Asymptotic enumeration and limit laws for graphs of fixed genus

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Abstract

It is shown that the number of labelled graphs with n vertices that can be embedded in the orientable surface $\mathbb{S}_g$ of genus g grows asymptotically like

$$c^{(g)} n^{5(g-1)/2-1} \gamma^n n!$$

where $c^{(g)} > 0$, and $\gamma \approx 27.23$ is the exponential growth rate of planar graphs. This generalizes the result for the planar case $g = 0$, obtained by Giménez and Noy.

An analogous result for non-orientable surfaces is obtained. In addition, it is proved that several parameters of interest behave asymptotically as in the planar case. It follows, in particular, that a random graph embeddable in $\mathbb{S}_g$ has a unique 2-connected component of linear size with high probability.

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1 Introduction and statement of main results

It has been shown by Giménez and Noy [20] that the number of planar graphs with n labelled vertices grows asymptotically as

$$c \cdot n^{-7/2}\gamma^n n!$$

where $c > 0$ and $\gamma \approx 27.23$ are well defined analytic constants. Since planar graphs are precisely those that can be embedded in the sphere, it is natural to ask about the number of graphs that can be embedded in a given surface.

In what follows, graphs are simple and labelled with $V = \{1, 2, \ldots, n\}$, so that isomorphic graphs are considered different unless they have exactly the same edges. Let $\mathbb{S}_g$ be the orientable surface of genus g, that is, a sphere with g handles, and let $a_n^{(g)}$ be the number of graphs with n vertices embeddable in $\mathbb{S}_g$. A first approximation to the magnitude of these numbers was given by McDiarmid [27], who showed that

$$\lim_{n\to\infty}\left(\frac{a_n^{(g)}}{n!}\right)^{1/n} = \gamma.$$

This establishes the *exponential growth* of the $a_n^{(g)}$, which is the same as for planar graphs and does not depend on the genus.

In this paper we provide a considerable refinement and obtain a sharp estimate, showing how the genus comes into play in the *subexponential growth*. In the next statements, γ is the exponential growth rate of planar graphs, and γ_μ is the exponential growth rate of planar graphs with n vertices and $\lfloor \mu n \rfloor$ edges. Both γ and the function γ_μ are determined analytically in [20].

Theorem 1.1. *For $g \geq 0$, the number $a_n^{(g)}$ of graphs with n vertices that can be embedded in the orientable surface $\mathbb{S}_g$ of genus g satisfies*

$$a_n^{(g)} \sim c^{(g)} n^{5(g-1)/2-1}\gamma^n n! \tag{1}$$

where $c^{(g)}$ is a positive constant and γ is as before.

For $\mu \in (1, 3)$, the number $a_{n,m}^{(g)}$ of graphs with n vertices and $m = \lfloor \mu n \rfloor$ edges that can be embedded in $\mathbb{S}_g$ satisfies

$$a_{n,m}^{(g)} \sim c_\mu^{(g)} n^{5g/2-4} (\gamma_\mu)^n n! \quad \text{when } n \to \infty,$$

where $c_\mu^{(g)}$ is a positive constant and γ_μ is as before.

We also prove an analogous result for non-orientable surfaces. Let $\mathbb{N}_h$ be the non-orientable surface of genus h, that is, a sphere with h crosscaps.

Theorem 1.2. *For $h \geq 1$, the number $b_n^{(h)}$ of graphs with n vertices that can be embedded in the non-orientable surface $\mathbb{N}_h$ of genus h satisfies*

$$b_n^{(h)} \sim \tilde{c}^{(h)} n^{5(h-2)/4-1}\gamma^n n!$$

where $\tilde{c}^{(h)}$ is a positive constant and γ is as before.

For $\mu \in (1,3)$, the number $b^{(h)}_{n,m}$ of graphs with n vertices and $m = \lfloor \mu n \rfloor$ edges that can be embedded in $\mathbb{N}_h$ satisfies

$$b^{(h)}_{n,m} \sim \tilde{c}^{(h)}_{\mu} n^{5h/4-4} (\gamma_\mu)^n n! \quad \textit{when } n \to \infty,$$

where $\tilde{c}^{(h)}_{\mu}$ is a positive constant and γ_μ is as before.

In theory, the constants $c^{(g)}$ can be computed via non-linear recursions. Indeed, our computations relate $c^{(g)}$ to the asymptotic number of maps embedded on the surface $\mathbb{S}_g$, and weighted by their number of vertices, which are shown to obey such recursions in [6]. However, these recursions are so intricate that in practice is not easy to compute even the first few of these numbers. The same is true for the numbers $\tilde{c}^{(h)}$. The case $\mu < 1$ of the above theorems has been treated in the planar case in [19], and gives results of a different nature. Also, it was shown in [20] that $\lim_{\mu \to 3^-} \gamma_\mu = \frac{256}{27}$, which is the exponential growth rate of triangulations embedded on a fixed surface.

There are three main ingredients in our proof. The first one is the theory of map enumeration, started by Tutte in his pioneering work on planar maps, and extended later by Arquès, Bender, Canfield, Gao, Richmond, Wormald, and others to arbitrary surfaces. Our main references in this context are [5], [6] and [9]. In particular, Bender and Canfield [5] showed that the number of rooted maps with n edges embeddable in $\mathbb{S}_g$ grows asymptotically as

$$t_g n^{5(g-1)/2} 12^n,$$

for some constant $t_g > 0$. If we compare it with Estimate (1) in Theorem 1.1, we see that they are very similar. This is no coincidence, since our counting of graphs relies in a fundamental way on the counting of maps (the extra factor of n in the estimate occurs because maps are rooted, and the absence of a factorial term is because they are unlabelled).

The second ingredient is topological graph theory, in particular the concept of face-width, which measures in some sense the local planarity of an embedding of a graph in a surface. According to Whitney's theorem, planar 3-connected graphs have a unique embedding in the sphere, but this is not true for arbitrary surfaces. The key result is that a 3-connected graph with large enough face-width has a unique embedding [25]. It turns out that almost all 3-connected graphs have large face-width and, as a consequence, the asymptotic enumeration of 3-connected *graphs* in a surface can be reduced to the enumeration of 3-connected *maps*. In order to enumerate 3-connected maps of genus g we start from the known enumeration of maps of genus g [5, 6] via associated quadrangulations.

The final step is to go from 3-connected graphs in a surface to 2-connected, connected and finally arbitrary graphs. Again, the face-width plays the main role in this reduction. A result of Robertson and Vitray [28] says that if a connected graph G of genus g has face-width at least two, then G has a unique block of genus g and the remaining blocks are planar. A similar result holds for 2-connected graphs and 3-connected components. As a consequence, the asymptotic enumeration of graphs of genus g can be reduced to the planar case, which was completely solved in [20].

There is a fundamental difference between the planar and non-planar cases. For planar graphs we have at our disposal *exact* counting generating functions, defined through functional and differential equations [20]. The reason is precisely that 3-connected graphs have a unique embedding, and there is a bijection with 3-connected maps, for which we know the exact generating function. For higher surfaces this is not the case and we have to *approximate* the counting series. If

$f(x)$ is the generating function of interest, we find series $f_1(x)$ and $f_2(x)$ that are computable and whose coefficients have the same leading asymptotic estimates, and such that $f_1(x)$ dominates $f(x)$ coefficient-wise from below, and $f_2(x)$ from above. If we can estimate the coefficients of the $f_i(x)$, then we can estimate those of $f(x)$. A key argument in the proofs is the following. If a graph G of genus g has a short non-contractible cycle C, then cutting G along C produces either a graph of genus $g-1$, or two graphs whose genera add up to g. In either case, induction on the genus g shows that there are few such graphs and that the probability of a graph having a short non-contractible cycle tends to zero as the size of the graph grows. This is precisely where approximate counting series come into play.

The paper is structured as follows. Section 2 contains the analytic tools used in the paper. In Section 3 we study the enumeration of a family of maps, near-irreducible quadrangulations, closely related to 3-connected graphs. In order to prove our main results, we need a careful analysis of quadrangulations according to their face-width. The proof of Theorems 1.1 and 1.2 comes in Section 4, again using results on face-width. On the way, we also characterize the asymptotic number of 2-connected graphs of genus g (Theorem 4.6). In Section 5 we apply the machinery we have developed to analyze random graphs of genus g.

After we announced the result of Theorem 1.1 at the conferences *AofA'09* and *Random Graphs and Maps on Surfaces (Institut Henri Poincaré, 2009)*, E. Bender and Z. Gao informed us of their ongoing project to address the same problem, with slightly different methods. They have recently issued a draft of their approach [7]. As far as we can tell, the main difference with our paper will be in the way of handling the singular analysis of bivariate generating functions.

