuy’08]
  - **This talk:** Schnyder woods can be extended to genus $g>0$ by allowing $\Theta(g)$ ‘special edges’ [Castelli, F, Lewiner’08]
Schnyder woods for planar triangulations
Planar triangulations

$n$ inner vertices

$3n$ inner edges
Definition of Schnyder woods

Each inner edge is directed and colored in red, green or blue.

Local rules:

• Every planar triangulation admits a Schnyder wood [Schnyder’89]
Fundamental property

• Schnyder wood $\rightarrow$ 3 spanning trees (one for each color)
Applications of Schnyder woods

**Graph drawing**
[Schnyder’90, Bonichon-Felsner-Mosbah’04]

**Coding**
[He-Kao-Lu’99, Bernardi-Bonichon’07, Poulalhon-Schaeffer’03]

**Planarity criterion**
[Schnyder’89, Felsner-Zickfeld’04]

\[ G = (V, E) \text{ planar } \iff (V \cup E, \subseteq) \text{ has dimension } \leq 3 \]
Computing a Schnyder wood

Traversal algorithm: the faces are conquered progressively

[Schnyder’89] reformulated by [Brehm’03]
Computing a Schnyder wood

First step: Conquer
The outer face
Computing a Schnyder wood

Conquest step:
Computing a Schnyder wood
Computing a Schnyder wood

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Computing a Schnyder wood

Conquest step:
Result

• At each step, take care that the chosen vertex is not incident to a chord (nor to the bottom outer edge)

![is forbidden](image)

![is accepted](image)

• There is always such a vertex

![admissible vertex](image)

• Hence the algorithm terminates, it outputs a Schnyder wood

[Schnyder’89, Brehm’03]
Triangulations in higher genus

A triangulation of genus 1
Triangulations in higher genus

A triangulation of genus 1, with a root-face.

\[ n \text{ inner vertices} \]

\[ 3(n+2g) \text{ inner edges} \]
Triangulations in higher genus

A triangulation of genus 1, with a root-face.

- $n$ inner vertices
- $3(n+2g)$ inner edges

No hope to have outdegree 3 everywhere
Conquest in higher genus

Conquest step:
Conquest in higher genus

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Can not extend conquered area C
Conquest in higher genus

Can not extend conquered area $C$

Special step:
- choose chord $e$
- make it fat
- add it to $C$
Conquest in higher genus

Can not extend conquered area $C$

Special step:
- choose chord $e$
- make it fat
- add it to $C$

$C : \text{disk}\rightarrow\text{cylinder}$

$S_g \backslash C : \text{torus}\rightarrow\text{cylinder}$
Conquest in higher genus

Continue!
Conquest in higher genus

Conquest step:
Conquest in higher genus

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Can not extend conquered area C
Conquest in higher genus

Can not extend conquered area $C$

Special step:
- choose chord $e$
- make it fat
- add it to $C$

$C : \text{cylinder} \rightarrow \text{torus}$
$S_g \setminus C : \text{cylinder} \rightarrow \text{disk}$
Conquest in higher genus

Continue and finish!
Conquest in higher genus

Conquest step:
Conquest in higher genus

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Finished!
Main result

- **Theorem** [Castelli, F, Lewiner’08]: The conquest (with 2g special steps) terminates. Running time is $O((n+g)g)$.

- The structure computed is called a $g$-Schnyder wood.
Main result

- **Theorem** [Castelli, F, Lewiner’08]: The conquest (with $2g$ special steps) terminates. Running time is $O((n+g)g)$.

- The structure computed is called a $g$-Schnyder wood

- Our traversal procedure is inspired by handlebody theory:

Handlebody decomposition of a torus

From [Rossignac et al’03]: ``EdgeBreaker” procedure
Properties in higher genus
Properties in higher genus

- $T_R=\{\text{red edges}\}+\{\text{R-B}\}+\{\text{R-G}\}$ is a spanning tree
Properties in higher genus

- $T_R = \text{red edges} + \{R-B\} + \{R-G\}$ is a spanning tree
- $G_R = T_R + \{2g \text{ special edges}\}$ is a spanning submap with 1 face
• $T_R = \{\text{red edges}\} + \{\text{R-B}\} + \{\text{R-G}\}$ is a spanning tree
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• $G_B = \{\text{blue edges}\} + \{\text{B-R}\} + \{\text{B-G}\}$ is a spanning submap with $1 + 2g$ faces
• $T_R = \{\text{red edges}\} + \{\text{R-B}\} + \{\text{R-G}\}$ is a spanning tree
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• $G_B = \{\text{blue edges}\} + \{\text{B-R}\} + \{\text{B-G}\}$ is a spanning submap with 1+2g faces

