Bijective counting of plane bipolar orientations

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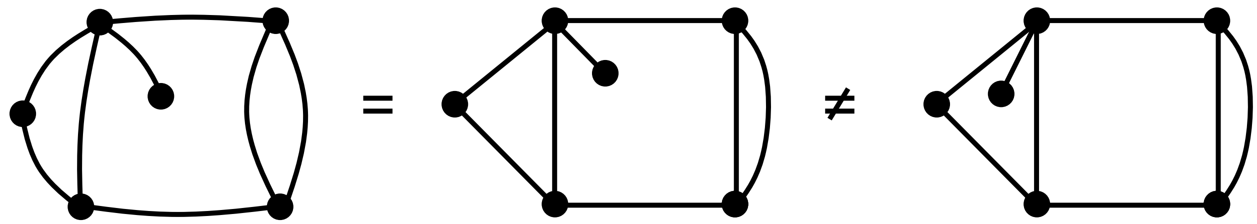
Planar maps

- Planar map = graph drawn in the plane without edge-crossing, taken up to continuous deformation



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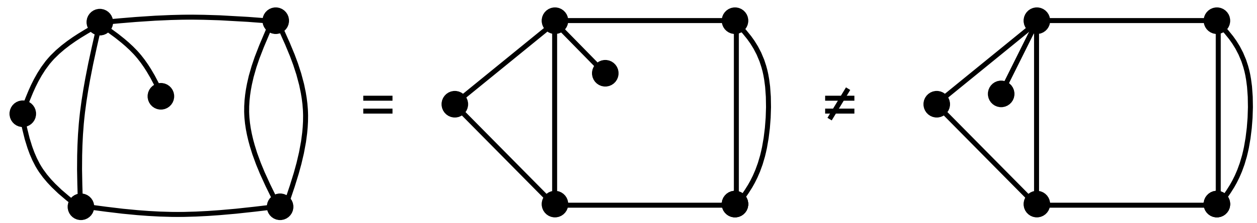

- Rooted map = map + root edge

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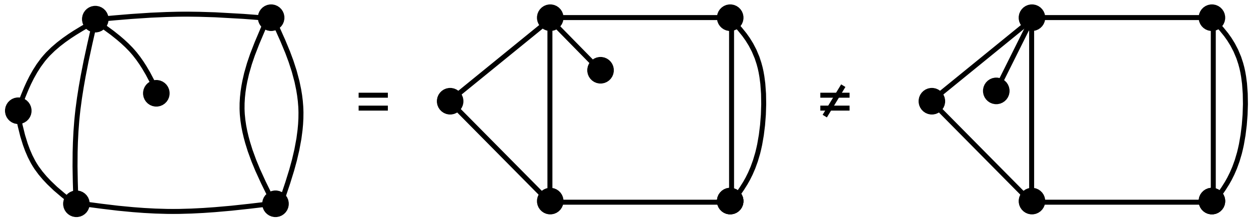
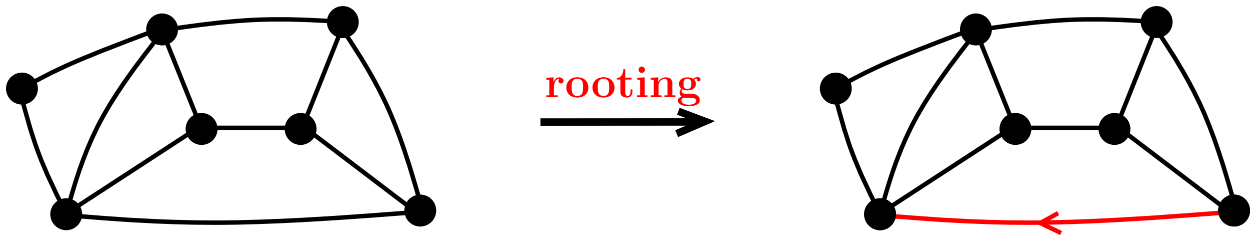
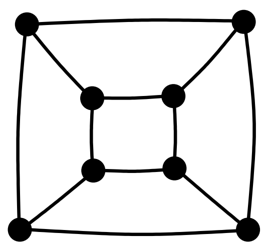


- Rooted map = map + root edge

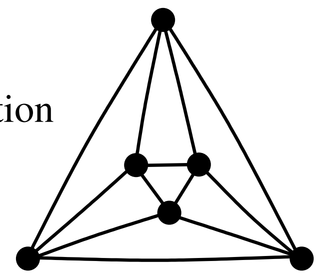


- Some classical families

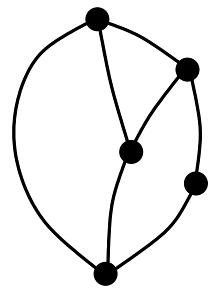
Quadrangulation



Triangulation



2–connected



Counting rooted maps

- Simple counting formulas

  (Planar rooted) maps with n edges: $$\frac{2 \cdot 3^n}{(n+2)(n+1)}\binom{2n}{n}$$