2 Singularity analysis

In our work singular expansions are expressed in terms both of square-roots *and* logarithms (indeed the leading terms in the series counting graphs embeddable on the torus and on the Klein bottle involve logarithms).

We define a set D of ordered pairs of integers as follows:

$$D = \mathbb{Z}^2 \setminus \{(2a, 0)\ ;\ a \in \mathbb{Z}\}.$$

Let $f(x)$ be a series with non-negative coefficients and finite positive radius of convergence ρ. We use throughout the following notations:

$$X = \sqrt{1 - x/\rho}, \quad L = \ln(1 - x/\rho).$$

Monomials of the form $X^a L^b$, with $(a, b) \in D$, are totally ordered as follows (essentially a lexicographic order): $X^a L^b < X^{a'} L^{b'}$ if either $a' < a$ or $\{a' = a, b' > b\}$. An *expansion-series* is a series $h(X, L)$ of the form

$$h(X, L) = X^d \, \widetilde{h}(X, L),$$

where $d \in \mathbb{Z}$, $\widetilde{h}(X, L)$ is analytic at $(0, 0)$, and $\widetilde{h}(X, L)$ is of the form $h_0(X)+h_1(X)L+\ldots+h_c(X)L^c$ (i.e., a polynomial in L) with the h_i's analytic at 0. The *type* of $h(X, L)$ is the largest $(a, b) \in D$ such that $[X^a L^b]h(X, L) \neq 0$.

The series $f(x)$ is said to admit a *singular expansion* of type $(a, b) \in D$ if, in a complex neighbourhood of $\rho > 0$ (except on $x - \rho \in \mathbb{R}_+$):

$$f(x) = h(L, X), \tag{2}$$

where $h(L, X)$ is an expansion-series of type (a, b) (we also say of type $X^a L^b$). The singular expansion is called *strong* if $f(x)$ is analytically continuable to a complex domain of the form $\Omega = \{|x| \leq \rho + \delta\} \backslash \{x - \rho \in \mathbb{R}_+\}$ for some $\delta > 0$. We consider mostly singular expansions of type X^a, with $a \in \mathbb{Z} \backslash (2\mathbb{Z}_{\geq 0})$ —such singular expansion are called *of order* $a/2$— and singular expansions of type $X^{2a}L$, with $a \in \mathbb{Z}_{\geq 0}$ —called *of order* a. In particular, if the order is 0, a singular expansion has L as leading monomial. With this definition, the order α can take values that are either integers or half-integers (nonnegative or negative), which we write as $\alpha \in \frac{1}{2}\mathbb{Z}$.

From a singular expansion of a series $f(x)$ one can obtain automatically an asymptotic estimate for the coefficients $[x^n]f(x)$.

Theorem 2.1 (Transfer theorem [16]). *Let $f(x)$ be a series with non-negative coefficients that admits a strong singular expansion of order α around its radius of convergence $\rho > 0$. Then $f_n = [x^n]f(x)$ satisfies*

$$f_n \sim c\, \rho^{-n} n^{-\alpha-1},$$

where c depends explicitly on the "dominating" coefficient of the expansion-series $h(X, L)$.

Let us now extend the concept to a bivariate series $f(x, u)$ with non-negative coefficients. For $u > 0$, let $\rho(u)$ be the radius of convergence of $x \mapsto f(x, u)$. The function $\rho(u)$ is called the *singularity function* of $f(x, u)$ with respect to x, and each point (x_0, u_0) such that $x_0 = \rho(u_0)$ is called a *singular point* of $f(x, u)$. We use the notations

$$X = \sqrt{1 - x/\rho(u)}, \quad L = \ln(1 - x/\rho(u)), \quad U = u - u_0.$$

A trivariate series $h(X, L, U)$ is called an *expansion series* if it is of the form

$$h(X, L, U) = X^d\, \widetilde{h}(X, L, U),$$

with $d \in \mathbb{Z}$, $\widetilde{h}(X, L, U)$ analytic at $(0, 0, 0)$ and of polynomial dependence in L, that is, $\widetilde{h}(X, L, U)$ is of the form $h_0(X, U) + h_1(X, U)L + \ldots + h_c(X, U)L^c$, with the h_i's analytic at $(0, 0)$. If there is some $(a, b) \in D$ such that $[X^a L^b]h(X, L, U)$ is non-zero at $U = 0$ and $[X^{a'} L^{b'}]h(X, L, U) = 0$ for $(a', b') > (a, b)$, then the expansion series is said to be *of type* (a, b).

Let (x_0, u_0) be a singular point of $f(x, u)$ such that $\rho'(u_0) \neq 0$ and such that $\rho(u)$ is analytically continuable to a complex neighbourhood of u_0. Then $f(x, u)$ is said to admit a *singular expansion* of type $(a, b) \in D$, or of type $X^a L^b$, if in a complex neighbourhood of (x_0, u_0) (except on the set $1 - x/\rho(u) \in \mathbb{R}_{\leq 0}$) we have

$$f(x, u) = h(X, L, U), \tag{3}$$

where $h(X, L, U)$ is an expansion series of type (a, b). The singular expansion is called *strong* if $f(x, u)$ is analytically continuable to a complex domain of the form $\Omega = \{|x| \leq x_0 + \delta, |u| \leq u_0 + \delta\} \backslash \{1 - x/\rho(u) \in \mathbb{R}_{\leq 0}\}$ for some $\delta > 0$. Again we consider mostly singular expansions of type X^a, with $a \in \mathbb{Z} \backslash (2\mathbb{Z}_{\geq 0})$ —called *of order* $a/2$— and singular expansions of type $X^{2a}L$, with $a \in \mathbb{Z}_{\geq 0}$ —called *of order* a. A singular expansion is called *log-free* if h does not involve L, that is, $\partial_L h(X, L, U) = 0$. Singular expansions of order $1/2$ are called *square-root* singular expansions.

At some points we will need the following easy lemma:

Lemma 2.2. *If two series $f(x, u)$, $g(x, u)$ with non-negative coefficients admit singular expansions of orders $\alpha \leq \alpha'$ around (x_0, u_0), then the product-series $h(x, u) = f(x, u) \cdot g(x, u)$ admits a singular expansion at (x_0, u_0),*

- *of order $\alpha+\alpha'$ if $\alpha<0$ and $\alpha'<0$,*
- *of order α if $\alpha<0<\alpha'$.*

Remark. Our definition of singular expansions may seem a bit technical, but it meets the following convenient requirements: they are closed under derivation and integration (see Lemma 2.5 in this section), closed under product, and they include the basic singular functions $\sqrt{1-x/\rho}$ which typically appear in series counting maps, and $\ln(1-x/\rho)$ which appear in series counting unrooted graphs in the torus and in the Klein bottle.

From bivariate singular expansions, it is possible to get asymptotic estimates for the coefficient $f_n=[x^n]f(x,1)$, (just apply Theorem 2.1 to $f(x,1)$), but also to get asymptotic estimates according to two parameters.

Theorem 2.3 (Local limit theorem [16]). *Suppose that $f(x,u)=\sum_{n,m} f_{n,m}x^n u^m$ admits a strong singular expansion of order α around each singular point $(\rho(u_0),u_0)$, and let $\rho(u)$ be the singularity function of $f(x,u)$. Suppose that the function $\mu(u)=-u\rho'(u)/\rho(u)$ is strictly increasing, and define $a=\mu(0^+)$, $b=\lim_{u\to\infty}\mu(u)$. For each $\mu\in(a,b)$, let u_0 be the positive value such that $\mu(u_0)=\mu$, and let $\gamma(\mu)=1/\rho(u_0)$. Then there exists a constant $c(\mu)>0$ such that*

$$f_{n,m}\sim c(\mu)\, n^{-\alpha-3/2}\gamma(\mu)^n \quad \text{when } n\to\infty \text{ and } m=\lfloor \mu n\rfloor.$$

Lemma 2.4 (Exchange of variables, from [14]). *With $f(x,u)$ and $\rho(u)$ as in the previous lemma, assume (x_0,u_0) is a singular point and $\rho'(u_0)\neq 0$.*

If $f(x,u)$ admits a singular expansion of order α around (x_0,u_0) with leading variable x, then $f(x,u)$ also admits a singular expansion of order α around (x_0,u_0) with leading variable u. In addition, if the singular expansion in x is strong, then the singular expansion in u is also strong.

Sketch of proof. Let $R(x)$ be the local inverse of $\rho(u)$ at u_0. Then, by the Weierstrass preparation theorem, there exists $Q(x,u)$ analytic at (x_0,u_0), with $Q(x_0,u_0)\neq 0$, such that $x-\rho(u)=(u-R(x))\cdot Q(x,u)$ in a neighbourhood of (x_0,u_0). Replacing $x-\rho(u)$ by $(u-R(x))Q(x,u)$ and rearranging terms, one obtains a singular expansion with leading variable u; and this singular expansion is strong if the one in x is strong (indeed, by definition, the domain Ω of analytic continuation required by strong expansions is symmetric in x and u). □

By Lemma 2.4, we can omit to mention which is the leading variable in a singular expansion, since the other variable could be chosen as well.

