# Periodic planar straight-frame drawings with polynomial resolution

Luca Castelli Aleardi, Éric Fusy, and Anatolii Kostrygin

LIX - École Polytechnique. amturing,fusy@lix.polytechnique.fr, anatoly.kostrygin@polytechnique.org

**Abstract.** We present a new algorithm to compute periodic (planar) straight-line drawings of toroidal graphs. Our algorithm is the first to achieve two important aesthetic criteria: the drawing fits in a straight rectangular frame, and the grid area is polynomial, precisely the grid size is $O(n^4 \times n^4)$. This solves one of the main open problems in a recent paper by Duncan et al. [3].

## 1 Introduction

The main goal of graph drawing algorithms is to compute a drawing which is easily readable. One basic problem consists in mapping the vertices and edges of a graph onto a region in the plane or a portion of a 3D surface. Most of the time edges are represented as smooth curves (very often as straight-line segments), and the drawing is required to be *crossing-free*. Sometimes the drawing is asked to satisfy some further aesthetic criteria, in order to obtain a pleasing and readable result. For example, one could seek for *good vertex resolution* for ensuring that vertices are not too close to one another. In the planar case, an elegant solution to this problem is provided by the *barycentric embedding* by Tutte [12]. Tutte showed how to compute vertex positions by solving a system of linear equations: the method applies to a 3-connected planar graph and the resulting drawing is guaranteed to be crossing-free, also allowing to fix the positions of outer vertices (which are mapped to the vertices of a given convex polygon). The solution can be also reformulated in terms of a system of springs converging to an equilibrium position, and has inspired a huge number of force-directed embedding algorithms. A drawback of Tutte's method is that one cannot achieve a good vertex resolution, since matrix computations lead to vertex coordinates of exponential size. A solution to this problem are the so-called *straight-line grid drawings* [4, 8, 11]: the graph is embedded on a regular grid whose area is typically polynomial with respect to the size of the graph. A further advantage of this approach is that the algorithm performs essentially arithmetic computations on integers of bounded magnitude (no roundings are needed). For the planar case, many classes of algorithms have been proposed to solve this task, achieving very good vertex resolution (quadratic area) and time complexity (running in linear time).

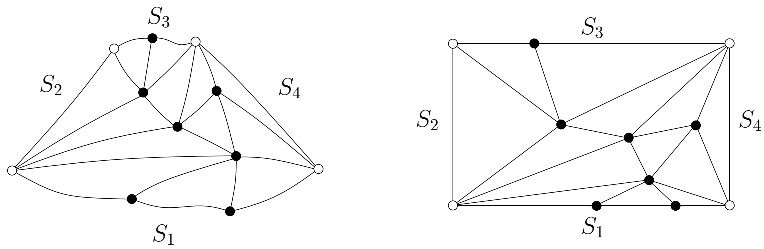


**Fig. 1.** Left: a 4-scheme triangulation $G$ (corners are white). Right: a straight-frame drawing of $G$.

*Statement of the main result.* A *quasi-triangulation* is a graph (topologically) embedded in the plane such that all inner faces are triangles and the outer face contour is a simple cycle. The vertices and edges are called *outer* or *inner* whether they are incident to the outer face or not. Define a *k-scheme triangulation* as a quasi-triangulation with $k$ marked outer vertices, called *corners*, such that each path of the outer face contour between two consecutive corners is chordless. Call $S_1, \ldots, S_k$ these $k$ outer paths (in clockwise order around the outer face), and denote by $|S_i|$ the length of $S_i$.

In this article we focus on the case $k = 4$. Given a 4-scheme triangulation $G$, a *straight-frame drawing* of $G$ is a planar straight-line drawing of $G$ such that the outer face contour is an axis-aligned rectangle whose corners are the corners of $G$, with $S_1$ and $S_3$ as horizontal sides and $S_2$ and $S_4$ as vertical sides, see Fig. 1. In a recent article, Duncan et al. [3] prove the following:

**Theorem 1 (Duncan et al. [3]).** *Each 4-scheme triangulation with $n$ vertices admits a straight-frame drawing on a (regular) grid of size $O(n^2 \times n)$.*

A motivation for such drawings is to draw toroidal graphs [9, 10]. Indeed, it is shown in [3] (more general results in higher genus are shown in [1]) that any triangulation $\tilde{G}$ on the torus can be cut along a subgraph (cut-graph) such that the resulting unfolded graph $G$ is naturally a 4-scheme triangulation, with the additional property that $|S_1| = |S_3|$ and $|S_2| = |S_4|$. A 4-scheme triangulation $G$ satisfying $|S_1| = |S_3|$ and $|S_2| = |S_4|$ is called *balanced*, and a straight-frame drawing of $G$ is called *periodic* if the abscissas of vertices of the same rank along $S_1$ and $S_3$ (ordered from left to right) coincide, and the ordinates of vertices of the same rank along $S_2$ and $S_4$ (ordered from bottom to top) coincide. As shown in Fig. 2, when $G$ arises from a toroidal triangulation $\tilde{G}$, this is exactly the condition to satisfy so that the drawing lifts to a periodic representation of $G$ in the plane (i.e., to a drawing of $G$ on the flat torus). The authors of [3, 1] cite it as an important open problem to compute periodic straight-frame drawings with polynomial grid size[1]. Our main result is the following.

