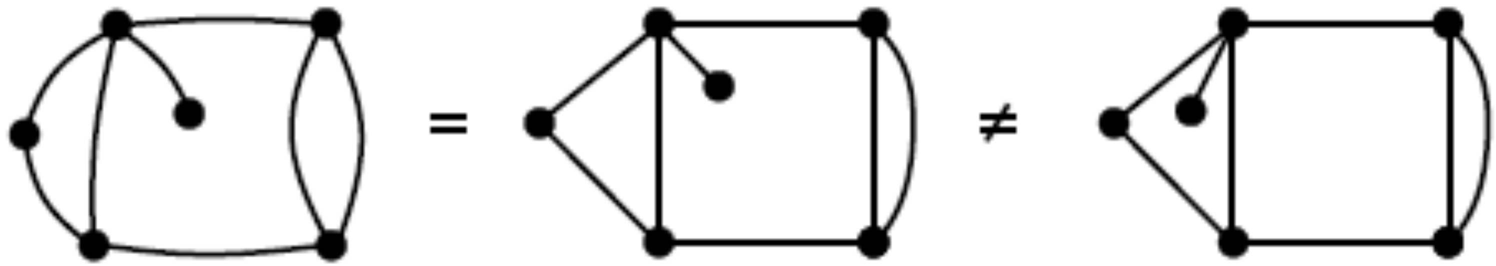
On the diameter of random planar graphs

Eric Fusy (LIX, Ecole Polytechnique)
joint work with Guillaume Chapuy, Omer Gimenez, Marc Noy

Planar maps / planar graphs

- Planar map = planar embedding of a graph



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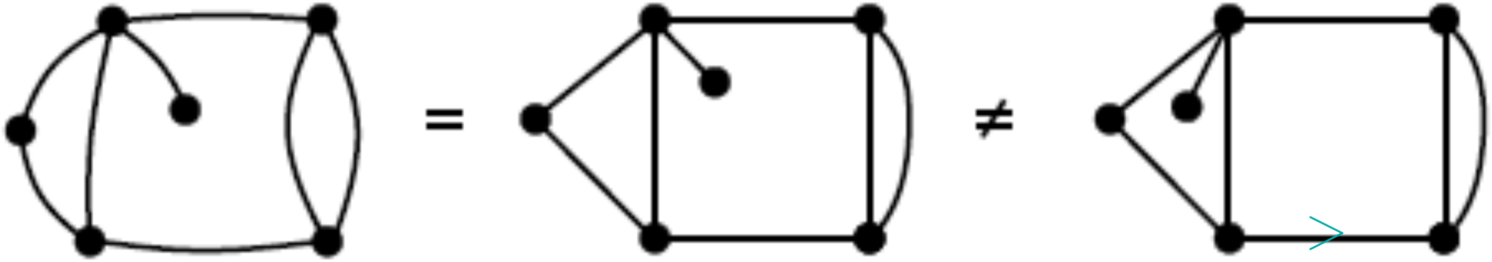

(rooting is enough to avoid symmetries)

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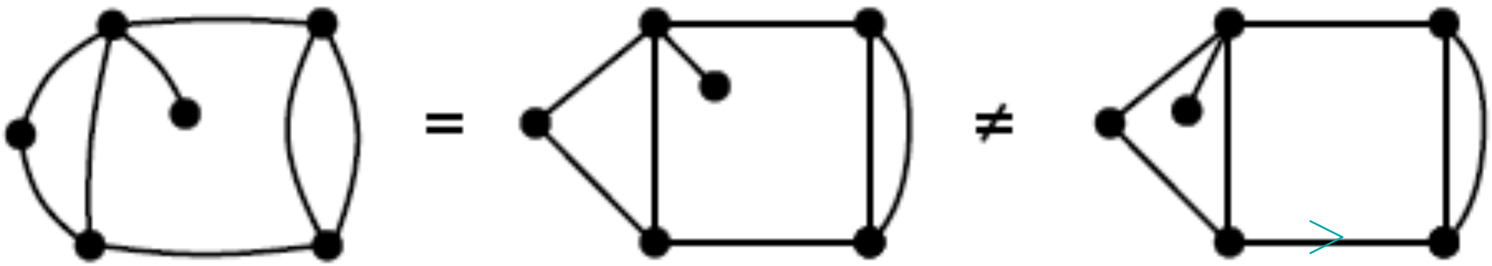

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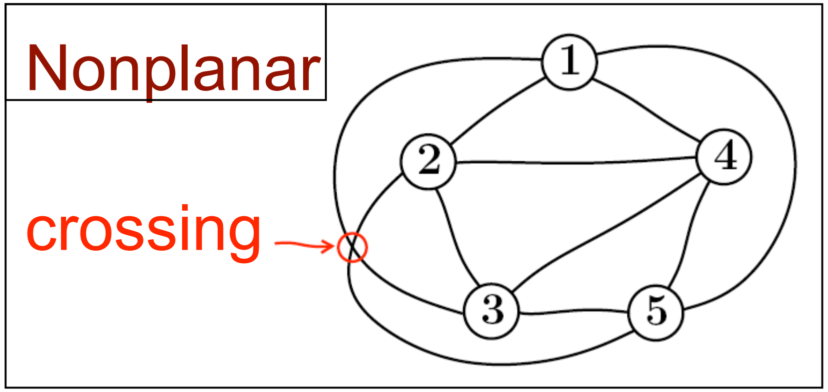
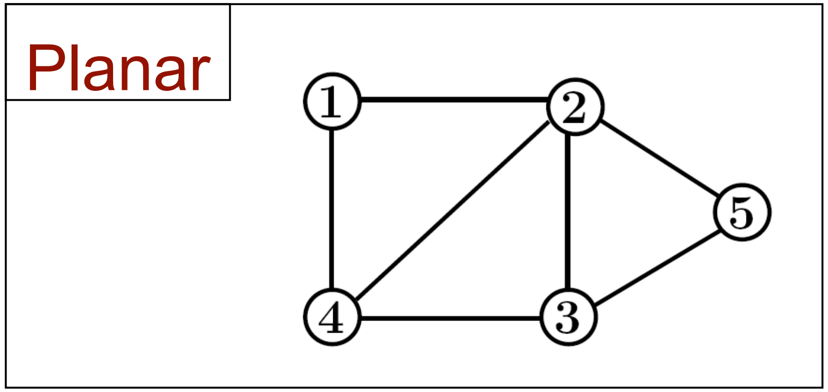
- Planar graph = graph with at least one planar embedding





(needs to label all vertices to avoid symmetries)

Planar maps / planar graphs

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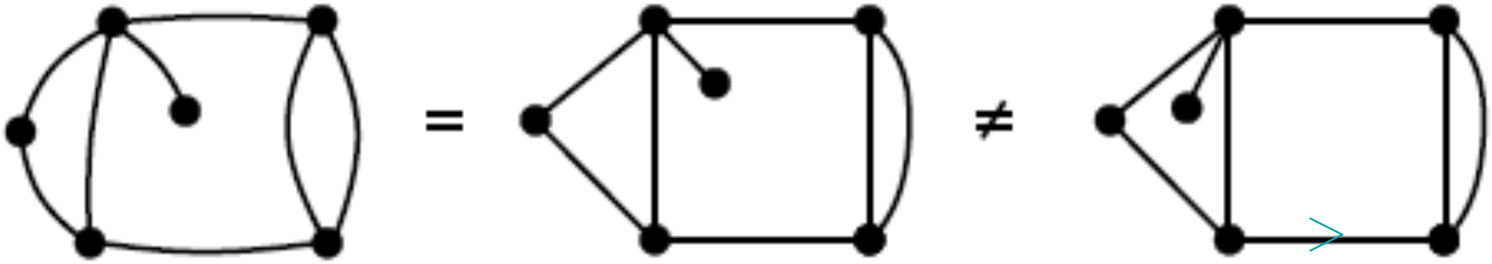

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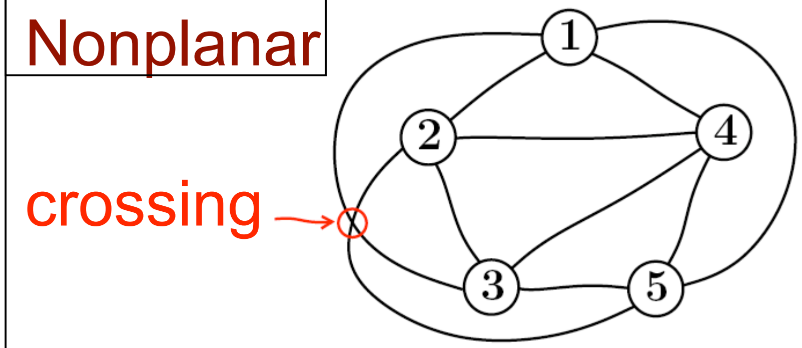
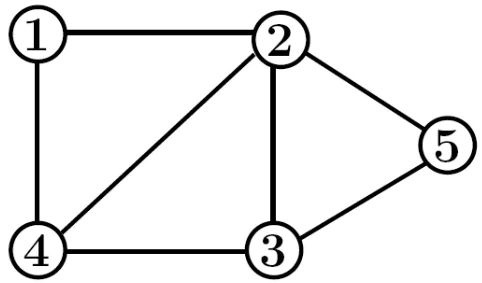
Nonplanar



Planar



(needs to label all vertices to avoid symmetries)

- Random planar graph $G_n \neq$ random planar map M_n

Main result

Theorem: Let G_n be the random (unembedded) planar graph with *n* vertices. Then

$$\mathbb{P}(\mathrm{Diam}(G_n) \notin [n^{1/4-\epsilon}, n^{1/4+\epsilon}]) = O(\exp(-n^{\Theta(\epsilon)}))$$

Conjecture:

$$\frac{1}{n^{1/4}}\mathrm{Diam}(G_n) \quad \textbf{converges in law}$$

Related results

Random planar structures

Random map M_n:

- $\mathrm{Radius}(M_n)/n^{1/4}$ converges in law

[Chassaing-Schaeffer'04], [Marckert-Miermont'06],...

Random tree T_n:

- $h(T_n)/n^{1/2}$ converges in law
- $\mathbb{P}(h(T_n) \notin [n^{1/2-\epsilon}, n^{1/2+\epsilon}]) = O(\exp(-n^{\Theta(\epsilon)}))$

[Flajolet et al'93]

Random graph (Erdös-Rényi)

For each fixed $\lambda > 1$

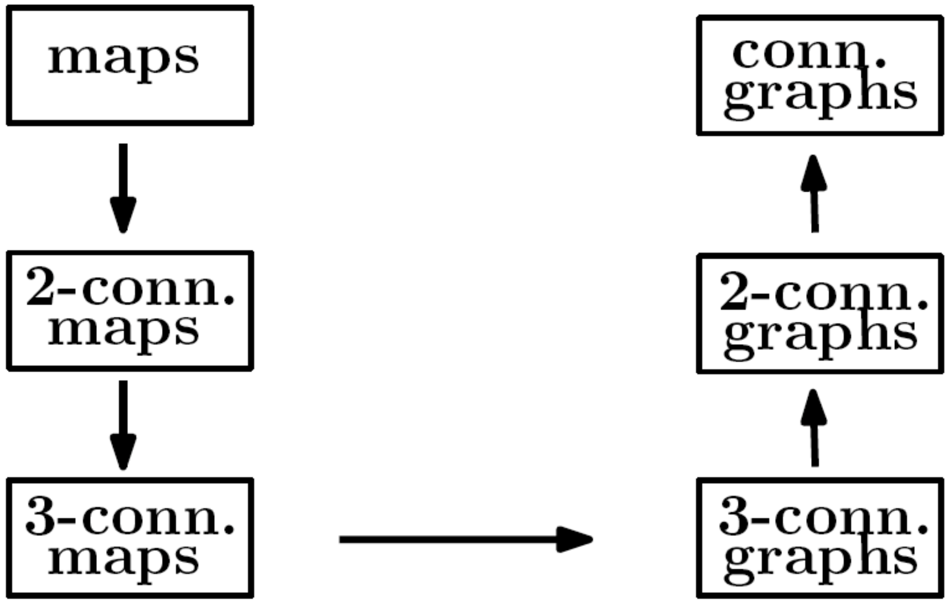
$$\mathrm{Diam}(G(n, p = \lambda/n) \sim c_\lambda \log(n)$$

with high probability

[Riordan-Wormald'09]

