A Hybrid of Darboux’s method and Singularity Analysis in Combinatorial Asymptotics

Philippe Flajolet, Éric Fusy, Xavier Gourdon, Daniel Panario, Nicolas Pouyanne

É.F.: Dept. Mathematics, Simon Fraser University (Vancouver)
Motivations

• We consider classes of objects
  Each object has a size (e.g. \#(vertices))

Graphs  Forests  Permutations
1 3 5 2 4

• Given a class $C$, let $c_n$ be the number of objects of size $n$ in $C$

• Our aim: find automatic methods for estimating the coefficients $c_n$ asymptotically.
Generating functions

• Let $\mathcal{C}$ be a class, with counting coefficients $c_n$.

**OGF (unlabelled class):** $C(z) = \sum_{n} c_n z^n$

**EGF (labelled class):** $C'(z) = \sum_{n} c_n \frac{z^n}{n!}$
Generating functions

• Let $\mathcal{C}$ be a class, with counting coefficients $c_n$.

  **OGF (unlabelled class):**  
  \[ C(z) = \sum_{n} c_n z^n \]

  **EGF (labelled class):**  
  \[ C(z) = \sum_{n} c_n \frac{z^n}{n!} \]

• Dictionary for computing GF:

<table>
<thead>
<tr>
<th>$\mathcal{C}$</th>
<th>$C(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{C} = \mathcal{A} + \mathcal{B}$</td>
<td>$C(z) = A(z) + B(z)$</td>
</tr>
<tr>
<td>$\mathcal{C} = \mathcal{A} \times \mathcal{B}$</td>
<td>$C(z) = A(z) \cdot B(z)$</td>
</tr>
<tr>
<td>$\mathcal{C} = \text{Set}(\mathcal{A})$</td>
<td>$C(z) = \exp(A(z))$</td>
</tr>
<tr>
<td>$\mathcal{C} = \text{Cyc}(\mathcal{A})$</td>
<td>$C(z) = \log(1/(1 - A(z)))$</td>
</tr>
</tbody>
</table>
Generating functions

• Let $\mathcal{C}$ be a class, with counting coefficients $c_n$.
  OGF (unlabelled class): $C(z) = \sum c_n z^n$
  EGF (labelled class): $C(z) = \sum c_n \frac{z^n}{n!}$

• Dictionary for computing GF:

| $\mathcal{C} = \mathcal{A} + \mathcal{B}$ | $C(z) = A(z) + B(z)$ |
| $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ | $C(z) = A(z) \cdot B(z)$ |
| $\mathcal{C} = \text{Set}(\mathcal{A})$ | $C(z) = \exp(A(z))$ |
| $\mathcal{C} = \text{Cyc}(\mathcal{A})$ | $C(z) = \log(1/(1 - A(z))))$ |

• Example: permutations with no fixed point:
  $\mathcal{C} = \text{Set}(\text{Cyc}_{\geq 2}(\mathcal{Z}))$
  $\Rightarrow C(z) = \exp \left( \log \left( \frac{1}{1 - z} - z \right) \right) = \frac{e^{-z}}{1 - z}$. 

  – p.3/14
**Complex analysis**

- **Generating function** $C(z)$ as a complex function
  \[
  \rho = \limsup ([z^n]C(z))^{1/n}
  \]
  $C(z)$ is singular at $\rho$
  (Pringsheim)

- **Asymptotic methods:**
  \[
  \begin{array}{c|c|c}
  \text{singular behaviour of } C(z) & \iff & \text{asymptotic estimate of } [z^n]C(z) \\
  \text{(i) Darboux} & & \text{(ii) Sing. analysis} \\
  \end{array}
  \]

- **Remark:** we can assume $\rho = 1$ without loss of generality, using $[z^n]C(z) = \rho^{-n}[z^n]C(\rho \cdot z)$
Coefficients of basic functions

- A log-power function at 1 is a linear combination $\sigma(z)$ of functions of the form

\[(1 - z)^\alpha \log^k \left( \frac{1}{1 - z} \right), \quad \alpha \in \mathbb{R}, \; k \in \mathbb{Z}_{\geq 0}\]
Coefficients of basic functions