  Triangulations with $n+2$ vertices: $$\frac{1}{2n(2n+1)}\binom{4n-2}{n-1}$$

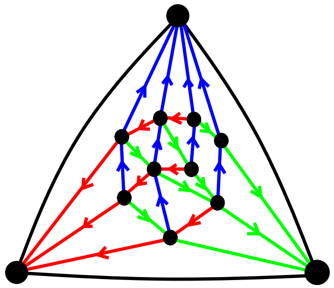
  Quadrangulations with $n+3$ vertices: $$\frac{2}{(n+1)(2n+1)}\binom{3n}{n}$$

- Two methods:
  - Recursive: Tutte 1963
  - Bijective: Cori 1981, Schaeffer 1997

Combinatorial structures on maps

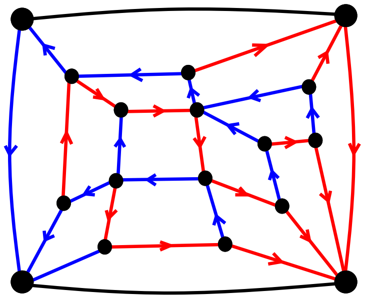
- Many families of maps are characterised by a structure

Triangulations



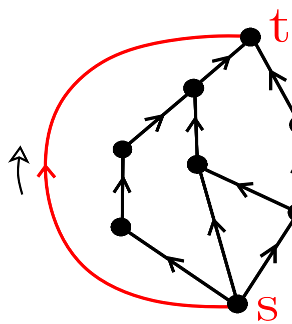
Schnyder woods
3 spanning trees

Quadrangulations



Separating decompositions
2 spanning trees

2-connected



plane bipolar orientations
acyclic with two poles
the root connects the poles

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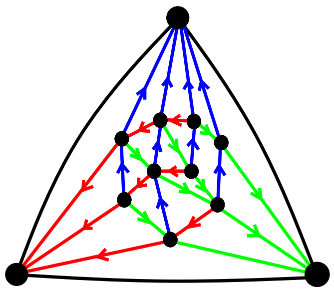
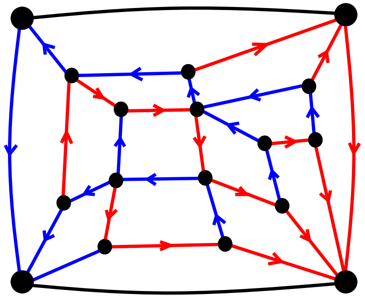


Schnyder woods
3 spanning trees

Quadrangulations



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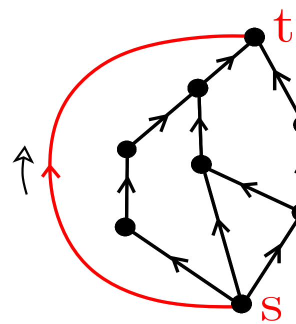

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- **Interests:**
  - **planarity conditions:** **Schnyder**
  - **tool for bijections:** **Poulalhon, Schaeffer, Bernardi**
  - **graph drawing:** **Schnyder, Felsner, de Fraysseix, Ossona de Mendez**

Combinatorial structures on maps

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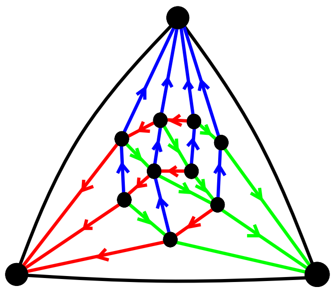

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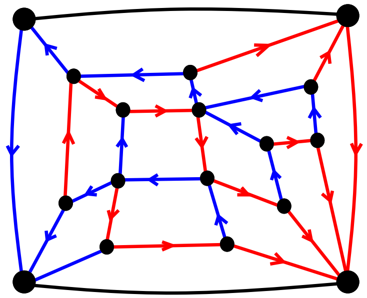

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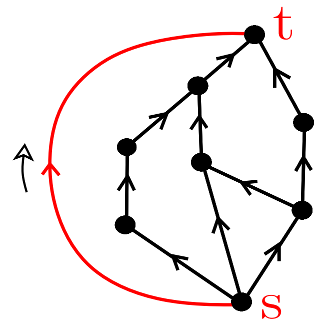

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- **Definition:** A structured map is a rooted map endowed with a structure.