We need the following two technical lemmas to manipulate singular expansions.

Lemma 2.5 (Derivation and integration, from [14]). *Let $f(x,u)$ be a series that admits a singular expansion of order $\alpha\in\frac{1}{2}\mathbb{Z}$ around (x_0,u_0). Then the series $\frac{\partial f}{\partial x}(x,u)$ admits a singular expansion of order $\alpha-1$ around (x_0,u_0) with the same singularity function; and the series $\int_0^x f(s,u)\mathrm{d}s$ admits a singular expansion of order $\alpha+1$ around (x_0,u_0) with the same singularity function.*

Moreover, if the expansion of $f(x,u)$ is strong, then so are the ones of $\frac{\partial f}{\partial x}(x,u)$ and $\int_0^x f(s,u)\mathrm{d}s$.

Proof. We use again the notations $X=\sqrt{1-x/\rho(u)}$, $L=\ln(1-x/\rho(u))$. The main point is that, for $a\in 2\mathbb{Z}_{>0}$, $\partial_x X^a L$ has dominating monomial $X^{a-2}L$, $\partial_x L$ has dominating monomial X^{-2}, and for $a\in\mathbb{Z}\backslash 2\mathbb{Z}_{\geq 0}$, $\partial_x X^a$ has dominating monomial X^{a-2}. We refer the reader to [14] for a detailed proof in the case of a square-root singular expansion, the arguments of which can be extended directly to any order. □

Lemma 2.6 (Critical composition schemes)**.** *Let $M(x,u)$, $H(x,u)$, $C(t,u)$ be non-zero series with non-negative coefficients linked by the equation*

$$M(x,u) = C(H(x,u),u).$$

Assume that $H(x,u)$ has a log-free strong singular expansion of order $3/2$ around (x_0,u_0), and let $\rho(u)$ be the singularity function of $H(x,u)$. Assume also that $C(t,u)$ has a strong singular expansion of order $\alpha \leq 0$ at (t_0,u_0), with $t_0 = H(x_0,u_0)$, and the singularity function $R(u)$ of $C(t,u)$ coincides with $H(\rho(u),u)$ (this means that the composition is critical*).*

Then $M(x,u)$ has a strong singular expansion of order α around (x_0,u_0) with singularity function $\rho(u)$.

Proof. Since $C(t,u)$ has a singular expansion of order α around (t_0,u_0), the series $M(x,u) = C(H(x,u),u)$ has an expansion of the form

$$M(x,u) = \overline{h}(\overline{X},\overline{L},U), \quad \text{with } t = H(x,u),\ T = 1 - \frac{t}{R(u)},\ \overline{X} = \sqrt{T},\ \overline{L} = \ln(T),\ U = u - u_0,$$

with $\overline{h}$ an expansion-series of type $(2\alpha, 0)$ if $\alpha \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}_{\geq 0}$, or of type $(2\alpha, 1)$ if $\alpha \in \mathbb{Z}_{\geq 0}$. Moreover, since $H(x,u)$ has a log-free singular expansion of order $3/2$, we have

$$T = Z \cdot \lambda(X,U), \text{ with } Z = 1 - \frac{x}{\rho(u)},\ X = \sqrt{Z},$$

where $\lambda(X,U)$ is analytic and non-zero in a neighbourhood of $(0,0)$, which implies that $\overline{X} = X \cdot \mu(X,U)$ and $\overline{L} = L \cdot \nu(X,U)$ with $\mu(X,U)$ and $\nu(X,U)$ analytic and non-zero in a neighbourhood of $(0,0)$. Hence, by easy rearrangements, $M(x,u)$ can be written as

$$M(x,u) = h(X,L,U), \text{ with } X = \sqrt{Z},\ L = \ln(Z),\ U = u - u_0,$$

where $h(X,L,U)$ is an expansion-series of the same type as $\overline{h}(X,L,U)$.

It remains to prove that the singular expansion of $M(x,u)$ is strong. For $z \in \mathbb{C}$ and $\epsilon > 0$, denote by $\mathcal{B}(z,\epsilon)$ the open ball of center z and radius ϵ. Since $M(x,u)$ has a singular expansion around (x_0,u_0) and has nonnegative coefficients, $M(x,u)$ is already analytically continuable to a domain of the form $D_1 \cup D_2$, with

$$D_1 = \{(x,u) \in \mathbb{C}^2 \,:\, |x| < x_0,\ |u| < u_0\},$$

and

$$D_2 : \{(x,u) \in \mathbb{C}^2 \,:\, x \in \mathcal{B}(x_0,\epsilon),\ u \in \mathcal{B}(u_0,\epsilon),\ x - \rho(u) \notin \mathbb{R}_+\},$$

for some $\epsilon > 0$. So we just have to show that $M(x,u)$ is analytically continuable to the domain

$$D_3 = \{(x,u) \in \mathbb{C}^2 \,:\, x_0 \leq |x| < x_0 + \delta,\ u_0 \leq |u| < u_0 + \delta\} \backslash (\mathcal{B}(x_0,\epsilon) \times \mathcal{B}(u_0,\epsilon)),$$

for some $\delta > 0$ to be adjusted. Note that, for δ sufficiently small (compared to ϵ), there exists $\phi > 0$ such that, for $(x,u) \in D_3$, the arguments of x and u are in $(\phi, 2\pi - \phi)$, that is, x and u are *away* from the positive real axis. Since $H(x,u)$ has nonnegative coefficients, for δ small enough, we have $|H(x,u)| < t_0 - \eta$ for some $\eta > 0$, where $t_0 = H(x_0,u_0)$). Since $C(t,u)$ has non-negative coefficients, its singularity function $R(u)$ satisfies $R(u) \geq R(u_0 + \delta)$ for $|u| \leq u_0 + \delta$. Reducing again δ so that $R(u_0 + \delta) > t_0 - \eta/2$, we conclude that $|H(x,u)| < R(u)$ for $(x,u) \in D_3$. Hence $C(t,u)$ is analytic at $(H(x,u),u)$, and so $M(x,u)$ is analytic at any $(x,u) \in D_3$. □

We need some further definitions. Given two series f_1 and f_2 in the same number of variables, write $f_1 \preceq f_2$ if $f_2 - f_1$ has non-negative coefficients. A bivariate series $f(x,u)$ is said to be of *approximate-singular order* α at (x_0, u_0) if there are two series $D(x,u)$ and $E(x,u)$ such that

$$D(x,u) - E(x,u) \preceq f(x,u) \preceq D(x,u),$$

where $D(x,u)$ has a singular expansion of order α and $E(x,u)$ has a singular expansion of order $\beta > \alpha$ around (x_0, u_0), and $D(x,u)$ and $E(x,u)$ have the same singularity function. This common singularity function is called the *singularity function* of $f(x,u)$. The function $f(x,u)$ is said to be of *strong* approximate-singular order α if the singular expansions of $D(x,u)$ and $E(x,u)$ are strong.

Since the singular order of $D(x,u)$ dominates the singular order of $E(x,u)$, the asymptotic estimate of the coefficients of $f(x,u)$ is dictated by the estimate of $D(x,u)$, and we have the following basic result.

Corollary 2.7. *Let $f(x,u)$ be a series of strong approximate-singular order $\alpha \in \frac{1}{2}\mathbb{Z}$ around each singular point of $f(x,u)$, and let $\rho(u)$ be the singularity function of $f(x,u)$. Then $f_n = [x^n]f(x,1)$ satisfies*

$$f_n \sim c\, n^{-\alpha-1}\gamma^n,$$

where $c > 0$ is a constant and $\gamma = 1/\rho(1)$.

Assume that $\mu(u) = -u\rho'(u)/\rho(u)$ is strictly increasing, and define $a = \mu(0^+)$, $b = \lim_{u\to\infty}\mu(u)$. For each $\mu \in (a,b)$, let u_0 be the positive value such that $\mu(u_0) = \mu$, and let $\gamma(\mu) = 1/\rho(u_0)$. Then there exists a constant $c(\mu) > 0$ such that

$$f_{n,\lfloor \mu n \rfloor} \sim c(\mu)\, n^{-\alpha-3/2}\gamma(\mu)^n \quad \text{when } n \to \infty.$$

We also have a corollary regarding approximate-singular orders of critical compositions. The following result follows directly from Lemma 2.6, but notice that the statement is not the same, since we are now dealing with approximate-singular orders (which is precisely what is needed later on).