Overview

- We recall the classical scheme:



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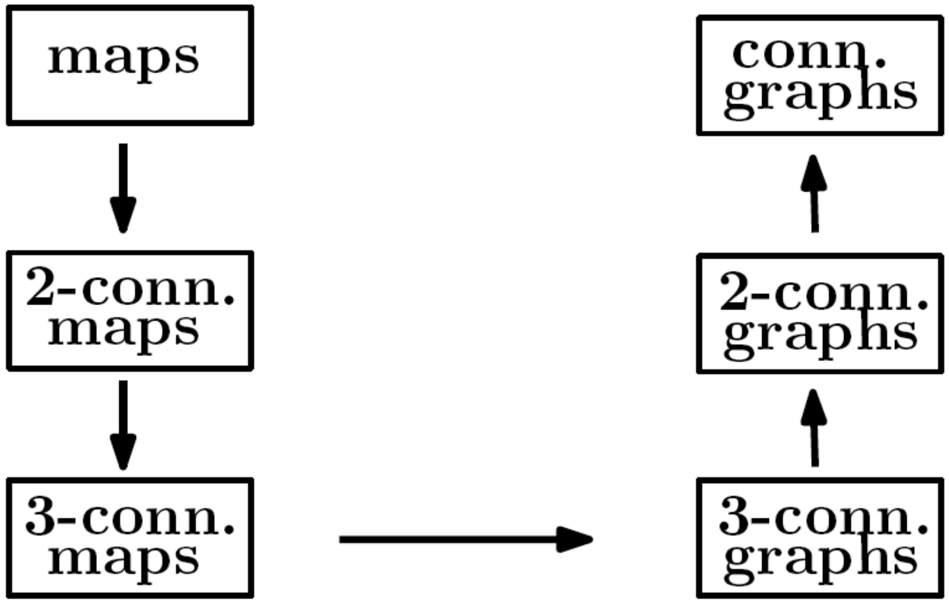

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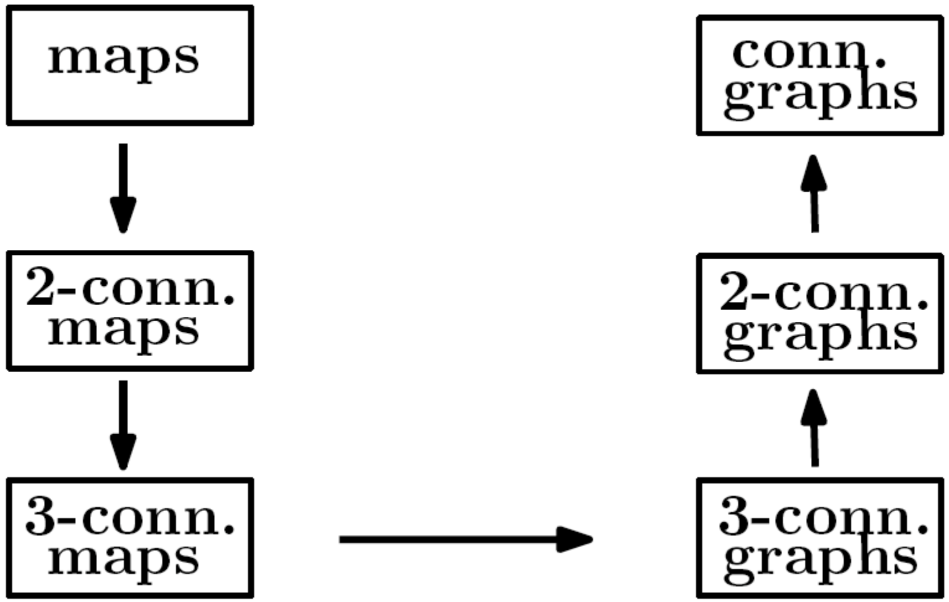

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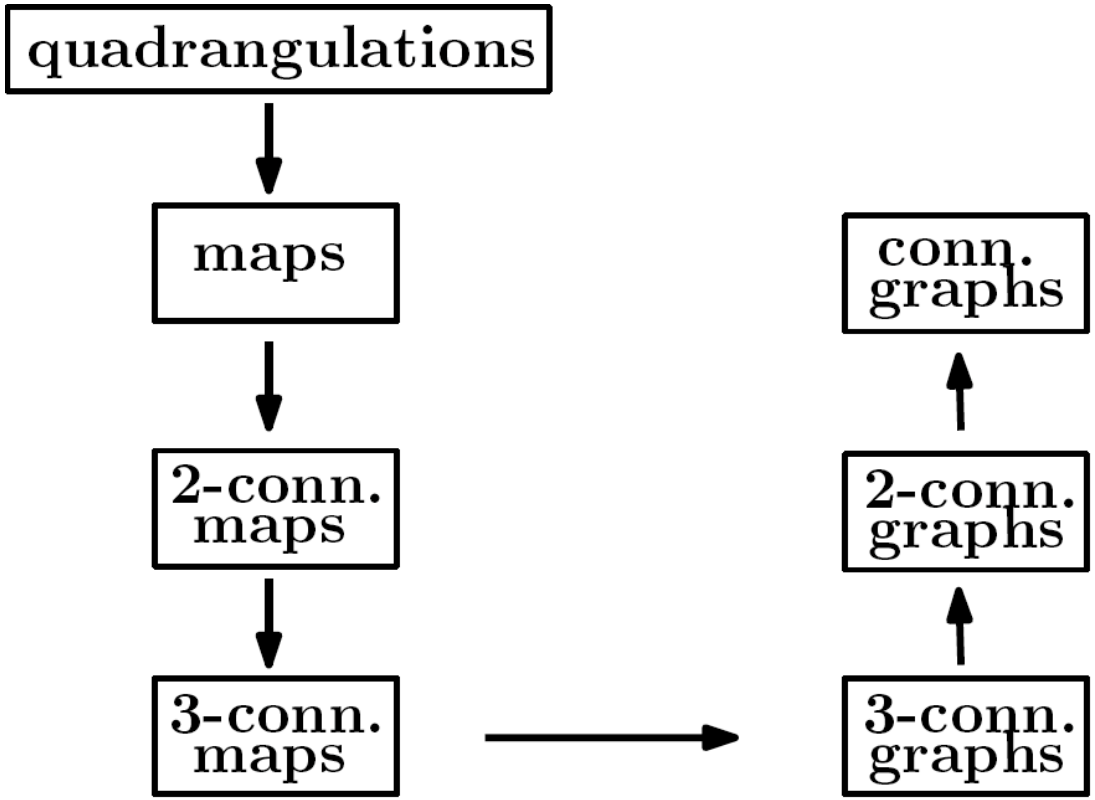

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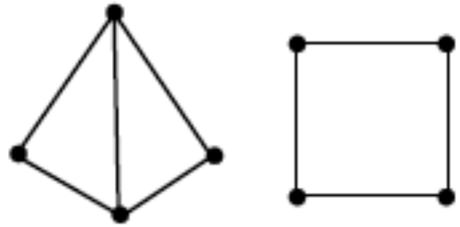
Part I: counting scheme for connected planar graphs

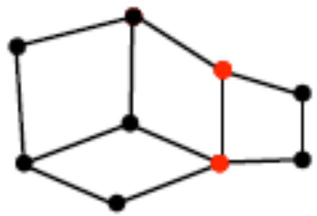
Connectivity in graphs

General

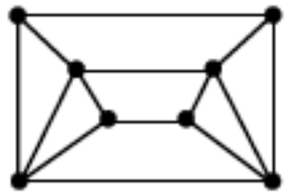


Connected

2-connected



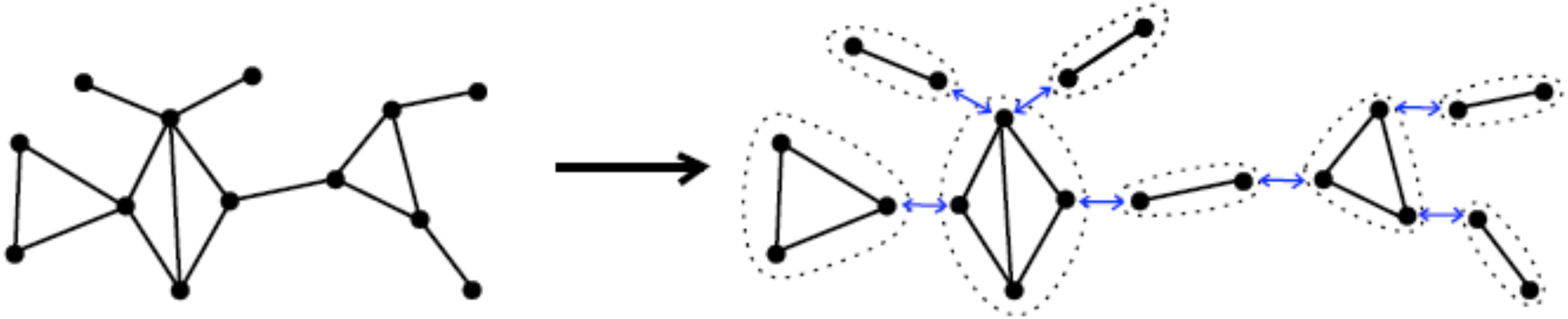
3-connected



- [Tutte'66]:

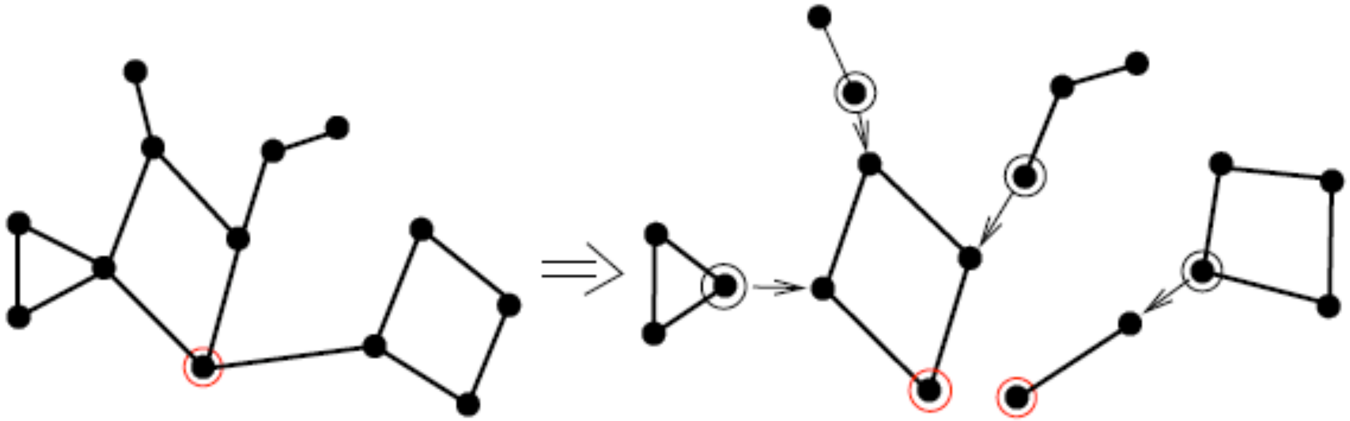
- a connected graph decomposes into 2-connected components
- a 2-connected graph decomposes into 3-connected components