Counting structured maps

Known results:

- The total number S_n of Schnyder woods with $n+3$ vertices is

$$S_n = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}.$$

  Bijective proofs by Bonichon'02, Bonichon-Bernardi'06

- The number B_{ij} of plane bipolar orientations with $i+1$ vertices and $j+1$ faces is

$$B_{ij} = 2\frac{(i+j-2)!(i+j-1)!(i+j)!}{(i-1)!i!(i+1)!(j-1)!j!(j+1)!}.$$

  Recursive proofs by Baxter'01, Bousquet-Mélou'03

New results

- Bijective proof of the formula

$$B_{ij} = 2\frac{(i+j-2)!(i+j-1)!(i+j)!}{(i-1)!i!(i+1)!(j-1)!j!(j+1)!}$$

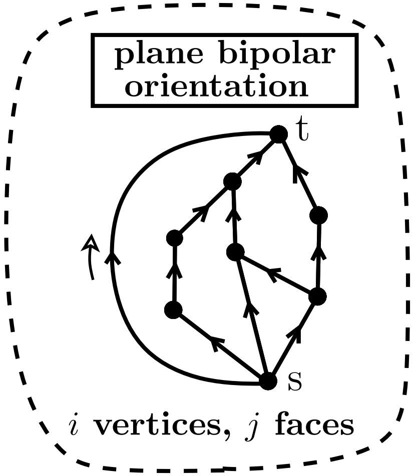
  for counting plane bipolar orientations.
  (similar principles as Bonichon-Bernardi'06)

- We recover the formula

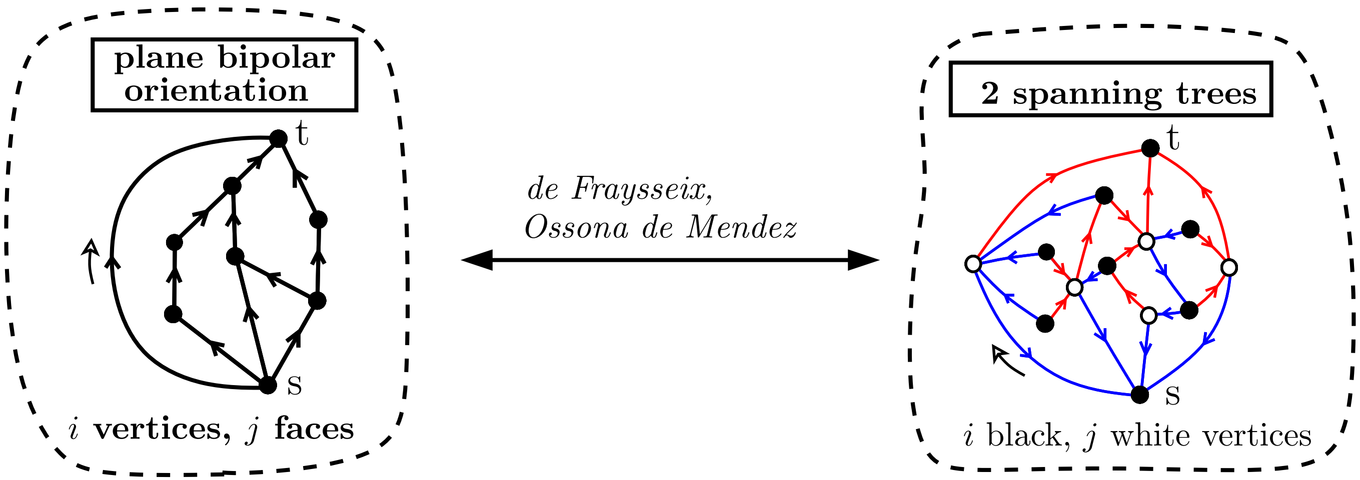
$$S_n = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!},$$

  for counting Schnyder woods, as a special case of our bijection.

