

# ASYMPTOTIC EXPANSION OF THE MULTI-ORIENTABLE RANDOM TENSOR MODEL

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ABSTRACT. Three-dimensional random tensor models are a natural generalization of the celebrated matrix models. The associated tensor graphs, or 3D maps, can be classified with respect to a particular integer or half-integer, the degree of the respective graph. In this paper we analyze the general term of the asymptotic expansion in  $N$ , the size of the tensor, of a particular random tensor model, the multi-orientable tensor model. We perform their enumeration and we establish which are the dominant configurations of a given degree.

**Keywords.** 3D maps, 4-regular maps, Eulerian orientations, schemes

## 1. INTRODUCTION AND MOTIVATION

Random tensor models (see [11] for a recent review) generalize in dimension three (and higher) the celebrated matrix models (see, for example, [3] for a review). Indeed, in the same way matrix models are related to combinatorial maps [14], tensor models in dimension three are related to tensor graphs or 3D maps. A certain case of random tensor models, the so-called colored tensor models, have been intensively studied in the recent years (see [7] for a review). The graphs associated to these models are regular edge-colored graphs. An important result is the asymptotic expansion in the limit  $N \rightarrow \infty$  ( $N$  being the size of the tensor), expansion which was obtained in [5]. The role played by the genus in the 2D case is played here by a distinct integer called the *degree*. The dominant graphs of this asymptotic expansion are the so-called melonic graphs, which correspond to particular triangulations of the three-dimensional sphere  $\mathcal{S}^3$ . Let us also mention here that a universality result generalizing matrix universality was obtained in the tensor case in [6].

A particularly interesting combinatorial approach for the study of these colored graphs was proposed recently by Gurău and Schaeffer in [9], where they analyze in detail the structure of colored graphs of fixed degree and perform exact and asymptotic enumeration. This analysis relies on the reduction of colored graphs to some terminal forms, called *schemes*. An important result proven in [9] is that the number of schemes of a given degree is finite (while the number of graphs of a given degree is infinite).

Nevertheless, a certain drawback of colored tensor models is that an important number of tensor graphs is discarded by the very definition of the model. Thus, a different type of model was initially proposed in [13], the 3D multi-orientable (MO) tensor model. This model is related to tensor graphs which correspond to 3D maps with a particular Eulerian orientation. The set of MO tensor graphs contains as a strict subset the set of colored tensor graphs. The asymptotic expansion in the

limit  $N \rightarrow \infty$  for the MO tensor model was studied in [2], where it was shown that the same class of tensor graphs, the melonic ones, are the dominant graphs in this limit. The sub-dominant term of this expansion was then studied in detail in [10].

In this paper we implement a Gurău-Schaeffer analysis for the MO random tensor model. We investigate in detail the general term of the asymptotic expansion in the limit  $N \rightarrow \infty$ . As in the colored case, this is done by defining appropriate terminal forms, the schemes. Nevertheless, our analysis is somehow more involved from a combinatorial point of view, since, as already mentioned above, a larger class of 3D maps has to be taken into consideration. Also an important difference with respect to the colored case is that the MO model allows for half-integer degrees, and not only integer degrees, as does the colored model. This leads to the fact that the dominant schemes are different from the ones identified in [9] for the colored model (interestingly, in both cases, dominant schemes are naturally associated to rooted binary trees).

Let us also mention that the analysis of this paper may further allow for the implementation of the so-called *double scaling limit* for the MO tensor model. This is a particularly important mechanism for matrix models (see again [3]), and which makes it possible to take, in a correlated way, the double limit  $N \rightarrow \infty$  and  $z \rightarrow z_c$  where  $z$  is a variable counting the number of vertices of the graph and  $z_c$  is some critical point of the generating function of schemes of a given degree.

## 2. PRELIMINARIES

In this section we recall the main definitions related to MO tensor graphs.

**2.1. Multi-orientable tensor graphs.** A *map* (also called a fat-graph) is a graph (possibly with loops and multiple edges, possibly disconnected) such that at each vertex  $v$  the cyclic order of the  $d$  incident half-edges ( $d$  being the degree of  $v$ ) in clockwise (cw) order around  $v$  is specified. A *corner* of a map is defined as a sector between two consecutive half-edges around a vertex, so that a vertex of degree  $d$  has  $d$  incident corners. A *4-regular map* is a map where all vertices have degree 4. A *multi-orientable tensor graph*, shortly called MO-graph thereafter, is a 4-regular map where each half-edge carries a sign,  $+$  or  $-$ , such that each edge has its two half-edges of opposite signs, and the two half-edges at each corner also have opposite signs. In addition, for convenience, the half-edges at each vertex are turned into 3 parallel strands, see Figure 1 for an example.

The strand in the middle is called *internal*, the two other ones are called *external*. An external strand is called *left* if it is on the left side of a positive half-edge or on the right side of a negative half-edge; an external strand is called *right* if it is on the right side of a positive half-edge or on the left side of a negative half-edge. A *face* of an MO-graph is a closed walk formed by a closed (cyclic) sequence of strands. External faces (faces formed by external strands) are the classical faces of the 4-regular map, while *internal faces* (faces formed by internal strands), also called *straight faces* thereafter, are not faces of the 4-regular map. Note also that external faces are either made completely of left external strands, or are made completely of right external strands; accordingly external faces are called either left or right. We finally define a *rooted MO-graph* as a connected MO-graph with a marked edge, which is convenient (as in the combinatorial study of maps) to avoid symmetry issues (indeed, as for maps, MO graphs are *unlabelled*).

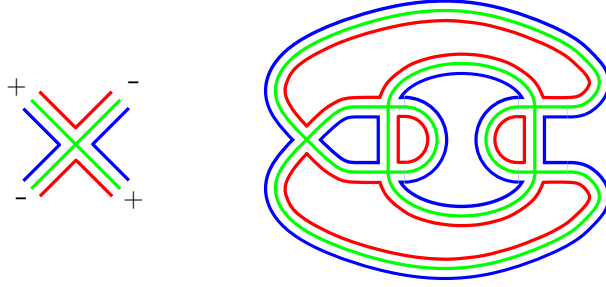


FIGURE 1. An MO-graph, with the 3 types of strands (left strands in blue, straight strands in green, right strands in red).

**2.2. The degree of an MO-graph.** The *degree* of an MO-graph  $G$  is the quantity  $\delta \in \frac{1}{2}\mathbb{Z}_+$  defined by

$$(1) \quad 2\delta = 6c + 3V - 2F,$$

where  $c, V, F$  are respectively the numbers of connected components, vertices, and faces (including internal faces) of  $G$ . For  $G_1$  and  $G_2$  two MO-graphs, denote by  $G_1 + G_2$  the MO-graph made of disconnected copies of  $G_1$  and  $G_2$ . Note that the quantities  $c, V, F$  for  $G$  result from adding the respective quantities from  $G_1$  and  $G_2$ . Hence we have:

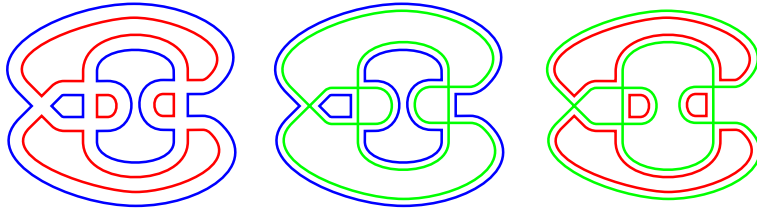
**Claim 2.1.** *For two MO-graphs  $G_1$  and  $G_2$ , the degree of  $G_1 + G_2$  is the sum of the degrees of  $G_1$  and  $G_2$ . In particular, the degree of an MO-graph is the sum of the degrees of its connected components.*

That  $\delta$  is nonnegative follows from the following observation [13]:  $G$  gives rise to three 4-regular maps  $G_{\ell,r}$  (resp.  $G_{\ell,s}, G_{r,s}$ ), called the *jackets* of  $G$ , which are obtained from  $G$  by deleting straight faces (resp. deleting right faces, deleting left faces), see Figure 2 for an example. Note that  $G_{\ell,r}$  is an orientable map, while  $G_{\ell,s}$  and  $G_{r,s}$  are typically only locally orientable (in a usual ribbon representation, edges have twists). Let  $g_{\ell,r}, g_{\ell,s}$  and  $g_{r,s}$  be the respective genera of  $G_{\ell,r}, G_{\ell,s}, G_{r,s}$  (since  $G_{\ell,r}$  is orientable,  $g_{\ell,r} \in \mathbb{Z}_+$ , while  $g_{\ell,s}$  and  $g_{r,s}$  are in  $\frac{1}{2}\mathbb{Z}_+$ ). Let  $F_\ell, F_r, F_s$  be the numbers of left faces, right faces, and straight faces of  $G$ . Denoting by  $E$  the number of edges of  $G$ , the Euler relation gives  $V - E + (F_\ell + F_r) = 2c - 2g_{\ell,r}$ ,  $V - E + (F_\ell + F_s) = 2c - 2g_{\ell,s}$ ,  $V - E + (F_r + F_s) = 2c - 2g_{r,s}$ . Since  $F = F_{\ell,r} + F_{\ell,s} + F_{r,s}$  and  $E = 2V$  (because the map is 4-regular) we conclude that

$$\delta = g_{\ell,r} + g_{\ell,s} + g_{r,s}.$$

We call  $g := g_{\ell,r} \in \mathbb{Z}_+$  the (*canonical*) *genus* of  $G$ . An MO-graph  $G$  is called *planar* if  $g = 0$ .

**2.3. MO-graphs as oriented 4-regular maps.** Seeing edges as directed from the positive to the negative half-edge, MO-graphs may be seen as 4-regular maps where the edges are directed such that each vertex has two ingoing and two outgoing half-edges, and the two ingoing (resp. two outgoing) half-edges are opposite, see Figure 3 for an example. We call *admissible orientations* such orientations of 4-regular maps. This point of view of MO-graphs as oriented 4-regular maps is quite convenient and will be adopted most of the time in the rest of the article (except when we need to locally follow the strand structure). Note that the left faces



bijection with rooted connected maps having genus  $g$  and  $m$  edges). In genus 0 (planar case), all rooted 4-regular maps have this property.

We conclude this subsection with a simple lemma relating the degree and the genus to the number of straight faces:

**Lemma 2.1.** *Let  $G$  be a connected MO-graph. Let  $V$  be the number of vertices,  $F_s$  the number of straight faces, and for each  $p \geq 1$ ,  $F_s^{(p)}$  the number of straight faces of length  $2p$  in  $G$ . Then the quantity  $\Lambda := \sum_{p \geq 1} (p-2) \cdot F_s^{(p)} + 2 = V - 2F_s + 2$  satisfies*

$$\Lambda = 2\delta - 4g,$$

where  $\delta$  and  $g$  are respectively the degree and the genus of  $G$ . Hence  $\Lambda \leq 2\delta$ .