[1] Tutte's spring embedding algorithm gives a solution, but with exponential grid size; the two recent articles [2, 5] yield periodic drawings for toroidal triangulations of

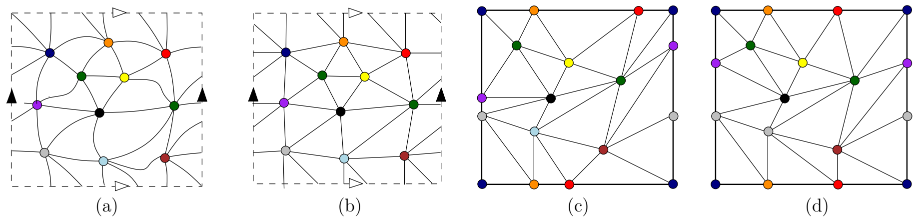


**Fig. 2.** (a) A toroidal triangulation $G$; (b) a periodic straight-line drawing of $G$ which is not straight-frame; (c) a non-periodic straight-frame drawing of $G$ (where $G$ is planarly unfolded using a cut-graph); (d) a periodic straight-frame drawing of $G$.

**Theorem 2.** *Each balanced 4-scheme-triangulation admits a periodic straight-frame drawing on a (regular) grid of size $O(n^4 \times n^4)$. The drawing can be computed in linear time.*

Our algorithm makes use of a new decomposition strategy for 4-scheme triangulations, we cut the triangulation into two components $A, B$ along a certain path "close to" $S_3$, the "lower part" $B$ is drawn using the algorithm of Duncan et al. [3] specially adapted for our purpose (to make the ordinates of matching vertices in the lower parts of $S_2$ and $S_4$ coincide), and then we draw the "upper part" $A$ with a suitably fixed outer frame (this is the most technical part, which requires a further decomposition of $A$ into several pieces). At first we will show that we can have grid-size $O(n^5 \times n^5)$, then we will argue in Section 5.2 that the grid-size can be improved to $O(n^4 \times n^4)$.

## 2 The Duncan et al. algorithm, adapted

### 2.1 Description of the algorithm in [3]

From now on, a 4-scheme triangulation is shortly called a 4ST. Define a *half-4-scheme triangulation*, shortly written H4ST, as a graph $H$ embedded in the plane satisfying all conditions of a 4ST, except that the paths $S_2$ and $S_4$ are allowed to be empty, and the path $S_3$ is allowed to have chords and to meet $S_1$; $S_1$ is called the *bottom-path* of $H$. Given a 4ST $G$, an important ingredient in [3] is the *river lemma*, which guarantees the existence of a so-called *river*, that is, a path $P$ in the dual graph of $G$, such that the first edge of $P$ crosses $S_1$, the last edge of $P$ crosses $S_3$, and the two components $H, H'$ of $G$ separated by $P$ are H4ST (with $S_2$ as bottom path of the left component and $S_4$ as bottom-path of the right component), see Fig. 3(a).

respective grid areas $O(n^{5/2})$ and $O(n^4)$, but these drawings do not fit inside a straight frame, so that when looking at an elementary cell, there is the aesthetic disadvantage of having edges crossing the boundary of the cell, as in Fig. 2(b).

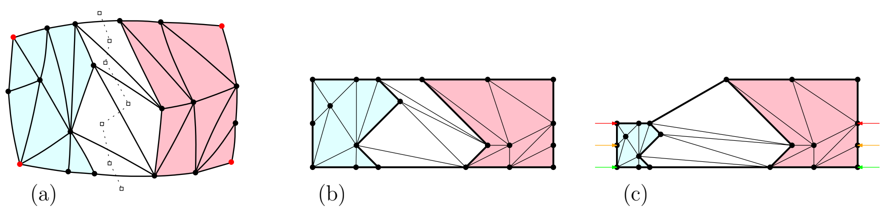


**Fig. 3.** (a) a 4ST $G$; the river between $S_1$ and $S_3$ decomposes $G$ into two H4ST $H, H'$ (shaded). (b) The straight-frame drawing of $G$ as given in [3], which results from the two drawings of $H, H'$ put together; (c) our modified version (with an oblique edge on $S_3$), where the ordinates of the vertices on $S_2$ coincide with the ordinates of the corresponding vertices on $S_4$.