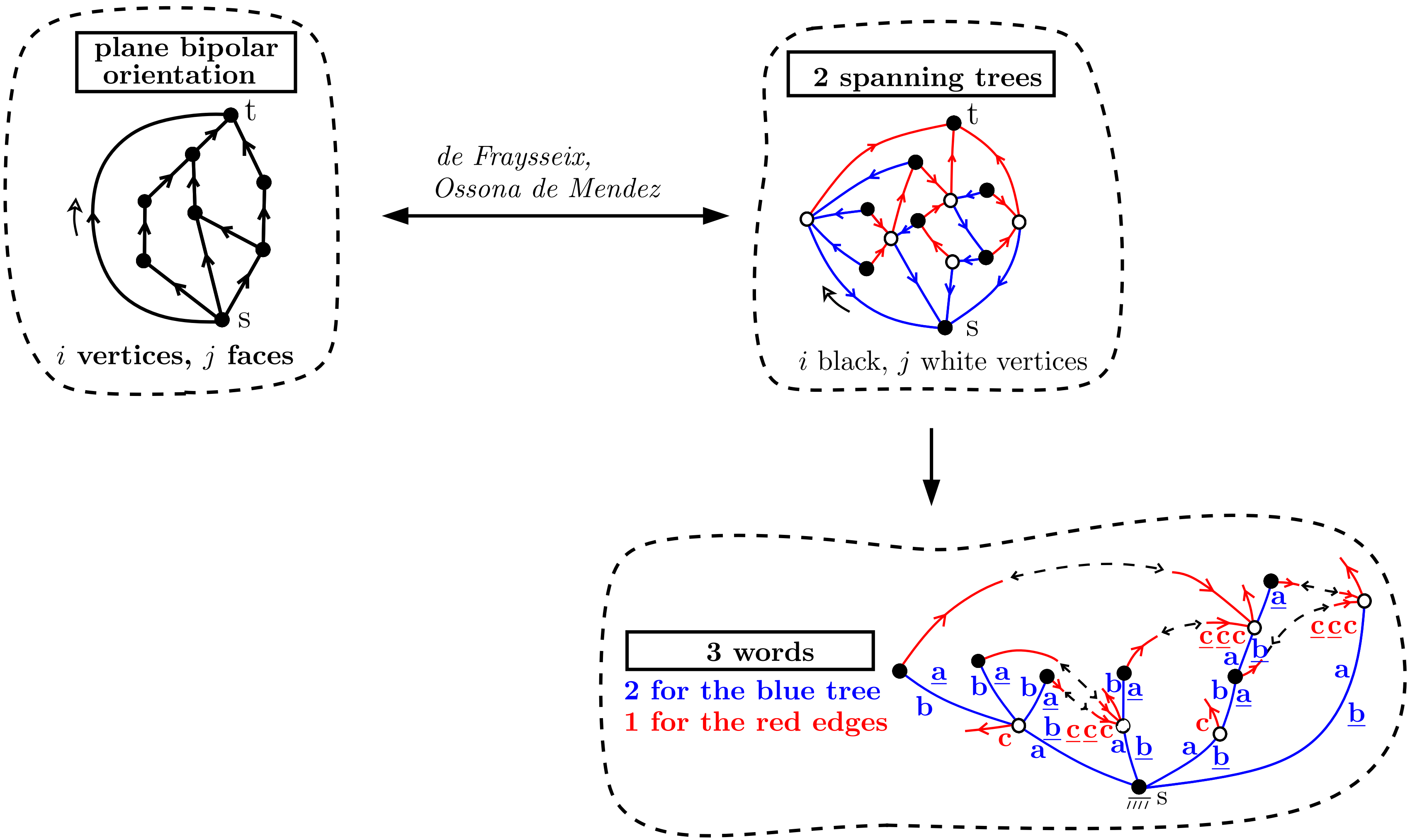
Principle of the bijection



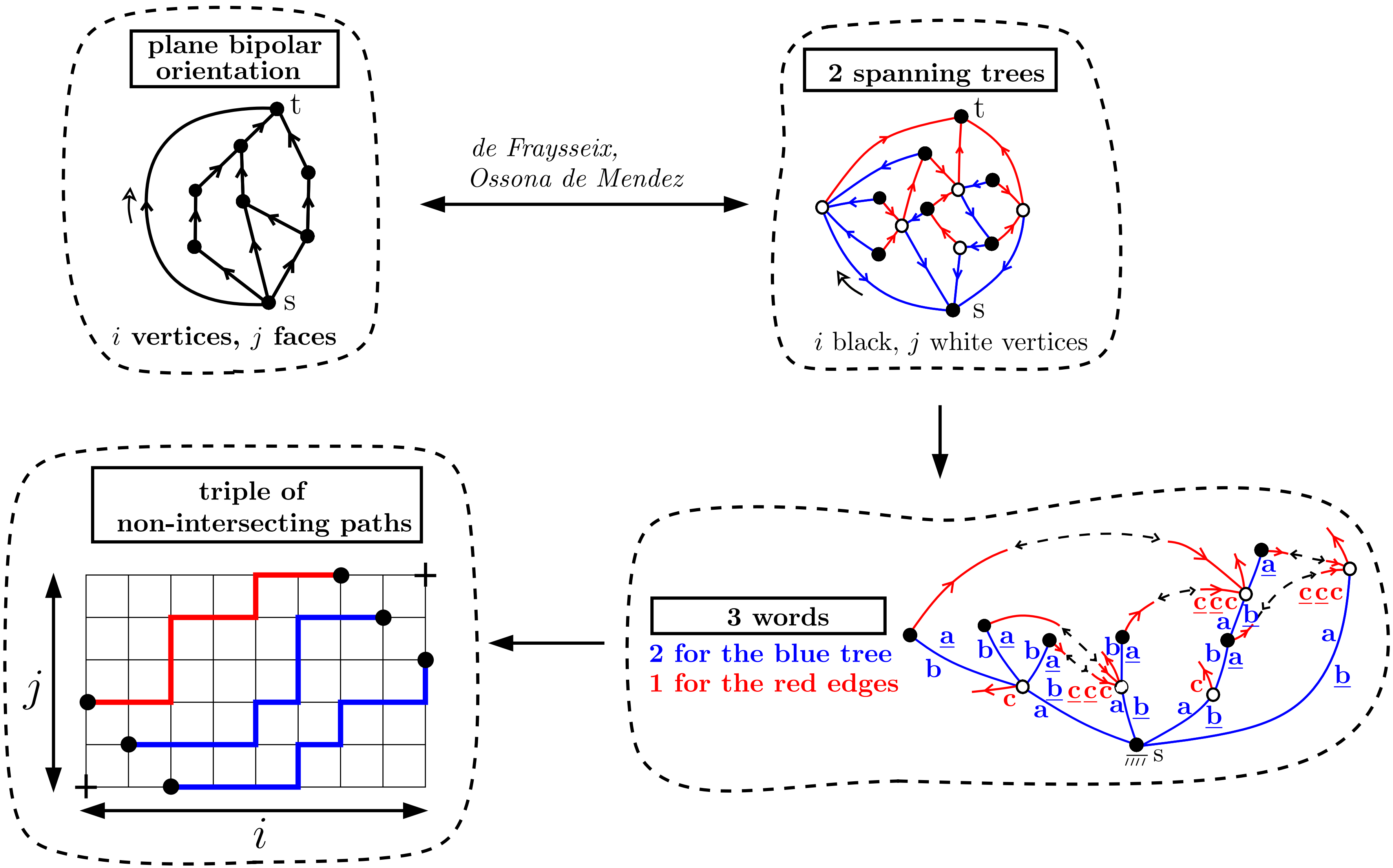
Principle of the bijection



Principle of the bijection

