Straight-line drawing of quadrangulations

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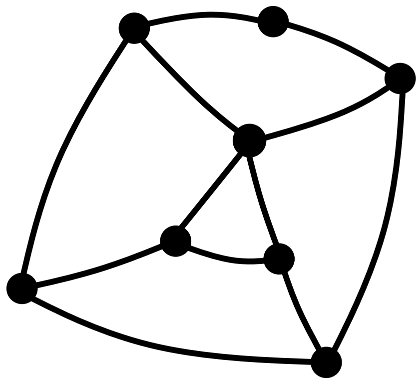
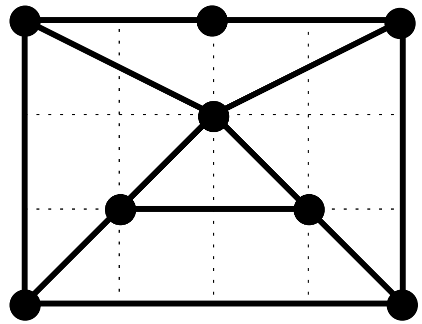
Plane graphs and straight-line drawings

- A plane graph is a graph drawn in the plane up to continuous deformation.



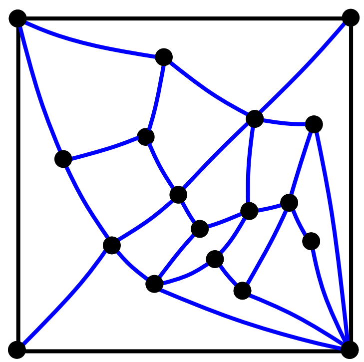
plane graph G

=



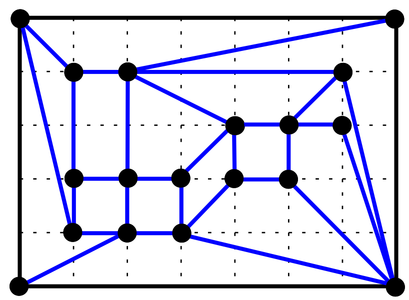
straight-line drawing of G

- Quadrangulation: plane graph with all faces of degree 4



quadrangulation Q

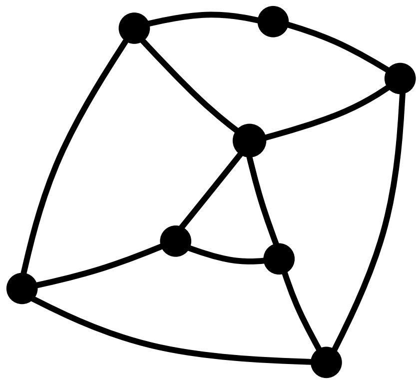
=



straight-line drawing of Q

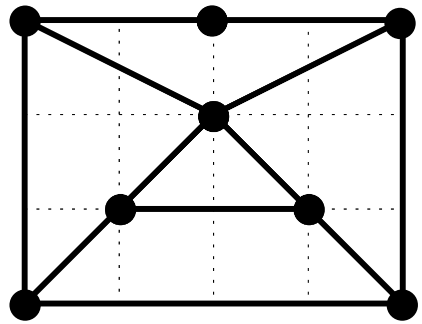
Plane graphs and straight-line drawings

- A plane graph is a graph drawn in the plane up to continuous deformation.



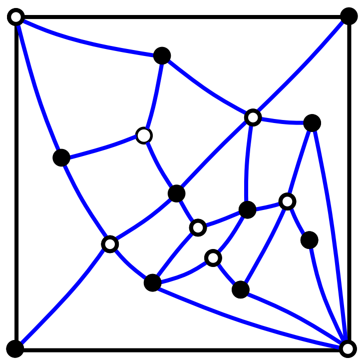
plane graph G

=



straight-line drawing of G

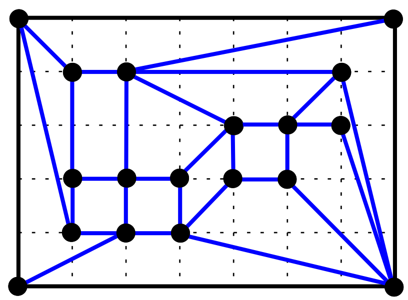
- Quadrangulation: plane graph with all faces of degree 4



quadrangulation Q

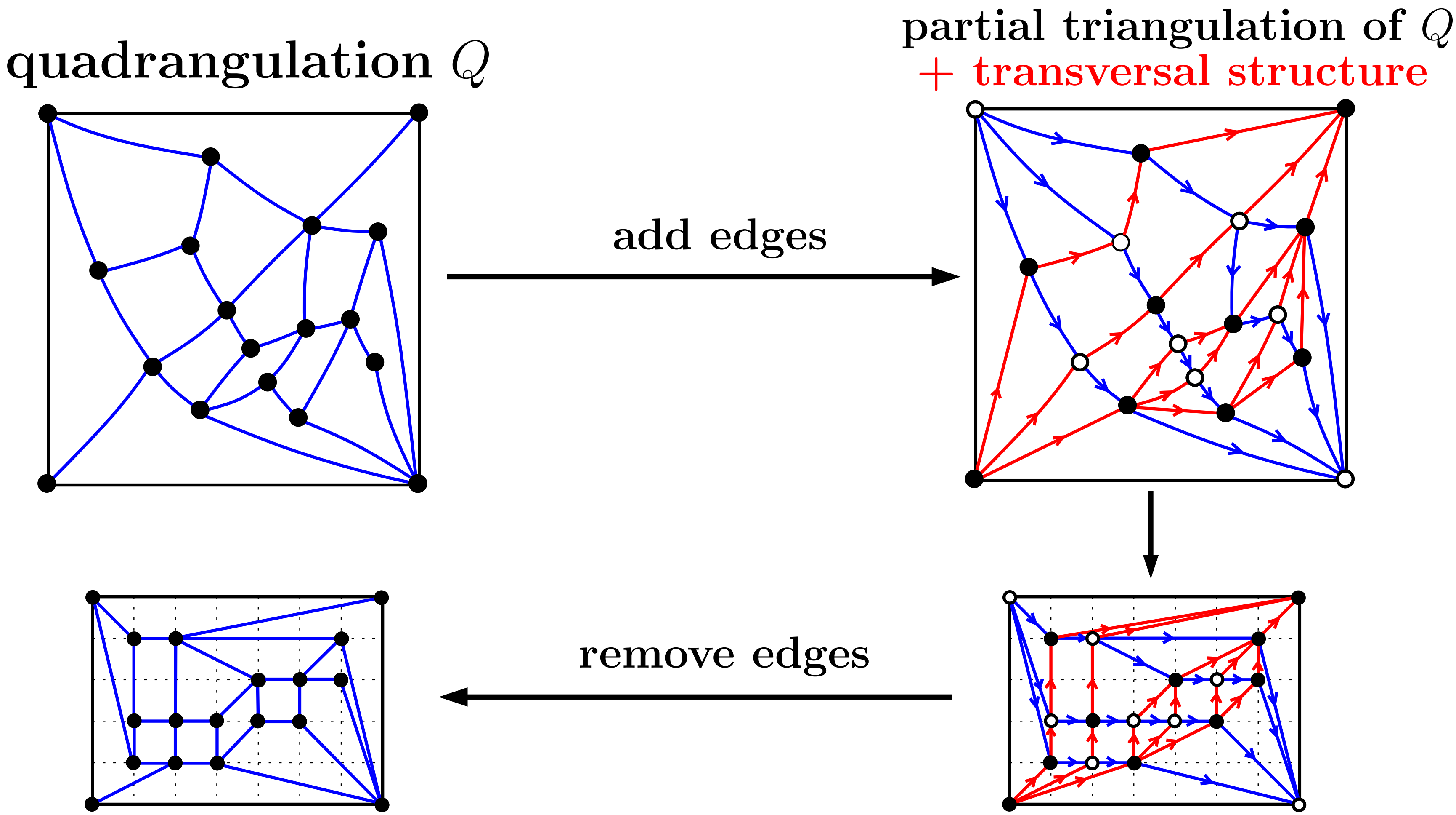
= maximal bipartite plane graph

=



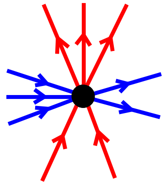
straight-line drawing of Q

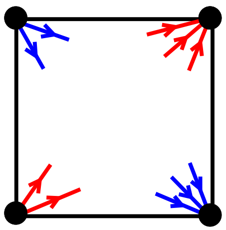
Principle of the algorithm

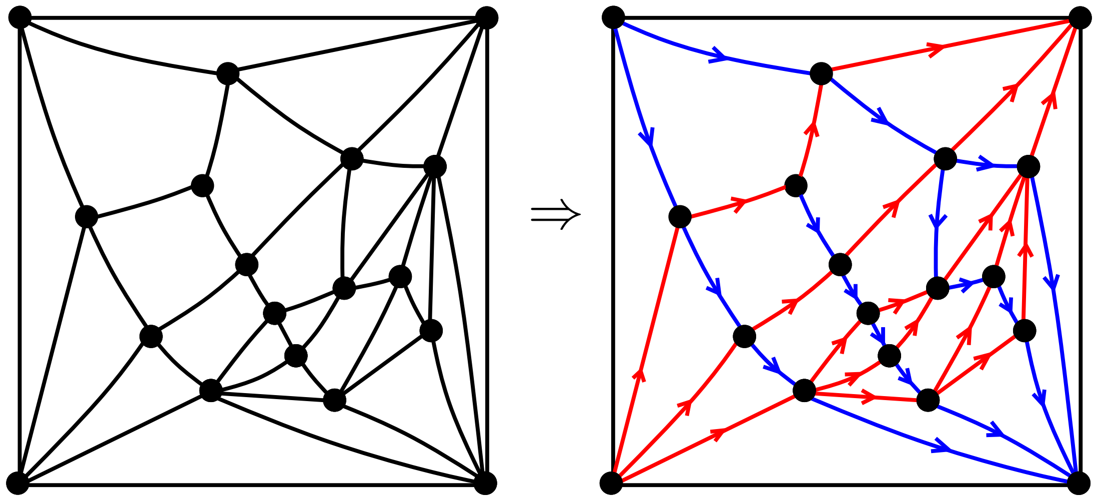


Transversal structures

- **Transversal structure**: each inner edge receives an orientation and a color (blue or red) such that