Corollary 2.8. *Let $M(x,u)$, $H(x,u)$, $C(t,u)$ be non-zero series with non-negative coefficients linked by the equation*

$$M(x,u) = C(H(x,u),u).$$

Assume that $H(x,u)$ has a log-free strong singular expansion of order $3/2$ around (x_0,u_0) —call $\rho(u)$ the singularity function of $H(x,u)$—, $C(t,u)$ is of strong approximate-singular order α at (t_0,u_0), with $t_0 = H(x_0,u_0)$, and the singularity function $R(u)$ of $C(t,u)$ coincides with $H(\rho(u),u)$ (the composition is critical*).*

Then $M(x,u)$ is of strong approximate-singular order α around (x_0,u_0) with singularity function $\rho(u)$.

3 Maps of genus g via quadrangulations

We recall that a map is a graph G embedded in a surface S in such a way that the faces are homeomorphic to topological disks. If $S = \mathbb{S}_g$ we say that the map is of genus g. We have a similar definition for the non-orientable surface $\mathbb{N}_h$ of genus h. Loops and multiple edges are allowed. A map is *rooted* if it has a marked edge that is given a direction. If $g = 0$ (planar maps), the face on the right of the root is called the *outer face*, and the other faces are called *inner faces*. Vertices and edges incident to the outer face are said to be *outer* vertices (edges, resp.), the other ones are called *inner* vertices (edges, resp.).

In this section we obtain the asymptotic enumeration of a family of maps $\mathcal{S}_g$ that is closely related to 3-connected graphs of genus g. To enumerate $\mathcal{S}_g$ we follow the scheme of Bender et al. [9], and use a classical correspondence between maps and quadrangulations (maps in which every face has degree four). We decompose quadrangulations successively at contractible 2-cycles and then at contractible 4-cycles. Along this way we encounter near-simple and near-irreducible quadrangulations.

However, in our treatment, quadrangulations are labelled at edges and are allowed to be unrooted (classically, in map enumeration maps are unlabelled and rooted). Since each rooted quadrangulation with $2n$ edges corresponds to exactly $(2n-1)!$ labelled quadrangulations, the ordinary generating function of rooted objects is given by the derivative of the exponential generating function of the labelled ones. We believe this approach yields simplified decompositions and equations. Note that the corresponding generating functions have to be exponential to take into account the labelling of the edges. Subsections 3.1 and 3.2 contain the basic concepts and definitions. In Subsection 3.3 we obtain singular expansions that are essential for the enumeration of graphs in Section 4.

3.1 Maps and quadrangulations, face-width and edge-width

A map is *bipartite* if there is a partition of the vertices into black and white vertices such that each edge connects a black vertex with a white vertex. Whenever we work with bipartite maps we consider the partition as given. A quadrangulation is a map such that all faces have degree four. In this article we consider only bipartite quadrangulations. (Note that in positive genus not all quadrangulations are bipartite; for instance, a toroidal grid $C_p \times C_q$ is a quadrangulation but it has odd cycles if either p or q are odd.) Here, by a quadrangulation we always mean a *bipartite* quadrangulation.

We recall here a well-known correspondence, for any fixed genus $g \geq 0$, between maps and quadrangulations. Given a map M, whose vertices are depicted as black, consider the following operations:

- insert a white vertex in each face of M,
- for each corner θ of M, with v the incident vertex and f the incident face of θ, insert an edge that connects v to the white vertex corresponding to f. Such an edge is called a *corner-edge*.

The embedded graph Q consisting of all vertices (black and white) and of all corner-edges is a (bipartite) quadrangulation, which is called the *quadrangulation* of M. A correspondence between the parameters of M and the parameters of Q is listed in the following table:

M	Q
corner	edge
edge	face
vertex	black vertex
face	white vertex

Let us now define the face-width and the edge-width of a map M in a surface S of genus $g \geq 0$. The *face-width* $\mathrm{fw}(M)$ of M is the minimum number of intersections of M with a simple non-contractible curve C on S. It is easy to see that this minimum is achieved when C meets M only at vertices. The *edge-width* $\mathrm{ew}(M)$ of M is the minimum length of a non-contractible cycle of M. (For genus 0, the edge-width and the face-width are defined to be $+\infty$ since every curve is contractible.) Let Q be the quadrangulation of M. Since a closed curve such that $|C \cap M| = \ell$ yields a non-contractible cycle of Q of length 2ℓ and vice versa, we have (see [25])

$$\mathrm{ew}(Q) = 2\,\mathrm{fw}(M). \tag{4}$$

3.2 Families of quadrangulations and links between them

Again, all quadrangulations are assumed to be bipartite. In addition, a quadrangulation is assumed to have its $2m$ edges labelled $\{1, 2, \ldots, 2m\}$. This is not the standard procedure with maps but, as explained earlier, in our context it is easier to work with unrooted maps, and in this case we must label the edges in order to avoid caring about automorphisms.

A quadrangulation is called *near-simple* if it has no contractible 2-cycle, and is called *simple* if it has no 2-cycle at all, that is, no multiple edges. A quadrangulation is called *near-irreducible* if it is simple and every contractible 4-cycle delimits a face, and is called *irreducible* if it is simple and every 4-cycle delimits a face. Note that a near-simple quadrangulation Q is simple if and only if $\mathrm{ew}(Q) > 2$, and a near-irreducible quadrangulation Q is irreducible if and only if $\mathrm{ew}(Q) > 4$. For $g \geq 0$, the families of (bipartite) quadrangulations, near-simple quadrangulations, and near-irreducible quadrangulations of genus g are denoted, respectively, by $\mathcal{Q}_g$, $\mathcal{R}_g$ and $\mathcal{S}_g$. If $\mathcal{F}$ is a family of quadrangulations with labelled edges, let $\mathcal{F}[n, m]$ be the set of elements in $\mathcal{F}$ with n black vertices and m faces, and let

$$F(x, u) = \sum_{n,m} \big|\mathcal{F}[n, m]\big|\, x^n \frac{u^m}{(2m)!}.$$

In other words, $F(x, u)$ enumerates the family $\mathcal{F}$ according to black vertices and faces, and the factor $(2m)!$ in the denominator takes account of the fact that the $2m$ edges are labelled. We also define $F'(x, u)$ as the series $\partial_u F(x, u)$, that is, differentiation with respect to the second variable.

Let us recall how one extracts, for $g > 0$, a near-simple core $R \in \mathcal{R}_g$ from $Q \in \mathcal{Q}_g$, and a near-irreducible core $S \in \mathcal{S}_g$ from $R \in \mathcal{R}_g$, as described in [9]. The contractible 2-cycles of $Q \in \mathcal{Q}_g$ are ordered by inclusion, that is, given two 2-cycles c and c', one has $c \leq c'$ if the disc inside c is included in the disc inside c'. The *near-simple core* R of Q is obtained by collapsing all maximal contractible 2-cycles into edges. Conversely, every $Q \in \mathcal{Q}_g$ is obtained from some $R \in \mathcal{R}_g$, where each edge is possibly “opened” into a 2-cycle that is to be filled with a (rooted) planar quadrangulation. Therefore, for $g > 0$ we have

$$Q_g(x, u) = R_g(x, V(x, u)), \quad \text{where } V(x, u) = u\big(1 + \tfrac{2u}{x} Q_0'(x, u)\big)^2. \tag{5}$$

Similarly, the contractible 4-cycles of $R \in \mathcal{R}_g$ are partially ordered according to the analogous inclusion relation. The *near-irreducible core* S of $R \in \mathcal{R}_g$ is obtained by emptying the disc within each maximal contractible 4-cycle, that is, the disc inside a maximal 4-cycle is replaced by a face. Conversely, every $R \in \mathcal{R}_g$ is obtained from some $S \in \mathcal{S}_g$, where one might insert in each face a planar simple quadrangulation (the "degenerate" simple planar quadrangulation with only 2 edges is not allowed for insertion). Therefore, for $g > 0$ we have

$$R_g(x, v) = S_g(x, W(x, v)), \quad \text{where } W(x, v) = \frac{1}{x^2 v}(2vR_0'(x, v) - xv - x^2 v). \tag{6}$$

3.3 Singular expansions

Throughout this section, we associate to genus g the value

$$\alpha = \tfrac{5}{2}(1 - g). \tag{7}$$

Rooted maps. For $g \geq 0$, let $\overrightarrow{M}_g(x, u)$ be the series counting rooted maps of genus g where x marks vertices and u marks edges. According to the correspondence between maps and quadrangulations, we have

$$2uQ_g'(x, u) = \overrightarrow{M}_g(x, u). \tag{8}$$

Let $p = p(x, u)$ and $q = q(x, u)$ be the algebraic bivariate series defined by the system

$$\left\{ \begin{array}{rcl} p & = & xu + 2pq + p^2, \\ q & = & u + 2pq + q^2, \end{array} \right. \tag{9}$$

and let $\Delta = \Delta(p, q) = (1 - 2p - 2q)^2 - 4pq$ be the Jacobian determinant of the system. Singular points of $p(x, u)$ and $q(x, u)$ are the same: for $x_0 > 0$ the corresponding singular point (x_0, u_0) is such that u_0 is the smallest positive u satisfying $\Delta(p, q) = 0$ where $p = p(x_0, u)$ and $q = q(x_0, u)$. Such a pair (x_0, u_0) is called a *singular point* of (9).

