Bijective counting of plane bipolar orientations

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Planar maps

- **Planar map** = graph drawn in the plane without edge-crossing, taken up to continuous deformation

![Graph diagrams](image)
Planar maps

- **Planar map** = graph drawn in the plane without edge-crossing, taken up to continuous deformation

- **Rooted map** = map + root edge
Planar maps

1. **Planar map** = graph drawn in the plane **without** edge-crossing, taken **up to continuous deformation**

2. **Rooted map** = map + root edge

3. **Some classical families**
   - Quadrangulation
   - Triangulation
   - 2-connected
Counting rooted maps

- **Simple counting formulas**
  
  (Planar rooted) maps with \( n \) edges: \( \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n} \)
  
  Triangulations with \( n+2 \) vertices: \( \frac{1}{2n(2n+1)} \binom{4n-2}{n-1} \)
  
  Quadrangulations with \( n+3 \) vertices: \( \frac{2}{(n+1)(2n+1)} \binom{3n}{n} \)

- **Two methods:**
  - **Recursive**: Tutte 1963
  - **Bijective**: Cori 1981, Schaeffer 1997
Combinatorial structures on maps

- Many families of maps are characterised by a structure.

- **Triangulations**: Schnyder woods
  - 3 spanning trees

- **Quadrangulations**: Separating decompositions
  - 2 spanning trees

- **2-connected**: Plane bipolar orientations
  - Acyclic with two poles
  - The root connects the poles
Combinatorial structures on maps

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  - Quadrangulations
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- Interests:
  - planarity conditions: Schnyder
  - tool for bijections: Poulalhon, Schaeffer, Bernardi
  - graph drawing: Schnyder, Felsner, de Fraysseix, Ossona de Mendez
Combinatorial structures on maps

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  - Schnyder woods
  - Separating decompositions
  - Plane bipolar orientations

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• Definition: A structured map is a rooted map endowed with a structure.
Counting structured maps

Known results:

- The total number $S_n$ of Schnyder woods with $n + 3$ vertices is

$$S_n = \frac{6(2n)! (2n+2)!}{n!(n+1)!(n+2)!(n+3)!}.$$

**Bijective proofs** by Bonichon’02, Bonichon-Bernardi’06

- The number $B_{ij}$ of plane bipolar orientations with $i + 1$ vertices and $j + 1$ faces is

$$B_{ij} = 2 \frac{(i+j-2)! (i+j-1)! (i+j)!}{(i-1)!i!(i+1)!(j-1)!j!(j+1)!}.$$

**Recursive proofs** by Baxter’01, Bousquet-Mélou’03
New results

• **Bijective proof** of the formula

\[ B_{ij} = 2 \frac{(i + j - 2)!(i + j - 1)!(i + j)!}{(i - 1)!i!(i + 1)!(j - 1)!j!(j + 1)!} \]

for counting **plane bipolar orientations**.
(similar principles as Bonichon-Bernardi’06)

• We **recover** the formula

\[ S_n = \frac{6(2n)!(2n + 2)!}{n!(n + 1)!(n + 2)!(n + 3)!}, \]

for counting **Schnyder woods**, as a **special case** of our bijection.
Principle of the bijection
Principle of the bijection

Plane bipolar orientation

2 spanning trees

i vertices, j faces

i black, j white vertices

de Fraysseix, Ossona de Mendez
Principle of the bijection

plane bipolar orientation

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3 words

2 for the blue tree
1 for the red edges

de Fraysseix, Ossona de Mendez
Principle of the bijection

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triple of non-intersecting paths

3 words

2 for the blue tree
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i vertices, j faces
Encoding the blue tree

Classical Dyck encoding \((n\) vertices\)

\[ W_i = \overset{n-1}{oooooo} \]

Dyck path

away from the root: discovers a vertex

toward the root: leaves a vertex

\[ n - 1 \]
Encoding the blue tree

Bicolored Dyck encoding ($i$ black, $j$ white vertices)

$W_i = abababababababababababababab$  

$W_a = aaaaaaaaaaaaaa$  

$W_b = bbbbbbbbbbbb$  

$P_a$  

$P_b$  

non-crossing pair of paths
Encoding the red edges

- Encode the **degrees in red** of the **white vertices**:
  \[ \text{deg}_r := (d_1, d_2, \ldots, d_k) \]
is the **sequence of degrees in red** of white vertices discovered in a **cw traversal** around the blue tree.
- The **red word** is
  \[ W_r := c^{d_1-1} c^{d_2-1} c \ldots c^{d_k-1} c \]
- **Example:**
  \[ \text{deg}_r = (1, 3, 1, 3, 3) \]
  \[ W_r = ccccccccccc \]

- **Lemma:** The **red path** shifted once top-left does not cross the two **blue paths**.
The bijection

Theorem:

Rooted plane bipolar orientations with \(i\) vertices and \(j\) faces

\[ \text{bijection} \]

triples of non crossing upright lattice paths with

\[ P_1 : (0, 2) \rightarrow (i-2, j) \]
\[ P_2 : (1, 1) \rightarrow (i-1, j-1) \]
\[ P_3 : (2, 0) \rightarrow (i, j-2) \]

Counting:

Using Gessel-Viennot’s formula, we recover

\[
B_{ij} = 2 \frac{(i+j-2)! (i+j-1)! (i+j)!}{(i-1)! i! (i+1)! (j-1)! j! (j+1)!}.
\]
The path for red edges is a staircase

We recover: [Bonichon’02, Bonichon-Bernardi’06]

Schnyder woods with \( n + 3 \) vertices

bijection

pairs of non-crossing Dyck paths of size \( n \)

Counting:

The number of Schnyder woods with \( n + 3 \) vertices is

\[
S_n = C_{n+2}C_n - C_{n+1}^2 = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}
\]