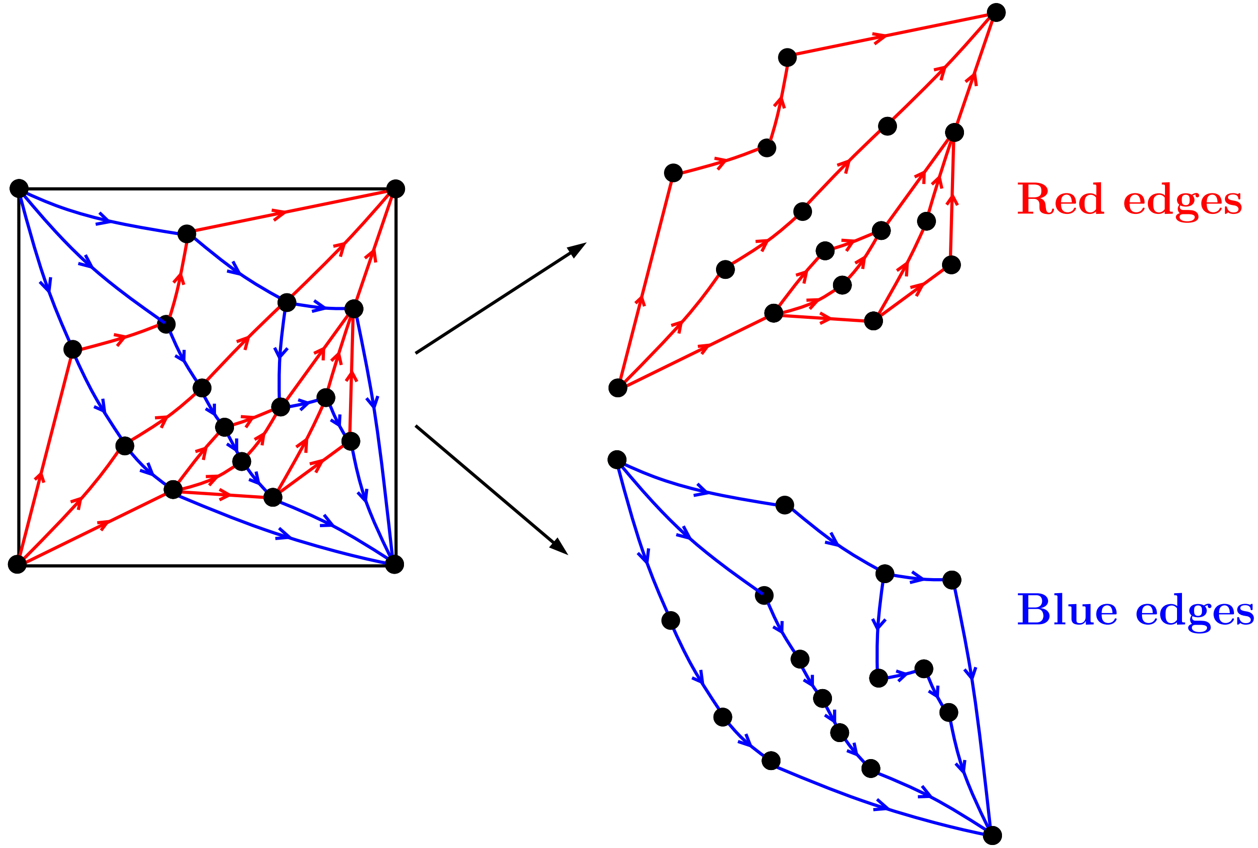
1) For each inner vertex:  the 4 bunches are not empty

2) Border vertices: 

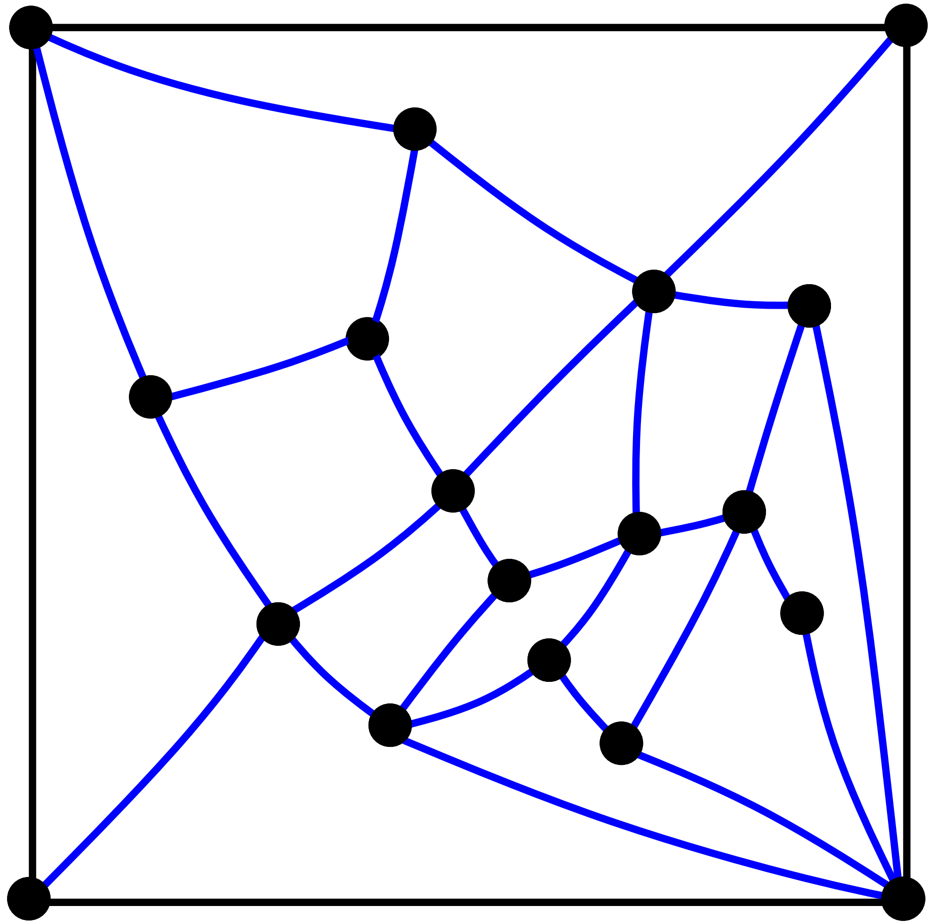
Example: 

Link with bipolar orientations

- **Bipolar orientation**= acyclic orientation with unique minimum (source) and unique maximum (sink)
- The blue (resp. red) edges form a bipolar orientation
- The two bipolar orientations are transversal

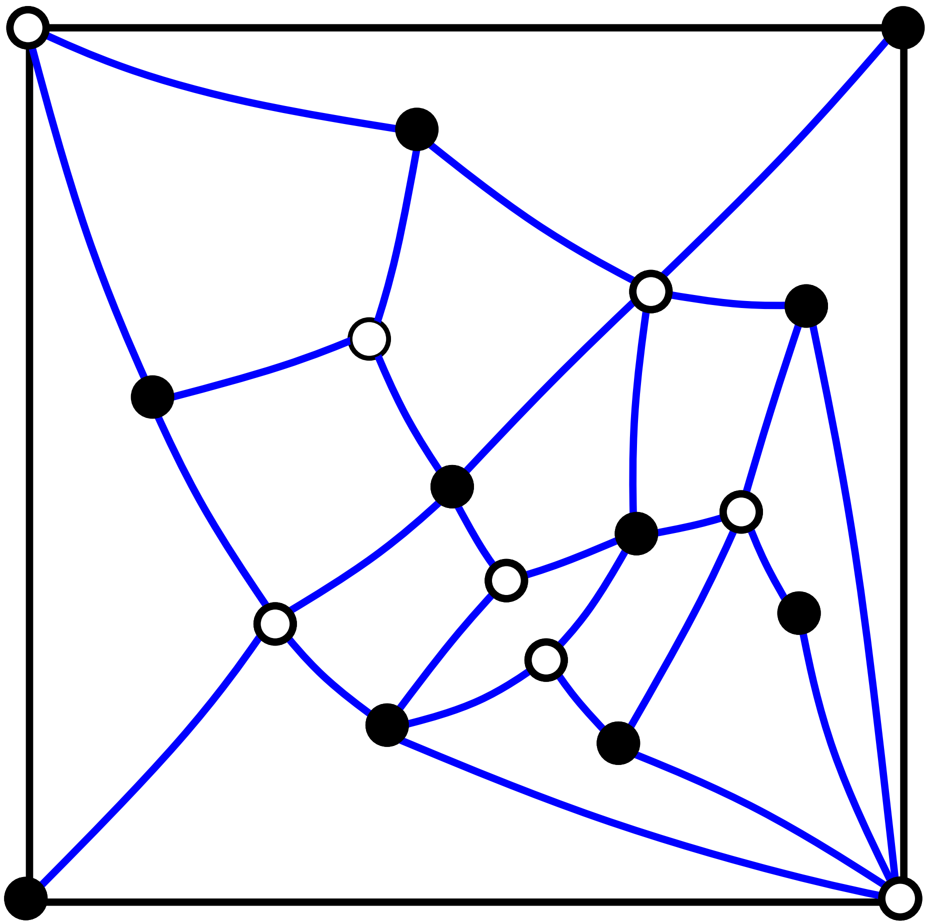


Partial triangulation of Q into t.s.