A straight-line drawing of a H4ST $H$ is called *admissible* if $S_1$ is horizontal, $S_2$ and $S_4$ are vertical, and all edges in $S_3$ have slope in $\{-1, 0, +1\}$. By an extension of the shift algorithm introduced by de Fraysseix, Pach and Pollack [4] (which treats the case where $S_2$ and $S_4$ are empty), Duncan et al. show that it is possible to obtain an admissible straight-line drawing of $H$ on a grid of size $O(n \times n^2)$, more precisely a grid of size $O(n \times (d+1)n)$, where $n$ is the number of vertices of $H$ and $d$ is the graph-distance between $S_1$ and $S_3$. To obtain a straight-frame drawing of a 4ST $G$, decompose $G$ along a river between $S_1$ and $S_3$ into two H4ST components $H$ and $H'$, draw $H$ and $H'$ using the shift algorithm (with $S_2$ as the bottom-path of $H$ and $S_4$ as the bottom-path of $H'$), and then shift the left boundary of the component whose drawing has the smaller width so that the widths of the drawings of $H$ and $H'$ coincide. Then rotate the drawing of $H$ by $\pi/2$ clockwise, rotate the drawing of $H'$ by $\pi/2$ counterclockwise, and put the two drawing in front of each other, leaving enough horizontal space between them so that the edges connecting $H$ to $H'$ have slope smaller than 1 in absolute value. Since the edges on the boundaries of (the rotated copies of) $H$ and $H'$ are either vertical or of slope in $\{-1, +1\}$, the edges between $H$ and $H'$ do not introduce crossings, so the resulting drawing of $G$ is planar, see Fig. 3(b). Overall the grid-size is $O(n^2 \times n)$, more precisely $O(n(d+1) \times n)$, with $d$ the graph-distance between $S_2$ and $S_4$.

### 2.2 Our modified version of the algorithm

A first simple adaptation we do is to do all (abscissa) shift operations by 2 instead of doing them by 1. Given a H4ST $G$, let $p$ be the number of edges of

$S_1$. For a vector $I$ of $p$ even integers, $I$ is called the *initial interspace vector*, consider the shift algorithm for $G$ starting with $S_1$ drawn as a horizontal line with interspaces given by $I$. Let $F(I) = (f_1, \ldots, f_p)$ be the vector representing the interspaces between consecutive vertices on $S_1$ at the end of the algorithm; $F(I)$ is called the *final interspace vector*. For our purpose, the advantage of doing the shift operations by 2 is to guarantee that the components of $F(I)$ are even.

*Property 1.* Let $F_0 = F(I_0)$ be the final interspace vector when using $I_0 = (2, 2, \ldots, 2)$ as initial interspace vector on $S_1$. Then for any vector $F$ of $p$ even integers such that $F \geq F_0$ (component-wise), it is possible to re-execute the shift-algorithm so as to have $F$ as final interspace vector.

*Proof.* In a similar way as in [2], one can adopt a reformulation of the shift algorithm in terms of vertical strips insertions (of width 2 here) and edge stretch. With this reformulation it is easy to see that $F(I) - I$ is an invariant (is the same for any vector $I$ of $p$ even integers). □

We can now give another strategy (suited in view of showing Theorem 2) for drawing a 4ST $G$. The drawing we obtain is not straight-frame, but is straight-frame except for an oblique edge along $S_3$, and is such that, with $m = \min(|S_2|, |S_4|)$, the $m$ first components of the interspace vectors along $S_2$ and $S_4$ (ordered from bottom to top) coincide. At first, similarly as in [3] we cut $G$ along a river (between $S_1$ and $S_3$) into two components $H$ and $H'$. Then we draw independently $H$ and $H'$ using the shift algorithm (with width 2 strip insertions). Let $F_0$ ($F_0'$) be the final interspace vector of $H$ (resp. of $H'$), starting with initial interspace vector $(2, \ldots, 2)$. For $1 \leq i \leq m$ let $u_i$ be the maximum of the $i$th components of $F_0$ and $F_0'$. By Property 1, one can redraw $H$ and $H'$ so that the $m$ first components of the final interspace vectors of $H$ and of $H'$ are $(u_1, \ldots, u_m)$. In addition one can check that the widths of both drawings are at most $8n$, and the two widths differ by at most $4n$. Similarly as in [3] we rotate $H$ (resp. $H'$) by $\pi/2$ clockwise (resp. counterclockwise) and place the drawings in front of each other, leaving horizontal space $8n$ between them, enough to draw the edges between $H$ and $H'$ crossing-free. We obtain (see Fig. 3(c)):