Define a bivariate series $f(x, u)$ to be (p, q)-*rational* if it admits a rational expression in terms of $p(x, u)$ and $q(x, u)$. Note that the elementary functions $(x, u) \mapsto x$ and $(x, u) \mapsto u$ are (p, q)-rational, since $u = q - q^2 - 2pq$ and $x = (p - 2pq - p^2)/(q - 2pq - q^2)$. Remarkably, the series counting rooted maps in any genus are (p, q)-rational. Precisely, Arquès has shown in [1] that

$$\overrightarrow{M}_0(x, u) = \frac{1}{xu^2} p\, q\, (1 - 2p - 2q), \tag{10}$$

a result obtained formerly by Tutte under an equivalent parametrization [30]. In addition, Arquès and Giorgetti [2] have proved that, for $g > 0$, there is a bivariate polynomial $P_g(X, Y)$ such that

$$\overrightarrow{M}_g(x, u) = \frac{P_g(p, q)}{\Delta^{5g-3}}. \tag{11}$$

Lemma 3.1. *For every singular point (x_0, u_0) of (9), the polynomial P_g in (11) is non-zero at (p_0, q_0), where $p_0 = p(x_0, u_0)$ and $q_0 = q(x_0, u_0)$.*

Proof. Bender and Canfield showed in [5] that, for $g \geq 0$, $u \mapsto \overrightarrow{Q}_g(1, u)$ admits a singular expansion of order $\alpha - 1$ around $1/12$. In [6, proof of Theorem 2], working with similar algebraic expressions as (11) they provide a sketchy justification that all arguments in [5] can be extended to the bivariate case. In our terminology it means that, for every singular point (x_0, u_0) of (9), the series $u \mapsto \overrightarrow{M}_g(x_0, u)$ admits a singular expansion of order $\alpha - 1$ around u_0. Note that, using for instance the main theorem in [13], one easily shows that

$$\Delta(p(x_0, u), q(x_0, u)) = \sqrt{U} \cdot h\big(\sqrt{U}\big), \text{ with } U = \sqrt{1 - u/u_0},$$

and with h analytic and non-zero at 0. Since $\overrightarrow{M}_g(x_0,u) = P_g(p(x_0,u),q(x_0,u))/\Delta^{5g-3}$ is of order $\alpha - 1 = -\frac{1}{2}(5g-3)$ at u_0 and since $P_g(p(x_0,u),q(x_0,u))$ converges to $P_g(p_0,q_0)$, we conclude that $P_g(p_0,q_0) \neq 0$. □

Near-irreducible quadrangulations. Composing Equation (5) with Equation (6), we obtain the following link between Q_g and S_g:

$$Q_g(x,u) = S_g(x,w), \quad \text{with } w = H(x,u) = W(x,V(x,u)). \tag{12}$$

Differentiating this equation with respect to u and multiplying by $2u$, we obtain

$$\overrightarrow{M}_g(x,u) = 2uH'(x,u)\cdot S_g'(x,w), \quad \text{where } w = H(x,u). \tag{13}$$

The series $H(x,u)$ has a combinatorial interpretation: it enumerates rooted planar quadrangulations (the outer face is the one to the right of the root edge) where the four outer vertices are distinct, and where the root-edge and the opposite outer edge are simple. These are only minor constraints on quadrangulations; and by elementary decompositions and calculations (see for instance [26] for similar calculations, and more recently [11, Sect.7]), $H(x,u)$ is (p,q)-rational. So is $2uH'(x,u)$ because (p,q)-rationality is stable under derivation, see [17]; call $A(p,q)$ the algebraic expression such that $2uH'(x,u) = A(p(x,u),q(x,u))$. Note also the singular points of $2uH'(x,u)$ are the same as the singular points of (p,q). (Indeed, $2uH'(x,u)$ has the same singular points as $H(x,u)$. The series $H(x,u)$ itself is dominated coefficientwise by the series counting rooted planar quadrangulations, whose singular points are those of (p,q), so the singular points of $H(x,u)$ can not occur before those of (p,q).)

Define $r = r(x,w)$ and $s = s(x,w)$ as the algebraic series defined by the system

$$\begin{cases} r &= xw(1+s)^2, \\ s &= w(1+r)^2. \end{cases} \tag{14}$$

As shown in [10], for $x_0 > 0$ the radius of convergence of $w \mapsto r(x_0,w)$ (and also of $w \mapsto s(x_0,w)$) is the value $w_0 > 0$ such that

$$1 + r + s - 3rs = 0. \tag{15}$$

Such a solution (x_0,w_0) is called a *singular point* of (14). Hence singular points of (14) are the singular points of $r(x,w)$ (and also of $s(x,w)$).

Lemma 3.2. *For x_0, let u_0 and w_0 be such that (x_0,u_0) is a singular point of (9) and (x_0,w_0) is a singular point of (14). Then $w_0 = H(x_0,u_0)$, i.e., the singular points of (9) are mapped to singular points of (14) by the change of variable $(x,u) \mapsto (x,H(x,u))$.*

In addition, since $u \mapsto H(x_0,u)$ is strictly increasing (by positivity of the coefficients), it maps the interval $(0,u_0)$ to the interval $(0,w_0)$.

Proof. From manipulations of algebraic series (see for instance [23, Sect. 2.9] for the univariate case and more recently [11, Sect. 7] for the bivariate case), one shows that, when (x,w) and (x,u) are related by $w = H(x,u)$, p and q are expressed as

$$p = \frac{(1+r)r}{rs+3s+2s^2+3r+2r^2+1}, \quad q = \frac{(s+1)s}{rs+3s+2s^2+3r+2r^2+1}, \tag{16}$$

which implies that

$$\Delta(p,q) = \widetilde{\Delta}(r,s) := \frac{(1+r)(1+s)(1+r+s-3rs)}{(1+3r+3s+2r^2+2s^2+rs)^2}.$$

Let $(x_0,u_0) \in \mathbb{R}_+^2$ be a singular point of (9) (a solution of $\Delta(p,q)=0$), and let $w_0 = H(x_0,u_0)$. Since r and s are positive at (x_0,w_0), $\Delta(p,q) = 0 = \widetilde{\Delta}(r,s)$ implies $1+r+s-3rs=0$. Since $u \mapsto H(x_0,u)$ is strictly increasing by positivity of the coefficients, any $w < w_0$ corresponds to some $u < u_0$. Since (x_0,u_0) is a singular point of (9), $\Delta(p(x_0,u),q(x_0,u)) > 0$. So $\widetilde{\Delta}(r(x_0,w),s(x_0,w)) > 0$. Hence (x_0,w_0) is a singular point of (14). □

Next we get an expression for $S_g{}'(x,w)$. Substituting expression (11) of $\overrightarrow{M}_g(x,u)$ into (13) and replacing p and q by their expressions (16), we obtain for $S_g{}'(x,w)$ a rational expression in terms of (r,s), of the form

$$S_g{}'(x,w) = \frac{1}{\widetilde{A}(r,s)} \frac{\widetilde{P}_g(r,s)}{(\widetilde{\Delta}(r,s))^{5g-3}}, \tag{17}$$

where $\widetilde{A}(r,s)$ ($\widetilde{P}_g(r,s)$, resp.) equals $A(p,q)$ ($P_g(p,q)$, resp.) after substitution of p and q by their expression (16) in terms of r and s. (Note that $\widetilde{P}_g(r,s)$ is a rational expression, the denominator being a power of $(1+3r+3s+2r^2+2s^2+rs)$.)