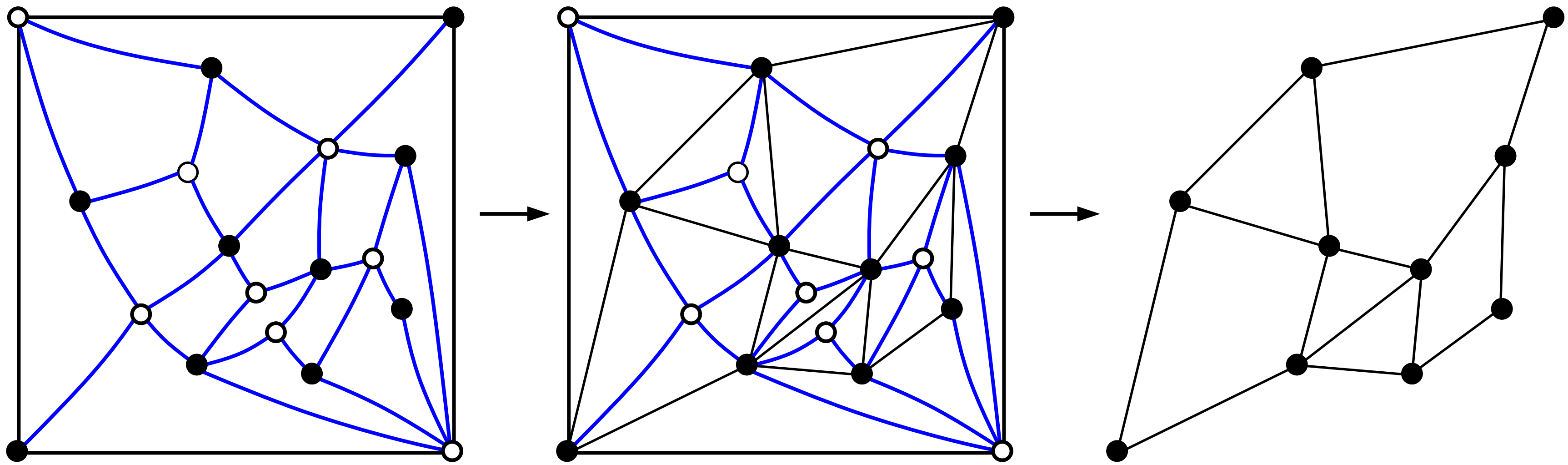
Partial triangulation of Q into t.s.

Bicolor the vertices of Q.



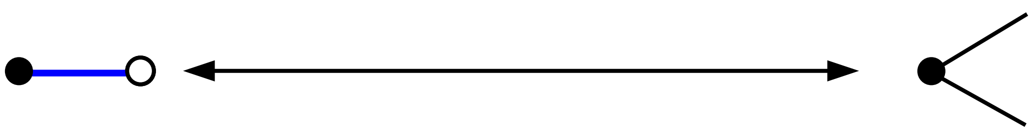
Partial triangulation of Q into t.s.

Associate plane graph M on black vertices.



Quadrangulation Q ⟷ **Plane graph M**

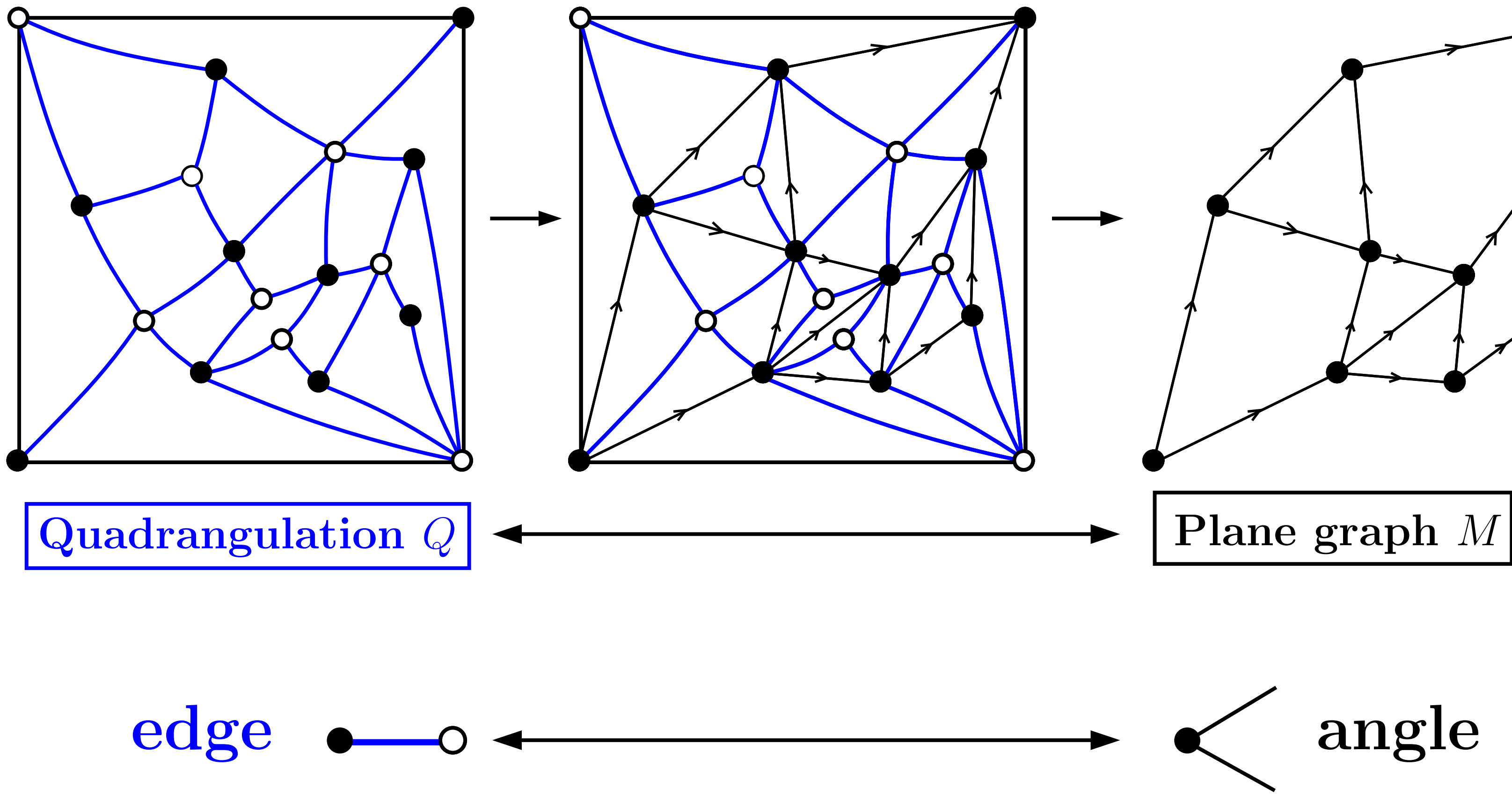
edge



angle

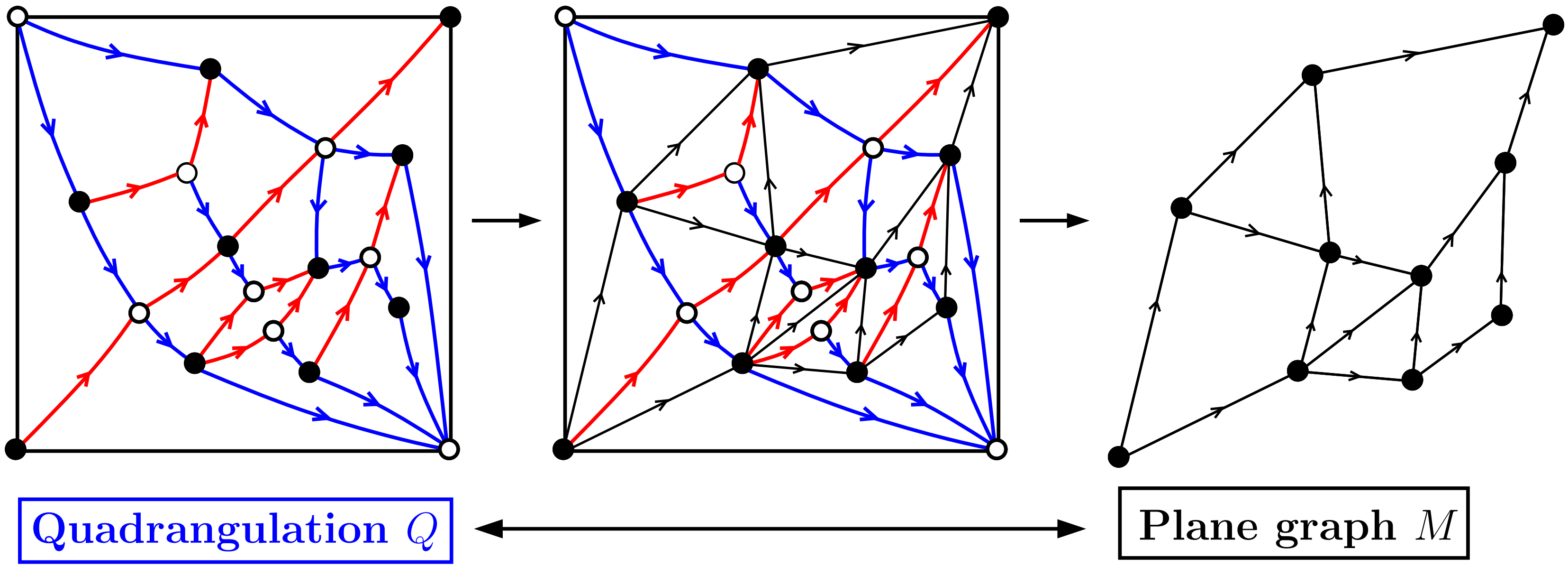
Partial triangulation of Q into t.s.