• $G_G = \{\text{green edges}\} + \{\text{G-R}\} + \{\text{G-B}\}$ is a spanning submap with 1+2g faces
Application to coding
Motivation: mesh compression

- Triangulations are the combinatorial part of triangular meshes
  - mesh of genus 0
  - mesh of genus 2

- Naïve encoding: vertices are labelled \( \{1,2,\ldots,n\} \)
  - store the faces (vertex-triples), takes memory of order \( n \log(n) \)

- This talk: Schnyder woods encoding in \( 4n+O(g \log(n)) \) bits
  - (extends encoding procedure of [He-Kao-Lu’99, Bernardi-Bonichon’07] to any genus)
Encoding a planar triangulation

- Reduces to encoding a Schnyder wood
Encoding a planar triangulation

• Reduces to encoding a Schnyder wood
Encoding a planar triangulation

- Some information is redundant
Encoding a planar triangulation

• Some information is redundant

can erase
blue edges
Encoding a planar triangulation

• Some information is **redundant**

(can erase blue edges)
Encoding a planar triangulation

- Some information is redundant
Encoding a planar triangulation

- Some information is **redundant**

(can cut green edges at the middle)
Encoding a planar triangulation

- Some information is redundant

(can cut green edges at the middle)
Encoding a planar triangulation

- Some information is **redundant**

**cw walk around red tree:** any **green edge** is met **first** at its **outgoing** end

**can cut green edges at the middle**
Encoding a planar triangulation

- Some information is **redundant**

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any **green edge** is met **first** at its **outgoing** end

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Encoding a planar triangulation

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Encoding a planar triangulation

- Some information is **redundant**

![](image.png)

**can erase green outer half-edges**
Encoding a planar triangulation

• Some information is **redundant**

can erase green outer half-edges
Encoding a planar triangulation

• Some information is redundant
Encoding a planar triangulation

- Some information is redundant

locate corners that can have ingoing green half-edges
Encoding a planar triangulation

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locate corners that can have ingoing green half-edges
Encoding a planar triangulation

What we obtain: is coded by **2 words**: 

![Diagram of a planar triangulation with numbers and arrows indicating connections between points.](image-url)
Encoding a planar triangulation

2 encoding words:

• $W_R$ codes the red tree (Dyck word)
Encoding a planar triangulation

1) $W_R = abaaabaabbbbaabbaabbbab$  \quad $W_R$ has length $2n-2$.

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Encoding a planar triangulation

2 encoding words:

• $W_R$ codes the red tree (Dyck word)

• $W_G$ codes green indegrees at framed corners

1) $W_R=$abaaabaabbbbaabbaabbbababab

$W_R$ has length $2n-2,$
Encoding a planar triangulation

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$W_R$ has length $2n-2$,

2) $W_G = 0,0,0,0,1,0,2,1,2,3$

$W_G \sim$ binary word length $2n-6$
Encoding a planar triangulation

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$W_R$ has length $2n-2$,

$W_G \sim \text{binary word length } 2n-6$

$\Rightarrow$ code length is $4n-8$

2) $W_G = 0,0,0,0,1,0,2,1,2,3$
Encoding in higher genus

- Everything works the same! (walk along red cut-graph)
Encoding in higher genus

- Everything works the same! (walk along red cut-graph)

- Code-length is $4n + O(g \log(n))$
Results (coding)

- **Theorem** [Castelli, F, Lewiner’08]: A genus \( g \) triangulation can be encoded with \( 4n + O(g \log(n)) \) bits. Coding and decoding take time \( O((n+g)g) \)
Results (coding)

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- Lower bound (entropy): $3.245n + O(g \log(n))$ bits [Gao]
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• In genus 0, **bijective coding** [Poulalhon-Schaeffer’03] optimal
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- In higher genus, best known rate is $4n + O(g \log(n))$ bits:
  - our encoding based on Schnyder woods
  - Edgebreaker of [Rossignac et al]
Conclusion

- We extend definition/computation of Schnyder woods to higher genus

- In genus $g > 0$, there are $2g$ `special' edges

- Schnyder wood $\rightarrow$ code triangulation of genus $g > 0$ in $4n + O(g \log(n))$ bits