Lemma 3.3. *For $g \geq 0$, the series $S_g{}'(x,w)$ admits a singular expansion of order $\alpha - 1$ around every singular point (x_0,w_0) of (14).*

Proof. Using again the main theorem in [13], one shows that r and s have square-root singular expansions around (x_0,w_0). Moreover, denoting by $R(x)$ the singularity function of $r(x,w)$ with respect to w, one easily checks, using the computer algebra system Maple, that $\widetilde{\Delta}(r,s)$ has around (x_0,w_0) an expansion of the form

$$\widetilde{\Delta}(r(x,w),s(x,w)) = W \cdot h(X,W), \text{ with } W = \sqrt{1-w/R(x)},\ X = x - x_0,$$

and where $h(X,W)$ is analytic at $(0,0)$ and $h(0,0) \neq 0$. Note also that the expressions of p and q in terms of r and s are positive; and $\widetilde{A}(r(x_0,w),s(x_0,w)) = 2uH'(x_0,u)$ (with w and u related by $w = H(x,u)$) is positive for $w \in (0,w_0)$ and we claim that it converges at w_0 (indeed $H(x,u)$ is dominated coefficientwise by the series counting rooted planar quadrangulations, satisfying the "universal" singular order $3/2$ at every singular point, so $2uH'(x,u)$ is dominated coefficientwise by a series of singular order $1/2$, which thus converges at its singular points). Hence, looking at Expression (17) of $S_g{}'(x,w)$ and using the fact that $\widetilde{P}_g(r(x_0,w_0),s(x_0,w_0)) = P_g(p(x_0,u_0),q(x_0,u_0)) \neq 0$, we conclude that the radius of convergence of $w \mapsto S_g{}'(x_0,w)$ must be w_0, and $S_g{}'(x,w)$ must have a singular expansion of order $(3-5g)/2$. □

Applying Lemma 2.5 (integration) to the series $S_g{}'(x,w)$, we obtain the following.

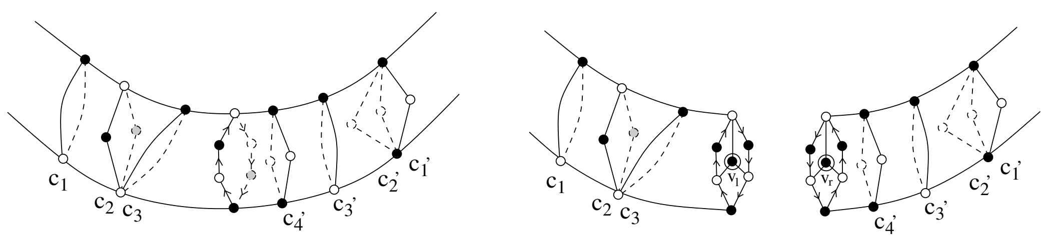
Lemma 3.4. *For $g \geq 0$, the series $S_g(x,w)$ enumerating near-irreducible quadrangulations of genus g admits a singular expansion of order α around every singular point of (14).*

Near-irreducible quadrangulations with fixed edge-width. For $g \geq 0$ and $i \geq 1$, let $\mathcal{S}_g^{\mathrm{ew}=2i}$ be the family of near-irreducible quadrangulations of genus g with edge-width $2i$. Define also $\mathcal{S}_g^{C=2i}$ as the family of quadrangulations from $\mathcal{S}_g$ that have a marked non-contractible cycle C of length $2i$ that is additionally given a direction and carries a marked white vertex, called *starting vertex* of C. Denote by $S_g^{\mathrm{ew}=2i}(x,w)$ and $S_g^{C=2i}(x,w)$ the series of $\mathcal{S}_g^{\mathrm{ew}=2i}$ and $\mathcal{S}_g^{C=2i}$. Note that $\mathcal{S}_g^{C=2i}$ is a superfamily of $\mathcal{S}_g^{\mathrm{ew}=2i}$, so that $S_g^{\mathrm{ew}=2i}(x,w) \preceq S_g^{C=2i}(x,w)$.

Given $S \in \mathcal{S}_g^{C=2i}$, let $L = \Phi(S)$ be obtained as follows:

- Cut the surface $\mathbb{S}_g$ along C (whether C is surface-separating or not, two cases); this yields two cylindric ends, and a *special face* of degree $2i$ appears at the boundary of each of the two cylindric ends; see Figure 1 (orienting C is needed to distinguish the cylindric ends; there is one on the left of C and one on the right of C).

- Add a marked black vertex, called a *special vertex*, inside each special face, and connect this vertex to all white vertices on the contour of the face, see Figure 1(b). The special vertex in the left (right, resp.) cylindric end is denoted v_l (v_r, resp.). Mark the two edges (v, v_l) and (v, v_r), where v is the starting vertex of C.



(a) (b)

Figure 1: Cutting along a non-contractible cycle cuts a handle and creates two cylindrical ends.

Let $\mathcal{L} = \Phi(\mathcal{S}_g^{C=2i})$ be the family of well-labelled quadrangulations (i.e., with the m edges relabelled so as to have distinct labels in $[1..m]$) obtained in this way, and let $L(x,w)$ be the series of $\mathcal{L}$. Note that

$$x^{i+2}u^{2i} \cdot S_g^{C=2i}(x,w) = L(x,w). \tag{18}$$

An object $L \in \mathcal{L}$ is of two possible types:

- Either L is disconnected (the original cycle C is surface-separating), in which case L has two connected components L_1 and L_2, each component carrying one special cycle (and one special vertex); in addition, the sum of genera of L_1 and L_2 is g;

- or L is connected of genus $g-1$.

We denote by $\mathcal{L}^{\mathrm{spl}}$ the subfamily of objects of $\mathcal{L}$ of the first type, and by $\mathcal{L}^{\mathrm{con}}$ the subfamily of objects of $\mathcal{L}$ of the second type.

Lemma 3.5. *For $g > 0$, let as before $\alpha = 5(1-g)/2$. Then $S_g^{C=2i}(x,w)$ is bounded coefficientwise by a series of singular order $\alpha + 1/2$ around every singular point (x_0, w_0) of (14).*

Proof. By Equation (18), it is enough to prove the result for the series $L(x, w)$ instead of working with $S_g^{C=2i}(x, w)$. Let $S \in \mathcal{S}_g^{C=2i}$, with $L = \Phi(S)$. Let $\sigma_S = (c_1, \ldots, c_\ell, c'_r, \ldots, c'_1)$ be the possibly empty sequence of non-contractible 2- or 4-cycles homotopic to the marked cycle C of S, where $c_1, \ldots, c_\ell$ are to the left of C and $c'_r, \ldots, c'_1$ are to the right of C; see Figure 1. Here we need to handle with a special care the case of the torus: for $g = 1$, the nested cycles form only one sequence (arranged along a unique cylinder), and by convention we decide that all these cycles are on the left part and form the sequence $(c_1, \ldots, c_\ell)$, while the right-sequence $(c'_r, \ldots, c'_1)$ is empty.

When cutting along C (cutting a handle), σ_S is cut into two (possibly empty) sequences $\sigma = (c_1, \ldots, c_\ell)$ and $\sigma' = (c'_1, \ldots, c'_r)$ of contractible 2- or 4-cycles of L, one at each cylindric end. Note that all cycles of σ and of σ' must enclose the special vertex at the corresponding cylindric end. Call $\mathcal{S}^{\|}$ the family of rooted planar bipartite maps —the root is an outer edge directed so as to have the outer face on its right and a black origin— with a secondary marked vertex v, an outer face of degree 2, all inner faces of degree 4, and such that each 2-cycle and each non-facial 4-cycle encloses v in its interior. Define similarly $\mathcal{S}^{\diamondsuit}$, but now with the condition that the outer face is of degree 4 with 4 distinct vertices.

Let $S^{\|}(x, w)$ and $S^{\diamondsuit}(x, w)$ be the series enumerating $\mathcal{S}^{\|}$ and $\mathcal{S}^{\diamondsuit}$, respectively, according to black inner vertices and inner faces. The operation of cutting the surface along the first cycle of σ (if σ is not empty) and the first cycle of σ' (if σ' is not empty) yields

$$L(x, w) \preceq K(x, w) \cdot (S^{\|}(x, w) + S^{\diamondsuit}(x, w) + 1)^2, \tag{19}$$

where

$$K(x, w) := S_{g-1}''(x, w) + \sum_{\substack{g_1+g_2=g \\ g_1>0, g_2>0}} S_{g_1}'(x, w) S_{g_2}'(x, w).$$

Indeed, each of the two sequences σ, σ', if not empty, yields a piece either in $\mathcal{S}^{\|}$ (say, if σ starts with a 2-cycle) or in $\mathcal{S}^{\diamondsuit}$ (if σ starts with a 4-cycle) at the corresponding cylindric end. Now let us obtain a singular expansion for $S^{\diamondsuit}(x, w)$ (the method for $S^{\|}(x, w)$ is similar)[1]:

Claim A. The series $S^{\diamondsuit}(x, w)$ admits a singular expansion of positive half-integer order around every singular point (x_0, w_0) of (14).