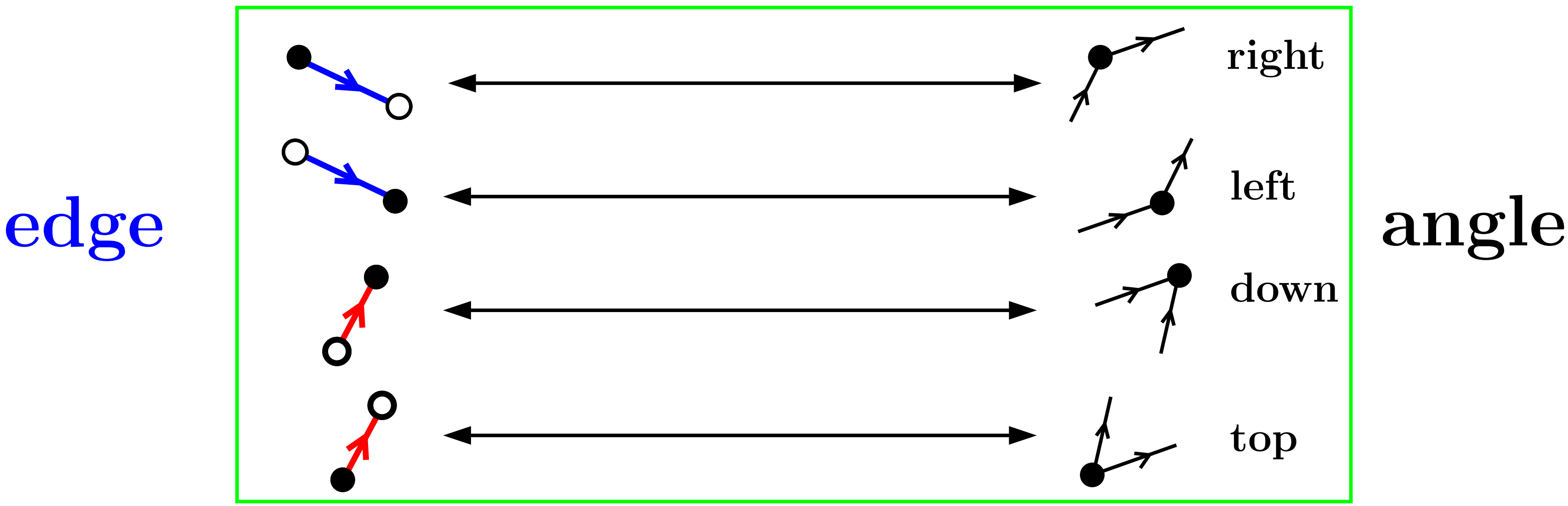
Compute a bipolar orientation of M.



Partial triangulation of Q into t.s.

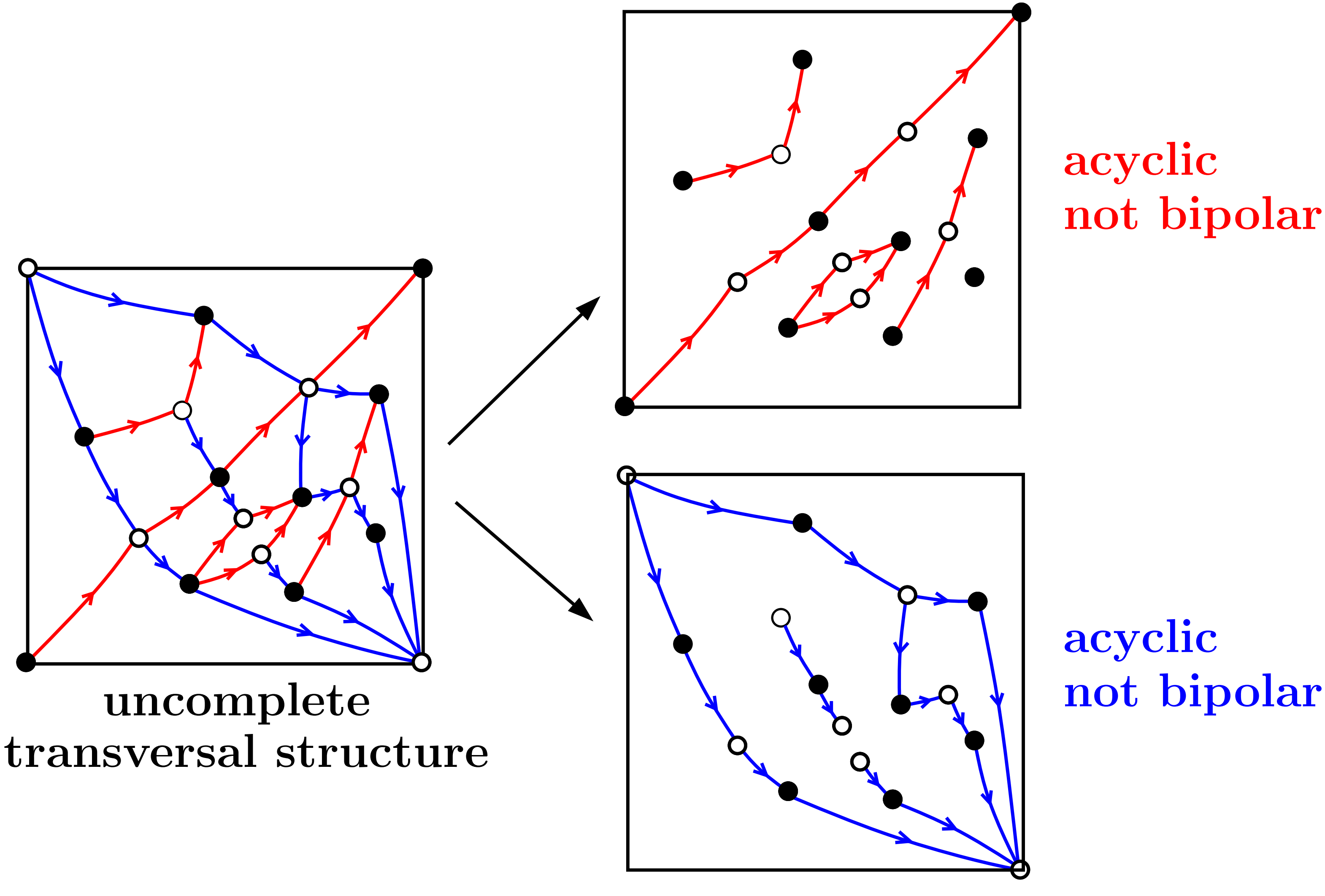
Bicolor and orient inner edges of Q.