Connected -> 2-connected



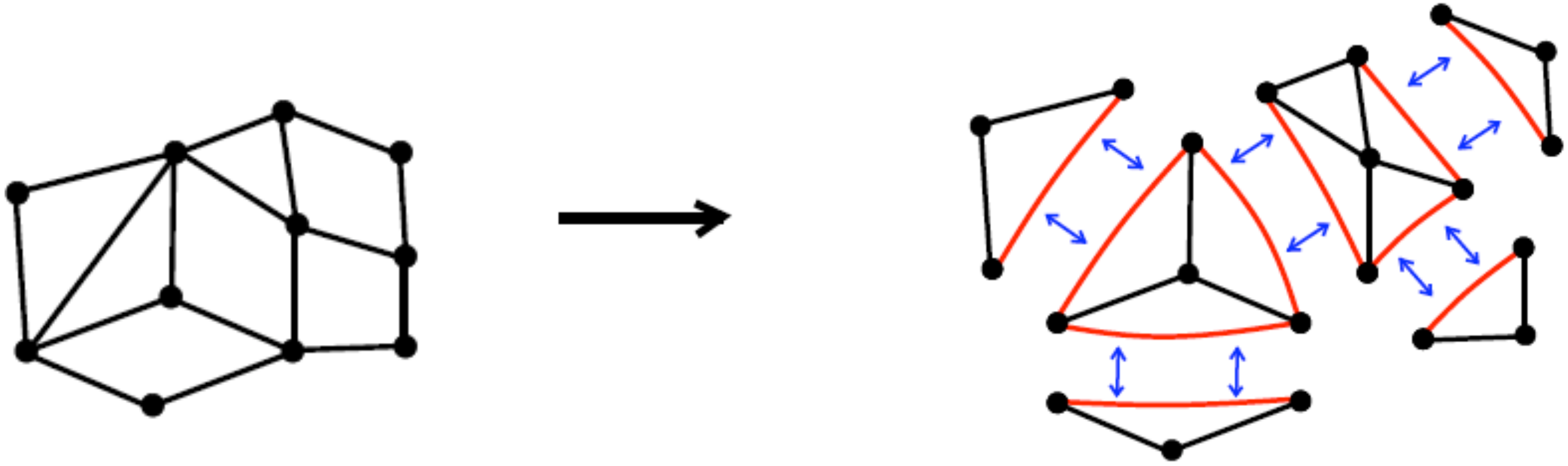
The incidences between cut-vertices and blocks forms a tree

- Vertex-rooted formulation:



$$\Rightarrow \boxed{C^{\bullet}(x,y) = x\exp(B'(C^{\bullet}(x,y),y))}$$

2-connected -> 3-connected



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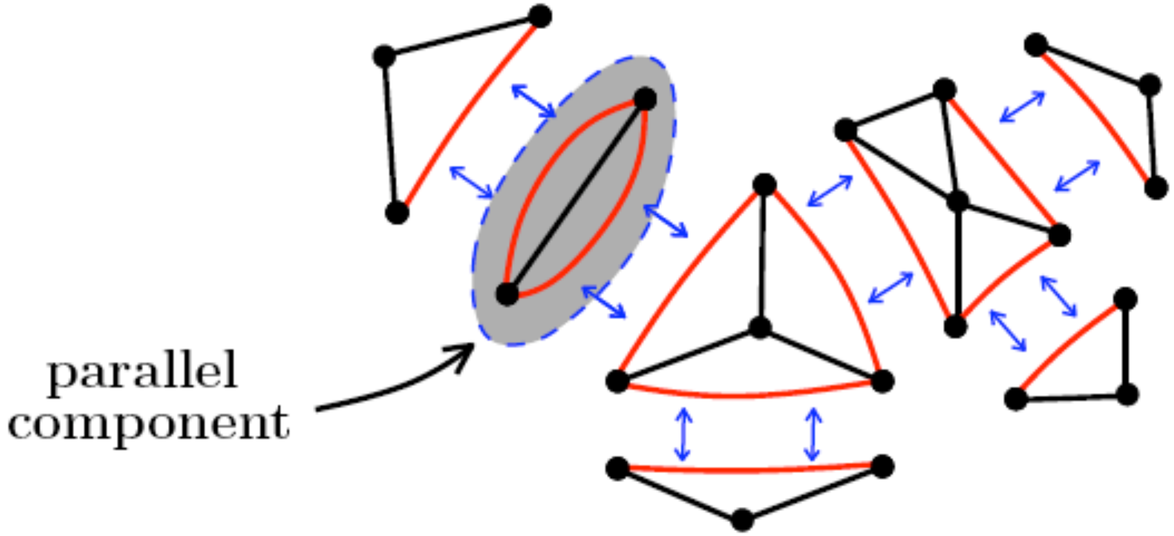

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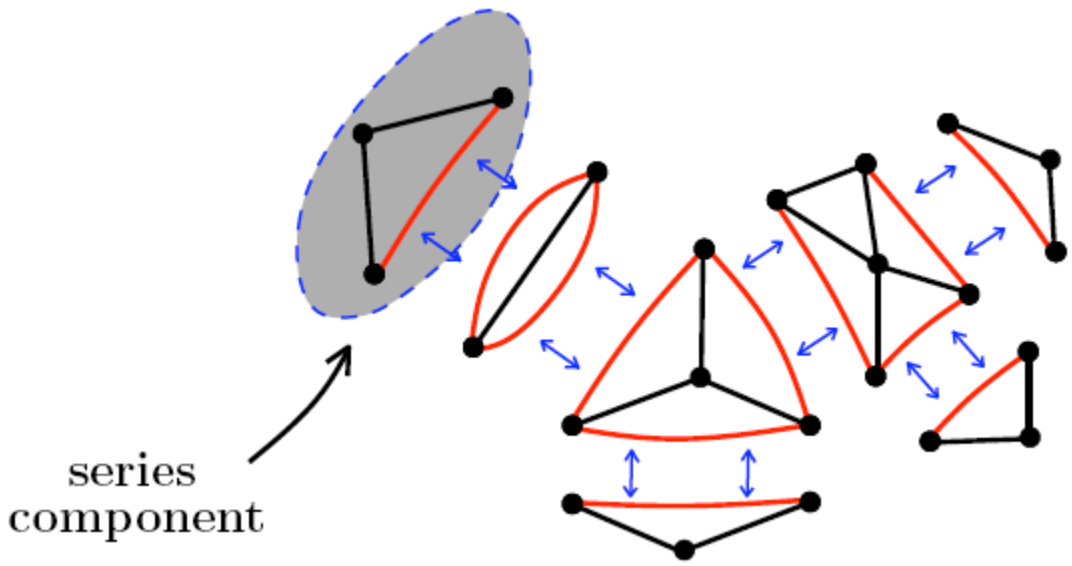

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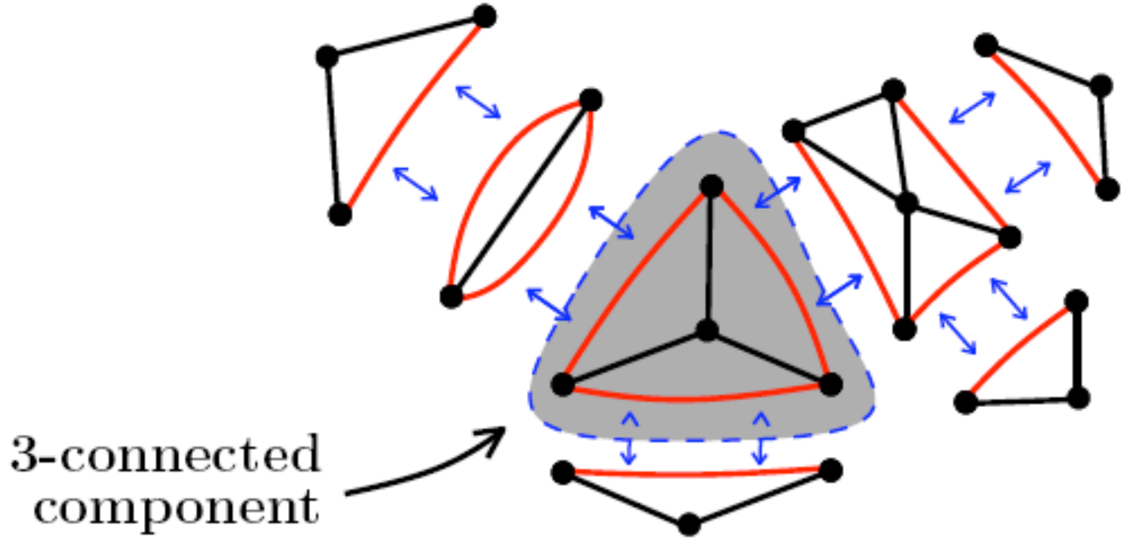

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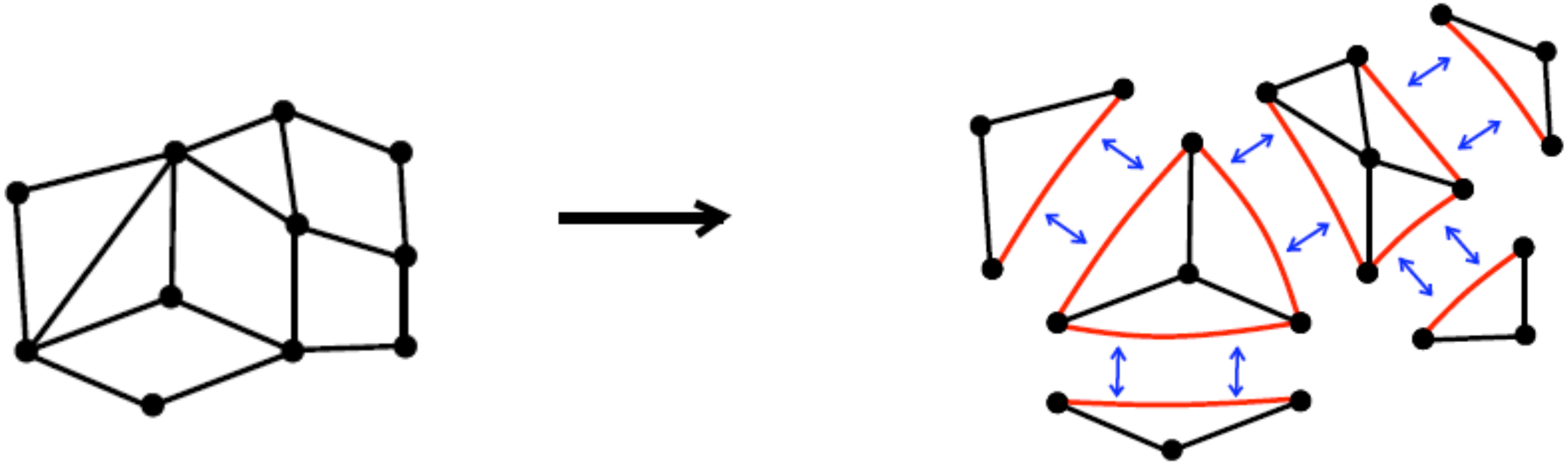
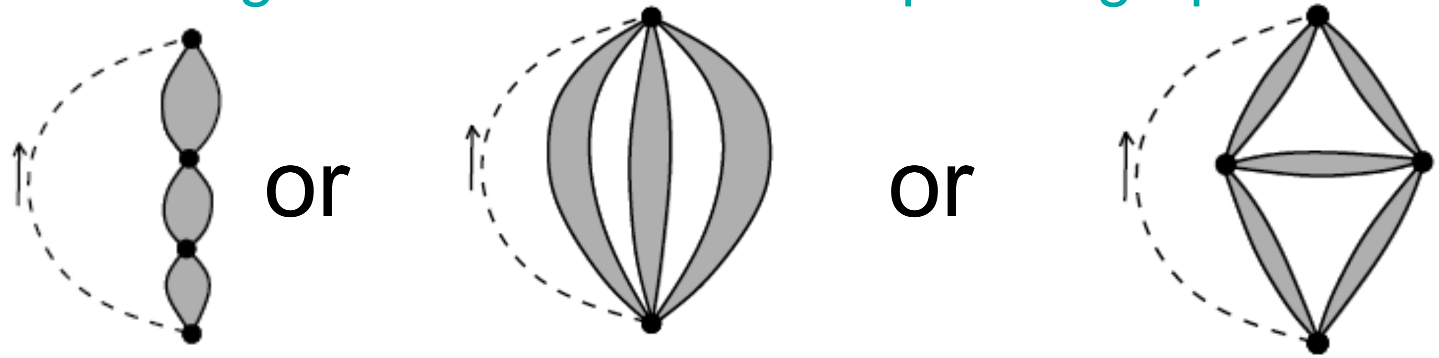


- Edge-rooted formulation [Trakhtenbrot'58]:

  An edge-rooted 2-connected planar graph is either:

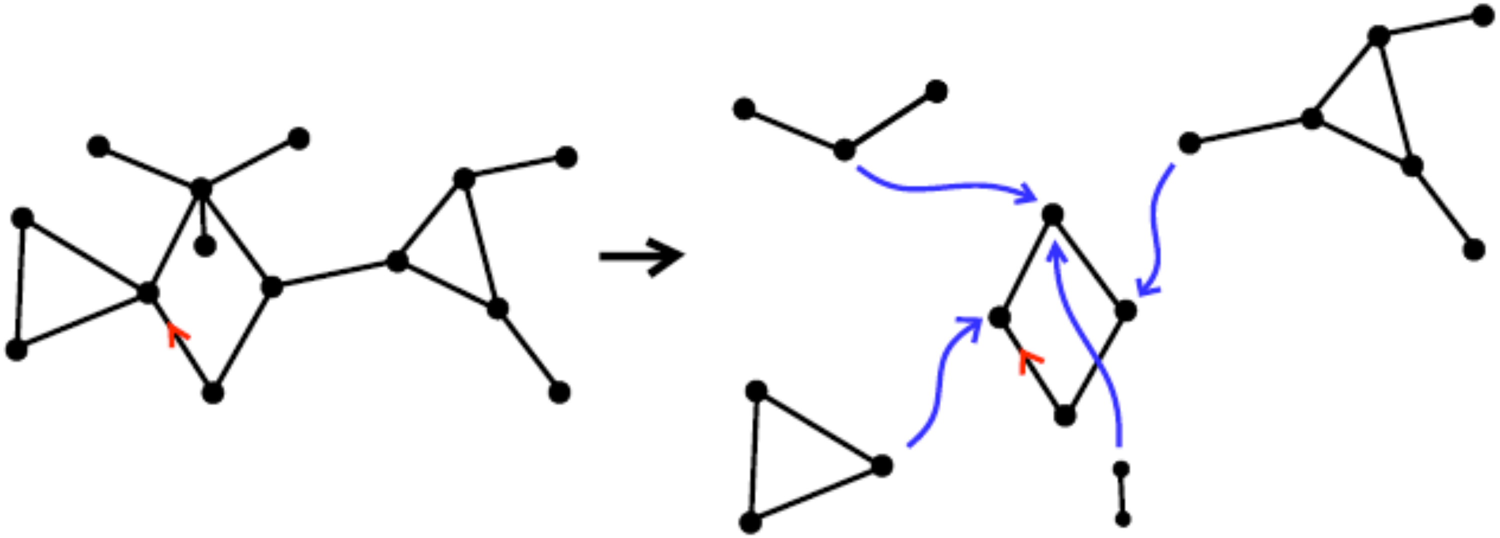


$$\overrightarrow{G_2}(x,y) \approx \overrightarrow{G_3}\left(x, \overrightarrow{G_2}(x,y)\right)$$

(modulo series-parallel operations)

3-connected graphs -> maps

- [Whitney]: 3-connected planar graphs $\simeq$ 3-connected maps
- [Tutte'63]: Use the decomposition the reverse way
  and take the embedding into account

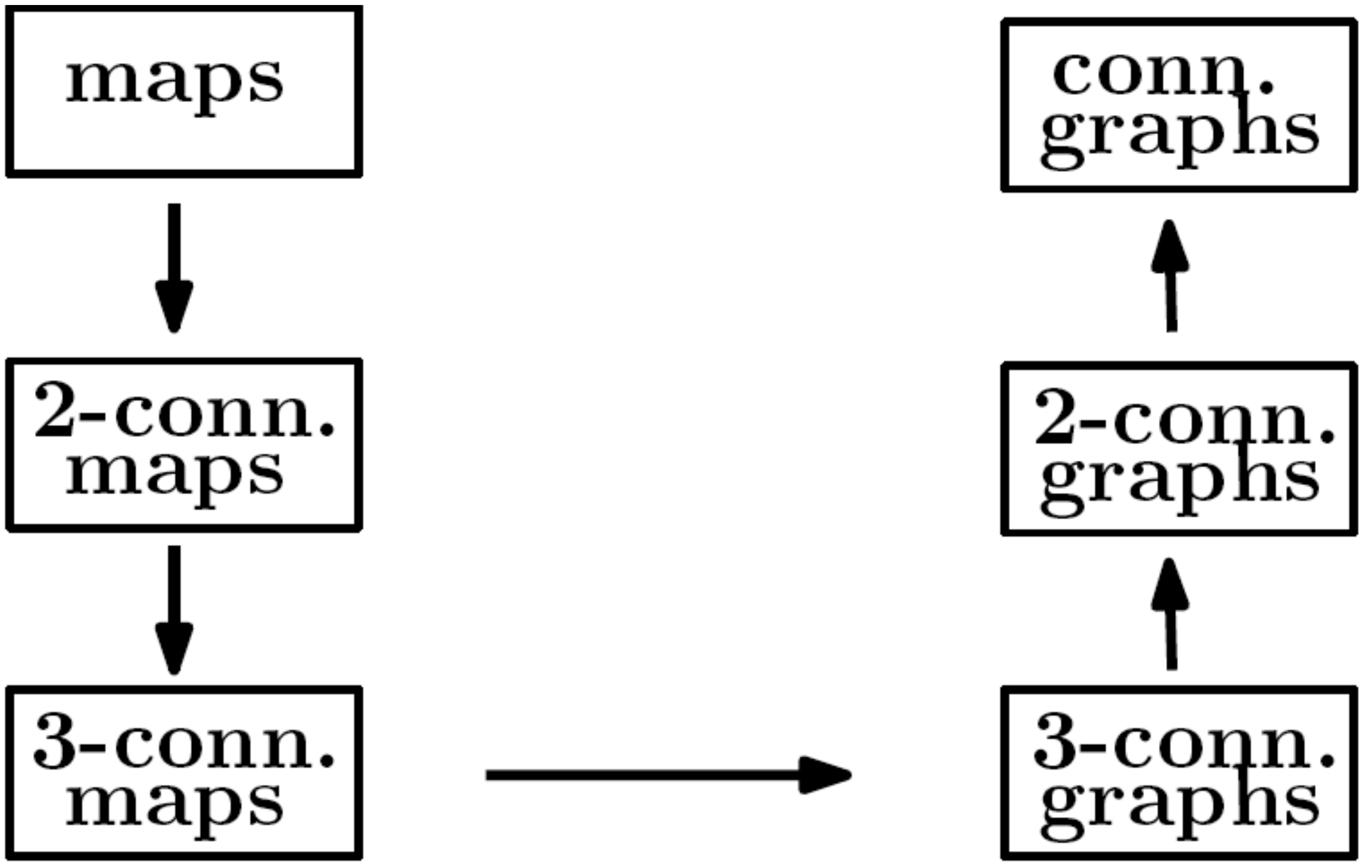


$$\overrightarrow{M_1}(y) = \overrightarrow{M_2}\left(y(1 + \overrightarrow{M_1}(y))^2\right)$$

$\Rightarrow$ **Can extract $\overrightarrow{M_1}(y)$ from $\overrightarrow{M_2}(y)$**

(similarly can extract $\overrightarrow{M_2}(y)$ from $\overrightarrow{M_3}(y)$)

Count graphs reduces to count maps



Count maps via quadrangulations

- There are several methods to count (rooted) maps:
  - recursive [Tutte'63]
  - bijective: either directly or via quadrangulations

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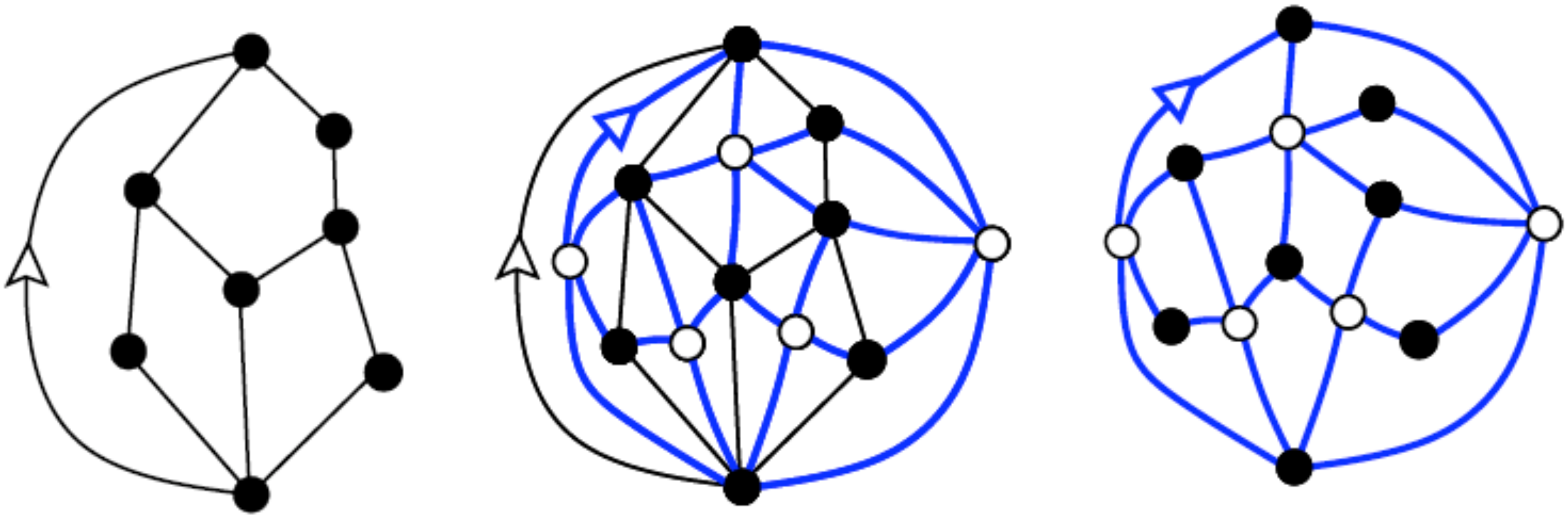
  - recursive [Tutte'63]
  - bijective: either directly or via quadrangulations
- Bijection between maps and quadrangulations ([Tutte]):

  1) Insert a star inside each face

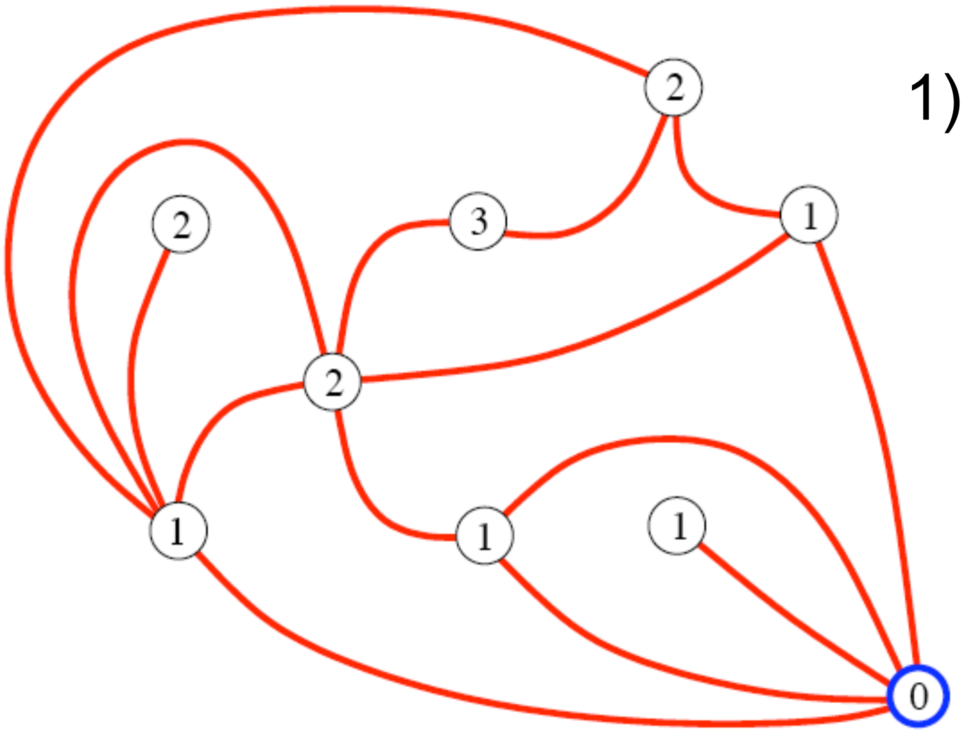
  2) Delete the edges of the original map