*Proof.* Since the number  $E$  of edges is twice the number of vertices, we have  $\sum_{p \geq 1} 2pF_s^{(p)} = E = 2V$ , so that  $\sum_{p \geq 1} pF_s^{(p)} = V$ , hence  $\Lambda = V - 2F_s + 2$ . Let  $F_\ell, F_r$  be the numbers of left faces and right faces of  $G$ . We have seen in Section 2.2 that  $\delta = g_{\ell,r} + g_{\ell,s} + g_{r,s}$ , where  $g = g_{\ell,r}$  and

$$-V + (F_\ell + F_r) = 2 - 2g_{\ell,r}, \quad -V + (F_\ell + F_s) = 2 - 2g_{\ell,s}, \quad -V + (F_r + F_s) = 2 - 2g_{r,s},$$

Subtracting the sum of the two last equalities from the first one, we obtain:

$$V - 2F_s = -2 + 2g_{\ell,s} + 2g_{r,s} - 2g_{\ell,r} = -2 + 2\delta - 4g,$$

so that  $\Lambda = 2\delta - 4g$ . □

*Remark 2.3.* For  $g = 0$ , according to Remark 2.2, rooted MO-graphs correspond to rooted 4-regular planar maps. Seeing such a map as the planar projection of an entangled link that lives in the 3D space (vertices of the map correspond to crossings, where the under/over information is omitted),  $F_s$  is classically interpreted as the number of *knot-components*. Lemma 2.1 gives  $F_s = V/2 + 1 - \delta$ . Since  $F_s \geq 1$  and  $\delta \geq 0$ , the extremal cases are: (1)  $\delta = 0$ , in which case  $F_s = V/2 + 1$ ; (2)  $F_s = 1$ , in which case  $\delta = V/2$  (the MO-graph of Figure 3, having  $V = 5$ ,  $F_s = 2$ , and thus  $\delta = 3/2$ , is intermediate). While the case  $\delta = 0$  is combinatorially well-understood (as will be recalled in Section 3.1), the case  $F_s = 1$  is notoriously difficult: in that case, 4-regular planar maps are projected diagrams of one-component links (i.e., knot diagrams), see [12] and references therein.

We will go back to this remark in Section 7 (on generating functions), where we will establish, for each fixed  $\delta \in \frac{1}{2}\mathbb{Z}_+$ , the asymptotic enumeration of rooted 4-regular planar maps with  $2n$  vertices and with  $n + 1 - \delta$  knot-components.

**2.4. Regular colored graphs for  $D = 3$  as a subfamily of MO-graphs.** As recalled here from [13], MO-graphs form a superfamily of the well studied *regular colored graphs* of dimension 3. For  $D \geq 2$  a *regular colored graph of dimension  $D$*  is defined as a  $(D + 1)$ -regular bipartite graph  $G$  (vertices are either black or white) where the edges have a color in  $\{0, \dots, D\}$ , such that at each vertex the  $D + 1$  incident edges have different colors. A *rooted colored graph* is a connected colored graph with a marked edge of color 0. For  $0 \leq i < j \leq D$ , a *face* of type  $(i, j)$  in a colored graph of dimension  $D$  is a cycle made of edges that alternate colors in  $\{i, j\}$ ; let  $F_{i,j}$  be the number of faces of type  $(i, j)$  in  $G$ . Let  $c$  be the number of connected components,  $V = 2k$  the number of vertices and  $F$  the total number of

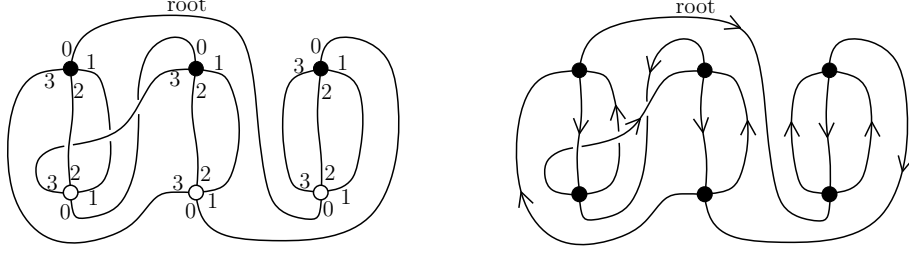


FIGURE 4. Left: a rooted colored graph in 3 dimensions. Right: the induced rooted MO-graph.



FIGURE 5. Left: a melon. Right: a triple edge that does not form a melon.

faces of  $G$ , i.e.,  $F = \sum_{0 \leq i < j \leq D} F_{i,j}$ . The *degree* of  $G$  is the integer  $\delta$  given by

$$(2) \quad \delta = \frac{1}{2}D(D-1)k + c \cdot D - F.$$

Note that a colored graph has a canonical realization as a map, where the edge colors in cw (resp. ccw) order around black (resp. white) vertices are  $(0, 1, \dots, D)$ . Let  $G$  be a rooted colored graph of dimension 3 that is canonically embedded. Orienting the edges of even color from the black to the white extremity and the edges of odd color from the white to the black extremity, we obtain a rooted MO-graph  $\tilde{G}$ , see Figure 4 for an example. Clearly this gives an injective mapping, since there is a unique way (when possible) to propagate the edge colors starting from the root-edge. Hence, rooted colored graphs of dimension 3 form a subfamily of rooted MO-graphs. In addition

$$F_r(\tilde{G}) = F_{0,1}(G) + F_{2,3}(G), \quad F_\ell(\tilde{G}) = F_{1,2}(G) + F_{0,3}(G), \quad F_s(\tilde{G}) = F_{0,2}(G) + F_{1,3}(G),$$

so that  $\tilde{G}$  and  $G$  have the same total number of faces. Hence the degree formula for colored graphs is consistent with the degree formula for MO-graphs.

### 3. FROM MO-GRAPHS TO SCHEMES

**3.1. Extracting the melon-free core of a rooted MO-graph.** From now on, it is convenient to consider that a “fake” vertex of degree 2, called the *root-vertex*, is inserted in the middle of the root-edge of any rooted MO-graph (so that the root-edge is turned into two edges). By convention also, it is convenient to introduce the following MO-graph: the *cycle-graph* is defined as an oriented self-loop carrying no vertex. The cycle-graph is connected, has  $V = 0$ ,  $F = 3$  (one face in each type), hence has degree 0. In its rooted version, the (rooted) *cycle-graph* is made of an oriented loop incident to the root-vertex. In a (possibly) rooted) MO-graph  $G$ , a *melon* is a triple edge  $(e_1, e_2, e_3)$  such that none of the 3 edges is the root-edge (if

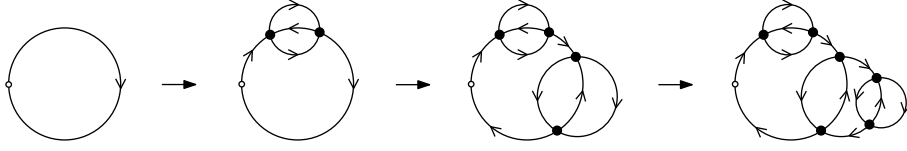


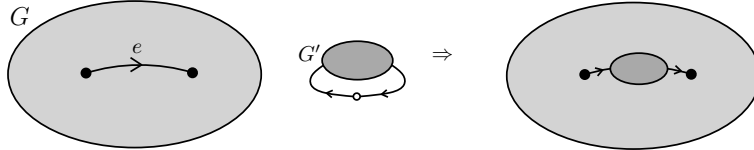
FIGURE 6. A rooted melonic graph is one that can be built from the cycle-graph, by successive insertions of melons at edges.

$G$  is rooted),  $(e_1, e_2)$  form a left face of length 2,  $(e_2, e_3)$  form a right face of length 2, and  $(e_1, e_3)$  form a straight face of length 2, see Figure 5. Define the *removal* of a melon as the operation below (where possibly  $u = v$ , and possibly  $u$  or  $v$  might be the root if the MO-graph is rooted):



The reverse operation (where  $e$  is allowed to be a loop, and is allowed to be incident to the root-vertex if the MO-graph is rooted) is called the *insertion* of a melon at an edge.

A rooted MO-graph  $G$  is called *melonic* if it can be reduced to the rooted cycle-graph by successive removals of melons, see Figure 6 for an example. An important remark is that, for a rooted melonic MO-graph, *any* greedy sequence of melon removals terminates at the cycle-graph. It is known [2] that rooted MO-graphs of degree 0 are exactly the rooted melonic graphs. For  $G$  a (possibly rooted) MO-graph,  $e$  an edge of  $G$  (possibly a loop, possibly incident to the root-vertex), and  $G'$  a rooted MO-graph, define the operation of *substituting  $e$  by  $G'$  in  $G$*  as in the following generic drawing:



We have the following bijective statement, which is the counterpart for MO-graphs of [9, Theo. 4]:

**Proposition 3.1.** *Each rooted MO-graph  $G$  is uniquely obtained as a rooted melon-free MO-graph  $H$  —called the melon-free core of  $G$ — where each edge  $e$  is substituted by a rooted melonic MO-graph (possibly the cycle-graph, in which case  $e$  is unchanged). In addition, the degree of  $G$  equals the degree of  $H$ .*

*Proof.* Let  $H$  be the rooted melon-free graph obtained from  $G$  after performing a greedy sequence of melon removals. Conversely,  $G$  is obtained from  $H$  where a sequence of melon insertions is performed. Hence  $G$  is equal to  $H$  where each edge  $e$  is substituted by a rooted melonic graph  $G_e$ . This gives the existence of a melon-free core. Uniqueness of the melon-free core is given by the observation that any greedy sequence of melon removals starting from  $G$  has to progressively shell the melonic components  $G_e$ , hence terminates at  $H$ . Finally, it is clear that  $G$  and

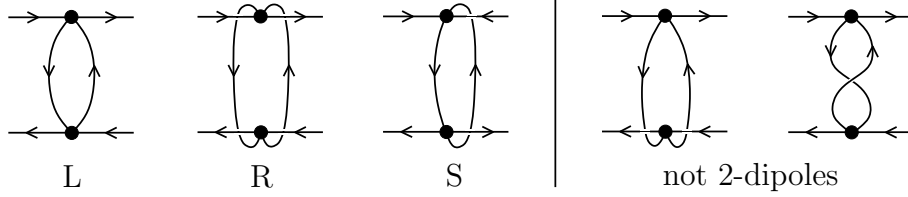


FIGURE 7. Left: the three types of dipoles. Right: two examples of double edges that do not form a dipole.



FIGURE 8. Left: the clockwise infinity graph. Right: the counterclockwise infinity graph. Both have two faces of length 2, two faces of length 1, and have degree  $1/2$ .

$H$  have the same degree, since a melon insertion preserves the degree (it preserves the number of connected components, increases the number of vertices by 2, and increases the number of faces by 3).  $\square$

**3.2. Extracting the scheme of a rooted melon-free MO-graph.** In this section we go a step further and define the process of extracting a *scheme* from a rooted melon-free MO-graph.