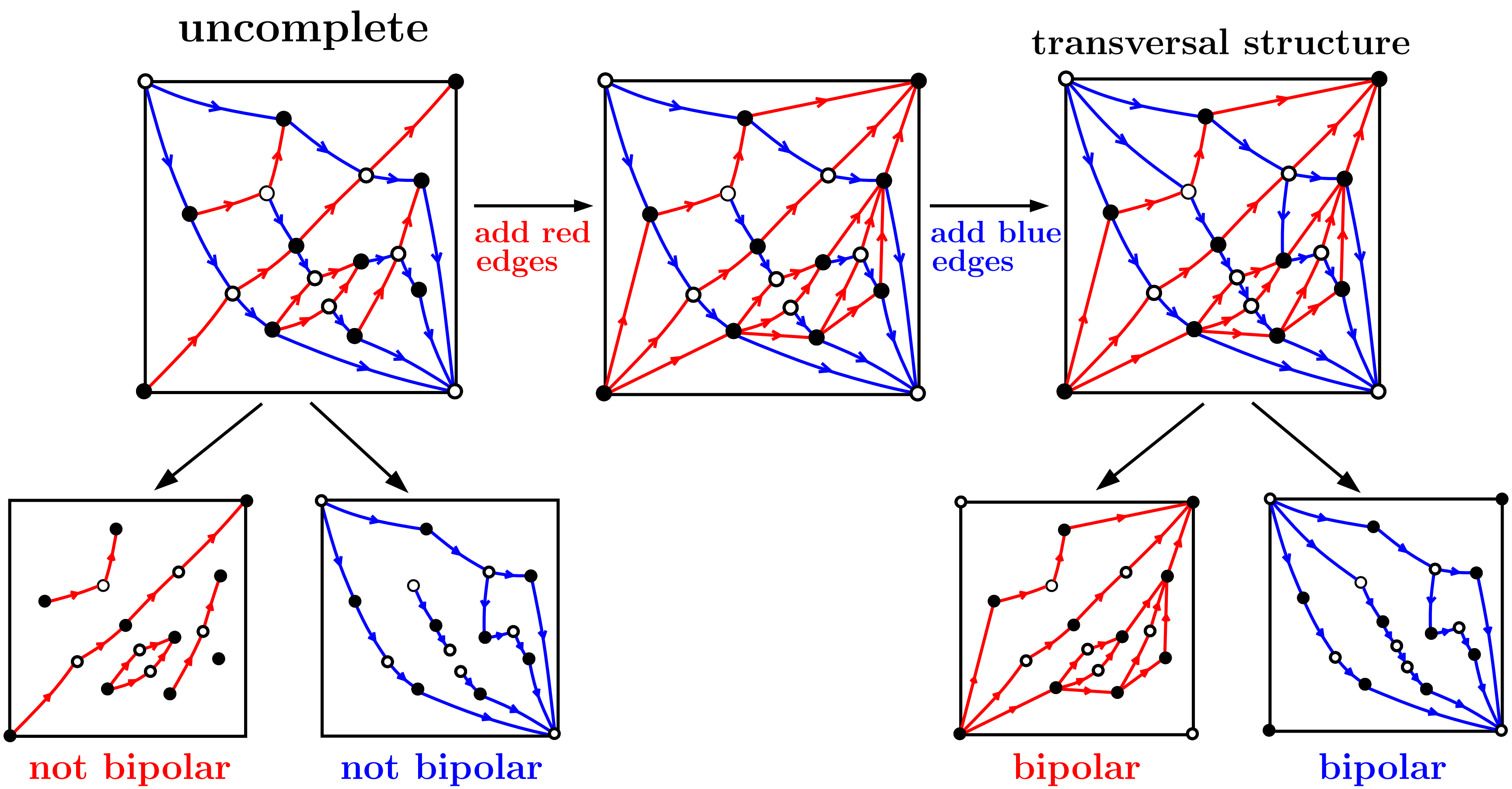
Partial triangulation of Q into t.s.

We obtain an uncomplete transversal structure.



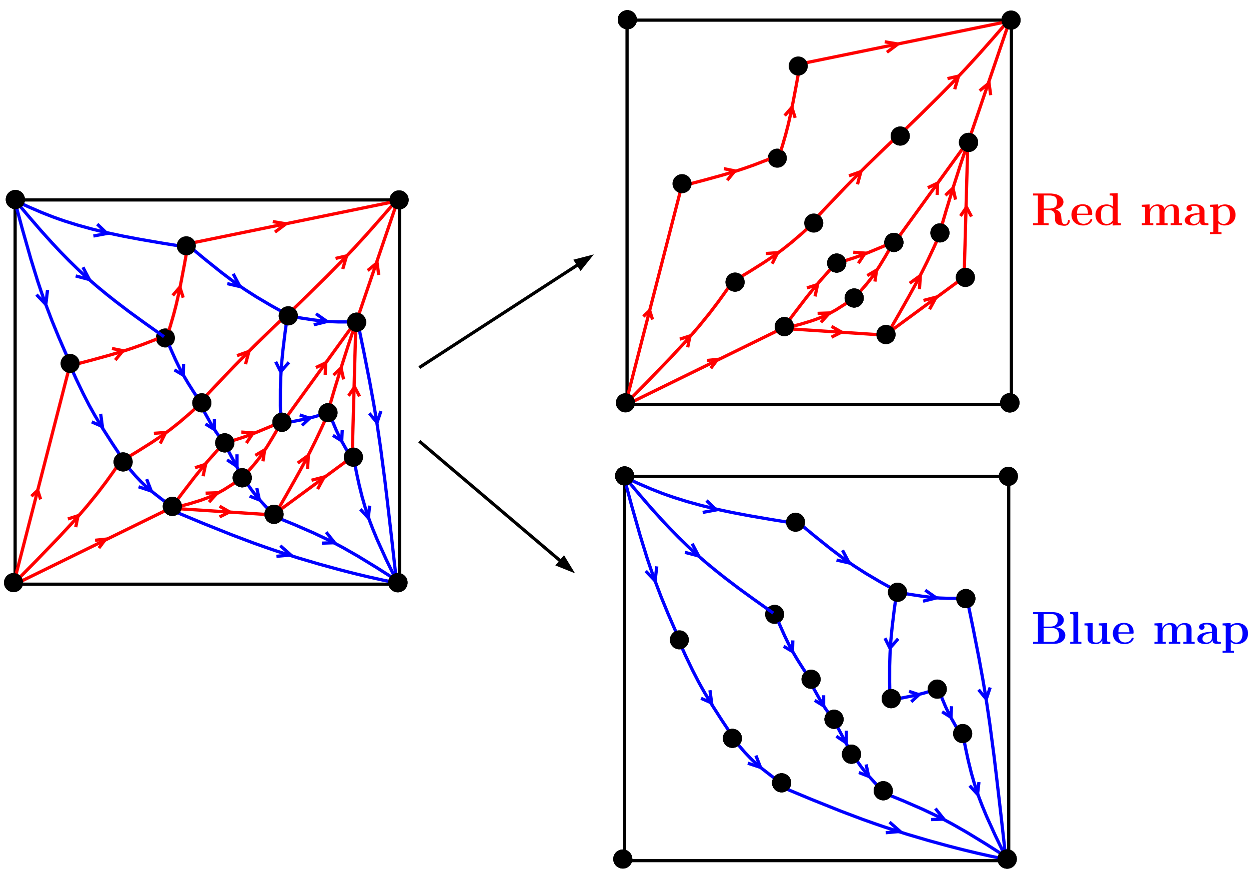
Partial triangulation of Q into t.s.

Add edges to complete the transversal structure.



Straight-line drawing using t.s.

- The blue (resp. red) edges form a bipolar orientation.
- Use the red edges to give abscissas and blue edges to give ordinates using face-counting operations

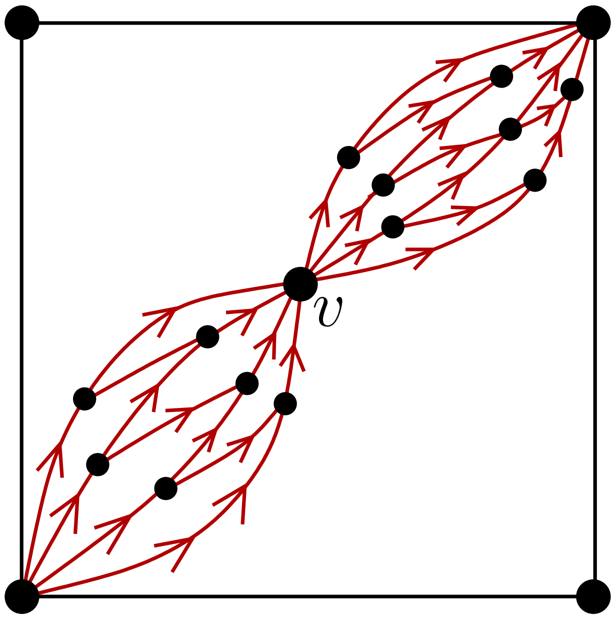


The red map gives abscissas (1)

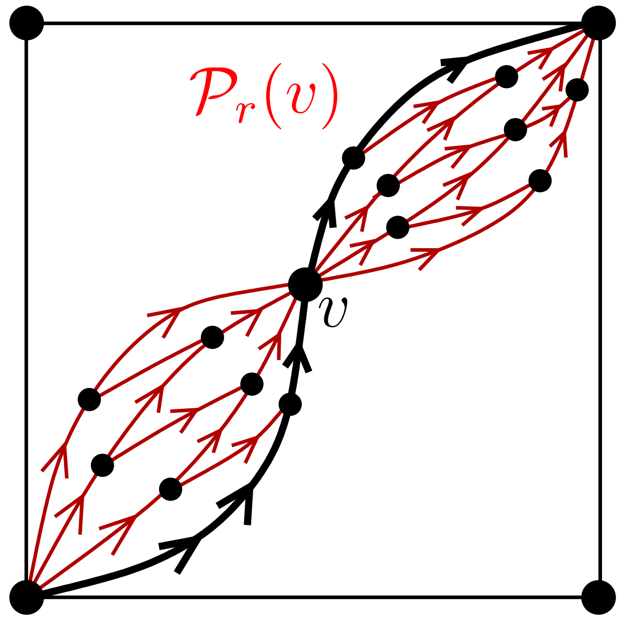
Let v be an inner vertex of T

Let $\mathcal{P}_r(v)$ be the unique path passing by v which is:

- the rightmost one before arriving at v
- the leftmost one after leaving v

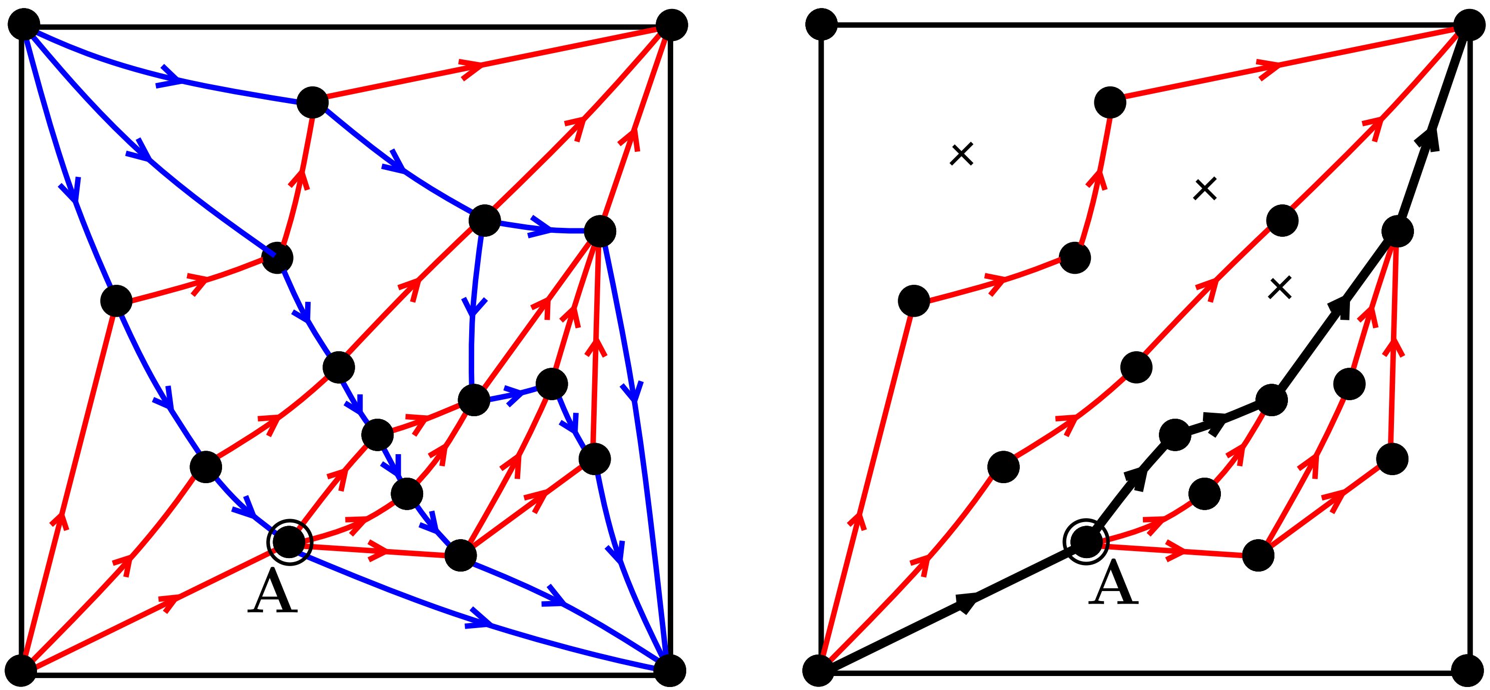


$\Rightarrow$



The red map gives abscissas (2)

The abscissa of v is the number of faces of the red map on the left of $\mathcal{P}_r(v)$

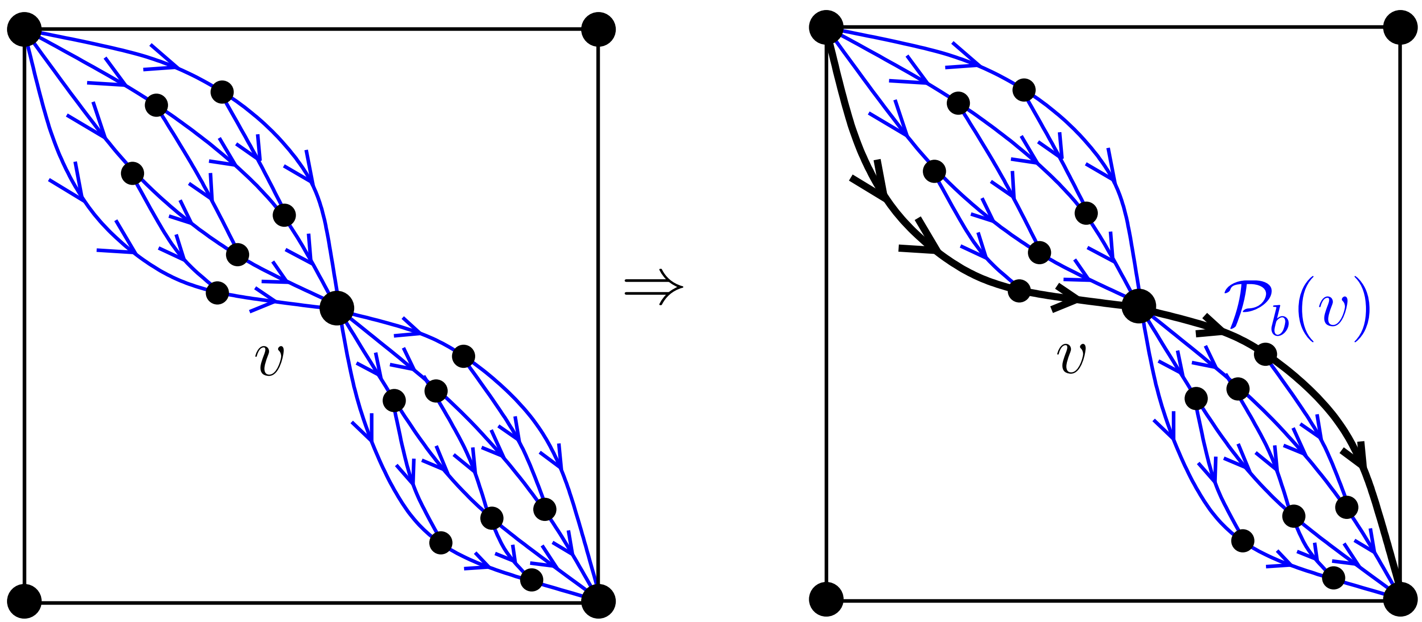


⇒ A has abscissa 3

The blue map gives ordinates (1)

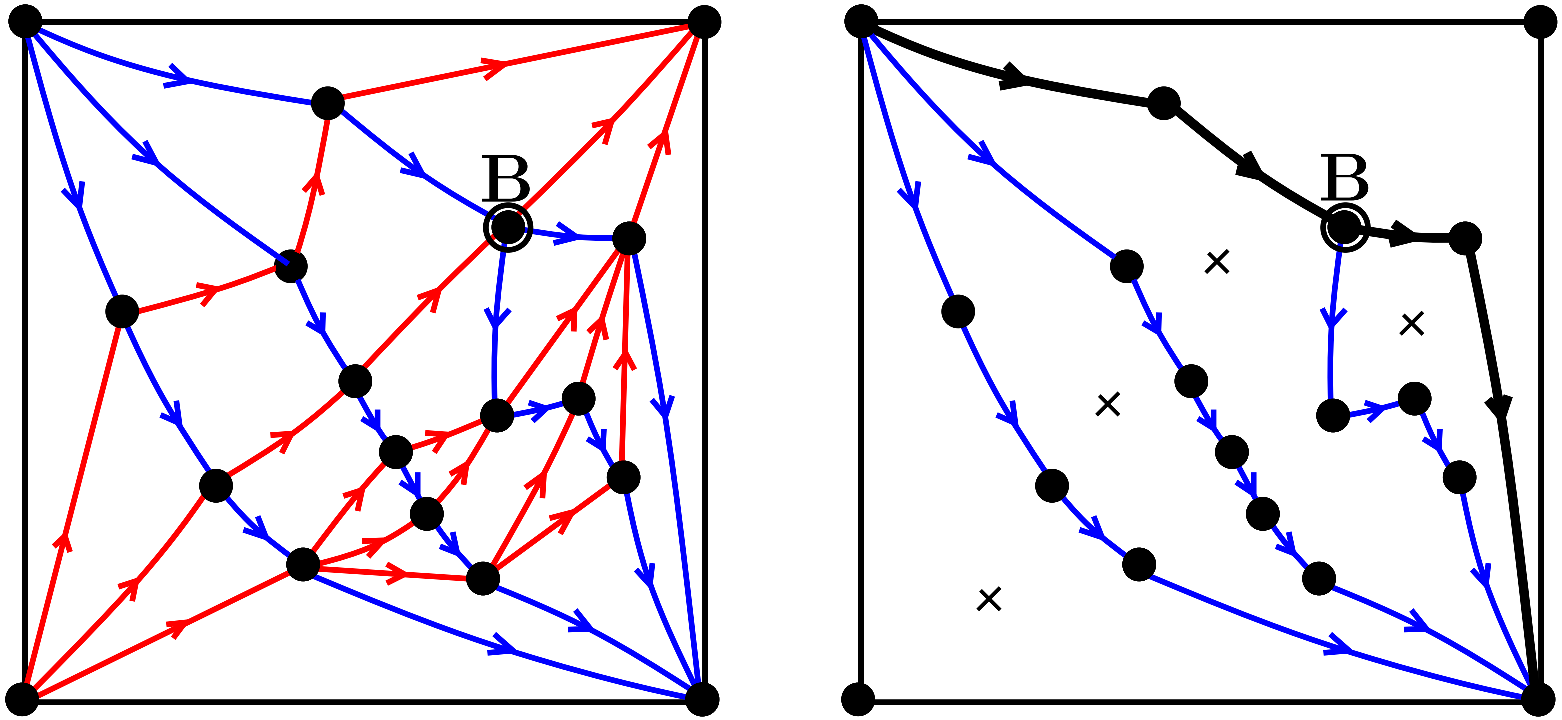
Similarly we define $\mathcal{P}_b(v)$ the unique blue path which is:

- the rightmost one before arriving at v
- the leftmost one after leaving v



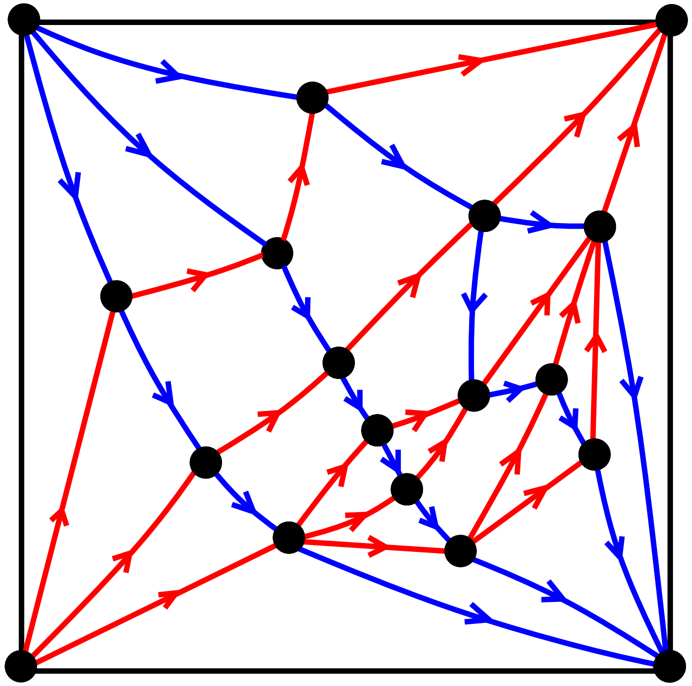
The blue map gives ordinates (2)

The ordinate of v is the number of faces of the blue map below $\mathcal{P}_b(v)$