- A log-power function at 1 is a linear combination $\sigma(z)$ of functions of the form

\[(1 - z)^\alpha \log^k \left( \frac{1}{1 - z} \right), \quad \alpha \in \mathbb{R}, \ k \in \mathbb{Z}_{\geq 0}\]

- Coefficients have a full asymptotic expansion

Example:

\[
[z^n] \log \left( \frac{1}{1-z} \right) \sim \frac{\log n + \gamma + 2\log 2}{\sqrt{\pi n}} - \frac{\log n + \gamma + 2\log 2}{8\sqrt{\pi n^3}} + \ldots
\]
Coefficients of basic functions

- A log-power function at 1 is a linear combination $\sigma(z)$ of functions of the form

$$ (1 - z)^\alpha \log^k \left( \frac{1}{1 - z} \right), \quad \alpha \in \mathbb{R}, \ k \in \mathbb{Z}_{\geq 0} $$

- Coefficients have a full asymptotic expansion

Example:

$$ [z^n] \log \left( \frac{1}{1 - z} \right) \sim \frac{\log n + \gamma + 2 \log 2}{\sqrt{\pi n}} - \frac{\log n + \gamma + 2 \log 2}{8\sqrt{\pi n^3}} + \ldots $$

- Applies for log-power with finitely many singularities $\zeta_1, \ldots, \zeta_\ell$, using $[z^n] \sigma(z/\zeta_i) = \zeta_i^{-n} [z^n] \sigma(z)$
Darboux’s method

• Key-remark: if $g(z)$ is $C_s$-smooth on the closed unit disk, then

$$[z^n] g(z) = o(n^{-s}).$$

(from Cauchy’s integral formula + int. by part)

• Application: given $C(z) = \sum_n c_n z^n$, decompose $C(z)$ as a sum

$$C(z) = \underbrace{\Sigma(z)} + \underbrace{g(z)}.$$

log-power $C_s$-smooth

Then

$$[z^n] C(z) = [z^n] \Sigma(z) + o(n^{-s}).$$
Singularity analysis

• There holds the transfer rule (Flajolet-Odlyzko’90)

\[ C'(z) = O \left( g(z) \right) \quad \xrightarrow{\ z \to \rho \ } \quad \text{transfer} \quad [z^n]C'(z) = O([z^n]g(z)) \]

(+ analytic continuation conditions to check)

• Applies for finitely many singularities \( \zeta_1, \ldots, \zeta_\ell \)
Singularity analysis

• There holds the **transfer rule** (Flajolet-Odlyzko’90)

\[
C'(z) = O\left(g(z)\right) \quad \xrightarrow{z \to \rho} \quad \.transfer{\left[z^n\right]C'(z) = O\left([z^n]g(z)\right)}
\]

(+ analytic continuation conditions to check)

• Applies for **finitely many singularities** \(\zeta_1, \ldots, \zeta_\ell\)

• **Application**: given \(C(z) = \sum_n c_n z^n\) with singularities \(\zeta_1, \ldots, \zeta_\ell\), decompose \(C(z)\) as a sum

\[
C(z) = \Sigma(z) + g(z)
\]

where \(\Sigma(z)\) is a log-power and \(g(z)\) is \(O(z - \zeta_i)^\alpha\) at \(\zeta_i\).

Then

\[
\left[z^n\right]C(z) = \left[z^n\right]\Sigma(z) + O(n^{-\alpha-1})
\]
Singularity analysis

• There holds the \textbf{transfer rule} (Flajolet-Odlyzko’90)

\[
C'(z) = \mathcal{O}\left(g(z)\right) \quad \overset{z \to \rho}{\longrightarrow} \quad [z^n]C'(z) = \mathcal{O}( [z^n]g(z) )
\]

(+ analytic continuation conditions to check)

• Applies for \textbf{finitely many singularities} $\zeta_1, \ldots, \zeta_\ell$

• \textbf{Application:} given $C(z) = \sum_n c_n z^n$ with singularities $\zeta_1, \ldots, \zeta_\ell$, decompose $C(z)$ as a sum

\[
[C(z) = \Sigma(z) + g(z)]
\]

where $\Sigma(z)$ is a log-power and $g(z)$ is $\mathcal{O}(z - \zeta_i)^\alpha$ at $\zeta_i$.

Then $[z^n]C'(z) = [z^n]\Sigma(z) + \mathcal{O}(n^{-\alpha-1})$.