Define a *2-dipole* (shortly a dipole) in a (possibly rooted) MO-graph  $G$  as a face of length 2 incident to two distinct vertices, and not passing by the root if  $G$  is rooted. Accordingly, one has three distinct types of dipoles: L, R, or S, see Figure 7 (left part). As the figure shows, each dipole has two exterior half-edges on one side and two exterior half-edges on the other side. Notice also that a double edge does not necessarily delimit a dipole, as shown in Figure 7 (right part).

*Remark 3.1.* A face of length 2 is actually always incident to two distinct vertices, except in the MO-graphs that are made of one vertex and two loops. These two graphs, shown in Figure 8, are called the clockwise and the counterclockwise infinity graph, respectively. They have no dipole but have two faces of length 2.

In a (possibly rooted) MO-graph, define a *chain* as a sequence of dipoles  $d_1 \dots, d_p$  such that for each  $1 \leq i < p$ ,  $d_i$  and  $d_{i+1}$  are connected by two edges involving two half-edges on the same side of  $d_i$  and two half-edges on the same side of  $d_{i+1}$ , see Figure 9. A chain is called *unbroken* if all the  $p$  dipoles are of the same type. A *broken chain* is a chain which is not unbroken. A *proper chain* is a chain of at least two dipoles. A proper chain is called *maximal* if it cannot be extended into a larger proper chain. By very similar arguments as in Lemma 8 of [9] one obtains the following result:

**Claim 3.1.** *In a rooted MO-graph, any two maximal proper chains are vertex-disjoint.*



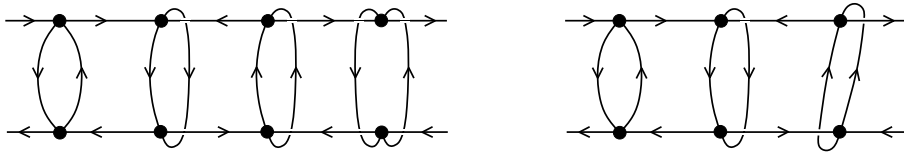


FIGURE 9. Left: a chain of 4 dipoles. Right: a sequence of 3 dipoles that do not form a chain (the 3rd element, a dipole of type  $S$ , is connected to the 2nd element by two half-edges on opposite sides).

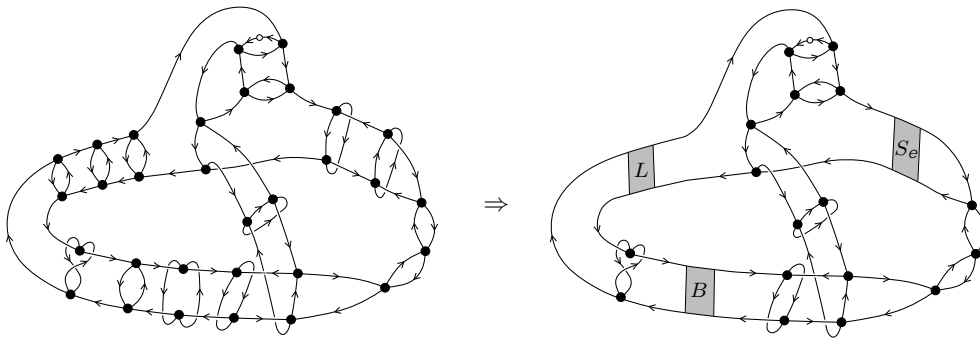


FIGURE 10. Left: a rooted melon-free MO-graph. Right: the associated scheme.

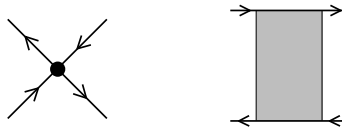


FIGURE 11. The two types of vertices (standard vertex or chain-vertex, which can be labelled by either  $\{L, R, S_e, S_o, B\}$ ) in an MO-graph with chain-vertices.

Let  $G$  be a rooted melon-free MO-graph. The *scheme* of  $G$  is the graph obtained by simultaneously replacing any maximal proper chain of  $G$  by a so-called *chain-vertex*, as shown in Figure 12, see also Figure 10 for an example.

Since a scheme is not an MO-graph (due to the presence of chain-vertices), we need to extend the definition and main properties seen so far. In the following, an *MO-graph with chain-vertices* is an oriented 4-regular map with two types of vertices: standard vertices and chain-vertices (each chain-vertex being labelled by either  $\{L, R, S_e, S_o, B\}$ ), so as to satisfy the local rules of Figure 11. Such a graph is possibly rooted, i.e., has a fake-vertex of degree 2 in the middle of some edge. In order to compute the number of faces of an MO-graph  $\tilde{G}$  with chain-vertices, we need to specify locally the *strand structure* at every type of chain-vertex; as for MO-graphs, we imagine here that each edge is turned into 3 parallel strands and we have to specify at each chain-vertex how the incident strands go through the vertex. The specification is given by Figure 13; in the case of an unbroken chain-vertex the

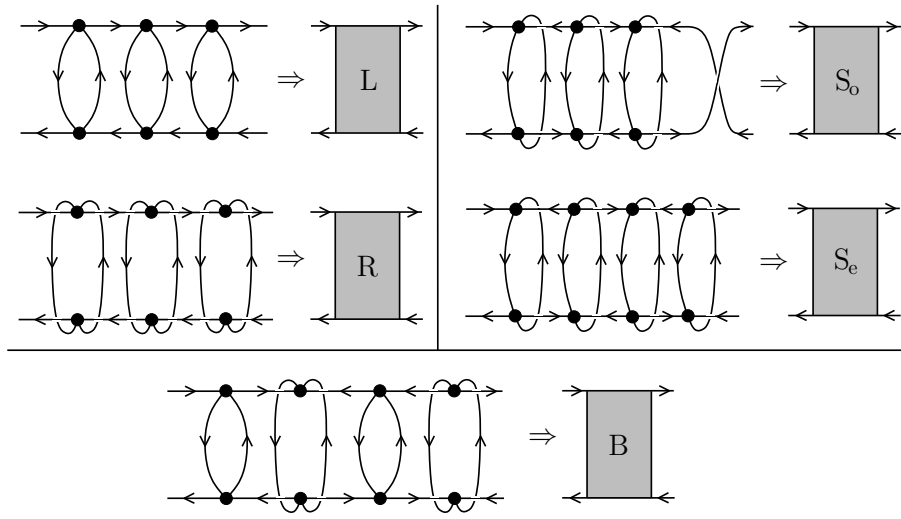


FIGURE 12. Top-part: examples of the 4 types of unbroken proper chains; Bottom-part: example of a broken proper chain. In each case, the graphical symbol for the chain-vertex (in the associated scheme) is shown.

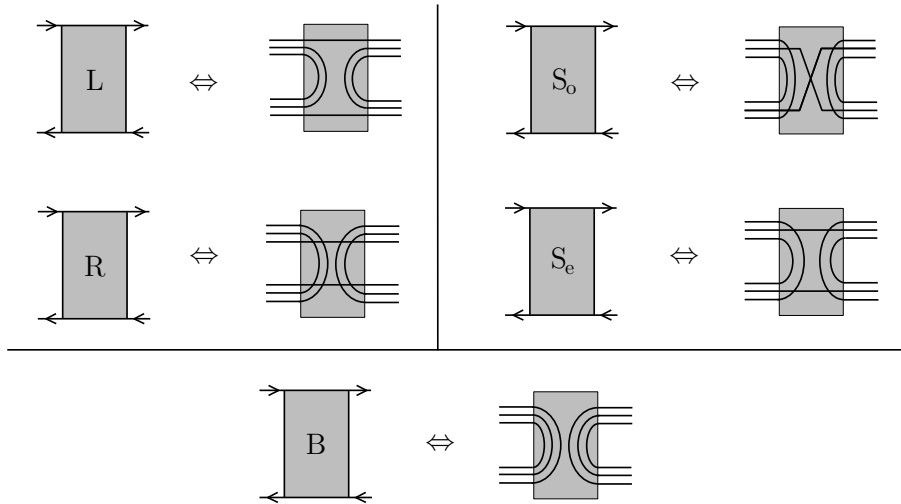


FIGURE 13. The configurations of the strands for each type of chain-vertex.

two strands that go through the chain-vertex are called the *crossing strands* at that chain-vertex. Then, as for classical MO-graphs, a *face* is a closed walk formed from strands. The *degree*  $\delta$  of  $\tilde{G}$  is defined as

$$(3) \quad 2\delta = 6c + 3V - 2F + 4U + 6B,$$

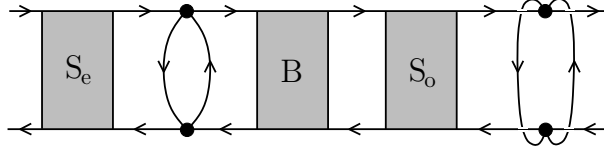


FIGURE 14. A chain of five elements in an MO-graph with chain-vertices: it consists of two dipoles and three chain-vertices.

where  $c$ ,  $V$ ,  $F$  are as usual the numbers of connected components, standard vertices, and faces, and where  $U$  and  $B$  stand respectively for the numbers of unbroken chain-vertices and broken chain-vertices.

An edge  $e$  of  $\tilde{G}$  is said to be *adjacent* to a chain-vertex, if the two half-edges of  $e$  are the two half-edges on the same side of a chain-vertex of  $\tilde{G}$ . Then  $\tilde{G}$  is said to be *melon-free* if it has no melon nor an edge adjacent to a chain-vertex. It is easy to see that an MO-graph is melon-free iff its scheme is melon-free.

In a melon-free (possibly rooted) MO-graph  $\tilde{G}$  with chain-vertices, define a *chain* as a sequence  $d_1 \dots, d_p$  of elements that are either dipoles (not passing by the root if  $\tilde{G}$  is rooted) or chain-vertices, such that for each  $1 \leq i < p$ ,  $d_i$  and  $d_{i+1}$  are connected by two edges involving two half-edges on the same side of  $d_i$  and two half-edges on the same side of  $d_{i+1}$ , see Figure 14. A *proper chain* is a chain of at least two elements.