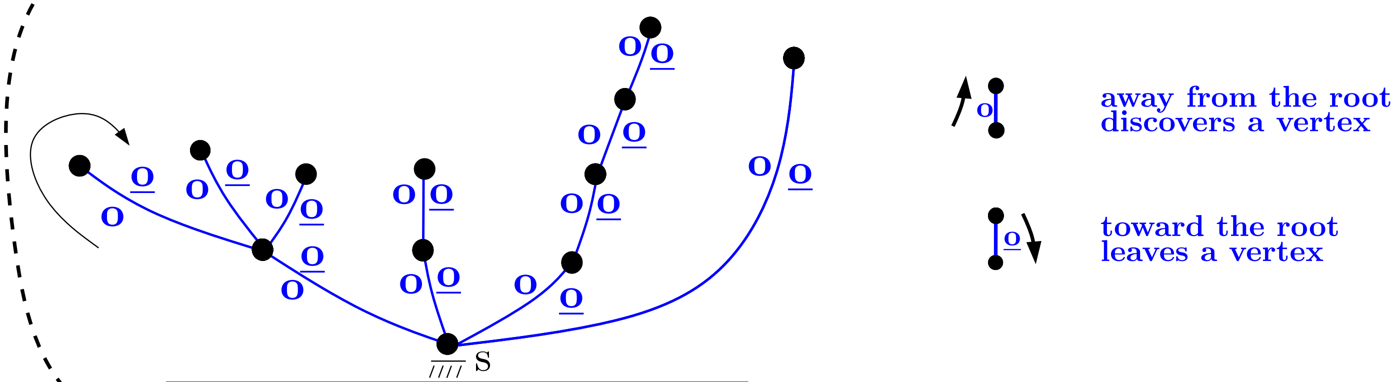


Principle of the bijection



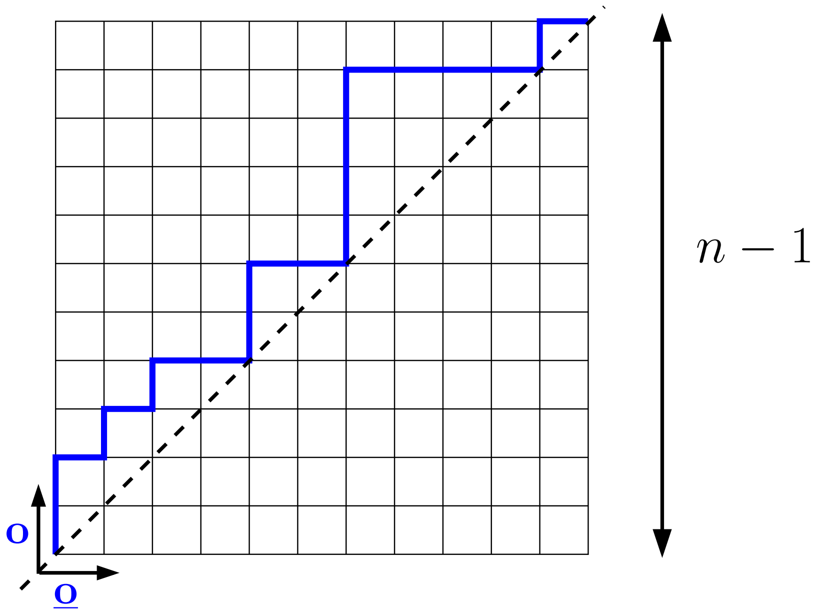
Encoding the blue tree

Classical Dyck encoding (n vertices)



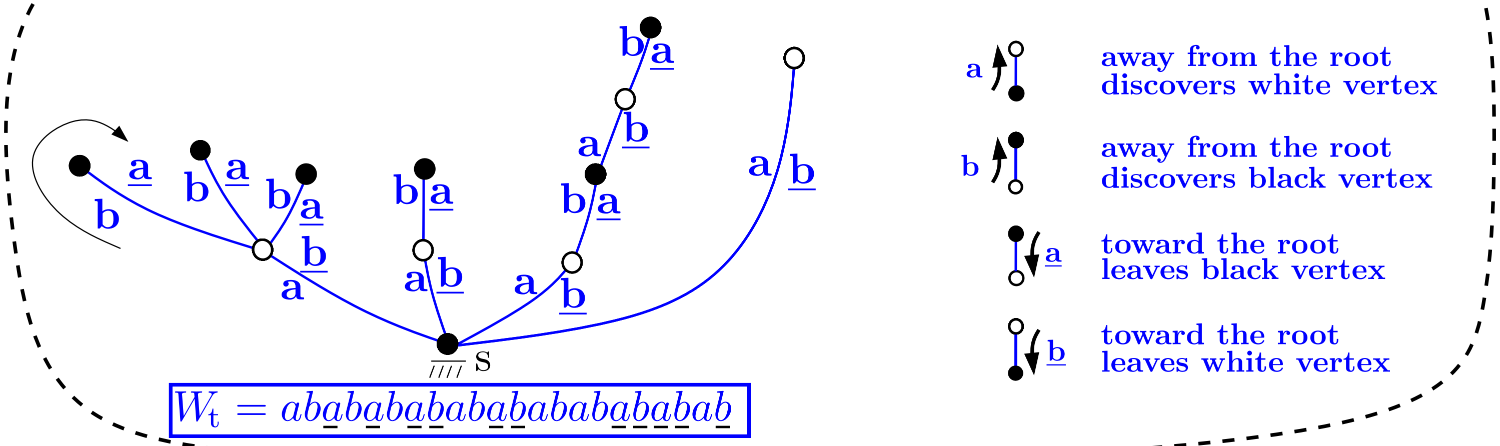
$W_{\mathrm{t}} = oo\underline{o}o\underline{o}o\underline{o}\underline{o}oo\underline{o}\underline{o}oooo\underline{o}\underline{o}\underline{o}\underline{o}o\underline{o}$