Proof of the claim. Let $\mathcal{R}^{\diamondsuit}$ be the family of rooted quadrangulations defined as $\mathcal{S}^{\diamondsuit}$, except that any type of non-facial 4-cycle is allowed; and let $\mathcal{Q}^{\diamondsuit}$ be the family of rooted quadrangulations with 4 distinct outer vertices and a secondary marked vertex v_0. One extracts from $Q \in \mathcal{Q}^{\diamondsuit}$ a core $R \in \mathcal{R}^{\diamondsuit}$ by collapsing all 2-cycles not strictly enclosing v_0. Hence, similarly as in Equation (5), the series $Q^{\diamondsuit}(x, v)$ and $R^{\diamondsuit}(x, u)$ enumerating $\mathcal{R}^{\diamondsuit}$ and $\mathcal{Q}^{\diamondsuit}$ by black inner vertices and inner faces are related by

$$Q^{\diamondsuit}(x, u) = \frac{V(x, u)}{u} R^{\diamondsuit}(x, V(x, u)),$$

where $V(x, u)$ is defined in (5). Similarly one extracts from $R \in \mathcal{R}^{\diamondsuit}$ a core $S \in \mathcal{S}^{\diamondsuit}$ by emptying all 4-cycles that are not strictly enclosing v_0. Hence, similarly as in Equation (6), the series $R^{\diamondsuit}(x, v)$ and $S^{\diamondsuit}(x, w)$ are related by

$$R^{\diamondsuit}(x, v) = S^{\diamondsuit}(x, W(x, v)),$$

[1] In the univariate case, Bender et al. [9] count $\mathcal{S}^{\|}$ and $\mathcal{S}^{\diamondsuit}$ by characterizing the pieces between two successive cycles of σ. Instead, our approach to count these classes relies on a composition scheme.

where $W(x,v)$ is defined in (6). Hence we obtain, with $H(x,u) = W(x, V(x,u))$,

$$\frac{u}{V(x,u)} Q^{\diamondsuit}(x,u) = S^{\diamondsuit}(x,w), \quad \text{where } w = H(x,u). \tag{20}$$

One easily shows (from elementary manipulations similar as in [17]) that $Q^{\diamondsuit}(x,u)$ and $V(x,u)$ are expressed rationally in terms of the series p and q defined in (9), and that the singular order of $Q^{\diamondsuit}(x,u)$ is 1/2 at each singular point, whereas the singular order of $V(x,u)$ is 3/2 at each singular point. The reason is that the series of rooted quadrangulations with four distinct outer vertices is of singular order 3/2, as any series counting a "natural" family of rooted maps; so $Q^{\diamondsuit}(x,u)$ is of singular order 1/2 by derivation effect. And $V(x,u)$ is equal to $2u(1+Q_0'(x,u))^2/x$, so has the same singular order as $Q_0'(x,u)$, i.e., 3/2.

Replacing p and q by their expressions (16), one gets a rational expression for $S^{\diamondsuit}(x,w)$ in terms of r and s. As already discussed in Section 3.3, the singular points of (9) are mapped by $H(x,u)$ into the singular points of (14). Since $Q^{\diamondsuit}(x,u)$ and $V(x,u)$ converge to positive constants at every singular point (x_0,u_0) of (9), $S^{\diamondsuit}(x,w)$ converges at every singular point (x_0,w_0) of (14). Since $S^{\diamondsuit}(x,w)$ is rationally expressed in terms of r and s, which have square-root singular expansion around (x_0,w_0), we conclude that $S^{\diamondsuit}(x,w)$ admits a singular expansion of order $(2k+1)/2$ around (x_0,w_0), with k a non-negative integer. △

Similarly, one proves that for every singular point (x_0,w_0) of (14), $S^{\|}(x,w)$ has a singular expansion of positive half-integer order around every singular point of (14).

Claim B. The series $K(x,w)$ admits a singular expansion of order $\alpha+1/2$ around every singular point (x_0,w_0) of (14).

Proof of the claim. Let $h = g-1$. Since $S_h(x,w)$ has a singular expansion of order $\frac{5}{2}(1-h)$, by Lemma 2.5 $S_h{}''(x,w)$ has a singular expansion of order $\frac{5}{2}(1-h)-2 = \alpha+1/2$ around (x_0,w_0). Moreover for each pair g_1, g_2 of positive integers adding up to g, the series $S_{g_1}{}'(x,w)$ and $S_{g_2}{}'(x,w)$ have singular expansions of respective orders $5(1-g_1)/2-1$ and $5(1-g_2)/2-1$ around (x_0,w_0). Hence, by Lemma 2.2, $S_{g_1}{}'(x,w)S_{g_2}{}'(x,w)$ has a singular expansion of order $3-5g/2 = \alpha+1/2$. △

We now conclude the proof of Lemma 3.5. Since $K(x,w)$ has a singular expansion of order $\alpha+1/2$ and $S^{\|}(x,w)+S^{\diamondsuit}(x,w)$ has a converging singular expansion around (x_0,w_0), then by Lemma 2.2 the product of these two series (which dominates coefficientwise $L(x,w)$ according to (19)) admits a singular expansion of order $\alpha+1/2$. □

4 Counting graphs of genus g asymptotically

The *genus* of a graph G is the minimum genus of a surface in which G can be embedded. So planar graphs are graphs of genus 0. The *face-width* of a graph G of genus $g \geq 0$ is the maximum face-width over all embeddings of G in $\mathbb{S}_g$. Note that the face-width is $+\infty$ in genus 0 since all curves in the sphere are contractible.

4.1 Notations for graph families and counting series

Graph families of connectivity 1, 2, 3 are denoted by the letters $\mathcal{C}$, $\mathcal{B}$, and $\mathcal{T}$, respectively. The genus g is indicated as a subscript, and the face-width is indicated as a superscript. For instance $\mathcal{C}_g^{\mathrm{fw}\geq 2}$ is the family of connected graphs of genus g with face-width at least 2. Unless explicitly mentioned, all generating functions count graph families according to vertices and edges, marked respectively by variables x and y. Since vertices are labelled, these series are exponential in the first variable. For instance, if $c_{n,k}$ is the number of connected graphs of genus g with n vertices and k edges, then

$$C_g(x,y) = \sum c_{n,k}\, y^k \frac{x^n}{n!}.$$

In the rest of the paper (as opposed to the previous section) the derivative of a series $G(x,y)$ according to the *first* variable is denoted by $G'(x,y)$.

Given a graph family $\mathcal{G}$, the *derived* class is the class $\mathcal{G}'$ of graphs from $\mathcal{G}$ with one marked vertex that is unlabelled and all the other n vertices being as usual labelled by $[1..n]$. Similarly the *doubly derived* class $\mathcal{G}''$ is the class of graphs from $\mathcal{G}$ with *two* marked vertices that are unlabelled (distinguished as a first and a second marked vertex), all the other n vertices being labelled in $[1..n]$. The counting series of $\mathcal{G}'$ and $\mathcal{G}''$ are exponential in the number of *labelled vertices*, that is, each graph with n labelled vertices and k edges has weight $x^n y^k/n!$ in the counting series. If $G(x,y)$ is the counting series of $\mathcal{G}$, it is well known that the counting series of $\mathcal{G}'$ is $G'(x,y)$ and the counting series of $\mathcal{G}''$ is $G''(x,y)$.

4.2 Tutte's decomposition of graphs

We recall here a well-known decomposition of a graph into 3-connected components, as formalized by Tutte [31]. For $k \geq 1$, a *k-connected graph* is a graph of order at least $k+1$ in which at least k vertices need to be deleted to disconnect it. A connected graph is decomposed into 2-connected components, also called blocks, that are articulated around cut vertices, so that the incidences between the 2-connected components and the separating vertices form a tree. Similarly, a 2-connected graph G is decomposed at its separating pairs of vertices into a tree with three kinds of nodes: polygons, multiple edges, and 3-connected graphs with at least four vertices. The latter type are called the 3-connected components of G. See [21] for a detailed discussion in the context of graph enumeration.

Robertson and Vitray proved the following result.

Theorem 4.1 (Robertson and Vitray [28]). *A connected graph G of genus $g > 0$ has face-width $k \geq 2$ if and only if G has a unique 2-connected component of genus g and face-width k, and all the other 2-connected components are planar.*

A 2-connected graph G of genus $g > 0$ has face-width $k \geq 3$ if and only if G has a unique 3-connected component of genus g and face-width k, and all the other 3-connected components are planar.

Let $F(x,y) = xC'(x,y)$ be the series counting vertex-pointed connected planar graphs (the pointed vertex being labelled). Then the first part of Theorem 4.1 yields, for $g > 0$,

$$C_g^{\mathrm{fw}\geq 2}(x,y) = B_g^{\mathrm{fw}\geq 2}(F(x,y),y). \tag{21}$$

A *network*, as defined in [33], is a connected planar graph with two special vertices, called the poles, such that adding an edge connecting the poles gives a 2-connected planar graph. Let $D(x,y)$ be the series counting networks (the poles are not labelled and not counted by the variable x). Then the second part of Theorem 4.1 yields, for $g > 0$,

$$B_g^{\mathrm{fw}\geq 3}(x,y) = T_g^{\mathrm{fw}\geq 3}(x, D(x,y)). \tag{22}$$

4.3 Singular expansions

Again, to genus $g \geq 0$ we associate the quantity

$$\alpha = \tfrac{5}{2}(1-g).$$

Our goal is to show that the series enumerating 3-, 2-, and 1-connected graphs of genus g are of approximate-singular order α.