Now define a *reduced scheme* as a rooted melon-free MO-graph with chain-vertices and with no proper chain. By construction, the scheme of a rooted melon-free MO-graph (with no chain-vertices) is a reduced scheme. Claim 3.1 then easily yields the following bijective statement:

**Proposition 3.2.** *Every rooted melon-free MO-graph is uniquely obtained as a reduced scheme where each chain-vertex is consistently substituted by a chain of at least two dipoles (consistent means that if the chain-vertex is of type  $L$ , then the substituted chain is an unbroken chain of  $L$ -dipoles, etc).*

The following result ensures that the degree definition for MO-graphs with chain-vertices is consistent with the replacement of chains by chain-vertices:

**Lemma 3.1.** *Let  $G$  be an MO-graph with chain-vertices. And let  $G'$  be a an MO-graph with chain-vertices obtained from  $G$  by consistently substituting a chain-vertex by a chain of dipoles. Then the degrees of  $G$  and  $G'$  are the same.*

*Proof.* Let  $m$  be the chain-vertex where the substitution takes place, and let  $k$  be the number of dipoles in the substituted chain. Denote by  $\delta, c, V, F, U, B$  the parameters (degree, numbers of connected components, standard vertices, faces, unbroken chain-vertices, broken chain-vertices) for  $G$ , and by  $\delta', c', V', F', U', B'$  the parameters for  $G'$ . Clearly  $c' = c$ ,  $V' = V + 2k$ . If  $m$  is unbroken, then  $U' = U - 1$  and  $B' = B$ . Say  $m$  is of type  $L$  (the arguments for types  $R, S_e, S_o$  are similar). Then the strand structure remains the same, except for  $k$  new faces of type  $L$  (of degree 2),  $k - 1$  new faces of type  $R$  (of degree 4) and  $k - 1$  new faces of type  $S$  (of degree 4) inside the substituted chain. Hence  $F' = F + 3k - 2$ . Consequently,  $\delta' = \delta$ . If  $m$  is broken, then  $U' = U$  and  $B' = B - 1$ . The strand structure remains the same, except for  $k - 1$  new faces of each type  $L, R, S$  inside the substituted chain (an easy case inspection ensures that substituting a chain of

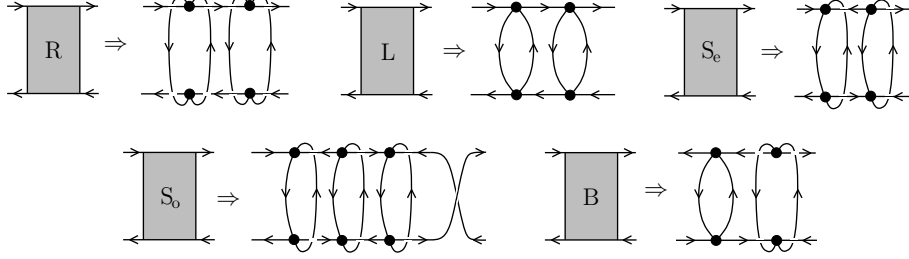


FIGURE 15. The drawings show a choice of canonical substitution of chain-vertices by proper chains that preserves the strand structure (hence the degree) and yields an injective mapping from MO-graphs with chain-vertices to MO-graphs without chain-vertices.

two dipoles of different types always bring 3 new faces, one of each type, and any additional dipole brings 3 new faces, one of each type). Hence  $F' = F + 3k - 3$ . Consequently,  $\delta' = \delta$ .  $\square$

**Corollary 3.1.** *A rooted melon-free MO-graph has the same degree as its scheme.*

#### 4. FINITENESS OF THE SET OF REDUCED SCHEMES AT FIXED DEGREE

The main result in this section is the following:

**Proposition 4.1.** *For each  $\delta \in \frac{1}{2}\mathbb{Z}_+$ , the set of reduced schemes of degree  $\delta$  is finite.*

Similarly as in [9], this result will be obtained from two successive lemmas, proved respectively in Section 4.2 and Section 4.3:

**Lemma 4.1.** *For each reduced scheme of degree  $\delta$ , the sum  $N(G)$  of the numbers of dipoles and chain-vertices satisfies  $N(G) \leq 7\delta - 1$ .*

**Lemma 4.2.** *For  $k \geq 1$  and  $\delta \in \frac{1}{2}\mathbb{Z}_+$ , there is a constant  $n_{k,\delta}$  (depending only on  $k$  and  $\delta$ ) such that any connected unrooted MO-graph (without chain-vertices) of degree  $\delta$  with at most  $k$  dipoles has at most  $n_{k,\delta}$  vertices.*

Let us show how Lemmas 4.1 and 4.2 imply Proposition 4.1. Let  $S$  be a reduced scheme of degree  $\delta$ , and let  $G(S)$  be the rooted MO-graph with no chain-vertices obtained by substituting each chain-vertex of  $S$  by a chain of dipoles of length 2 or 3 as shown in Figure 15. As already seen in Lemma 3.1, such substitutions preserve the degree, and the mapping  $S \rightarrow G(S)$  is injective. In addition, the number of dipoles of  $G(S)$  is at most three times the total number of dipoles and chain-vertices of  $S$ , hence  $G(S)$  has at most  $3 \cdot (7\delta - 1)$  dipoles according to Lemma 4.1. Unrooting  $G(S)$  might increase the number of dipoles by up to 3 (recall that dipoles are not counted in rooted MO-graphs if they pass by the root, and at most 3 dipoles can pass by the root, the case of 3 additional dipoles happening when the root-edge belongs to a melon). Hence, writing  $k := 3 \cdot (7\delta - 1) + 3$ , Lemma 4.2 ensures that  $G(S)$  has at most  $n_{k,\delta}$  vertices. Since there is an injective mapping from reduced schemes of degree  $\delta$  to rooted MO-graphs of size bounded by the fixed quantity  $n_{k,\delta}$ , we conclude that the number of reduced schemes of degree  $\delta$  is finite.



FIGURE 16. Removing a loop in an MO-graph.

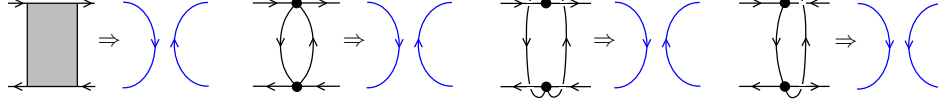


FIGURE 17. The first drawing shows the removal of a chain-vertex (whatever its type). The next drawings show the removal of a dipole in each of the three types (L,R,S).

We finally state the following “loop-removal” lemma, which will be useful at some points in the following:

**Lemma 4.3.** *Let  $G$  be an MO-graph of degree  $\delta \in \frac{1}{2}\mathbb{Z}_+$ , let  $\ell$  be a loop of  $G$ , and let  $G'$  be the MO-graph obtained from  $G$  by erasing the loop and its incident vertex, as shown in Figure 16. Then  $G'$  has degree  $\delta - 1/2$ . Hence  $G$  has at most  $2\delta$  loops.*

*Proof.* Clearly  $G'$  has the same number of connected components as  $G$ , has one vertex less, and has one face less (which has length 1).  $\square$

**4.1. Analysis of the removal of dipoles and of chain-vertices.** We analyze here how the degree of an MO-graph with chain-vertices (not necessarily a reduced scheme, possibly with melons) evolves when removing a chain-vertex. We then have a similar analysis for the removal of a dipole.

Let  $G$  be an MO-graph with chain-vertices, and let  $m$  be a chain-vertex of  $G$ . The *removal* of  $m$  consists of the following operations (see the left part of Figure 17):

- (i) delete  $m$  from  $G$ ;
- (ii) on each side of  $m$ , connect together the two detached legs (without creating a new vertex).

Denote by  $G'$  the resulting graph. The chain-vertex  $m$  is said to be *non-separating* if  $G'$  has the same number of connected components as  $G$ , and *separating* otherwise (in which case  $G'$  has one more connected component). In the separating case, it might be that one of the two resulting connected components is the cycle-graph (this happens if two half-edges on one side of  $m$  belong to the same edge). Let  $\delta, c, V, F, U, B$  be the parameters for  $G$  (degree, numbers of connected components, standard vertices, faces, unbroken chain-vertices, broken chain-vertices), and let  $\delta', V', F', U', B'$  be the parameters for  $G'$ .

Assume first that  $m$  is a broken chain-vertex. We clearly have  $V' = V$ ,  $F' = F$  (the strand structure is the same before and after removal),  $U' = U$ ,  $B' = B - 1$  and  $c' = c$  (resp.  $c' = c + 1$ ) if  $m$  is non-separating (resp. separating). Hence, if  $m$  is separating, then  $2\delta' = 2\delta + 6 - 6 = 2\delta$ , so that  $\delta' = \delta$ ; and if  $m$  is non-separating, then  $2\delta' = 2\delta - 6$ , so that  $\delta' = \delta - 3$ .

Assume now that  $m$  is an unbroken chain-vertex. We clearly have  $V' = V$ ,  $U' = U - 1$ ,  $B' = B$ , and  $c' = c$  (resp.  $c' = c + 1$ ) if  $m$  is non-separating (resp.

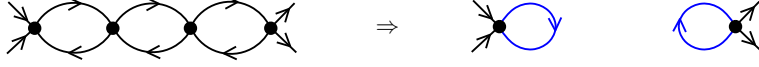


FIGURE 18. In the worst case, removing a dipole might decrease by 3 the total number of dipoles.

separating). If the two crossing strands at  $m$  belong to a same face, then that face splits into two faces when removing  $m$ , hence  $F' = F + 1$ ; as opposed to that, if the two crossing strands belong to different faces, then these two faces are merged when removing  $m$ , hence  $F' = F - 1$ . Moreover, the two crossing strands are easily seen to always be in the same face if  $m$  is separating. We conclude that, if  $m$  is separating, then  $2\delta' = 2\delta + 6 - 2 - 4 = 2\delta$ , so that  $\delta' = \delta$ ; and if  $m$  is non-separating then  $2\delta' = 2\delta - 2\sigma - 4$ , where  $\sigma \in \{-1, +1\}$ , hence either  $\delta' = \delta - 1$  or  $\delta' = \delta - 3$ .

We now consider the operation of removing a dipole  $d$  from  $G$ , removal which consists in the following operations (the 3 cases are shown in the right-part of Figure 17):

- (i) delete the two vertices and the two edges of  $d$  from  $G$ ;
- (ii) on each side of  $d$ , connect together the two detached legs (without creating a new vertex).

Let  $G'$  be the resulting graph. Again,  $d$  is called *non-separating* if  $G'$  has as many connected components as  $G$ , and *separating* otherwise. Observe that the operation of removing  $d$  is the same as substituting  $d$  by an unbroken chain-vertex  $m$  (of the same type as  $d$ , so that the substitution does not change the degree according to Lemma 3.1) and then removing  $m$ . From the discussion on the degree variation when removing  $m$ , we conclude that, if  $d$  is separating, then the degree is the same in  $G'$  as in  $G$ , and if  $d$  is non-separating then the degree of  $G'$  equals the degree of  $G$  minus a quantity that is either 1 or 3.

This analysis thus leads to the following statement:

**Lemma 4.4.** *The degree is unchanged when removing a separating chain-vertex or a separating dipole (hence, by Claim 2.1, the degree is distributed among the resulting components). The degree decreases by 3 when removing a non-separating broken chain-vertex. The degree decreases by 1 or 3 when removing a non-separating unbroken chain-vertex or a non-separating dipole.*

**Corollary 4.1.** *An MO-graph with chain-vertices and with degree  $\delta \in \{0, 1/2\}$  has no non-separating dipole nor non-separating chain-vertex.*

**4.2. Proof of Lemma 4.1.** We show here Lemma 4.1, i.e., that for each reduced scheme  $G$  of degree  $\delta$ , the sum  $N(G)$  of the numbers of dipoles and chain-vertices in  $G$  satisfies  $N(G) \leq 7\delta - 1$ .