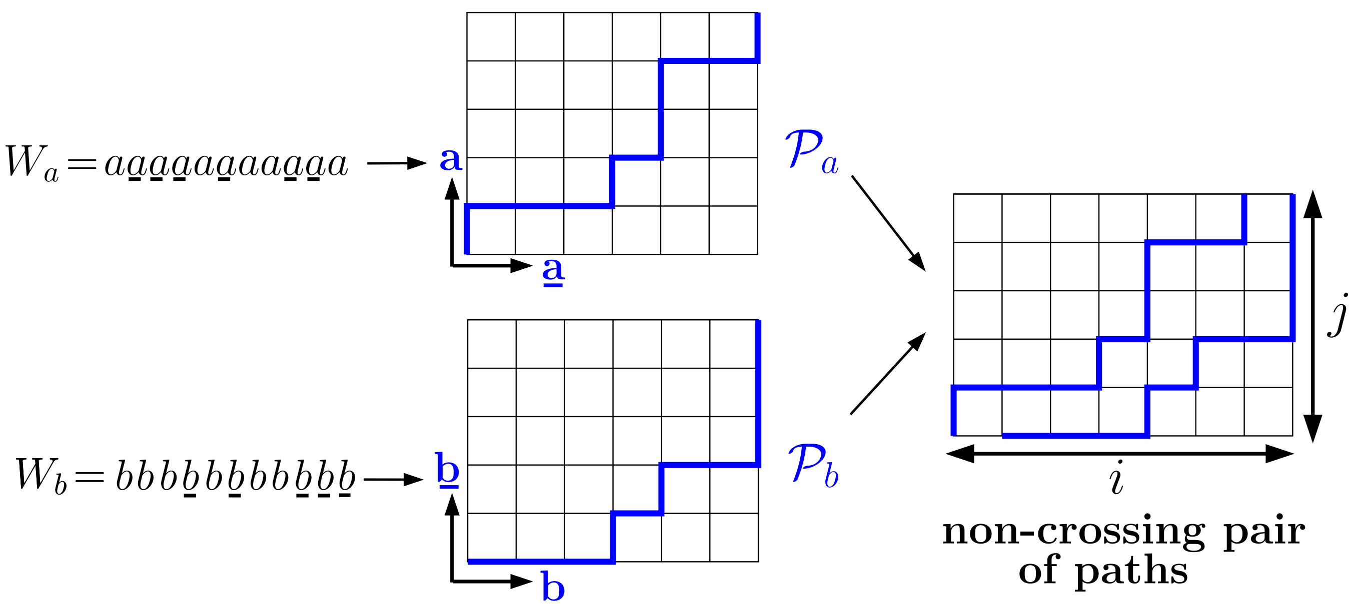
Dyck path



Encoding the blue tree

Bicolored Dyck encoding (i black, j white vertices)





Encoding the red edges

- Encode the degrees in red of the **white vertices**:

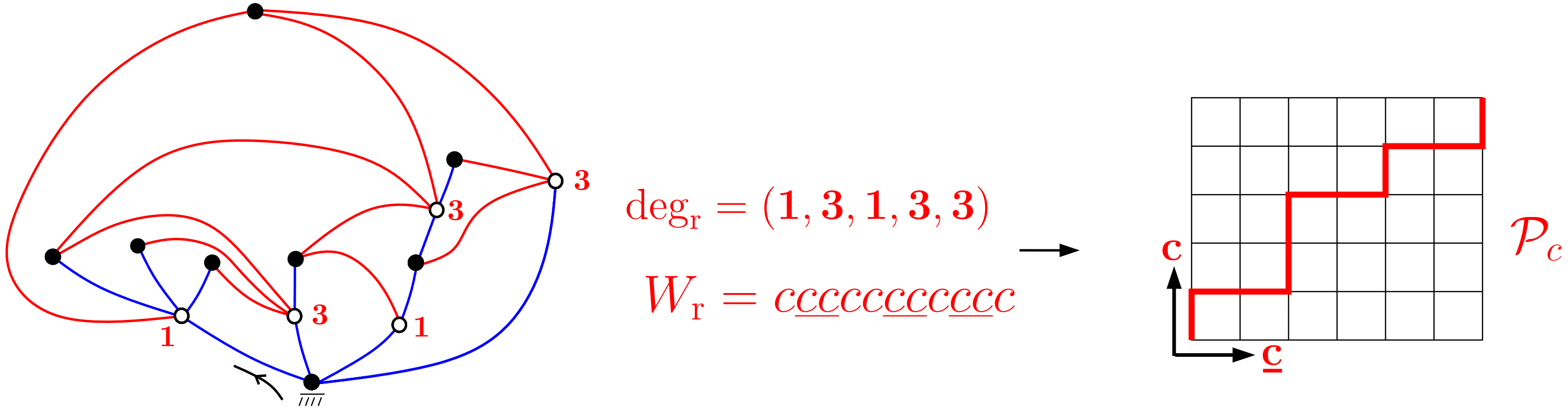
$$\deg_{\mathrm{r}} := (d_1, d_2, \ldots, d_k)$$

is the sequence of degrees in red of white vertices discovered in a cw traversal around the blue tree.