• \textbf{Remark:} $g(z)$ is $C_{[\alpha]}$-smooth, as $[z^n]g(z) = \mathcal{O}(n^{-\alpha-1})$.

By Darboux, this gives only $[z^n]g(z) = o(n^{-[\alpha]})$.
Example

Let $\mathcal{C}$ be the class of labelled **2-regular graphs**.
Example

Let $\mathcal{C}$ be the class of labelled 2-regular graphs.

2-regular graph = Set of undirected cycles of length $\geq 3$

$\Rightarrow C(z) = \exp \left( \frac{1}{2} \log \left( \frac{1}{1-z} \right) - \frac{z}{2} - \frac{z^2}{4} \right) = \frac{e^{-z/2-z^4/4}}{\sqrt{1-z}}$
Example

Let $C$ be the class of labelled 2-regular graphs.

2-regular graph = Set of undirected cycles of length $\geq 3$

$\Rightarrow C(z) = \exp \left( \frac{1}{2} \log(\frac{1}{1-z}) - \frac{z}{2} - \frac{z^2}{4} \right) = \frac{e^{-z/2-z^4/4}}{\sqrt{1-z}}$

- Singularity analysis:

$C(z) \sim \frac{e^{-3/4}}{\sqrt{1-z}} \quad \Rightarrow \quad [z^n]C(z) \sim \frac{e^{-3/4}}{\sqrt{\pi n}}$
Example

Let $C$ be the class of labelled 2-regular graphs.

2-regular graph = Set of undirected cycles of length $\geq 3$

$$\Rightarrow C(z) = \exp \left( \frac{1}{2} \log \left( \frac{1}{1-z} \right) - \frac{z}{2} - \frac{z^2}{4} \right) = \frac{e^{-z/2-z^4/4}}{\sqrt{1-z}}$$

- Singularity analysis:
  $$C(z) \sim e^{-3/4} \frac{e^{-3/4}}{\sqrt{1-z}} \quad \Rightarrow \quad [z^n]C(z) \sim \frac{e^{-3/4}}{\sqrt{\pi n}}$$

- Darboux’s method:
  $$C(z) = \frac{e^{-3/4}}{\sqrt{1-z}} + *\sqrt{1-z} + O((1-z)^{3/2})$$

$$\Rightarrow \quad [z^n]C(z) \sim \frac{e^{-3/4}}{\sqrt{\pi n}} + \frac{*}{n^{3/2}} + o(n^{-1}) \sim \frac{e^{-3/4}}{\sqrt{\pi n}}$$
The Hybrid method

• Classical Darboux’s method: sum-decomposition of

\[ C(z) = \sum_n c_n z^n \] as

\[ C(z) = \underbrace{\Sigma(z)}_{\text{log-power}} + \underbrace{g(z)}_{C_s-\text{smooth}}. \]
The Hybrid method

• Classical Darboux’s method: sum-decomposition of 
  \( C(z) = \sum_n c_n z^n \) as

  \[
  C(z) = \sum (z) + g(z),
  \]

  log–power \quad \text{Cs–smooth}

• For many classes, \( C(z) \) is an infinite product
  \( \Rightarrow \) sum-decomposition does not easily apply

• Example: The EGF of permutations with distinct cycle lengths is

  \[
  C(z) = \prod_{k \geq 1} \left( 1 + \frac{z^k}{k} \right)
  \]
The Hybrid method

• Classical Darboux’s method: sum-decomposition of
  \[ C(z) = \sum_n c_n z^n \]
  as
  \[ C(z) = \underbrace{\Sigma(z)}_{\text{log–power}} + \underbrace{g(z)}_{\text{C}_s–\text{smooth}}. \]

• For many classes, \( C(z) \) is an infinite product
  \[ \Rightarrow \text{ sum-decomposition does not easily apply} \]

• Example: The EGF of permutations with distinct cycle lengths is
  \[ C(z) = \prod_{k \geq 1} \left( 1 + \frac{z^k}{k} \right) \]

• Idea: in such cases, decompose \( C(z) \) as a product,
  where the first factor gathers the most salient singularities
Hybrid methodology

Given is a function $C(z) = \sum_n c_n z^n$ of finite order $a$, $C(z) = O(1 - |z|)^a$. 
Hybrid methodology

Given is a function $C(z) = \sum_{n} c_n z^n$ of finite order $a$, $C(z) = O(1 - |z|)^a$.