As a first step, starting from  $G$ , as long as there is at least one non-separating dipole or one non-separating chain-vertex, delete it (as in Figure 17) and color the two resulting edges. Let  $G'$  be the resulting graph with chain-vertices, and let  $q$  be the number of removals from  $G$  to  $G'$ . According to Lemma 4.4, each non-separating removal decreases the degree by at least 1, hence  $q \leq \delta$ ,  $G'$  has degree at most  $\delta - q$ , and  $G'$  has at most  $2q$  blue edges. During the transition from  $G$  to  $G'$ , denote by  $N$  the sum of the numbers of chain-vertices and dipoles. As shown in Figure 18, each

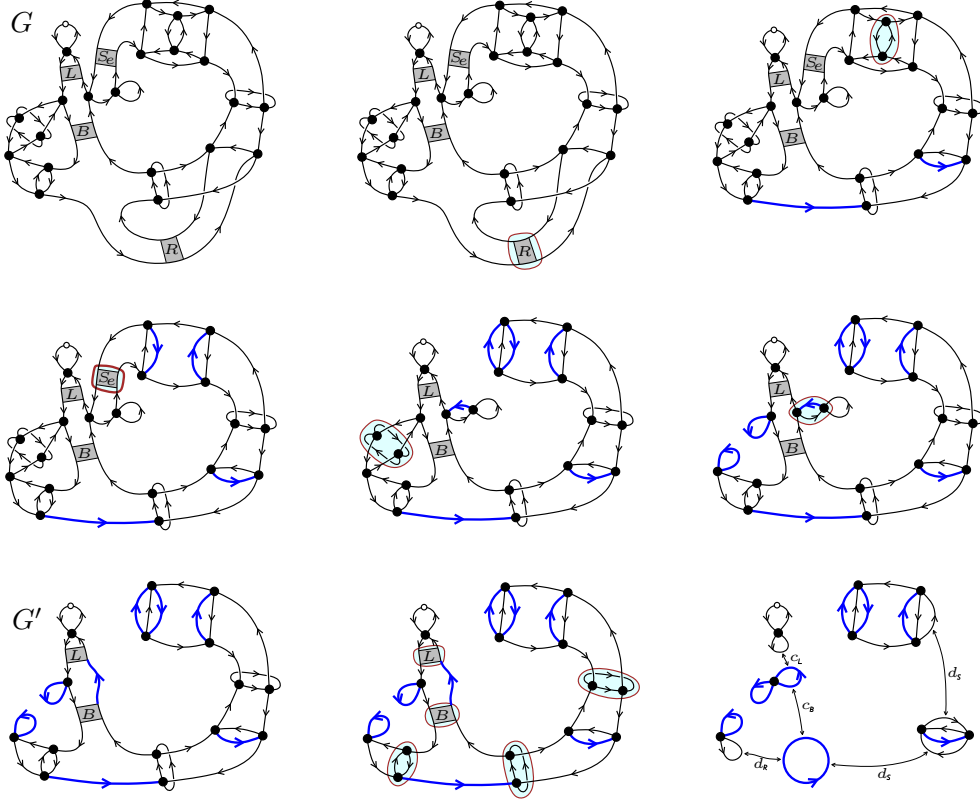


FIGURE 19. The first drawing shows a reduced scheme  $G$ . Then, iteratively, one removes at each step a non-separating dipole or a non-separating chain-vertex (at each step the non-separating dipole or chain-vertex to be removed next is surrounded). Let  $G'$  be the MO-graph with chain-vertices thus obtained (where colored edges are drawn bolder). As the last two drawings show, the removal of uncolored dipoles and chain-vertices (which are all separating) of  $G'$  yields a tree of components (a tree edge is labelled  $c_x$  if it comes from a chain-vertex of type  $x$  and is labelled  $d_x$  if it comes from an uncolored dipole of type  $x$ ).

removal might decrease  $N$  by 3, and this is actually the worst case. Hence, if we denote by  $N(G')$  the value of  $N$  for  $G'$ , we have  $N(G') \geq N(G) - 3q \geq N(G) - 3\delta$ .

Since all dipoles and chain-vertices of  $G'$  are separating,  $G'$  can be seen as a tree  $T$  of “components” (see the last row of Figure 19 for an example): the edges of  $T$  correspond to the  $N(G')$  chain-vertices and uncolored dipoles (dipoles with no colored edge) of  $G'$ , and the nodes of  $T$  correspond to the connected components of  $G'$  left after these removals (note that some of these connected components might be cycle-graphs, including the component carrying the root). For a component  $C$  of  $T$ , the *adjacency order* of  $C$  is the number of adjacent components in  $T$ . An edge  $e$  of  $C$  resulting from a (removed) connection is called a *marked edge*. Note that a marked edge can be involved in several connections (for instance in Figure 19 there

is an edge of a cycle-graph involved in 3 connections, and there is an edge of an infinity graph involved in 2 connections), hence the adjacency order of a component is at least as large as its number of marked edges. A marked edge  $e$  results from the merger of edges of  $G'$  incident to separating chain-vertices or uncolored dipoles of  $G'$ . Then  $e$  is called *colored* if at least one of these edges is a colored edge of  $G'$ . A component is called *uncolored* if it bears neither a colored edge of  $G'$  left untouched by removals, nor a colored marked edge. Since  $G'$  has at most  $2q$  colored edges,  $T$  has at most  $2q$  colored components.

Similarly as in [9], an important remark is the following:

**Claim 4.1.** *An uncolored component  $C$  of  $T$  of degree 0 and not containing the root has adjacency order at least 3.*

*Proof.* The first unrooted MO-graphs of degree 0 are the cycle-graph (no vertex) and the so-called “quadruple-edge graph” (two vertices connected by 4 edges). And any other MO-graph of degree 0 can be obtained from the quadruple-edge graph by successive melon insertions. This easily implies that, if  $C$  is not the cycle-graph and has adjacency order smaller than 3 (hence has less than 3 marked edges), then  $C$  has a dipole not passing by any marked edge, a contradiction. If  $C$  is the cycle-graph and has adjacency order 1, then it clearly yields a melon in  $G$ , a contradiction. Finally, if  $C$  is the cycle-graph and has adjacency order 2, then the two edges of  $T$  at  $C$ —each of which is either a separating dipole or a chain-vertex of  $G$ —together form a proper chain (of two elements) of  $G$ , a contradiction.  $\square$

Denote by  $n_0$  the number of uncolored components in  $G'$  of degree 0 and not containing the root,  $n'_0$  the number of other components of degree 0 (colored or containing the root), and  $n_+$  the number of components of positive degree. The degrees of the components add up to the degree of  $G'$  (recall that, for separating removals, the degree is distributed among the components), and each component of positive degree has degree at least  $1/2$ , hence  $n_+/2 \leq \delta - q$ . The components counted by  $n_0$  have at least 3 neighbours in  $T$ , hence  $3n_0 + n'_0 + n_+ \leq 2N(G')$ . Since  $T$  has at most  $2q$  colored components, we have  $n'_0 \leq 2q + 1$  (the  $+1$  accounting for the component containing the root). And since  $T$  is a tree, we have

$$N(G') = n_0 + n'_0 + n_+ - 1,$$

hence, if we eliminate  $n_0$  using  $3n_0 + n'_0 + n_+ \leq 2N(G')$ , we obtain

$$N(G') \leq \frac{2}{3}(N(G') + n'_0 + n_+) - 1 \leq \frac{2}{3}(N(G') + 2q + 1 + 2\delta - 2q) - 1,$$

so that  $N(G') \leq 4\delta - 1$ . Since  $N(G) \leq N(G') + 3\delta$ , we conclude that  $N(G) \leq 7\delta - 1$ .

**4.3. Proof of Lemma 4.2.** We prove here Lemma 4.2, i.e., that for any  $\delta \in \frac{1}{2}\mathbb{Z}_+$  and  $k \geq 1$ , there exists some constant  $n_{k,\delta}$  such that any connected unrooted MO-graph (with no chain-vertices) of degree  $\delta$  with at most  $k$  dipoles has at most  $n_{k,\delta}$  vertices, so that there are finitely many MO-graphs of degree  $\delta$  with  $k$  dipoles. The bound  $n_{k,\delta}$  we obtain is linear in  $k$  and  $\delta$ , with quite large multiplicative constants (we have not pushed to improve the bound, also to avoid making the proof too complicated).

The proof is a bit more technical than the analogous statement for colored graphs in [9], because in the MO model, only straight faces have even length, while left and right faces are allowed to have odd lengths. Precisely, our proof strategy is



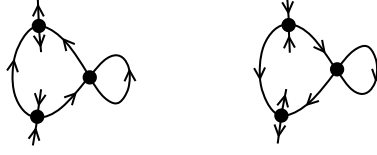


FIGURE 20. The two possible configurations of a self-intersecting straight face of length 4.

to show in a series of claims that some properties (e.g. being incident to a dipole) are satisfied only by a limited (bounded by a quantity depending only on  $k$  and  $\delta$ ) number of vertices. Hence if  $G$  could be arbitrarily large, it would have vertices not satisfying these properties. As we will see, this would yield the contradiction that  $G$  has to be reduced to a certain MO-graph with 6 vertices.

In all the proof,  $G$  denotes a connected unrooted MO graph with no chain-vertices, of degree  $\delta$  and with at most  $k$  dipoles;  $V, E, F$  are respectively the numbers of vertices, edges, and faces of  $G$ . Moreover, for  $p \geq 1$ ,  $F_s^{(p)}$  is the number of straight faces of length  $2p$ . According to Remark 3.1, the only MO-graphs where the set of dipoles is not equal to the set of faces of length 2 are the two infinity graphs (both having one vertex). Hence we can assume that  $G$  is not equal to these two graphs. Thus  $F_s^{(1)} \leq k$ , which gives:

**Claim 4.2.** *There are at most  $2k$  vertices of  $G$  that are incident to a dipole.*

Also, Lemma 2.1 directly implies

$$\sum_{p \geq 3} (p-2)F_s^{(p)} \leq 2\delta - 2 + k.$$

Since  $p-2 \geq p/3$  for  $p \geq 3$ , the total length  $L = \sum_{p \geq 3} 2pF_s^{(p)}$  of straight faces of length larger than 4 is bounded by  $12\delta - 12 + 6k$ , so that we obtain:

**Claim 4.3.** *There are at most  $12\delta - 12 + 6k$  vertices of  $G$  incident to a straight face of length larger than 4.*

A cycle is called *self-intersecting* if it passes twice by a same vertex.