$\Longrightarrow$ #rooted maps n edges = #rooted quadrangulations n faces

Quad. <-> well-labelled trees

Bijection [Schaeffer'98] (also [Cori,Vauquelin'84]):



1) Start from a vertex-pointed quadrangulation with vertices labelled by the distance from the pointed vertex

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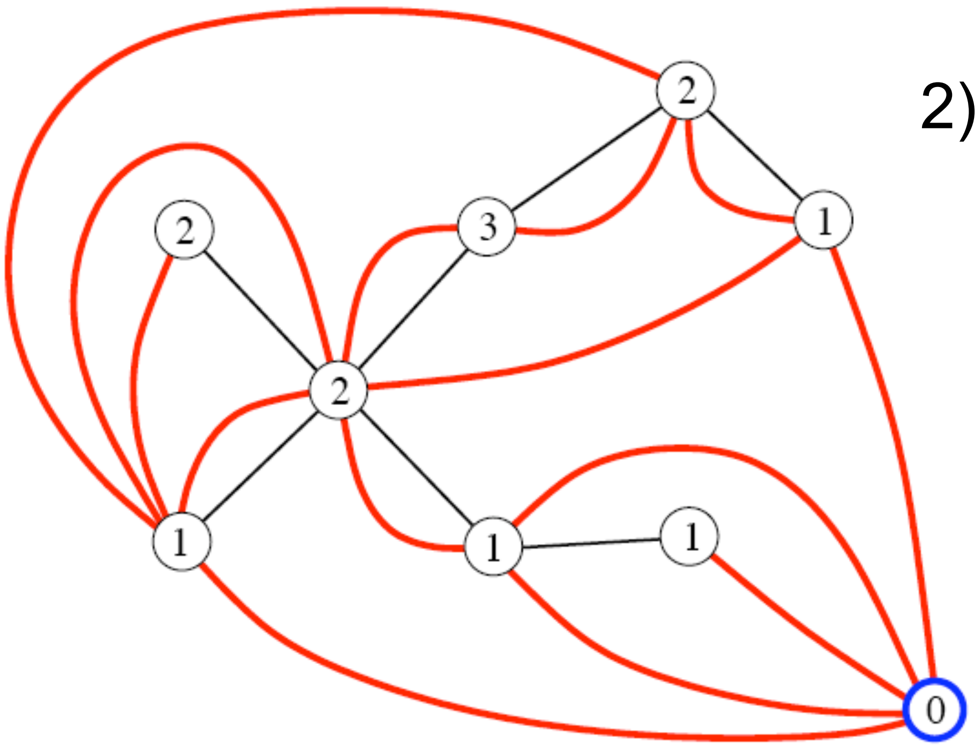
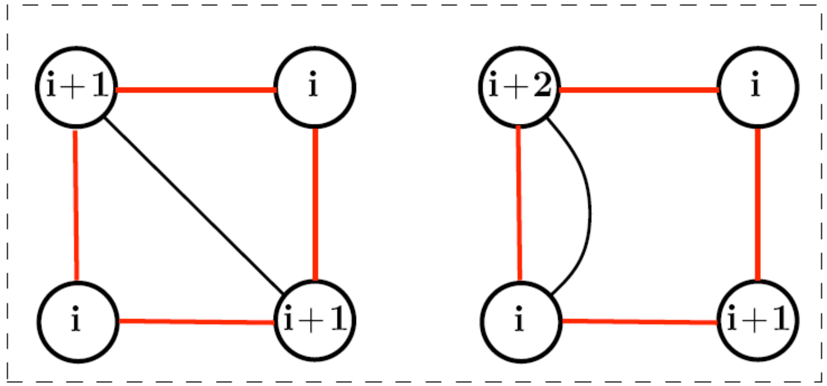


2) Draw a black edge in each face according to the rule:



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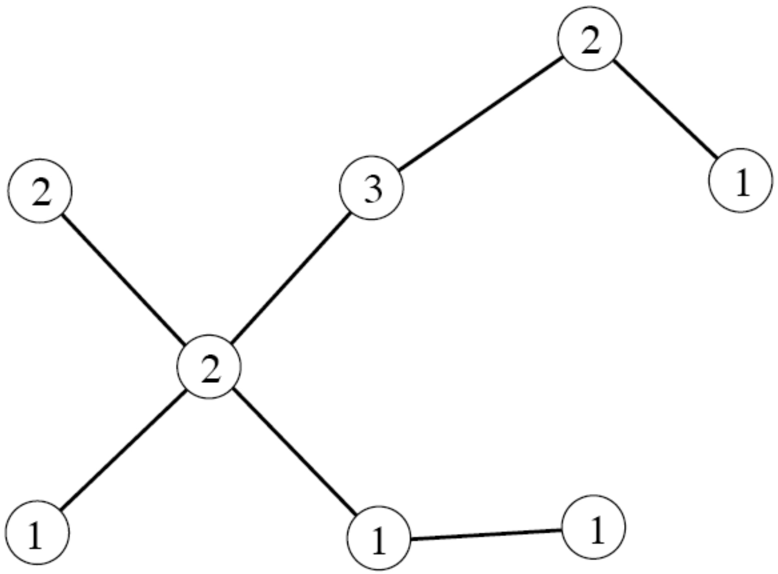


3) Delete the red edges and the pointed vertex

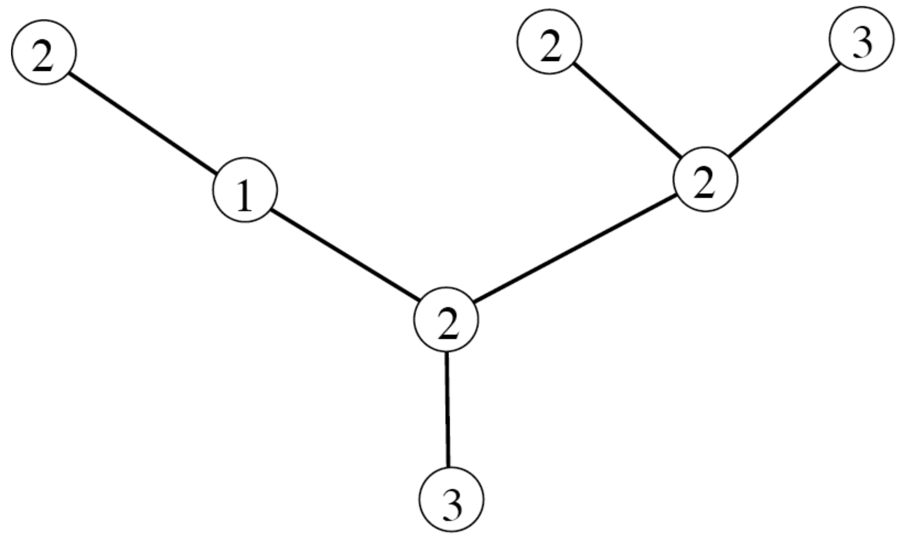
Quad. <-> well-labelled trees

- A well-labelled tree is a plane tree where:

  - each vertex v has a positive label

  - the labels at each edge (v,v') differ by at most 1

  - at least one vertex has label 1



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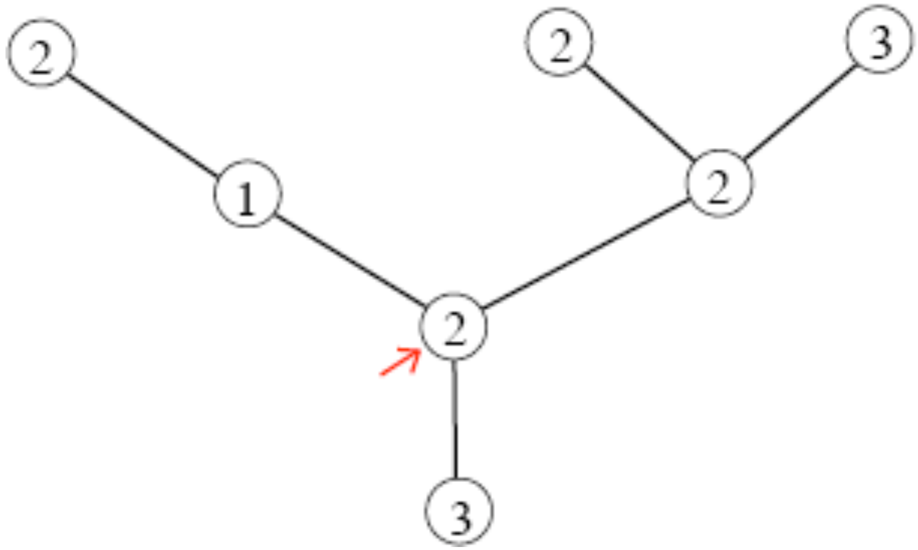
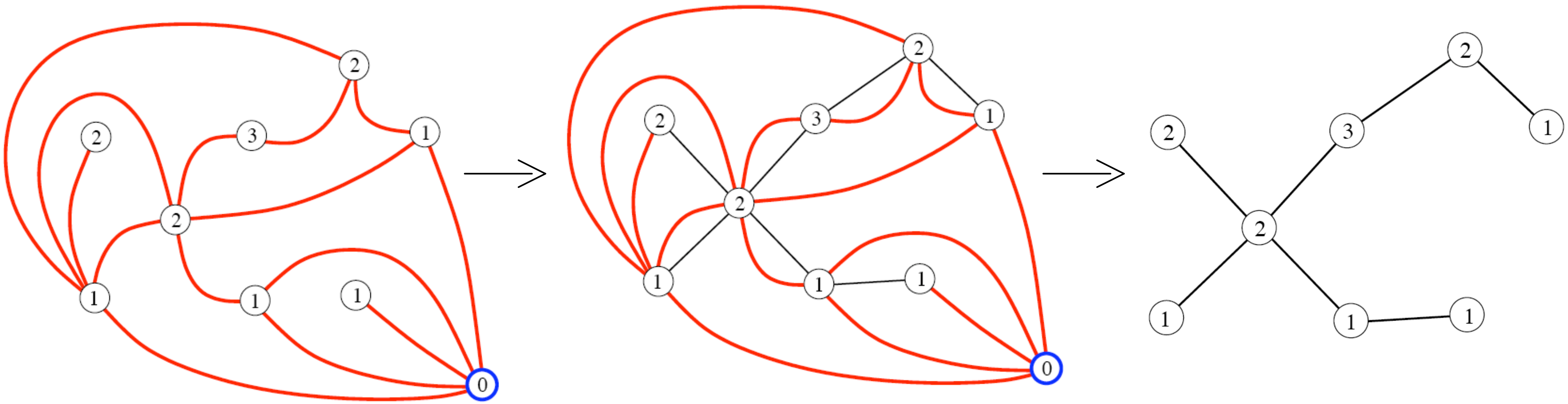


- Rooted well-labelled tree = well-labelled tree + marked corner

  (there are $3^n \frac{(2n)!}{n!(n+1)!}$ such trees with n edges)

The mapping is a bijection

Theorem [Schaeffer'98]: The mapping is a bijection between pointed quadrangulations and well-labelled trees



vertex label i	⟷	vertex at distance i
corner label i	⟷	edge at level i
edge	⟷	face