- The red word is

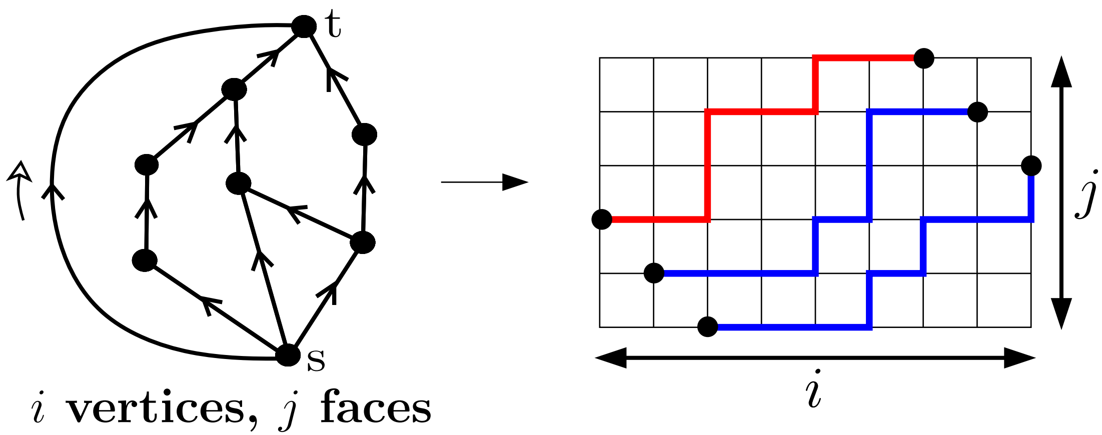
$$W_{\mathrm{r}} := \underline{c}^{d_1-1} c \underline{c}^{d_2-1} c \ldots \underline{c}^{d_k-1} c$$

- **Example:**



- **Lemma:** The red path shifted once top-left does not cross the two blue paths.

The bijection



Theorem:

Rooted plane bipolar orientations with i vertices and j faces

$\updownarrow$ ***bijection***

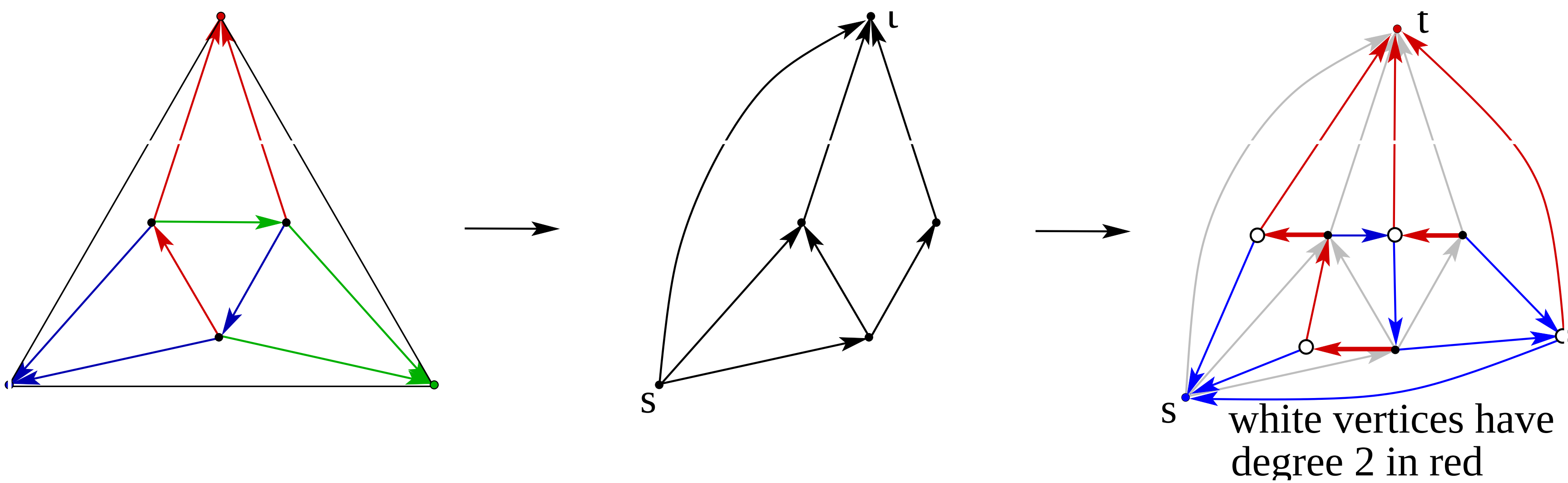
triples of non crossing upright lattice paths with $\mathbf{P_1 : (0,2) \to (i{-}2, j)}$, $\mathbf{P_2 : (1,1) \to (i{-}1, j{-}1)}$, $\mathbf{P_3 : (2,0) \to (i, j{-}2)}$

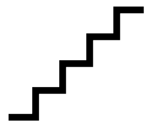
Counting:

Using Gessel-Viennot's formula, we recover

$$B_{ij} = 2\frac{(i+j-2)!(i+j-1)!(i+j)!}{(i-1)!i!(i+1)!(j-1)!j!(j+1)!}.$$

Counting Schnyder woods



$\Rightarrow$ **The path for red edges is a staircase** 

- **We recover:** **(Bonichon'02, Bonichon-Bernardi'06)**

Schnyder woods with $\mathbf{n+3}$ vertices

↕ ***bijection***

pairs of non-crossing Dyck paths of size n

- **Counting:**

The number of Schnyder woods with $n+3$ vertices is

$$S_n = C_{n+2}C_n - C_{n+1}^2 = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$