**Claim 4.4.** *There are at most  $2\delta$  self-intersecting straight faces of length 4 in  $G$ . Hence there are at most  $6\delta$  vertices incident to such faces.*

*Proof.* It is easy to see that the only possibility for a straight face of length 4 to be self-intersecting is the configuration of Figure 20, in which case, there is a loop included in the straight face. The statement then just follows from the fact that  $G$  has at most  $2\delta$  loops, according to Lemma 4.3.  $\square$

Given a non-self-intersecting cycle  $C$  of  $G$ , a *chordal path* for  $C$  is a path that starts from  $C$ , ends at  $C$  (possibly at the same vertex), and is outside of  $C$  in-between. In the context of maps, we call a chordal path *faulty* if it starts on one side of  $C$  and ends on the other side of  $C$ . A cycle  $C$  is called *faulty* if it has a faulty chordal path, and, for  $k \geq 1$ , is called  *$k$ -faulty* if it has a chordal path of length at most  $k$ . For  $C$  a faulty cycle, we denote by  $\widehat{C}$  the union of  $C$  and of a (canonically chosen) shortest possible faulty chordal path of  $C$ ;  $\widehat{C}$  is called the *faulty extension* of  $C$ .

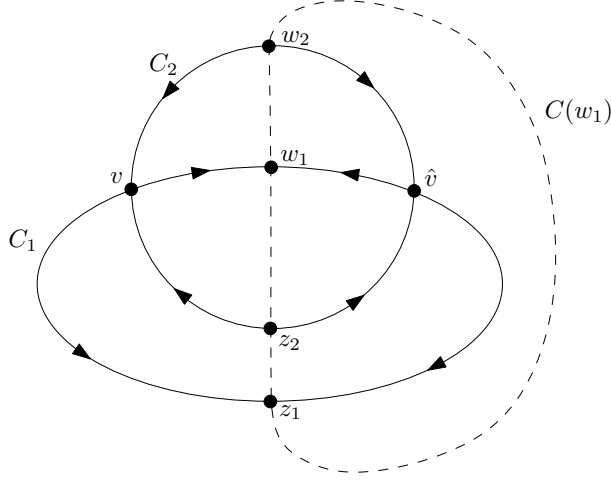


FIGURE 21. The situation in the proof of Lemma 4.2

**Claim 4.5.** *Let  $d := 96768$ . There are at most  $14d\delta$  4-faulty cycles of length 4 in  $G$ . Hence there are at most  $56d\delta$  vertices incident to such cycles.*

*Proof.* For  $C$  a faulty cycle,  $\widehat{C}$  is clearly a topological minor of  $G$  of genus 1. Hence, if we have  $s \geq 1$  vertex-disjoint faulty extensions of faulty cycles, these yield a topological minor of genus  $s$ . Since the genus of  $G$  is bounded by  $2\delta$ , we conclude that there can not be more than  $2\delta$  vertex-disjoint faulty extensions of faulty cycles. In the case of a 4-faulty cycle  $C$  of length 4,  $\widehat{C}$  has at most 7 vertices. An easy calculation ensures that any given vertex  $v$  of  $G$  belongs to at most  $d := 96768$  different faulty extensions of 4-faulty cycles (this can probably be improved, but we just aim at a certain fixed bound). Hence  $\widehat{C}$  can intersect at most  $7d$  other faulty extensions of 4-cycles. We easily conclude that there are at most  $14d\delta$  4-faulty cycles of length 4 (otherwise one could construct incrementally a set of  $2\delta+1$  4-faulty cycles whose faulty extensions are vertex-disjoint).  $\square$

Let  $V_{\text{exc}}$  be the set of vertices of  $G$  that are either incident to a dipole, or incident to a straight face of length larger than 4, or incident to a self-intersecting straight face of length 4, or incident to a 4-faulty cycle of length 4. The four claims above give

$$(4) \quad |V_{\text{exc}}| \leq (18 + 56d)\delta + 8k - 12 =: h.$$

And let  $V_0$  be the set of vertices in  $V_{\text{exc}}$  or adjacent to a vertex in  $V_{\text{exc}}$ . Since  $G$  is 4-regular,  $|V_0| \leq 5h$ . We are now going to show that if  $G$  has a vertex  $v$  not in  $V_0$ , then  $G$  is actually reduced to having only 6 vertices. Since  $v \notin V_{\text{exc}}$ , it is incident to two non-intersecting straight faces  $C_1$  and  $C_2$ , both of length 4. Since  $C_1$  is not faulty, it has to meet  $C_2$  at a vertex different from  $v$  (otherwise  $C_2$  would be a faulty chordal path for  $C_1$ ). Let  $\hat{v}$  be an intersection of  $C_1$  and  $C_2$  different from  $v$ . For  $i \in \{1, 2\}$ , let  $d_i$  be the distance of  $\hat{v}$  from  $v$  on  $C_i$  (note that  $d_i \in \{1, 2\}$ ). If  $d_1 = d_2 = 1$  then either the two corresponding edges  $e_1, e_2$  of  $C_1$  and  $C_2$  form an external dipole, or otherwise  $e_2$  is a faulty chordal edge for  $C_1$ . Both cases are excluded since  $v \notin V_{\text{exc}}$ , hence  $d_1 = d_2 = 2$ . We write the cycle  $C_1$  as  $(v, w_1, \hat{v}, z_1)$

and the cycle  $C_2$  as  $(v, w_2, \hat{v}, z_2)$ , see Figure 21. Let  $C(w_1)$  be the straight face (non-intersecting and of length 4, since  $w_1 \notin V_{\text{exc}}$ ) passing by  $w_1$  and different from  $C_1$ . Since  $w_1 \notin V_{\text{exc}}$ ,  $C(w_1)$  is not faulty, hence it intersects the 4-cycle  $(v, w_1, \hat{v}, w_2)$  at a vertex different from  $w_1$ ; this other intersection must be at  $w_2$  because  $v$  and  $\hat{v}$  have their 4 incident edges already in  $C_1 \cup C_2$ . Similarly  $C(w_1)$  intersects the 4-cycle  $(v, w_1, \hat{v}, z_2)$  at a vertex different from  $w_1$ , and this other intersection must be  $z_2$ ; and  $C(w_1)$  intersects the 4-cycle  $C_1$  at a vertex different from  $w_1$ , and this other intersection must be  $z_1$ , see Figure 21. Hence  $G' = C_1 \cup C_2 \cup C(w_1)$  forms a connected induced subgraph that saturates all its vertices (each of the 6 vertices of  $G'$  is incident to 4 edges of  $G'$ ). Hence  $G' = G$ . In other words, if  $G$  has a vertex outside of  $V_0$ , then  $G$  has 6 vertices.

We can now easily conclude the proof. Define  $n_{k,\delta} := \max(6, 5h)$ , and assume  $G$  has more than  $n_{k,\delta}$  vertices. Since  $|V_0| \leq 5h$ ,  $G$  has a vertex not in  $V_0$ , which implies that  $G$  is reduced to a graph with 6 vertices, giving a contradiction.

## 5. MO-GRAPHS OF DEGREE 1/2

In this section we recover the results of [10] about the structure of MO-graphs of degree 1/2, using ingredients from the preceding sections.

**Proposition 5.1.** *The only unrooted connected melon-free MO-graphs of degree 1/2 are the cw and the ccw infinity graph.*

*Proof.* Assume there is a connected melon-free MO-graph  $G$  of degree 1/2 different from the cw or the ccw infinity graph. According to Remark 3.1 the set of dipoles of  $G$  equals the set of its faces of length 2. Note that in degree 1/2, the genus  $g$  has to be 0 (since  $g$  is a nonnegative integer bounded by  $\delta$ ). Hence Lemma 2.1 gives  $-F_s^{(1)} + \sum_{p \geq 3} (p-2)F_s^{(p)} = -1$ , so that  $F_s^{(1)} \geq 1$ , i.e.,  $G$  has a dipole  $d$  (of type  $S$ ). Since  $\delta < 1$ ,  $d$  has to be separating according to Corollary 4.1. Let  $G_1$  and  $G_2$  be the connected components resulting from the removal of  $d$ . Since  $d$  is separating, the degrees of  $G_1$  and  $G_2$  add up to 1/2, hence one has degree 0 and the other has degree 1/2. By convention let  $G_1$  be the one of degree 0, considered as rooted at the edge resulting from the deletion of  $d$ . There are two cases: (i) if  $G_1$  is the cycle-graph, then  $d$  is part of a melon of  $G$ , giving a contradiction; (ii) if  $G_1$  is not the cycle-graph, then it has a melon (since rooted MO-graphs of degree 0 are melonic), giving again a contradiction.  $\square$

This yields the following bijective result, where we recover [10, Theo. 3.1]:

**Corollary 5.1.** *Every unrooted connected MO-graph of degree 1/2 is uniquely obtained from the cw or the ccw infinity graph where each of the two edges is substituted by a rooted melonic graph.*

*Proof.* All the arguments in the proof of Proposition 3.1 (extraction of a unique melon-free core) can be directly recycled to give the following statement: “for  $\delta \in \frac{1}{2}\mathbb{Z}_+^*$ , every unrooted connected MO-graph of degree  $\delta$  is uniquely obtained as an unrooted connected melon-free MO-graph of degree  $\delta$  where each edge is substituted by a rooted melonic graph”, which —together with Proposition 5.1— yields the result.  $\square$

Another direct consequence of Proposition 5.1 is the following statement that will prove useful in the next section:

**Corollary 5.2.** *Every rooted connected MO-graph of degree  $1/2$  different from the (rooted) cw or ccw infinity graph has a dipole.*

*Proof.* According to Proposition 5.1, the underlying unrooted MO-graph (obtained by erasing the fake-vertex of degree 2) has a melon, hence there is a dipole not including the marked edge; by definition this dipole is also a dipole of the rooted MO-graph (recall that dipoles of rooted MO-graphs are required not to pass by the root).  $\square$

## 6. DOMINANT SCHEMES

A reduced scheme  $S$  of degree  $\delta \in \frac{1}{2}\mathbb{Z}_+$  is called *dominant* if it maximizes (over reduced schemes of degree  $\delta$ ) the number  $b$  of broken chain-vertices (as we will see in Section 7,  $b$  determines the singularity order of the generating function of rooted MO-graphs of degree  $\delta$  and reduced scheme  $S$ , the larger  $b$  the larger the singularity order). In this section we show that, in each degree  $\delta$ , the maximal number of broken chain-vertices is  $4\delta - 1$ , and we precisely determine what are the dominant schemes.