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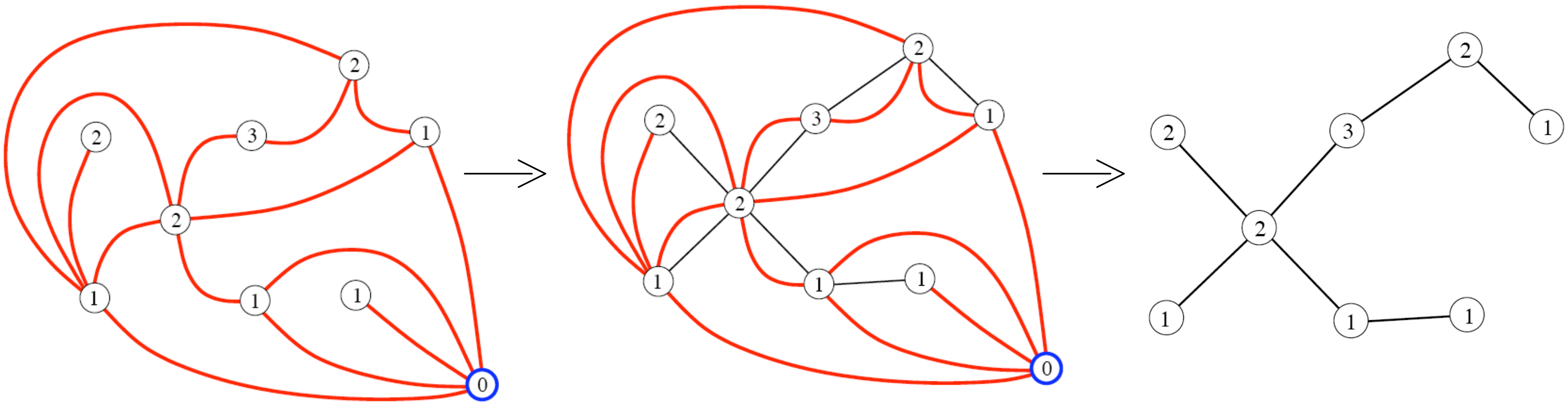

vertex label i	⟷	vertex at distance i
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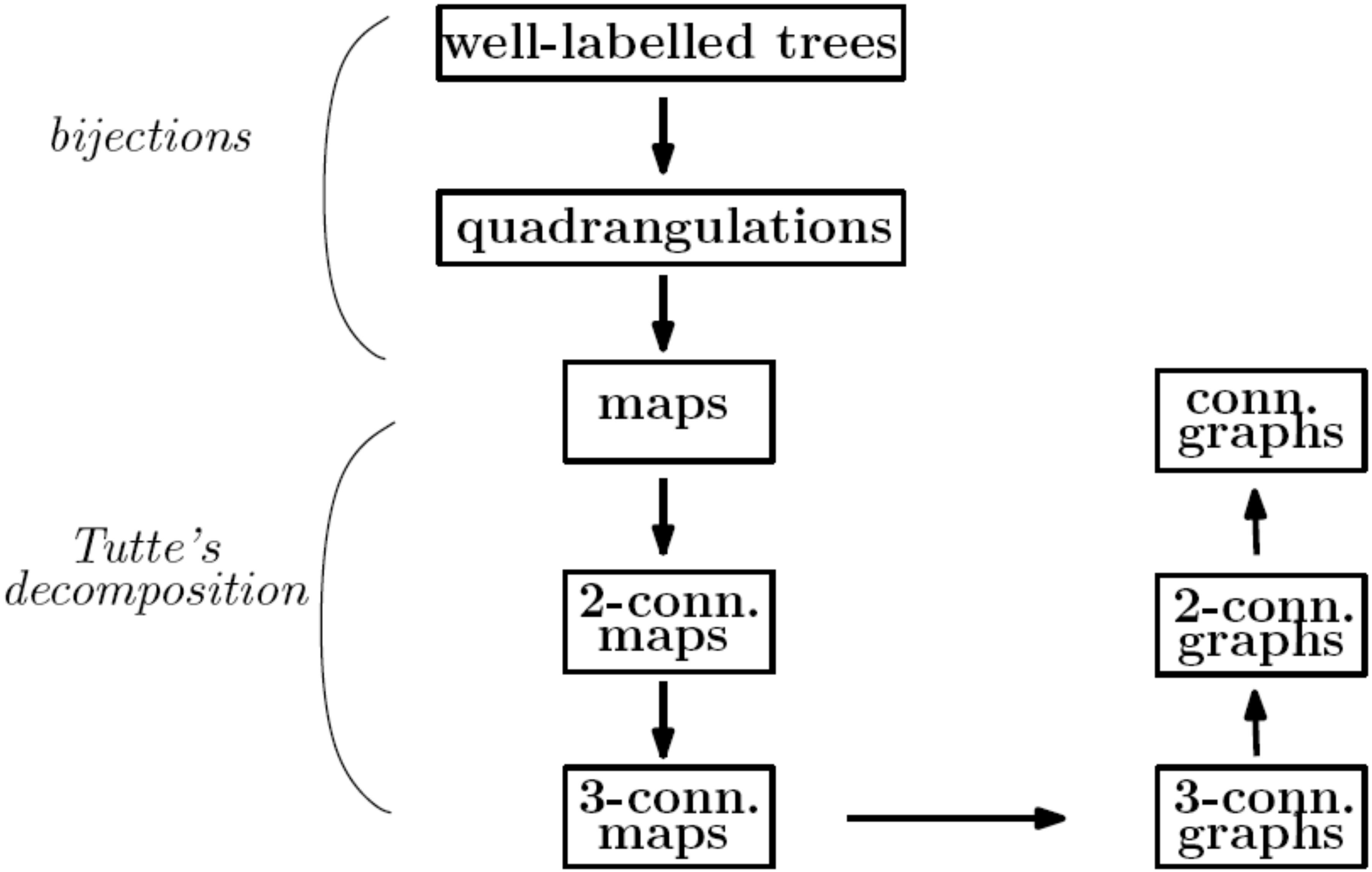
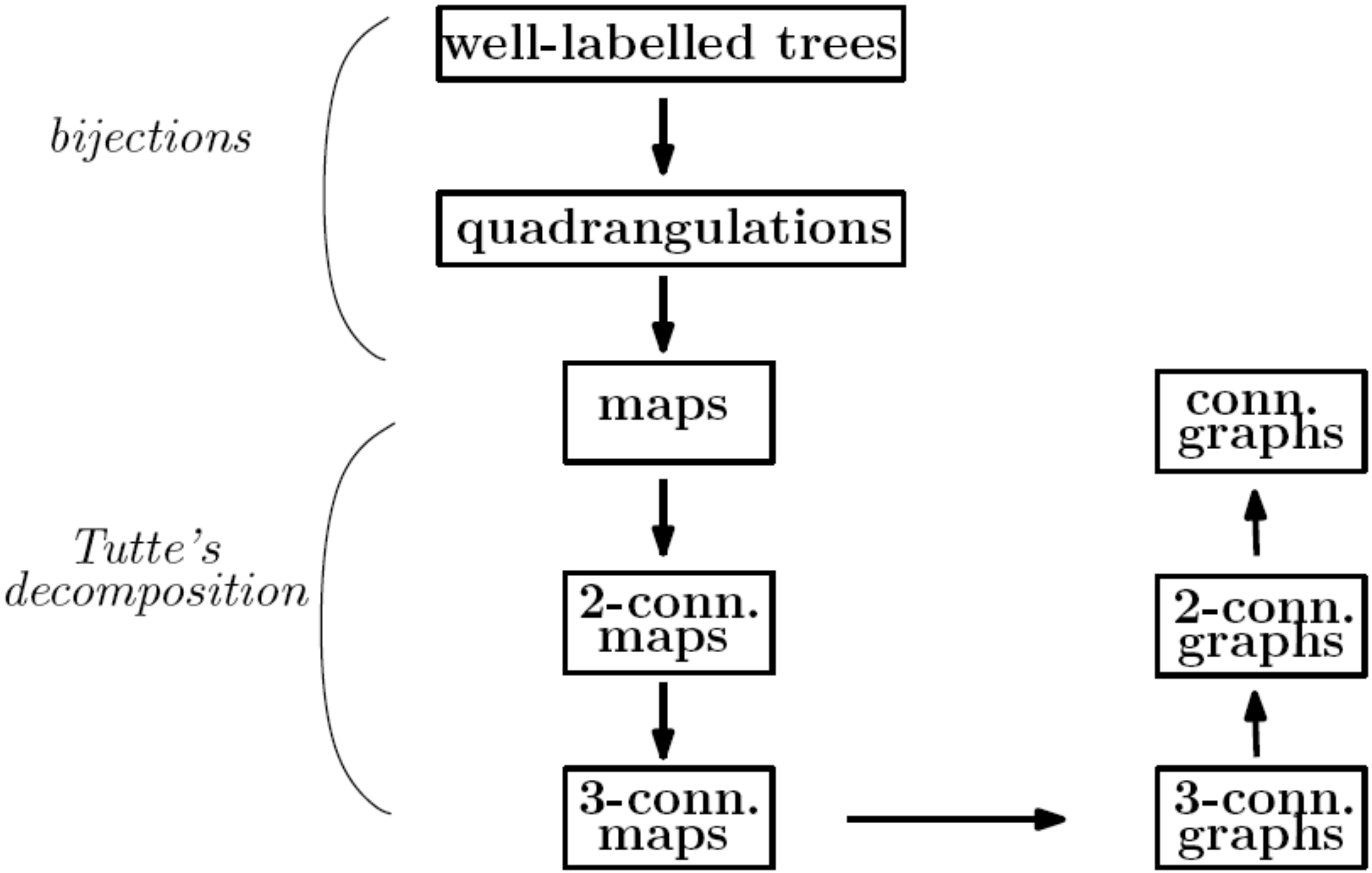
Corollary: there are $3^n \frac{(2n)!}{n!(n+1)!}$ quadrangulations with n faces, a marked vertex, and a marked edge

Summary of counting scheme

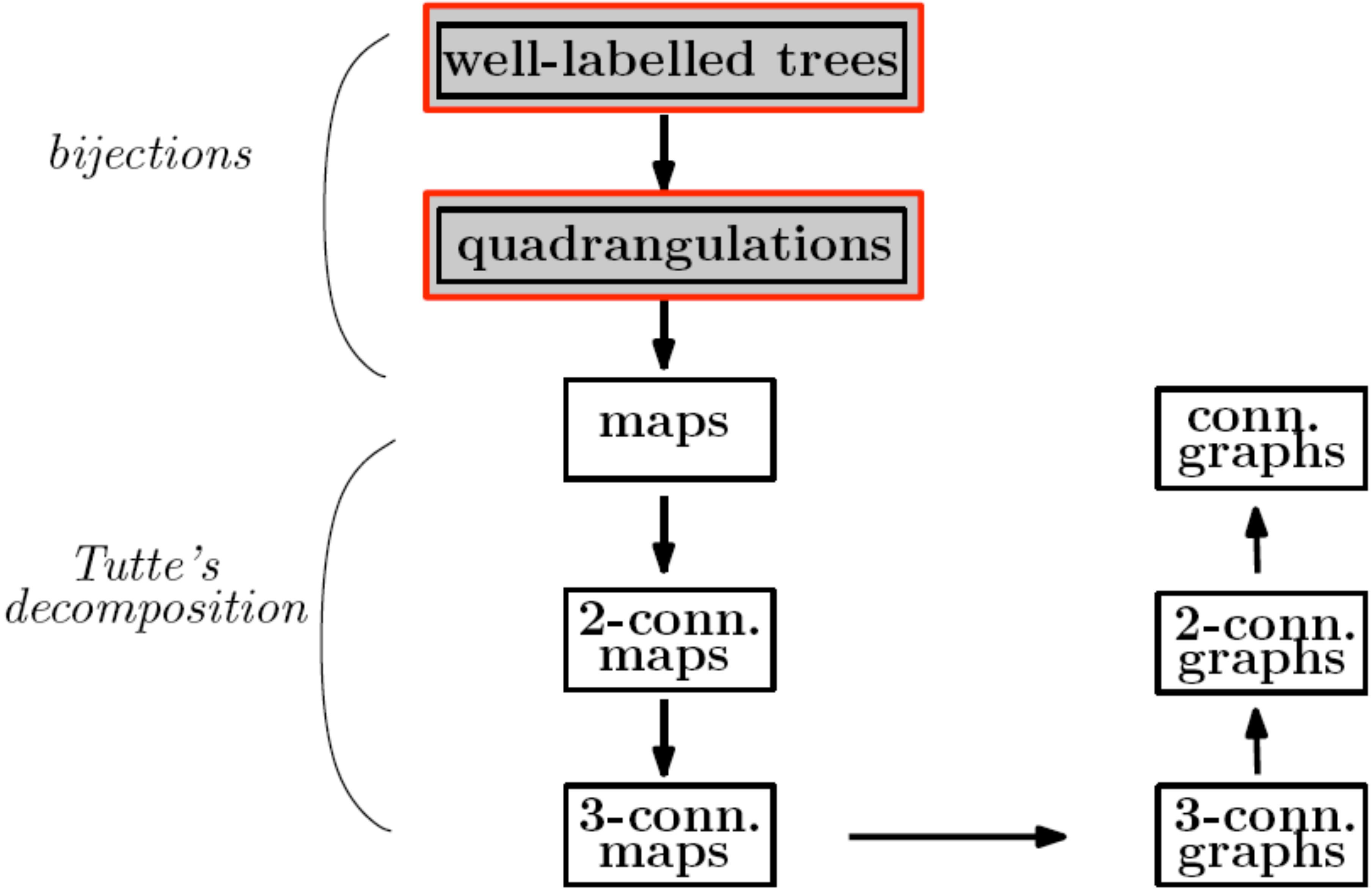


Part II: carry large deviation estimates for the diameter along the counting scheme