1. **Product-decomposition:**

$$C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1, \ldots, \zeta_\ell} \cdot \underbrace{Q(z)}_{C_s-\text{smooth}}$$
Hybrid methodology

Given is a function $C(z) = \sum_n c_n z^n$ of finite order $a$, $C(z) = O(1 - |z|)^a$.

1. Product-decomposition:

$$C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1, \ldots, \zeta_\ell} \cdot \underbrace{Q(z)}_{C_s-\text{smooth}}$$

2. Sum-decompositions:

$$P(z) = \underbrace{\Sigma(z)}_{\text{log-power}} + \underbrace{R(z)}_{C_t-\text{smooth}} \quad || \quad Q(z) = \underbrace{\overline{Q}(z)}_{\text{interpolation polynomial of order } c} + \underbrace{S(z)}_{\text{high contact (order } c) \text{ at } \zeta_1, \ldots, \zeta_\ell}$$
Hybrid methodology

Given is a function $C(z) = \sum_n c_n z^n$ of finite order $a$, $C(z) = O(1 - |z|)^a$.

1. **Product-decomposition:**

$$C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1, \ldots, \zeta_\ell} \cdot \underbrace{Q(z)}_{C_s-\text{smooth}}$$

2. **Sum-decompositions:**

$$P(z) = \underbrace{\Sigma(z)}_{\text{log-power } C_t-\text{smooth}} + \underbrace{R(z)}_{C_t-\text{smooth}} \mid \mid Q(z) = \underbrace{\overline{Q}(z)}_{\text{interpolation polynomial of order } c} + \underbrace{S(z)}_{\text{high contact (order } c) \text{ at } \zeta_1, \ldots, \zeta_\ell}$$

3. **Combine the terms:**

$$C(z) = \underbrace{\Sigma(z)\overline{Q}(z)}_{\text{log-power } C_u-\text{smooth } u=\min([c+a], s-c)} + \underbrace{\Sigma(z)S(z)}_{C_u-\text{smooth } u=\min([c+a], s-c)} + \underbrace{R(z)\overline{Q}(z)}_{C_s-\text{smooth}} + \underbrace{R(z)S(z)}_{C_{\min(s,t)}-\text{smooth}}$$
Hybrid methodology

Given is a function \( C(z) = \sum c_n z^n \) of finite order \( a \),
\( C(z) = O(1-|z|)^a \).

1. **Product-decomposition:**

\[
C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1, \ldots, \zeta_\ell} \cdot \underbrace{Q(z)}_{C_s-\text{smooth}}
\]

2. **Sum-decompositions:**

\[
P(z) = \underbrace{\Sigma(z)}_{\text{log-power}} + \underbrace{R(z)}_{C_t-\text{smooth}} \quad \| \quad Q(z) = \underbrace{\overline{Q}(z)}_{\text{interpolation polynomial of order } c} + \underbrace{S(z)}_{\text{high contact (order } c \text{) at } \zeta_1, \ldots, \zeta_\ell}
\]

3. **Combine the terms:**

\[
C(z) = \underbrace{\Sigma(z)\overline{Q}(z)}_{\text{log-power } C_u-\text{smooth}} + \underbrace{\Sigma(z)S(z)}_{C_u-\text{smooth}} + \underbrace{R(z)\overline{Q}(z)}_{C_s-\text{smooth}} + \underbrace{R(z)S(z)}_{C_{\min(s,t)}-\text{smooth}}
\]

4. **Adjust the parameters:** with \( t = \left\lfloor \frac{s+|a|}{2} \right\rfloor \), \( c = \left\lfloor \frac{s-|a|}{2} \right\rfloor \),

we obtain \( [z^n]C(z) = [z^n]\Sigma(z)\overline{Q}(z) + o(n^{-t}) \)
Hybrid in practice

Given is a function \( C(z) = \sum_n c_n z^n \) of order \( a \),
\[ C(z) = O(1 - |z|)^a. \]
Hybrid in practice

Given is a function \( C(z) = \sum_n c_n z^n \) of order \( a \), \( C(z) = O(1 - |z|)^a \).