Let  $S$  be a reduced scheme of degree  $\delta$ , and let  $b$  be the number of broken chain-vertices of  $S$ . We now determine a bound for  $b$  in terms of  $\delta$ . The discussion is very similar to the one of Section 4.2, with the difference that only the broken chain-vertices are removed. First, as long as there is a non-separating broken chain-vertex, we remove it (as shown in Figure 17) and color the two resulting edges. Let  $q$  be the number of such removals until all broken chain-vertices are separating; at this stage, denote by  $S'$  the resulting MO-graph with chain-vertices, which has degree  $\delta' = \delta - 3q$  according to Lemma 4.4, and has at most  $2q$  colored edges. Then  $S'$  can be seen as a tree  $T$  of components which are obtained by removing all separating broken chain-vertices. It is also convenient to consider the tree  $\hat{T}$  of components which are obtained by removing all separating chain-vertices and all separating uncolored dipoles —dipoles with no colored edge— of  $S'$  (in  $T$  and  $\hat{T}$  some components of degree 0 might be cycle-graphs). Note that  $\hat{T}$  is a refinement of  $T$ , i.e., each component  $C$  of  $T$  “occupies” a subtree  $T_C$  of  $\hat{T}$ . For a component  $C$  of  $T$  (resp. of  $\hat{T}$ ), the *adjacency order* of  $C$  is defined as the number of adjacent components in  $T$  (resp. in  $\hat{T}$ ). An edge  $e$  of  $C$  resulting from a separating removal for  $T$  (resp.  $\hat{T}$ ) is called a *marked edge*. Such an edge  $e$  results from the merger of edges of  $S'$ . Then  $e$  is called *colored* if at least one of these edges is a colored edge of  $S'$ . And a component  $C$  of  $T$  (resp. of  $\hat{T}$ ) is called *uncolored* if it bears no colored edge (marked or unmarked). Since  $S'$  has at most  $2q$  colored edges,  $T$  (resp.  $\hat{T}$ ) has at most  $2q$  colored components.

Consider an uncolored component  $C$  of  $T$  of degree 0 and not containing the root. Note that  $C$  has no non-separating dipoles nor non-separating chain-vertices, by Corollary 4.1. In addition, by the arguments of Claim 4.1, any component of  $T_C$  must have adjacency order at least 3, otherwise it would yield a melon or a proper chain in  $S$ . This clearly implies that  $C$  can not have adjacency order smaller than 3, and if it has adjacency order 3, then  $T_C$  consists of a unique component.

Let  $n_0$  be the number of components of  $T$  of degree 0, uncolored, and not containing the root. Let  $n'_0$  be the number of other components of degree 0. Let  $n_+$  be the number of components of positive degree. Moreover, let  $p$  be the number of edges of  $T$ , which is also the number of broken chain-vertices of  $S'$ , so that  $b = q + p$ .

The degrees of the components add up to the degree  $\delta' = \delta - 3q$  of  $G'$  (recall that, for separating removals, the degree is distributed among the components), and each component of positive degree has degree at least  $1/2$ , hence

$$n_+/2 \leq \delta - 3q.$$

The components counted by  $n_0$  have at least 3 neighbours in  $T$ , hence

$$3n_0 + n'_0 + n_+ \leq 2p.$$

The graph  $S'$  has at most  $2q$  blue edges, hence

$$n'_0 \leq 2q + 1,$$

where the  $+1$  accounts for the component containing the root. Since  $T$  is a tree, we have

$$p = n_0 + n'_0 + n_+ - 1,$$

hence, eliminating  $n_0$  using  $3n_0 + n'_0 + n_+ \leq 2p$ , we obtain

$$p \leq \frac{2}{3}(p + n'_0 + n_+) - 1 \leq \frac{2}{3}(p + 2q + 1 + 2\delta - 6q) - 1,$$

so that  $p \leq 4\delta - 8q - 1$ . Hence  $b = p + q \leq 4\delta - 7q - 1 \leq 4\delta - 1$ .

If a scheme  $S$  is such that  $b$  reaches the upper bound  $4\delta - 1$ , then it implies that all the above inequalities are tight, hence  $q = 0$  (all broken chain-vertices are separating),  $n'_0 = 1$  (the component containing the root has degree 0),  $3n_0 + n'_0 + n_+ = 2p$  (all the components of positive degree and the component containing the root are leaves of  $T$ , and the other components of degree 0 have 3 neighbours in  $T$ ), and  $n_+ = 2\delta$  (all positive degree components have degree  $1/2$ ).

We take a closer look at the components of degree 0. Let  $C$  be such a component not containing the root; note that  $C$  has no non-separating dipole nor non-separating chain-vertex according to Corollary 4.1. Since  $C$  has adjacency order 3 and is uncolored (because  $q = 0$ ), the discussion above ensures that  $T_C$  has a single component, hence  $C$  has no chain-vertex and any dipole of  $C$  must pass by a marked edge of  $C$ . Recall that an unrooted MO-graph of degree 0 is either the cycle-graph, or can be obtained from the ‘‘quadruple-edge’’ MO-graph (two vertices connected by 4 edges) by successive insertions of melons. Note that, as soon as at least one melon is inserted in the quadruple-edge MO-graph, there are two vertex-disjoint melons. Hence, an MO-graph of degree 0 with at most 3 marked edges and more than 2 vertices has a dipole that avoids the marked edges. It follows that  $C$  has to be either the cycle-graph or the quadruple-edge graph. Now, if  $C$  is the component containing the root, since it has only one adjacent component  $C'$ , it has to be the (rooted) cycle-graph. Indeed, the case of a rooted quadruple-edge MO-graph is excluded; in that case, among the 4 edges of  $C$ , one carries the root, one is marked, and the two other ones form a dipole that, together with the chain-vertex connecting  $C$  to  $C'$ , would form a proper chain of  $S$ , a contradiction.

We now take a closer look at components of positive degrees (of degree  $1/2$ ). Let  $C$  be such a component (which has a unique marked edge since it is at a non-root leaf of  $T$ ). Since  $C$  has degree  $1/2$  it can not have non-separating chain-vertices or non-separating unbroken chain-vertices, according to Corollary 4.1. And by the arguments of Claim 4.1, components of degree 0 of  $T_C$  must have adjacency order at least 3. Since the degrees of components in  $T_C$  add up to  $1/2$ , this easily implies that the unique possibility is  $T_C$  having a single node. Hence  $C$  has no dipole

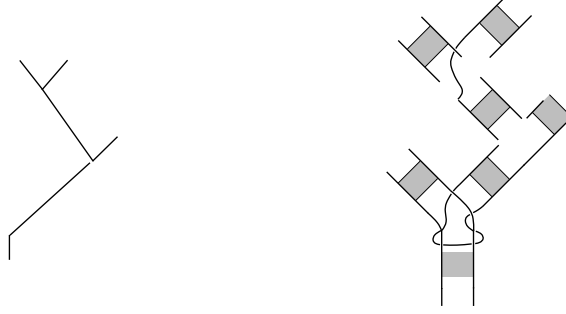


FIGURE 22. (a) A rooted binary tree  $\beta$  with 5 leaves. (b) One of the  $2^{10}$  dominant schemes of degree 2 arising from  $\beta$ .

avoiding the marked edge nor chain-vertices, so that  $C$  must be the cw or the ccw infinity graph, according to Corollary 5.2.

**Proposition 6.1.** *For  $\delta \in \frac{1}{2}\mathbb{Z}_+^*$ , the dominant schemes arise from rooted binary trees (see Figure 22 for an example) with  $2\delta+1$  leaves,  $2\delta-1$  inner nodes, and  $4\delta-1$  edges, where the root-leaf is occupied by the rooted cycle-graph, the  $2\delta$  other leaves are occupied by (cw or ccw) infinity graphs, the  $2\delta-1$  inner nodes are occupied either by the cycle-graph or by the quadruple-edge graph, and the  $4\delta-1$  edges are occupied by separating broken chain-vertices.*

*Each rooted binary tree with  $2\delta+1$  leaves yields  $2^{6\delta-2}$  dominant schemes.*

*Proof.* In the analysis above we have seen that  $b \leq 4\delta-1$  and that any scheme with  $b = 4\delta-1$  must be of the stated form. Conversely any scheme of the stated form is a valid scheme (no melon nor proper chain) of degree  $\delta$  with  $b = 4\delta-1$  chain-vertices. Hence these schemes are exactly the dominant schemes at degree  $\delta$ . Each rooted binary tree with  $2\delta+1$  leaves gives rise to  $4^{2\delta-1}2^{2\delta} = 2^{6\delta-2}$  dominant schemes. Indeed, at each inner node, one has to decide whether it is occupied by the cycle-graph or by the quadruple-edge graph, and in the second case, one has to choose among the three possible configurations for the free edge (the unique edge not involved in any connection) since the straight face passing by the free edge can go either toward the left child, or the right child, or the parent. Finally, one has to decide at each non-root leaf if it is occupied by the cw or the ccw infinity graph.  $\square$

*Remark 6.1.* As we have recalled in Section 2.4, regular colored graphs in dimension 3 naturally form a subfamily of MO-graphs. However, since half-integer degrees are not possible in the model of regular colored graphs, the dominant schemes differ; as shown in [9], in degree  $\delta \in \mathbb{Z}_+$ , the dominant schemes are associated to rooted binary trees with  $\delta+1$  leaves (and  $\delta-1$  inner nodes), where the root-leaf is occupied by a root-melon, while the  $\delta$  non-root leaves are occupied by the unique scheme of degree 1.

## 7. GENERATING FUNCTIONS AND ASYMPTOTIC ENUMERATION

Let  $\delta \in \frac{1}{2}\mathbb{Z}_+$ , and let  $\mathcal{S}_\delta$  be the (finite) set of reduced schemes of degree  $\delta$ . For each  $S \in \mathcal{S}_\delta$ , let  $G_S^{(\delta)}(u)$  be the generating function of rooted melon-free MO-graphs of reduced scheme  $S$ , where  $u$  marks half the number of non-root vertices (i.e., for  $p \in \frac{1}{2}\mathbb{Z}_+$ , a weight  $u^p$  is given to rooted MO graphs with  $2p$  non-root vertices). Let  $p$  be half the number of non-root standard vertices of  $S$ ,  $b$  the number of broken chain-vertices,  $a$  the number of unbroken chain-vertices of type L or R,  $s_e$  the number of even straight chain-vertices, and  $s_o$  the number of odd straight chain-vertices. The generating functions for unbroken chains of type L (resp. R) is clearly  $u^2/(1-u)$ , the one for even straight chains is  $u^2/(1-u^2)$ , the one for odd straight chains is  $u^3/(1-u^2)$ , and the one for broken chains is  $(3u)^2/(1-3u) - 3u^2/(1-u) = 6u^2/((1-3u)(1-u))$ . Therefore

$$G_S^{(\delta)}(u) = u^p \frac{u^{2a}}{(1-u)^a} \frac{u^{2s_e}}{(1-u^2)^{s_e}} \frac{u^{3s_o}}{(1-u^2)^{s_o}} \frac{6^b u^{2b}}{(1-3u)^b (1-u)^b}.$$

Denoting by  $c$  the total number of chain-vertices and by  $s = s_e + s_o$  the total number of straight chain-vertices, this simplifies as

$$(5) \quad G_S^{(\delta)} = \frac{6^b u^{p+2c+s_o}}{(1-u)^{c-s} (1-u^2)^s (1-3u)^b}.$$