- We carry large deviation statements of the form:

$$\mathbb{P}(\mathrm{Diam}(G_n) \notin [n^{1/4-\epsilon}, n^{1/4+\epsilon}]) = O(\exp(-n^{\Theta(\epsilon)}))$$



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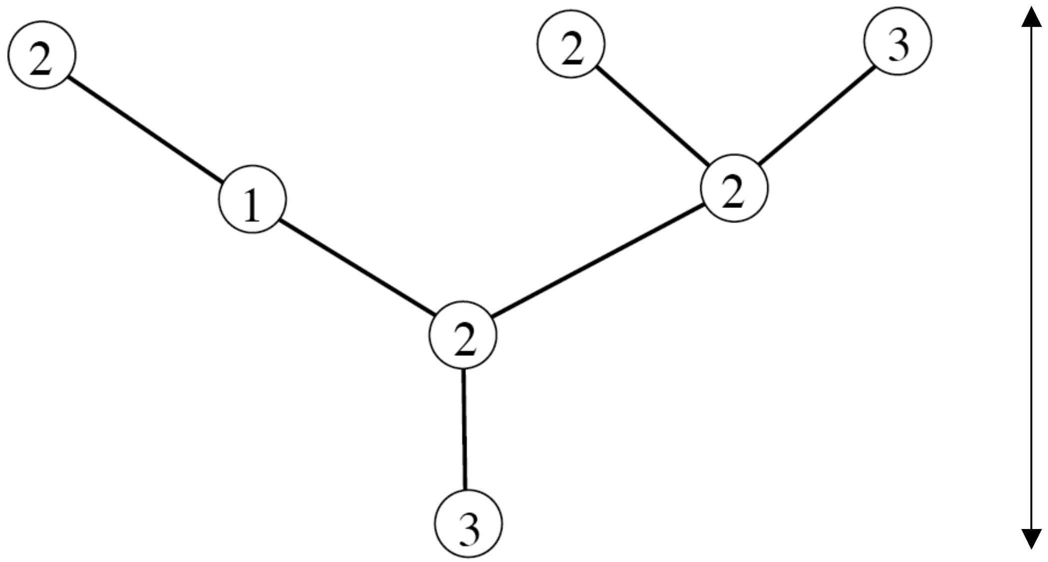
Quad. via well-labelled trees

Let Q_n random pointed quadrangulation with n+1 vertices

$Q_n \simeq T_n$ the random well-labelled tree with n vertices



$h(T_n)$ of order $n^{1/2}$

+ discrepancy of labels along k-path is of order $k^{1/2}$

Radius(Q_n) ``of order'' $n^{1/4}$

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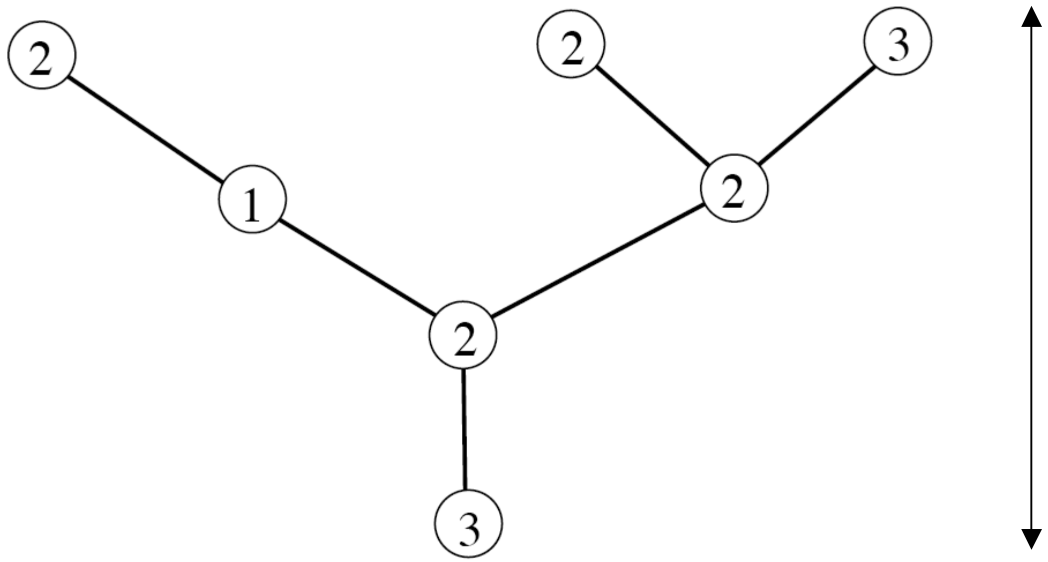

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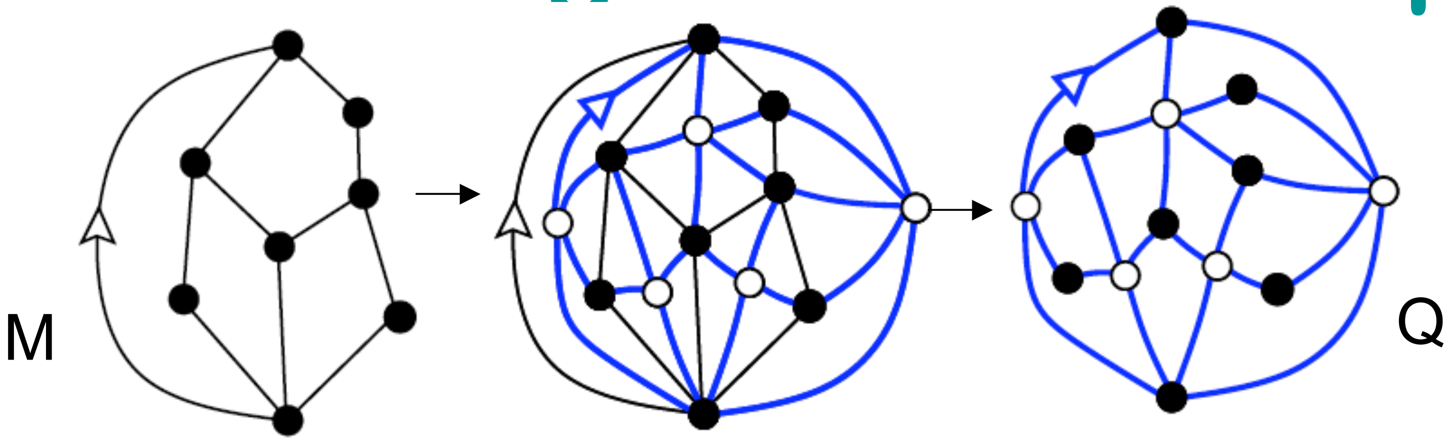
easily: $\mathrm{Radius}(Q_n) \in [n^{1/4-\epsilon}, n^{1/4+\epsilon}]$ a.a.s.

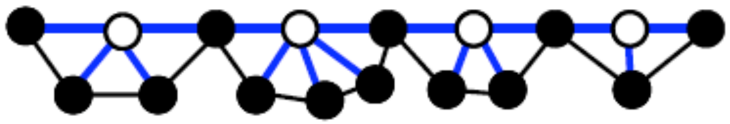
(Precise convergence results in [Chassaing Schaeffer'04])

Quadrangulations -> Maps



$\mathrm{dist}_Q \approx \mathrm{dist}_M$ modulo max face-degree Δ:

$\mathbf{dist}_Q(v, v') \leq 2 \cdot \mathbf{dist}_M(v, v')$



$\mathbf{dist}_M(v, v') \leq \Delta \cdot \mathbf{dist}_Q(v, v')$

Δ is small: $\mathbf{P}_n(\Delta \geq k) = O(\exp(-ck))$

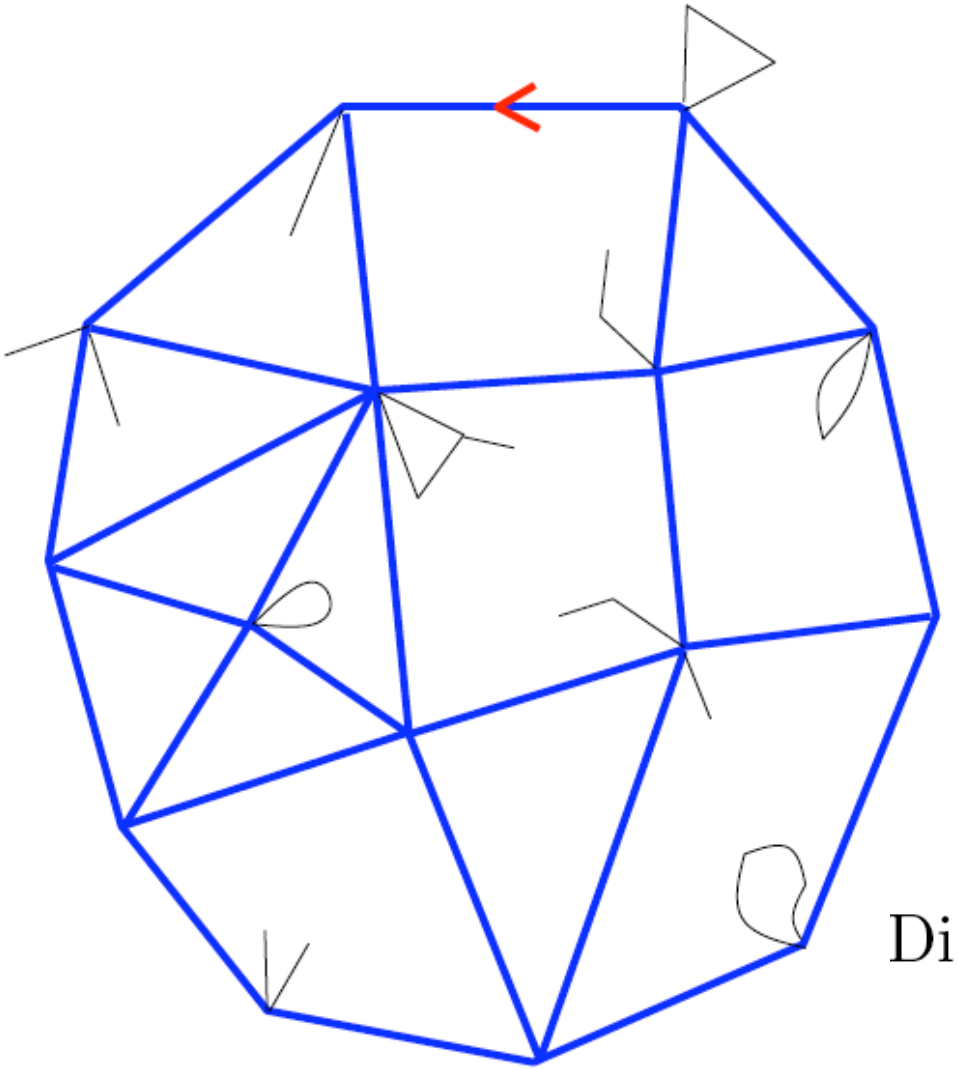
$\Rightarrow \quad \mathbf{Diam}(M_n) \in [n^{1/4-\epsilon}, n^{1/4+\epsilon}]$ a.a.s.