**Lemma 1.** *For any 4ST $G$ with $n$ vertices, and $m = \min(|S_2|, |S_4|)$, there is a straight-line drawing of $G$, where $S_2$ and $S_4$ are vertical with their interspace vectors equal at the $m$ first components, $S_1$ is horizontal, and $S_3$ is horizontal except for an oblique edge of slope in $[-1/2, 1/2]$. The grid size is $O(n^2 \times n)$, more precisely $O(n(d+1) \times n)$ with $d$ the graph-distance between $S_2$ and $S_4$, and the drawing can be computed in linear time.*

## 3 A new binary decomposition for 4ST

We now introduce a new way to decompose a 4ST $G$ into two components $A, B$, in such a way that proving Theorem 2 will reduce to drawing $B$ using Lemma 1, and then drawing $A$ using a certain fixed outer frame. A path $P$ is said to be *just*

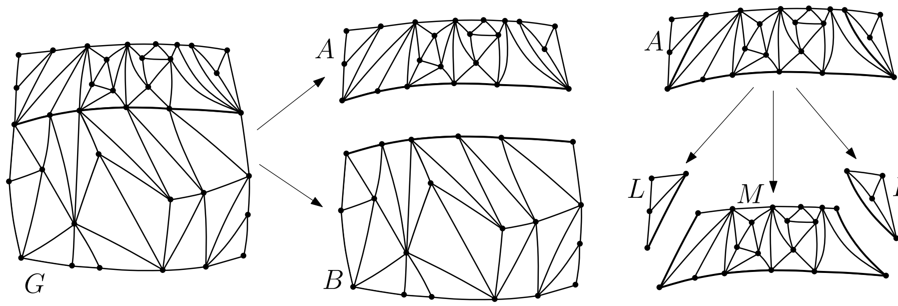


**Fig. 4.** Left: a 4ST $G$, where we distinguish a chordless path $P$ just below $S_3$ (shown bolder); middle: cutting along $P$ yields two components $A, B$; right: the band-graph $A$ is further decomposed into 3 pieces $L, M, R$.

*below* $S_3$ if $P$ connects a vertex of $S_2$ to a vertex of $S_4$, all non-extremal vertices of $P$ avoid $S_2 \cup S_3 \cup S_4$, and each vertex on $P$ (including the extremities) has at least one neighbour on $S_3$.

**Lemma 2.** *Each 4ST $G$ has a chordless path just below $S_3$.*

*Proof.* A path $P$ is said to be *below* $S_3$ if it satisfies the same conditions as "just below", but dropping the condition that each vertex of $P$ must have a neighbour on $S_3$. Let $E$ be the non-empty (since it contains $S_1$) set of paths below $S_3$. For $P, P' \in E$, write $P \leq P'$ if no edge of $P$ is above $P'$. It is easy to see that $E$ admits a unique maximal element $P_0$, and the fact that $G$ is triangulated ensures that all vertices of $P_0$ have at least one neighbour on $S_3$. Then one can extract a chordless subpath with same extremities out of $P_0$. □

Let $P$ be a chordless path just below $S_3$. Cutting $G$ along $P$ yields two 4ST denoted $A$ and $B$, with $B$ below $P$ and $A$ above $P$. We draw $B$ using Lemma 1. Then we have to draw $A$ —which is called the *band-graph*— in such a way that the drawing obtained by pasting the drawing of $B$ with the drawing of $A$ yields a periodic drawing of $G$. To state the drawing result for $A$, we introduce the notion of *fixed frame*. Given a quasi-triangulation $G$, with $C$ its outer cycle, a fixed frame $\kappa$ for $G$ is a crossing-free drawing of $C$ on a regular grid $w \times h$ ($w$ and $h$ are the width and height of the fixed frame). For $\gamma \geq 1$ the grid is *refined by factor* $\gamma$ by replacing each unit cell by a $\gamma \times \gamma$ regular grid. We say that the factor $\gamma$ is *suitable* for the pair $(G, \kappa)$ if, after refining the grid of $\kappa$ by factor $\gamma$, $G$ admits a straight-line drawing with $\kappa$ as the outer face contour of the drawing, see Fig. 5 for an example. We will prove the following result in Section 4:

**Lemma 3.** *For each $K \geq 1$ and any fixed frame $\kappa$ for $A$ ($A$ is the band-graph of $G$, which has $n$ vertices) of the form shown in the left drawing of Figure 6, there is $\gamma_0$ in $O(n \cdot \max(n, K))$ such that any even factor $\gamma \geq \gamma_0$ is suitable for $(A, \kappa)$.*