It is well-known that the generating function of rooted melonic graphs is given by

$$T(z) = 1 + zT(z)^4.$$

In addition, a rooted melon-free MO-graph with  $2p$  non-root vertices has  $4p + 1$  edges (recall that the root-edge is split into two edges) where one can insert a rooted melonic graph. Therefore, defining  $U(z) := zT(z)^4 = T(z) - 1$ , the generating function  $F_S^{(\delta)}(z)$  of rooted MO-graphs of reduced scheme  $S$  is given by

$$(6) \quad F_S^{(\delta)}(z) = T(z) \frac{6^b U(z)^{p+2c+s_o}}{(1-U(z))^{c-s} (1-U(z)^2)^s (1-3U(z))^b},$$

and the generating function  $F^{(\delta)}(z)$  of rooted MO-graphs of degree  $\delta$  is simply given by

$$(7) \quad F^{(\delta)}(z) = \sum_{S \in \mathcal{S}_\delta} F_S^{(\delta)}(z).$$

As discussed in [9],  $T(z)$  has its main singularity at  $z_0 := 3^3/2^8$ ,  $T(z_0) = 4/3$ , and  $1-3U(z) \sim_{z \rightarrow z_0} 2^{3/2} 3^{-1/2} (1-z/z_0)^{1/2}$ . Therefore, the dominant terms are those for which  $b$  is maximized. As shown in Section 6, these schemes are naturally associated to rooted binary trees with  $2\delta - 1$  inner nodes, as stated in Proposition 6.1. For a fixed rooted binary tree  $\gamma$  with  $2\delta - 1$  inner nodes, the total contribution of schemes arising from  $\gamma$  to the generating function of rooted melon-free MO-graphs of degree  $\delta$  is

$$\frac{(2u^{1/2})^{2\delta} (1+3u)^{2\delta-1} 6^{4\delta-1} u^{8\delta-2}}{(1-u)^{4\delta-1} (1-3u)^{4\delta-1}}.$$

Indeed, such schemes have  $b = 4\delta - 1$ ,  $c = b$ ,  $s = s_o = 0$ , each of the  $2\delta$  non-root leaves is occupied either by the cw or ccw infinity graph, and each of the  $2\delta - 1$  inner nodes is either occupied by the cycle-graph or the quadruple-edge graph that has 3 possible configurations.

Hence the total contribution to the generating function of rooted MO-graphs is

$$\begin{aligned} & T(z) \frac{U(z)^\delta 2^{2\delta} (1 + 3U(z))^{2\delta-1} 6^{4\delta-1} U(z)^{8\delta-2}}{(1 - U(z))^{4\delta-1} (1 - 3U(z))^{4\delta-1}} \\ &= z^\delta T(z)^{1+4\delta} \frac{2^{6\delta-1} 3^{4\delta-1} (1 + 3U(z))^{2\delta-1} U(z)^{8\delta-2}}{(1 - U(z))^{4\delta-1} (1 - 3U(z))^{4\delta-1}}. \end{aligned}$$

This is to be multiplied by the Catalan number  $\text{Cat}_{2\delta-1}$  of rooted binary trees with  $2\delta - 1$  inner nodes. Note that, by design, the dominant schemes contribute only to graphs with  $2n$  non-root vertices such that  $n \in \delta + \mathbb{Z}$ , which is consistent here with the prefactor  $z^\delta$ . Graphs with  $2n$  non-root vertices such that  $n \in \delta + 1/2 + \mathbb{Z}$  have schemes with strictly less than  $4\delta - 1$  broken chain-vertices, hence have a singular behaviour of type  $(1 - z/z_0)^{-2\delta+1+\alpha}$ , with  $\alpha \in \frac{1}{2}\mathbb{Z}_+$ . After elementary calculations and applying transfer theorems of analytic combinatorics [4], we obtain:

**Proposition 7.1.** *For  $\delta$  and  $n$  in  $\frac{1}{2}\mathbb{Z}_+$ , let  $a_n^{(\delta)}$  be the number of rooted MO-graphs with  $2n$  vertices and degree  $\delta$ . Then,  $\delta$  being fixed, for  $n \in \delta + \mathbb{Z}$  (and  $\Gamma(\cdot)$  denoting the Euler gamma function),*

$$(8) \quad a_n^{(\delta)} \sim \text{Cat}_{2\delta-1} \cdot \frac{3^{\delta-3/2}}{2^{2\delta-5/2}} \cdot \frac{n^{2\delta-3/2}}{\Gamma(2\delta-1/2)} \cdot (2^8/3^3)^n \text{ as } n \rightarrow \infty.$$

and  $a_{n+1/2}^\delta = O(a_n^\delta/\sqrt{n})$  as  $n \rightarrow \infty$ .

*Remark 7.1.* According to Proposition 7.1, for  $\delta \in \mathbb{Z}$  and  $n \in \mathbb{Z}$ ,  $a_n^{(\delta)}$  is asymptotically  $\Theta(n^{2\delta-3/2}(2^8/3^3)^n)$ , where the multiplicative constant involves  $\text{Cat}_{2\delta-1}$ . As a comparison, as shown in [9], for  $\delta \in \mathbb{Z}$  the number  $c_n^{(\delta)}$  of rooted colored graphs (in dimension 3) of degree  $\delta$  with  $2n$  vertices is asymptotically  $\Theta(n^{\delta-3/2}(2^8/3^3)^n)$ , where the multiplicative constant involves  $\text{Cat}_{\delta-1}$ . We also deduce from these estimates that for fixed  $\delta \in \mathbb{Z}_+$ , the probability that a random rooted MO-graph of degree  $\delta$  with  $2n$  vertices (where  $n \in \mathbb{Z}_+$ ) is a regular colored graph is  $\Theta(n^{-\delta})$ .

Now, going back to Remark 2.3, we can also easily obtain the asymptotic enumeration under the planarity constraint, based on the following:

**Lemma 7.1.** *Each rooted melon-free MO-graph whose reduced scheme is dominant is planar.*

*Proof.* As shown in Section 6, a rooted melon-free MO-graph  $G$  of degree  $\delta \in \frac{1}{2}\mathbb{Z}$  has a total of  $2\delta$  loops (one at each non-root leaf of the associated binary tree, see Figure 22). According to Lemma 4.3, erasing all these  $2\delta$  loops yields a (rooted) MO-graph  $G'$  of degree 0, hence a planar MO-graph. Clearly the  $2\delta$  erased loops can be inserted back without breaking planarity, from which we conclude that  $G$  is planar.  $\square$

*Remark 7.2.* Given any dominant scheme such as shown in Figure 22 (naturally drawn following the shape of a rooted binary tree), one can easily figure out how to retract/extend some parts of the drawing so as to obtain a planar redrawing, see Figure 23.

**Corollary 7.1.** *For each fixed  $\delta \in \frac{1}{2}\mathbb{Z}_+$  and for  $n \in \delta + \mathbb{Z}$ , the probability that a rooted MO-graph of degree  $\delta$  with  $2n$  vertices is planar tends to 1 as  $n \rightarrow \infty$ .*



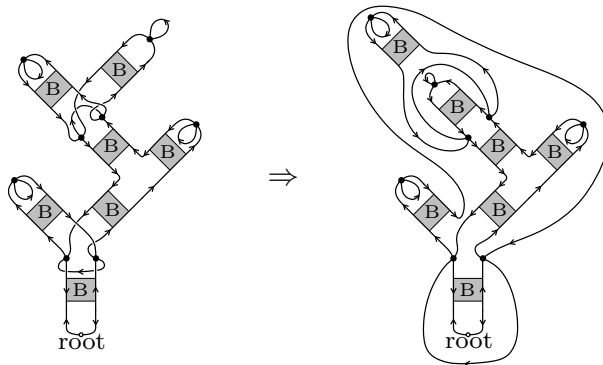


FIGURE 23. Planar redrawing of the scheme of Figure 22.

*Proof.* It just follows from the observation that the edge-substitution by melonic components clearly preserves planarity, so that all rooted MO-graphs arising from a dominating reduced scheme are planar. This concludes the proof since these rooted MO-graphs are the ones that dominate the asymptotic expansion.  $\square$

According to Remark 2.3, in the planar case, rooted MO-graphs correspond (bijectively) to rooted 4-regular maps, the straight faces of the MO-graph identify to the knot-components of the map, and the number of straight faces is equal to  $V/2 + 1 - \delta$ , with  $V$  the number of vertices and  $\delta$  the degree. Under this rephrasing, Proposition 7.1 and Corollary 7.1 yield the following:

**Proposition 7.2.** *For  $n \in \frac{1}{2}\mathbb{Z}_+$  and  $k \in \mathbb{Z}_+$ , let  $b_n^{(k)}$  be the number of rooted 4-regular planar maps with  $2n$  vertices and  $k$  knot-components. Then  $b_n^{(k)} = 0$  for  $k > n + 1$ . In addition, for each fixed  $\delta \in \frac{1}{2}\mathbb{Z}_+$ , and for  $n \in \delta + \mathbb{Z}$ ,  $b_n^{(n+1-\delta)}$  has the same asymptotic estimate as  $a_n^{(\delta)}$ , given by (8).*

## 8. CONCLUDING REMARKS AND PERSPECTIVES

In this article we have shown that, similarly as in the colored model [9], the combinatorial study of MO-graphs can be done by extraction of so-called *schemes*, such that there are *finitely many* schemes in each fixed degree  $\delta \in \frac{1}{2}\mathbb{Z}_+$ . We have also identified the dominant schemes in each degree, whose shapes are naturally associated to rooted binary trees. Having determined the dominant schemes we have obtained an explicit asymptotic estimate for the number of (rooted) MO-graphs of fixed degree as the number of vertices tends to infinity.

As already mentioned in the introduction, a first perspective for future work is the implementation of the double scale limit for the MO tensor model. For the sake of completeness, let us mention that the double scaling mechanism was implemented, for a different type of colored model, in [1], using quantum field theoretical-inspired methods (the so-called intermediate field method). From a probabilistic point a view, a different perspective for future work is to investigate whether the new dominant schemes we have exhibited in this paper correspond or not to phases different of the one of branched polymers (it was recently proved that the melon graphs correspond, from this point of view, to branched polymers

[8]). Another appealing perspective is to extend the MO model (currently only developed in dimension  $D = 3$ ) and results of this paper to higher dimensions. Finally, one can address the issue of proving counting theorems for 3D maps using tensor integral techniques, generalizing the fact that counting theorems for maps can be obtained using matrix integral techniques.

#### ACKNOWLEDGEMENTS

The authors acknowledge Răzvan Gurău and Gilles Schaeffer for very stimulating and instructive discussions. Adrian Tanasa is partially supported by the grants ANR JCJC “CombPhysMat2Tens” and PN 09 37 01 02. Éric Fusy is partially supported by the ANR grant “Cartaplus” 12-JS02-001-01, and the ANR grant “EGOS” 12-JS02-002-01. The authors also acknowledge the Erwin Schrödinger International Institute for the valuable work environment provided to them during the “Combinatorics, Geometry and Physics” programme.

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