Maps -> 2-connected maps

[Gao,Wormald'99], [Banderier,Flajolet,Schaeffer,Soria'01]:

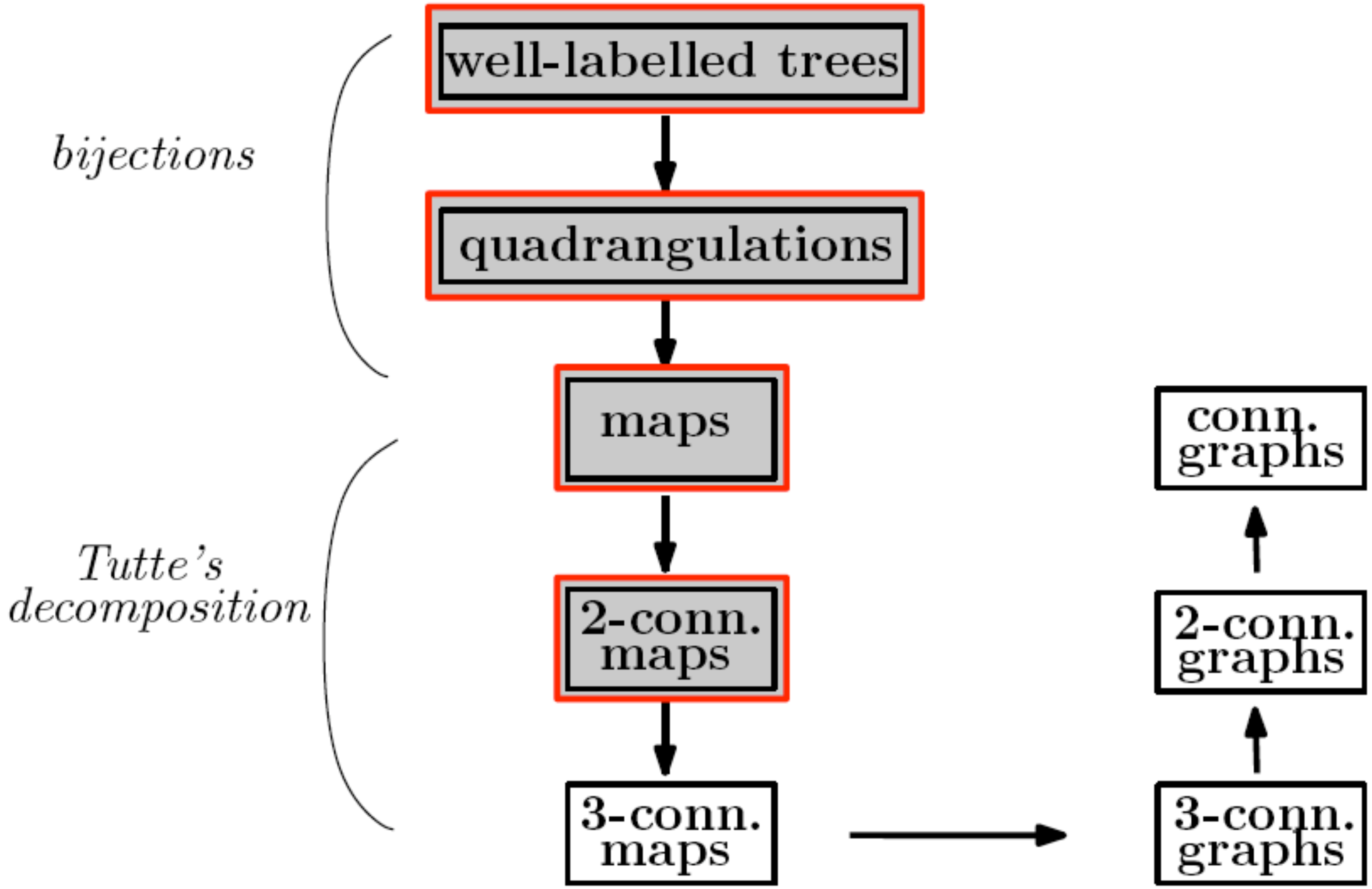
- The random map M_n has typically a core of size $n/3 + n^{2/3}\theta$
- The other components are small (size $O(n^{2/3})$)

Airy law



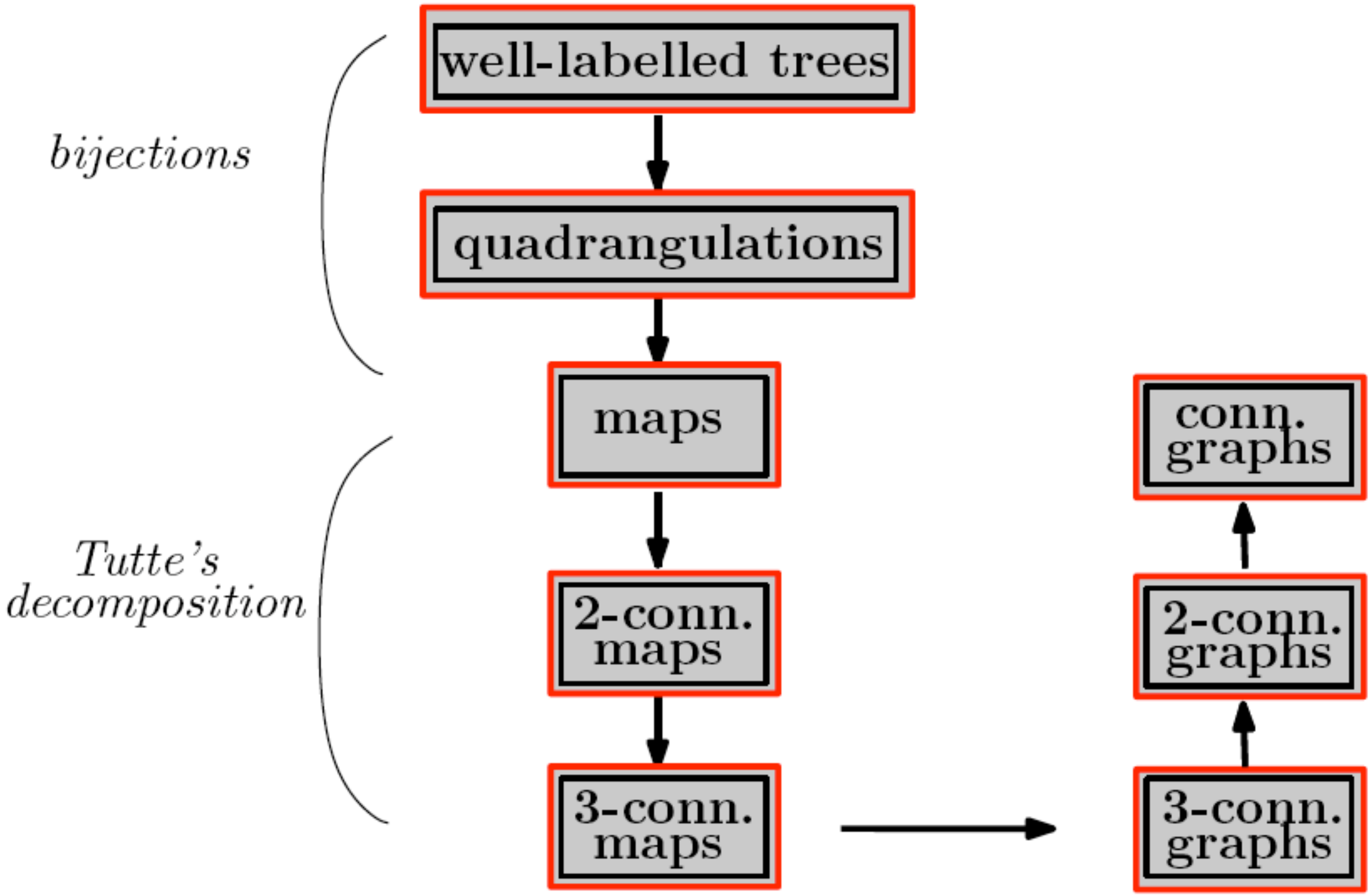
Diam(M_n) "≈" Diam($B_{n/3}$)

$$\mathrm{Diam}(B_n) \in [n^{1/4-\epsilon}, n^{1/4+\epsilon}] \quad \text{a.a.s.}$$



- For each of the highlighted families:

$$\mathbb{P}(\mathrm{Diam}(G_n) \notin [n^{1/4-\epsilon}, n^{1/4+\epsilon}]) = O(\exp(-n^{\Theta(\epsilon)}))$$



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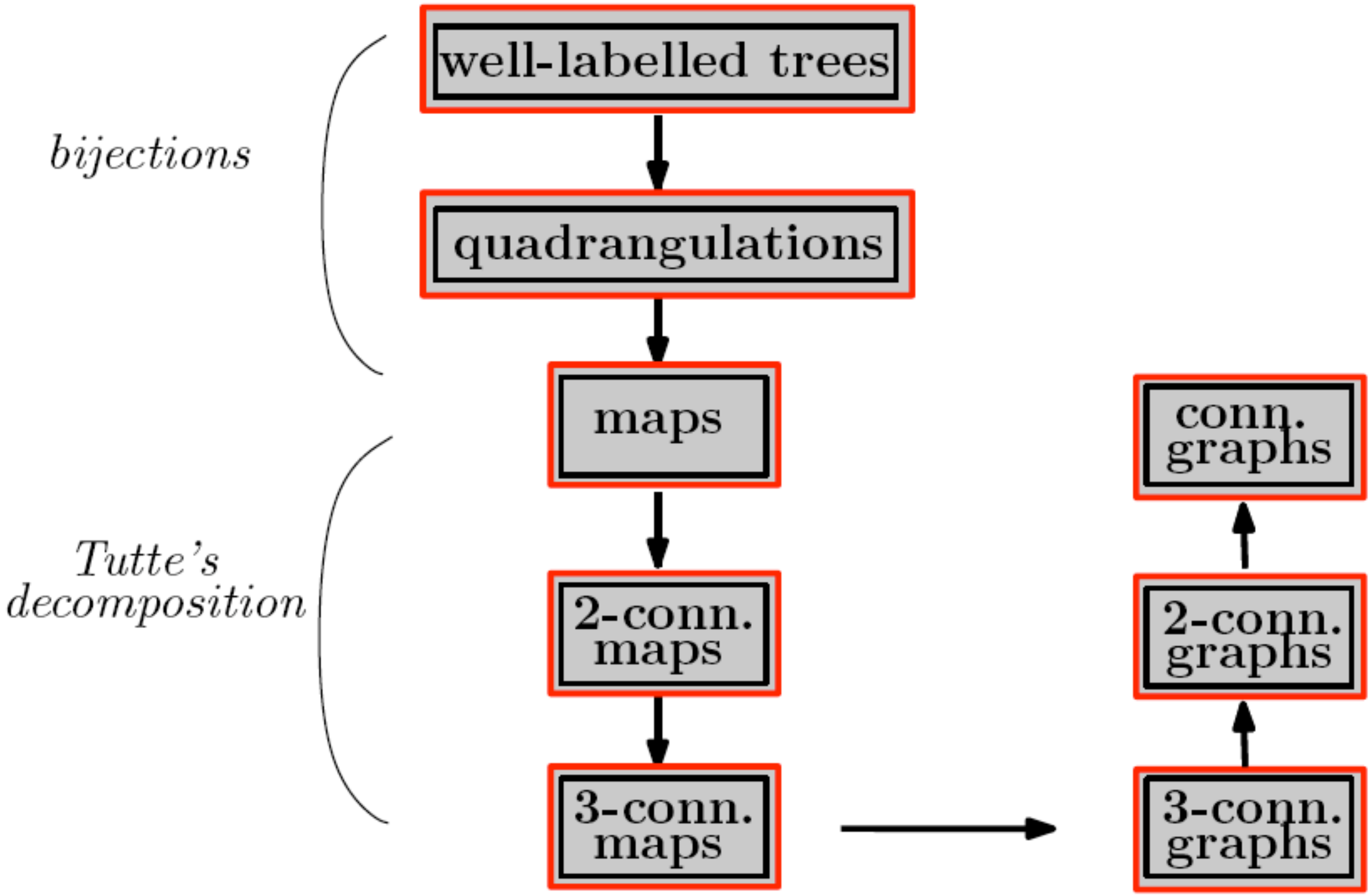
- Can carry it until connected planar graphs



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- Can carry it until connected planar graphs
- Is it possible to carry a statement such as **E**(Diam(G_n))= $\Theta(n^{1/4})$?