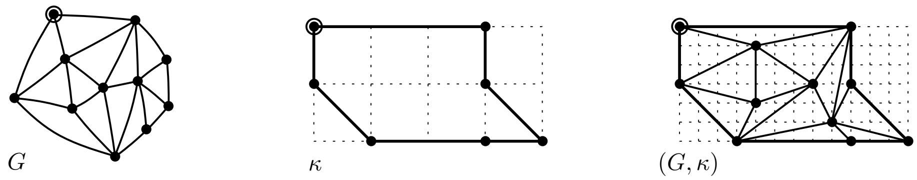


**Fig. 5.** Left: a quasi-triangulation $G$; middle: a fixed frame $\kappa$ for $G$ on a grid of size $4 \times 2$; right: a drawing of $G$ fitting in $\kappa$, using refinement factor 3 for the grid (hence the factor 3 is suitable for $(G, \kappa)$).

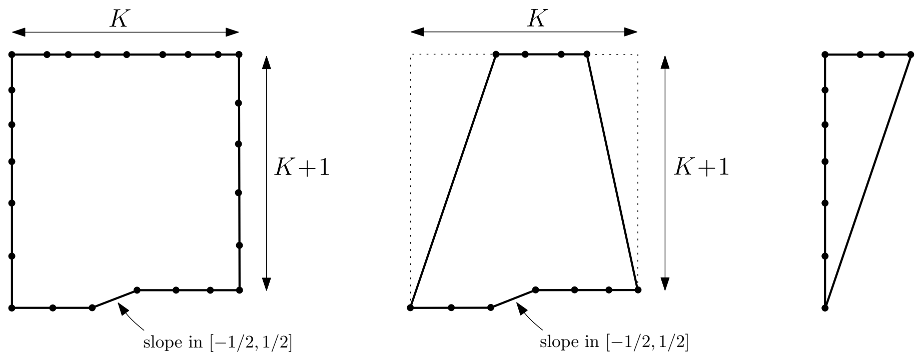


**Fig. 6.** Left: frame for the band-graph $A$, middle: frame for the middle piece $M$ of $A$, right: frame for the left piece $L$ of $A$.

We claim that Lemma 1 together with Lemma 3 imply Theorem 2 (at the moment with grid-size $O(n^5 \times n^5)$). Indeed, once $B$ is drawn using Lemma 1, one easily designs a fixed frame of the form of Fig. 6 and such that pasting a drawing of $A$ in this frame with the drawing of $B$ yields a periodic drawing of $G$ (note that the lower part of $\kappa$ is determined by the property of fitting with the boundary-path $S_3(B)$, and the upper part of $\kappa$ is determined by the property of fitting with the boundary-path $S_1(B)$ in order to get the periodicity property). Note that the parameter $K$ equals the width of the drawing of $B$, which is $O(n^2)$. Since the refinement factor for $A$ is in $O(n \cdot \max(n, K))$, the grid-size of the resulting drawing of $G$ is $O(K^2 n \times K^2 n)$, which is $O(n^5 \times n^5)$.

*Remark 1.* For convenience and to keep it short, we have written the proof in worst-case style, but there is room for improvements in favorable instances, in particular to reduce the height (which is $K+1$ in the worst case, as in Figure 6) of the frames for the band-graph.

## 4 Proof of Lemma 3

At first we state a useful lemma that easily follows from Property 1:

**Lemma 4.** *Let $G$ be a H4ST with $n$ vertices, with $p$ the length of the bottom-boundary. Let $V$ be a vector of $p$ positive integers. Then, for any even $\gamma \geq 4n$, there is an admissible drawing of $G$ whose (bottom-)interspace vector is $\gamma \cdot V$.*

*Proof.* With initial interspace vector $(2, 2, \ldots, 2)$, the final interspace vector is an even vector $F_0$ whose components are bounded by $4n$. Hence $\gamma \cdot V$ dominates $F_0$ for any $\gamma \geq 4n$, so the result follows from Property 1. □

Let $A$ be the band-graph of a 4ST $G$ with $n$ vertices, i.e., $A$ is obtained as the upper component after cutting along a chordless path $P$ just below $S_3(G)$. Let $a$ be the extremity of $P$ on $S_2$ and $b$ the extremity of $P$ on $S_4$. Let $u_a$ be the leftmost neighbour of $a$ along $S_3$ and let $u_b$ be the rightmost neighbour of $b$ along $S_3$. Then cutting $A$ along the edges $\{a, u_a\}$ and $\{b, u_b\}$ yields three pieces: a left-piece $L$, a middle piece $M$, and a right-piece $R$ (see the right drawing of Fig. 4). Note that $L$ (and similarly $R$) is either empty (if $u_a$ is the top-left corner of $G$) or otherwise is naturally a 3-scheme triangulation (shortly called a 3ST), whose three corners are $a$, $u_a$ and the topleft corner of $G$. And $M$ is naturally a 4ST such that $|S_2(M)| = |S_4(M)| = 1$. If $\kappa$ is a fixed frame for $A$ as in Lemma 3 (left drawing of Fig. 6), let $\kappa_M$ be the fixed frame for $M$ inherited from $\kappa$ (i.e., drawing the chords $\{a, u_a\}$, $\{b, u_b\}$, and deleting what is top-left of $\{a, u_a\}$ and top-right of $\{b, u_b\}$), of the form shown in the middle drawing of Fig. 6. We have:

**Lemma 5.** *Any even factor $\gamma \geq 4n$ is suitable for $(M, \kappa_M)$.*

*Proof.* If we decompose $M$ into two components using a river (between $S_2$ and $S_4$), the fact that $|S_2| = |S_4| = 1$ ensures that the two resulting H4ST $H$ and $H'$ have no left nor right vertical boundary: such H4ST are called *flat*. We draw the upper component $H'$ so as to respect the interspaces of the upper boundary of $\kappa$; according to Lemma 4 this is possible for any even refinement factor $\gamma \geq 4n$. Next we draw $H$, which is a bit more difficult due to the presence of the oblique edge at the bottom-boundary. We use the property that, since every vertex on $H$ is adjacent to a vertex on $S_3(G)$, then $H$ is even more constrained: there is a flat H4ST "attached to" each edge of the bottom-path of $H$. Denote by $G_e$ the flat H4ST at the oblique edge $e$. By an easy modification of the shift drawing algorithm in [4], for any even refinement factor $\gamma \geq 4n$, one can draw $G_e$ so that $e$ fits with the oblique edge of $\kappa$, and as usual with the slopes of the upper boundary of $G_e$ in $\{-1, 0, +1\}$. Finally we have to draw the flat H4ST $H_\ell$ to the left of $e$ (and similarly the flat H4ST $H_r$ to the right of $e$) so as to respect the positions of corresponding vertices of $\kappa$. By Lemma 4, this is possible for any even refinement factor $\gamma \geq 4n$. Finally note that all the H4ST considered are flat (they have no lateral path); and because of the shape of the frame (see Fig. 6), all edges connecting $H$ to $H'$ have slope greater than 1 in absolute value. Hence these edges can be added crossing-free. □

It remains to draw the left piece $L$ (and similarly the right piece $R$) with a fixed frame as shown in the right drawing of Fig. 6.

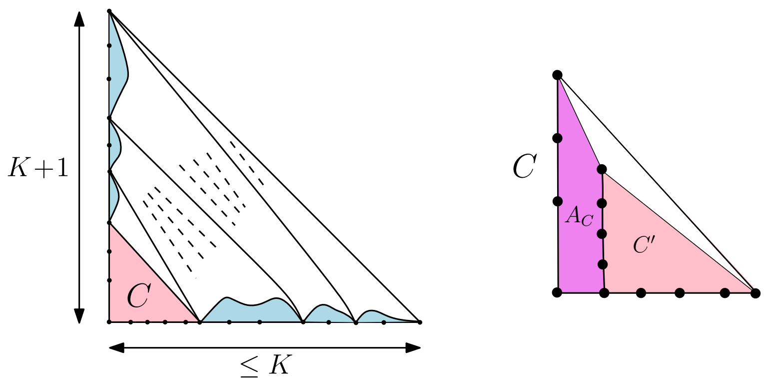


**Fig. 7.** Left: generic situation for a 3ST with one side (hypotenuse) reduced to an edge. Right: decomposition of $C$.

**Lemma 6.** *Let $\kappa$ be a fixed frame of the form in the right drawing of Fig. 6, and let $T$ be a 3ST (with the 3 side lengths compatible with $\kappa$). Then, for some $\gamma_0$ in $O(n \cdot \max(n, K))$, any even factor $\gamma \geq \gamma_0$ is suitable for $(T, \kappa)$.*