1. Check that there exists a product-decomposition

\[
C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1, \ldots, \zeta_\ell} \cdot \underbrace{Q(z)}_{\mathcal{C}_s-\text{smooth}}
\]
Hybrid in practice

Given is a function \( C(z) = \sum_n c_n z^n \) of order \( a \),
\( C(z) = O(1 - |z|)^a \).

1. Check that there exists a product-decomposition

\[
C(z) = \begin{cases} \text{P}(z) & \text{sing. } \zeta_1, \ldots, \zeta_\ell \\ \text{Q}(z) & \text{C}_s\text{-smooth} \end{cases}
\]

(then, with \( t = \left\lfloor \frac{s + |a|}{2} \right\rfloor \), \( c = \left\lfloor \frac{s - |a|}{2} \right\rfloor \),
\[
[z^n]C(z) = [z^n] \begin{cases} \text{\Sigma}(z) & \text{log-power expansion of } P \\ \overline{\text{Q}}(z) & \text{interpolation polynomial of } Q \end{cases} + o(n^{-t}))
\]
Given is a function \( C(z) = \sum_n c_n z^n \) of order \( a \),
\( C(z) = O(1 - |z|)^a \).

1. Check that there exists a product-decomposition
\[
C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1, \ldots, \zeta_\ell} \cdot \underbrace{Q(z)}_{\text{\(c_s\)-smooth}}
\]
(then, with \( t = \left\lfloor \frac{s + |a|}{2} \right\rfloor \), \( c = \left\lfloor \frac{s - |a|}{2} \right\rfloor \),
\[
[z^n]C(z) = [z^n] \underbrace{\Sigma(z)}_{\text{log-power expansion of } P} \cdot \underbrace{\bar{Q}(z)}_{\text{interpolation polynomial of } Q} + o(n^{-t})
\]

**Theorem:** \( \Sigma(z)\bar{Q}(z) \) is the sum of the radial singular expansions of order \( t \) of \( C(z) \) at \( \zeta_1, \ldots, \zeta_\ell \).
Hybrid in practice

Given is a function $C(z) = \sum_n c_n z^n$ of order $a$, $C(z) = O(1 - |z|)^a$.

1. Check that there exists a product-decomposition
   
   $$C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1, \ldots, \zeta_\ell} \cdot \underbrace{Q(z)}_{C_s-\text{smooth}}$$
   
   (then, with $t = \left\lfloor \frac{s + |a|}{2} \right\rfloor$, $c = \left\lfloor \frac{s - |a|}{2} \right\rfloor$,
   
   $$[z^n]C(z) = [z^n] \underbrace{\Sigma(z)}_{\text{log-power expansion of } P} \cdot \underbrace{\overline{Q}(z)}_{\text{interpolation polynomial of } Q} + o(n^{-t})$$

   Theorem: $\Sigma(z)\overline{Q}(z)$ is the sum of the radial singular expansions of order $t$ of $C(z)$ at $\zeta_1, \ldots, \zeta_\ell$.

2. Compute the radial expansions of $C(z)$ of order $t$ (all computations within the disk of convergence)
Example (computation steps)

$C(z)$ is the EGF of permutations with distinct cycle lengths

$$C(z) = \prod_{k \geq 1} \left( 1 + \frac{z^k}{k} \right)$$
Example (computation steps)

$C(z)$ is the EGF of permutations with dist. cycle lengths

$$C(z) = \prod_{k \geq 1} \left(1 + \frac{z^k}{k}\right)$$

1. **exp-log** transformation gives

$$C(z) = (1 + z) \exp\left(\sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell} (A_{\ell}(z))\right),$$

where $A_{\ell}(z) = \text{Li}_{\ell}(z^{\ell}) - z^{\ell}$, $\text{Li}_{\ell}(t) = \sum_{n \geq 1} \frac{t^n}{n^\ell}$ (polylog.)
Example (computation steps)

$C(z)$ is the EGF of permutations with dist. cycle lengths

$$C(z) = \prod_{k \geq 1} \left( 1 + \frac{z^k}{k} \right)$$

1. exp-log transformation gives

$$C(z) = (1 + z) \exp(\sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell} (A_{\ell}(z))),$$

where $A_{\ell}(z) = \text{Li}_\ell(z^\ell) - z^\ell$, $\text{Li}_\ell(t) = \sum_{n \geq 1} \frac{t^n}{n^\ell}$ (polylog.)