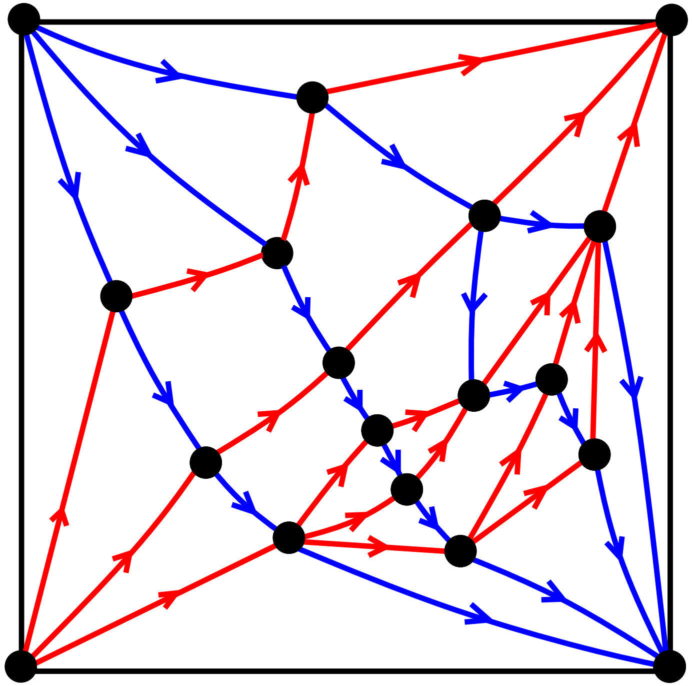
$\Rightarrow$ **B has ordinate 4**

Execution of the algorithm

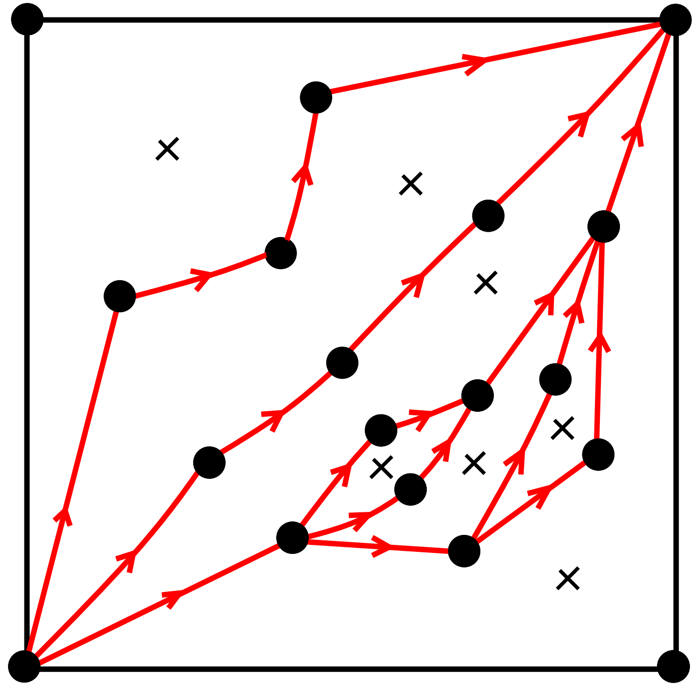


Execution of the algorithm

Let f_r be the number of faces of the red map

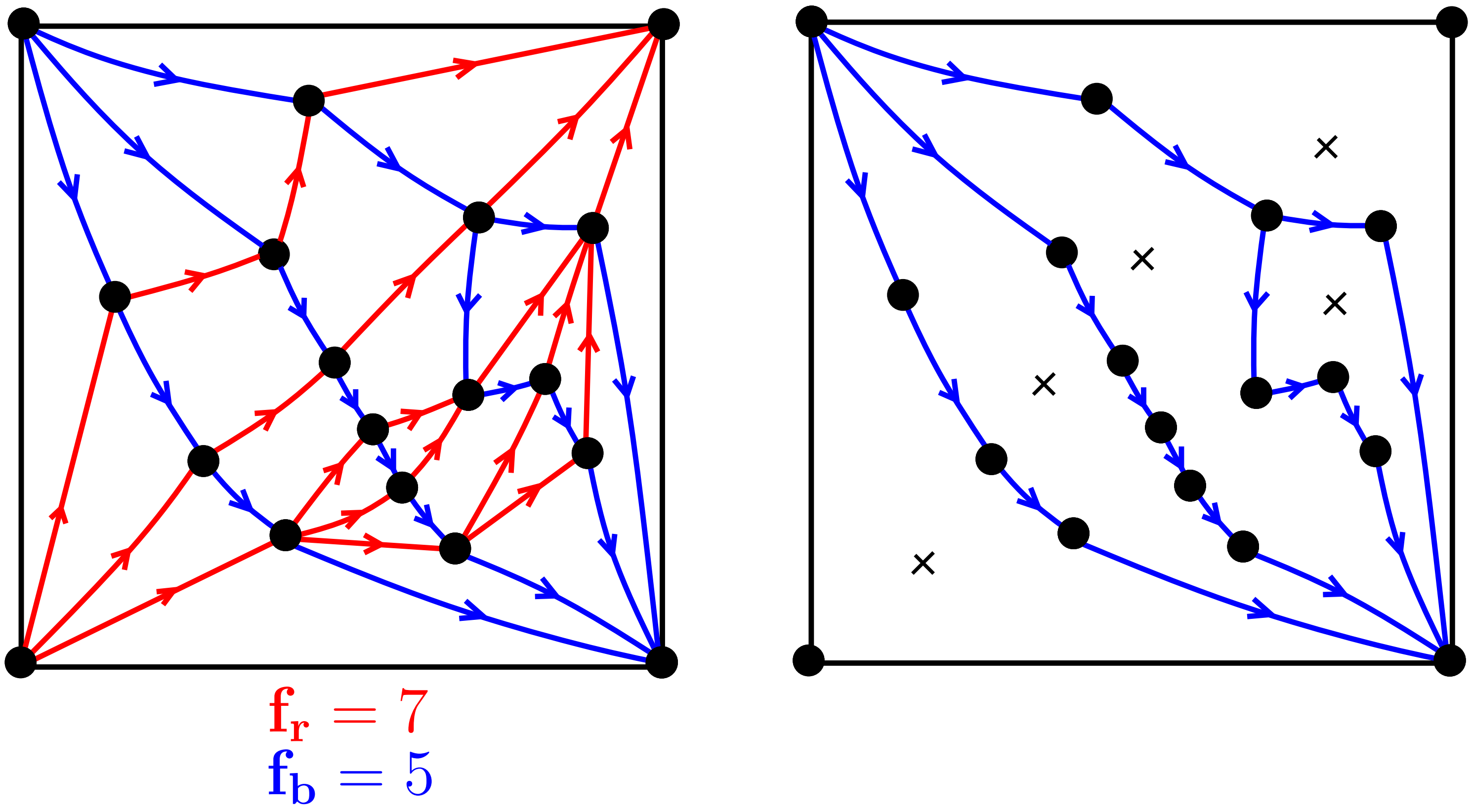


$\mathbf{f_r = 7}$



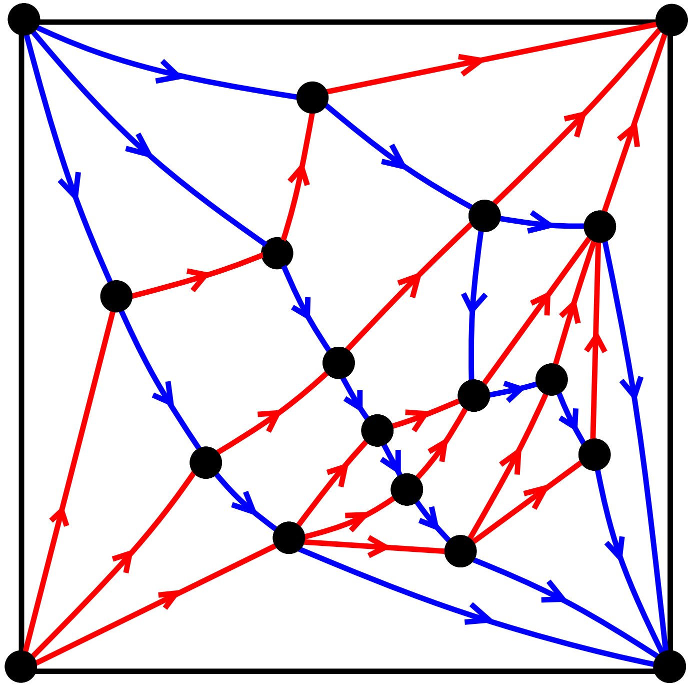
Execution of the algorithm

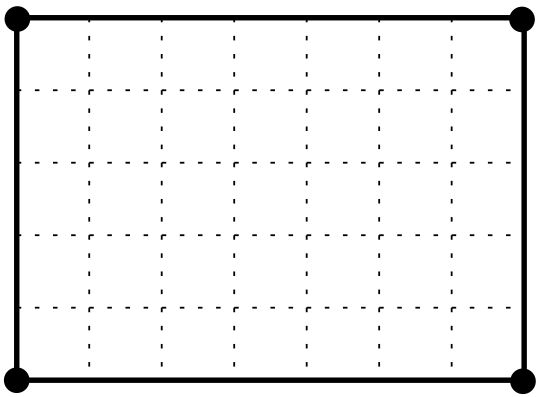
Let f_b be the number of faces of the blue map



Execution of the algorithm

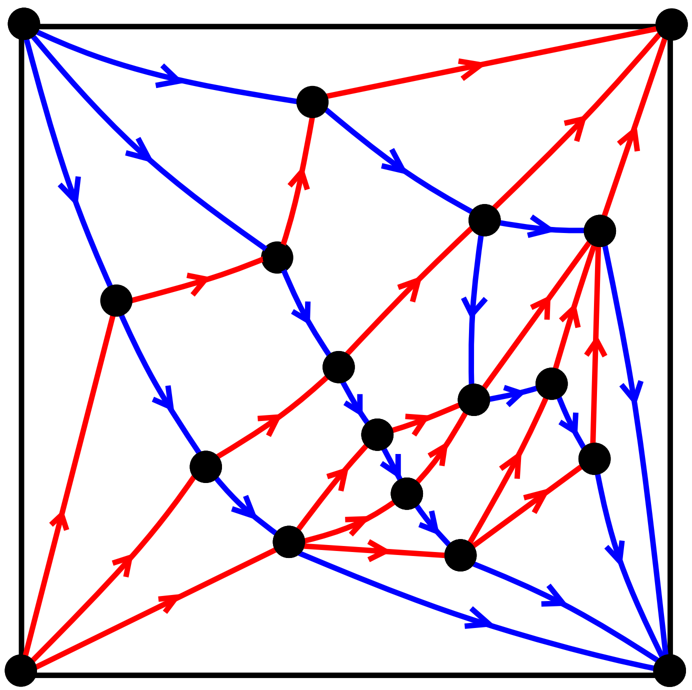
Take a regular grid of width f_r and height f_b and place the 4 border vertices of T at the 4 corners of the grid

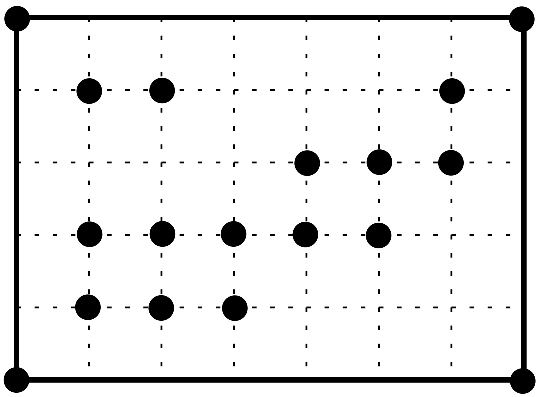




Execution of the algorithm

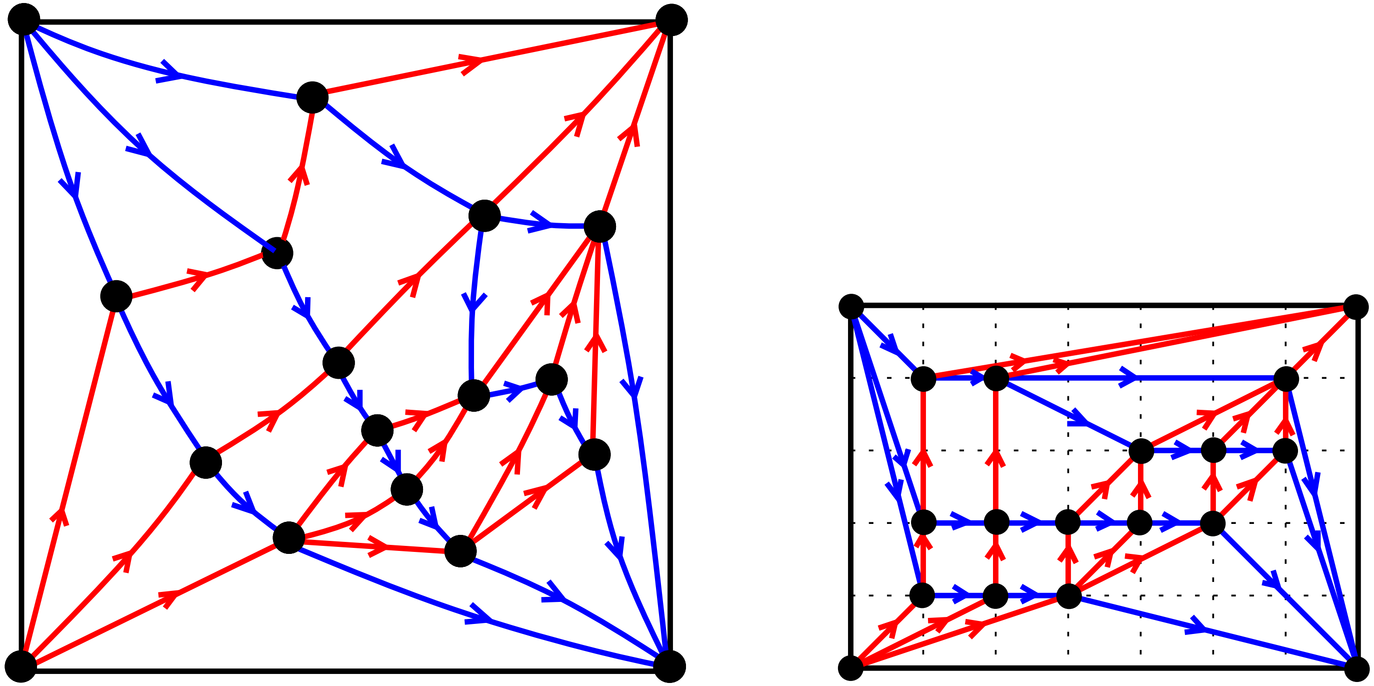
Place all other points using the red path for absciss and the blue path for ordinate



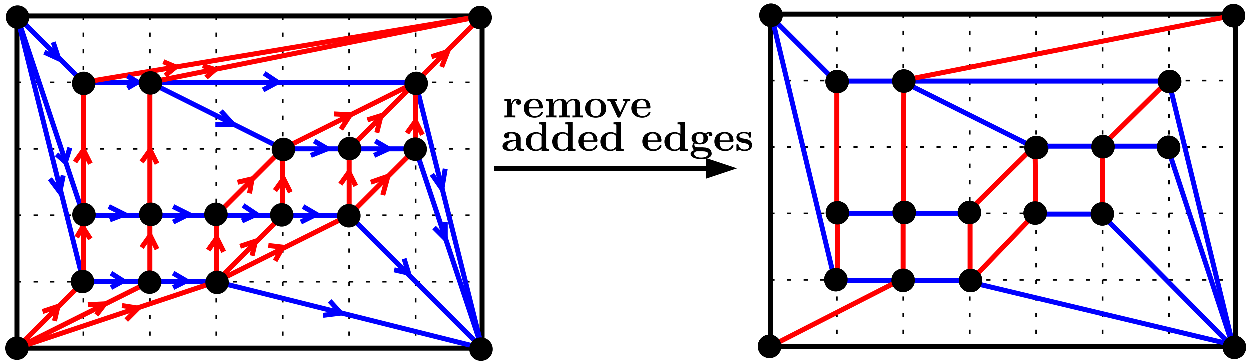


Execution of the algorithm

Link each pair of adjacent vertices by a segment



Results



- The drawing is a straight-line drawing of Q
- If Q has n vertices, the semi-perimeter verifies

$$W + H = n - 1 - \Delta,$$

where Δ is the number of alternating 4-cycles of Q.