*Proof.* The generic situation is shown in the left drawing of Fig. 7 (where for convenience, the right drawing of Fig. 6 has undergone an horizontal mirror). A crucial role is played by the most downleft chord $e$; there is a flat H4ST $H_r$ on the right of $e$ aligned along the horizontal side, and similarly there is a flat H4ST $H_a$ above $e$ aligned along the vertical side. By Lemma 4 we can draw these two H4ST so as to respect the interspaces of the outer frame, at the price of a refinement factor $O(n)$. To draw the chords in a crossing-free way, we have to do a further operation: refine by factor $K+2$ and divide the ordinates of $H_r$ by $K+2$ and the abscissas of $H_a$ by $K+2$. The effect is to make the slopes of the upper contour of $H_r$ strictly smaller than $1/(K+1)$ in absolute value (and similarly for $H_a$). We can now draw the chords in a crossing-free way (indeed, since the chords connect two points in a $(K+1) \times (K+1)$ grid they have slope not smaller than $1/(K+1)$). We now have to draw the piece $C$ downleft of $e$, and assume without loss of generality that $e$ has slope smaller than $-1$. Note that (except for the trivial case where $C$ has just one inner face) we get a 4ST $\tilde{C}$ when deleting the topright (hypotenuse) edge. Let $P$ be a chordless path "just to the right" of the left vertical boundary path of $\tilde{C}$. Denote by $A_C$ (band-graph) the part of $\tilde{C}$ to the left of $P$ and by $C'$ the part of $\tilde{C}$ to the right of $P$. Using Lemma 4 we draw $C'$ (with $P$ as left vertical boundary) so that its bottom path fits with the interspaces prescribed by $\kappa$. Next we draw the band-graph $A_C$ (note that the drawing of $C'$, together with the prescription of $\kappa$, completely fix the outer frame of $A_C$). An important point is that, in the 3-piece decomposition of $A_C$ (as shown in the right drawing of Fig. 6), the left-piece $L$ and the right-piece $R$ are trivial, due to the absence of chords in $C$. This means that we do not have to recurse, and can easily conclude by Lemma 5 (in fact a simpler version of Lemma 5, without bothering about an oblique edge). □

# 5 Finishing the proof of Theorem 2

## 5.1 Remaining cases

In order to carry out the decomposition strategy based on the path just below $S_3$ we have assumed that there is no chord between $S_1$ and $S_3$. Let $G$ be a 4ST. If there is a chord between $S_1$ and $S_3$ but there is no chord between $S_2$ and $S_4$, then we can just rotate $G$ by $\pi/2$ (so as to exchange $S_1, S_3$ with $S_2, S_4$). If there is also a chord between $S_2$ and $S_4$, then it is easy to see that there must be a chord incident to a corner of $G$. So we just have to find a drawing strategy for any 4ST $G$ that has a chord incident to a corner, say without loss of generality that there is a chord $e$ that connects the bottom-left corner to a vertex on $S_3$. If we cut along $e$ we obtain two components $L$, $R$, where $L$ is a 3ST and $R$ is a 4ST (possibly a 3ST if $e$ goes to the top-right corner). Then draw $R$ using Lemma 1 (with $e$ as oblique edge); since $|S_2(R)| = 1$, the drawing of $R$ has grid-size $O(n \times n)$. If we want a periodic straight-frame drawing of $G$, the drawing of $R$ completely fixes the outer frame for $L$ (hence the outer frame of $L$ is of size $O(n \times n)$). And using Lemma 6, we can refine the grid by factor $O(n^2)$ to fit in the fixed frame. We conclude that, when there is a chord incident to a corner of $G$, then $G$ has a periodic straight-frame drawing on a grid of size $O(n^3 \times n^3)$.

## 5.2 Getting grid-size $O(n^4 \times n^4)$

We now argue that, in the case where there is no chord incident to a corner of $G$, then $G$ actually admits a periodic straight-frame drawing of grid-size $O(n^4 \times n^4)$. For $G$ a 4ST with $n$ vertices, let $d_h$ be the graph-distance between $S_1$ and $S_3$, and let $d_v$ be the graph-distance between $S_2$ and $S_4$. Then it easily follows from the Menger vertex-disjoint theorem and the fact that $G$ is innerly triangulated that $d_h \times d_v \leq n$, hence $\min(d_h, d_v) \leq n^{1/2}$ (it follows from this observation that, up to possibly rotating $G$ by $\pi/2$ to ensure that $d_h \leq d_v$, the Duncan et al. algorithm gives a grid-size $O(n \times n^{3/2})$). We have seen in Section 5.1 that, if there is no chord incident to a corner, then either there is no chord between $S_1$ and $S_3$ or no chord between $S_2$ and $S_4$. Let us assume witout loss of generality that there is no chord between $S_1$ and $S_3$.