2. $\text{Li}_\ell(t)$ is $C_s$-smooth for $\ell \geq s + 2$, so prod.-decomp. is

$$C(z) = \underbrace{P(z)}_{(1+z) \exp(\sum_{\ell \leq s+1} A_{\ell}(z))} \cdot \underbrace{Q(z)}_{\exp(\sum_{\ell \geq s+2} A_{\ell}(z))}$$
Example (computation steps)

$C(z)$ is the EGF of permutations with dist. cycle lengths

$$C(z) = \prod_{k \geq 1} \left(1 + \frac{z^k}{k}\right)$$

1. **exp-log** transformation gives

$$C(z) = (1 + z) \exp\left(\sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell} (A_{\ell}(z))\right),$$

where $A_{\ell}(z) = \text{Li}_\ell(z^\ell) - z^\ell$, $\text{Li}_\ell(t) = \sum_{n \geq 1} \frac{t^n}{n^\ell}$ (polylog.)

2. $\text{Li}_\ell(t)$ is $C_s$-smooth for $\ell \geq s + 2$, so prod.-decomp. is

$$C(z) = \underbrace{(1+z) \exp(\sum_{\ell \leq s+1} A_{\ell}(z))}_{P(z)} \cdot \underbrace{\exp(\sum_{\ell \geq s+2} A_{\ell}(z))}_{Q(z)}$$

3. **Singularities of $P(z)$** are roots of unity till order $s+1$,

$\Rightarrow$ compute the radial expansions of $C(z)$ at these roots,

using the singular expansions of polylogarithms

(Zagier-Cohen’91, Flajolet’99)
Example (results)

**Proposition:** The probability that a permutation is made of cycles of distinct lengths admits a **full asymptotic expansion** of the form

\[ f_n \sim e^{-\gamma} + \frac{e^{-\gamma}}{n} + \frac{e^{-\gamma}}{n^2} (-\log n - 1 - \gamma + \log 2) + \frac{1}{n^3} \left[ e^{-\gamma} (\log^2 n + c_{3,1} \log n + c_{3,0}) + 2(-1)^n + \Re\left( \frac{18\Gamma(\frac{2}{3}) \omega^{-n}}{\Gamma(\frac{1}{6} + \frac{i\sqrt{3}}{6})\Gamma(\frac{1}{2} - \frac{i\sqrt{3}}{6})} \right) \right] + \sum_{r \geq 4} \frac{P_r(n)}{n^r}, \]

where \(c_{3,1}\) and \(c_{3,0}\) are explicit constants. Each \(P_r(n)\) is a **polynomial** of degree \(r - 1\) in \(\log n\) with coefficients that are **periodic functions** of \(n\) with period \(D(r) = \text{lcm}(2, 3, \ldots, r)\).
Scope of applications

- The hybrid method typically applies for a function $C(z)$ of the form

$$C(z) = \prod_{n \geq 1} (1 + c_n z^n), \text{ with } c_n \sim n^{-\alpha}, \alpha \geq 1$$

<table>
<thead>
<tr>
<th>$Q_{-1}$</th>
<th>$Q_{-3/2}$</th>
<th>$Q_{-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\prod_{n \geq 1} \left(1 + \frac{z^n}{n}\right)$</td>
<td>$\prod_{n \geq 1} \left(1 + \frac{z^n}{n^{3/2}}\right)$</td>
<td>$\prod_{n \geq 1} \left(1 + \frac{z^n}{n^2}\right)$</td>
</tr>
<tr>
<td>Perm. dist. cycle lengths</td>
<td>Forest dist. comp. sizes</td>
<td>Perm. same cycle type</td>
</tr>
<tr>
<td>$\alpha \sim 1$</td>
<td>$\alpha \sim n^{-3/2}$</td>
<td>$\alpha \sim n^{-2}$</td>
</tr>
</tbody>
</table>

- Outside of scope:
  - Partitions dist. summands: $\prod_{n \geq 1} (1 + z^n)$
  - Set partitions dist. sizes. $\prod_{n \geq 1} \left(1 + \frac{z^n}{n!}\right)$