If there is also no chord between $S_2$ and $S_4$, consider a chordless path $P$ just below $S_3$, and a chordless path $P'$ "just to the left" of $S_4$. Let $G'$ be the 4ST obtained by deleting the band above $P$ and the band to the right of $P'$. Let $d_h'$ be the graph-distance between $S_1(G')$ and $S_3(G')$ and let $d_v'$ be the graph-distance between $S_2(G')$ and $S_4(G')$. By the argument just above, either $d_h'$ or $d_v'$ is bounded by $n^{1/2}$. If $d_v' \leq n^{1/2}$ we do the binary decomposition using $P$, yielding two components $A, B$ (with $B$ a 4ST and $A$ a band-graph). Recall that the grid-size of the resulting drawing of $G$ is $O(nK^2 \times nK^2)$, where $K$ is $O(nd)$, with $d$ the graph-distance in $B$ between $S_2(B)$ and $S_4(B)$. In addition, since $G'$ is obtained from $B$ by removing a band-graph on the right side (recall that every vertex on the left side of the band-graph is adjacent to a vertex on the right side of the band-graph), then $d \leq d'_h + 1$. Hence $d$ is $O(n^{1/2})$, so that $K$ is $O(n^{3/2})$, hence the grid-size is $O(n^4 \times n^4)$. Finally, if there is a chord between $S_2$ and $S_4$, then this chord can not be above $P$ (otherwise the chord would be incident to the top-left corner or to the top-right corner). Hence $d = 1$, so that $K$ is $O(n)$, which guarantees a grid-size $O(n^3 \times n^3)$ for the drawing of $G$.

## 6 Application to spherical drawings

*Computing geodesic spherical drawings* In this section we consider the problem of drawing a planar triangulation on the sphere so that faces are mapped to non-overlapping spherical triangles. This problem is closely related to the *surface parameterization* problem, which has several applications (such as texture mapping, morphing and remeshing) and has attracted a great attention in the computer graphics and geometric modeling communities [6, 7]. In the spherical case, the goal is to define a bijective correspondence between the surface of a sphere (the parameter domain) and a surface mesh. A *geodesic spherical drawing* of a graph $G$ is a drawing such that vertices are mapped to distinct points on the unit sphere $\mathcal{S}^2$, and edges are drawn as non-crossing minor arcs of great circles (geodesics on $\mathcal{S}^2$). Given a graph of size $n$, we say that a spherical drawing has *polynomial vertex resolution* whether the geodesic length of the shortest edge is bounded by $\Omega(\frac{1}{n^c})$ (for some constant $c$).

As stated by the result below, we are able to compute spherical drawings guaranteeing linear time performace and polynomial vertex resolution (these requirements could not be achieved by prior works). The idea is quite simple and consists in constructing a convex mesh representation $\mathcal{M}$ of the input graph $G$ (refer to Fig. 8). First compute a special partition of the faces of $G$: each sub-graph is drawn in the plane using previous results (boundary vertices have to preserve inter-path distances). Then we glue all drawings together in order to obtain a polyhedron $\mathcal{M}$ contained in the interior of unit sphere. Finally perform a central projection (from the origin) of the vertices of $\mathcal{M}$ on $\mathcal{S}^2$: this bijectively maps edges to geodesic arcs on $\mathcal{S}^2$.

**Theorem 3.** *Given a planar triangulation $G$ with $n$ vertices, we can compute in linear time a geodesic spherical drawing of $G$, having resolution $\Omega(\frac{1}{n})$.*

A simple way to achieve this result is to pick a vertex $v^N$ of bounded degree $d$ (Euler's relations ensures the existence of a vertex of degree at most 5). Assuming $v^N$ has degree at least 4, we can construct a pyramid with rectangular base having $v^N$ as apex, and whose base corners are four vertices among the neighbors of $v^N$ (refer to Fig. 8). This leads to a partition of the faces of $G$ into five pieces: four triangulations and one 4-scheme triangulation (the base). Each piece, the lateral and bottom faces of the pyramid, admits a planar grid drawing of size $O(n) \times O(n)$: just apply the Duncan et. al algorithm to the base, recalling that

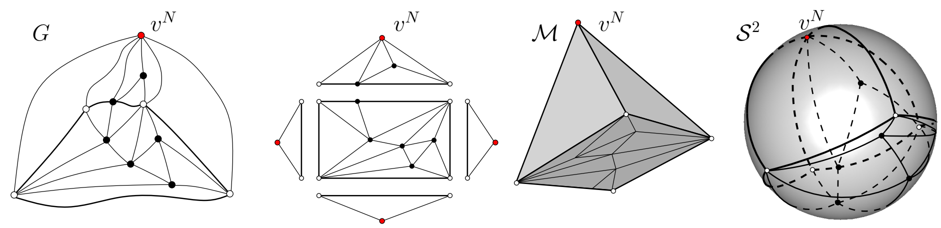


**Fig. 8.** These pictures illustrate the computation of a geodesic spherical drawing.

there are a constant number of boundary vertices (as $v^N$ has bounded degree). If $v^N$ has degree 3 then construct a triangular pyramid in a similar way: this time it remains only to draw the four faces of the pyramid using the shift algorithm.

We can also use a similar strategy with other polyhedral shapes, in order to obtain a better distribution of the vertices on the sphere. For example, picking two vertices of bounded degree (and at distance at least 3) it is possible to obtain a prism representation of $G$: lateral faces are then drawn using Property 1.

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