4.3.1 3-connected graphs of genus g

We start with 3-connected graphs and use the result for maps from the previous section. In particular, we use the fact [28] (see [24] for an alternative proof) that if a 3-connected graph of genus g has face-width at least $2k+3$, then it has a unique embedding in the surface $\mathbb{S}_g$.

Lemma 4.2 (3-connected graphs of genus g and fw ≥ 3)**.** *For $g \geq 0$, the series $T_g^{\mathrm{fw}\geq 3}(x,w)$ is approximate-singular of order α around every singular point (x_0, w_0) of* (14)*.*

Proof. The proof relies again on results by Robertson and Vitray [28]. We use the letter E to denote the series of embedded 3-connected graphs, that is, 3-connected maps. In this case we label maps at vertices and use variable w for marking edges, so that

$$E(x,w) = \sum E_{n,m} w^m \frac{x^n}{n!}.$$

Note that a 3-connected graph yields at least two embeddings (an embedding and its reflection), hence

$$2\, T_g(x,w) \preceq E_g(x,w).$$

It is shown in [28] that the quadrangulation of a 3-connected map is near-irreducible, hence

$$E_g(x,w) \preceq S_g(x,w).$$

There is a technical point here that we clarify. In the previous section, near-irreducible quadrangulations (counted by the series S_g) were labelled at edges, which correspond to corners of the 3-connected maps. But here we are labelling 3-connected maps at vertices, which correspond to black vertices in quadrangulations. But in near-irreducible quadrangulations labelling at edges is equivalent to labelling at black vertices, in the following sense. If $S_{n,m}$ is the number of near-irreducible quadrangulations with n black vertices and m faces, labelled at edges with $\{1,2,\ldots,2m\}$, and $\widetilde{S}_{n,m}$ is the number of near-irreducible quadrangulations labelled at black vertices with $\{1,2,\ldots,n\}$, then we have

$$S_{n,m} n! = \widetilde{S}_{n,m}(2m)!,$$

since the previous quantity is the number of quadrangulations labelled both at vertices and edges. The important point is that labelling near-irreducible quadrangulations at either black vertices or edges is enough to avoid symmetries. In order to avoid unnecessary complications, we use the notation S_g as in Section 3, but taking into account that $S_g = \sum S_{n,m} x^n w^m/(2m)! = \sum \widetilde{S}_{n,m} w^m x^n/n!$.

It is also shown in [28] that a map is 3-connected of face-width $k \geq 3$ if and only if the associated quadrangulation is irreducible of edge-width $2k$. Hence, for $k \geq 3$, we have

$$E_g^{\mathrm{fw}\geq k} = S_g - \sum_{i<k} S_g^{\mathrm{ew}=2i} \succeq S_g - \sum_{i<k} S_g^{C=2i}.$$

Finally it is shown in [28] that if the face-width of a 3-connected graph is at least $2g+3$, then the embedding is unique up to reflection, hence

$$2\,T_g^{\mathrm{fw}\geq 2g+3} = E_g^{\mathrm{fw}\geq 2g+3} = S_g - \sum_{i<2g+3} S_g^{C=2i}.$$

Since

$$T_g^{\mathrm{fw}\geq 2g+3} \preceq T_g^{\mathrm{fw}\geq 3} \preceq T_g,$$

we obtain

$$S_g - \sum_{i<2g+3} S_g^{C=2i} \preceq 2\,T_g^{\mathrm{fw}\geq 3} \preceq S_g. \tag{23}$$

By Lemma 3.4, S_g has a singular expansion of order α around every singular point (x_0, w_0) of (14); and by Lemma 3.5, for $i \geq 1$, $S_g^{C=2i}$ is bounded coefficient-wise by a series P_g that admits a singular expansion of order $\alpha + 1/2$ around (x_0, w_0). Hence $T_g^{\mathrm{fw}\geq 3}$ is of approximate-singular order α at (x_0, w_0). □

4.3.2 2-connected graphs of genus g

Lemma 4.3 (2-connected graphs of genus g and fw ≥ 3)**.** *For $g \geq 0$, the series $B_g^{\mathrm{fw}\geq 3}(x,y)$ is of strong approximate-singular order α around every singular point (x_0, y_0) of the series $D(x,y)$ of planar networks.*

Proof. By Equation (22) we have $B_g^{\mathrm{fw}\geq 3}(x,y) = T_g^{\mathrm{fw}\geq 3}(x, D(x,y))$. Bender et al. [8] have shown that, if (x_0, y_0) is a singular point of $D(x,y)$, then (x_0, w_0) —where $w_0 = D(x_0, y_0)$— is a singular point of (14), i.e., is a singular point of $T_g^{\mathrm{fw}\geq 3}(x,w)$. Hence the composition scheme is critical. In addition, as proved in [8], $D(x,y)$ admits a log-free singular expansion of order $3/2$ around (x_0, y_0). Hence by Corollary 2.8, $B_g^{\mathrm{fw}\geq 3}(x,y)$ is of strong approximate-singular order α around every singular point (x_0, y_0) of $D(x,y)$. □

Lemma 4.4 (2-connected graphs of genus g and fw ≥ 2)**.** *For every singular pair (x_0, y_0) of $D(x,y)$, the series $B_g^{\mathrm{fw}=2}(x,y)$ is bounded coefficient-wise by a series of strong singular order $\alpha + 1/2$ at (x_0, y_0).*

And the series $B_g^{\mathrm{fw}\geq 2}(x,y)$ is of strong approximate-singular order α around (x_0, y_0).

Proof. The proof works by induction on g. The first assertion is clearly true for $g = 0$ since the face-width is infinite by convention. The second assertion has been shown by Giménez and Noy for planar graphs in [20].

Let $g > 0$, and assume the property is true up to $g - 1$. Let $\mathcal{E}_g$ be the family of well-labelled graphs with two connected components: the first one in $\mathcal{B}_g^{\mathrm{fw}=2}$, the second one an oriented edge (the second component is only an artifact to have a reserve of two labelled vertices). Note that the series of $\mathcal{E}_g$ is

$$E_g(x,y) = x^2 y B_g^{\mathrm{fw}=2}(x,y)$$

Let $\mathcal{F}_g$ be the family of graphs with two ordered connected components of positive genera adding up to g, each connected component being a 2-connected graph of face-width at least 2 with additionally a marked directed edge. Note that the series of $\mathcal{F}_g$ satisfies

$$F_g(x,y) = \sum_{\substack{g_1+g_2=g \\ g_1>0, g_2>0}} 2y\frac{\partial}{\partial y} B_{g_1}^{\mathrm{fw}\geq 2}(x,y) \cdot 2y\frac{\partial}{\partial y} B_{g_2}^{\mathrm{fw}\geq 2}(x,y).$$

And let $\mathcal{H}_g$ be the family of 2-connected graphs of genus at most g and face-width at least 2 and having two (a first and a second) marked directed edges. Note that the series of $\mathcal{H}_g$ satisfies

$$H_g(x,y) = \sum_{h \le g} \frac{\partial^2}{\partial y^2} B_g^{\mathrm{fw} \ge 2}.$$

We are going to show an injection from $\mathcal{E}_g$ to $\mathcal{F}_g + \mathcal{H}_{g-1}$ (since $\mathcal{F}_g$ and $\mathcal{H}_{g-1}$ only involve family of 2-connected graphs of smaller genus, this will allow us to conclude the proof by induction).

Let $(G, e) \in \mathcal{E}_g$. Let M be an embedding of G of face-width 2 and let C be a non-contractible cycle intersecting M at two vertices v and v', C being additionally directed. Cut the surface along C as in Figure 1, making two cylindric ends appear (one on the left of C, on one the right of C). In the tip of each cylindric end, add an edge directed from v to v'. So v (v', resp.) is split into two vertices: the one on the left cylindric end keeps the label of v (of v', resp.), the one on the right cylindric end receives the label of the origin (end, resp.) of the edge-component e [2]. Let M' be the resulting embedded graph and G' the underlying graph. Note that if C is surface-separating G' has two ordered connected components (the first one contains the left cylindric end) of positive genera adding up to g, each connected component having additionally a marked directed edge. If C is not surface-separating, G' has one connected components of genus at most $g-1$ and has additionally two marked directed edges. Clearly the mapping from (G, e) to G' is injective, since G is recovered from G' by identifying the two marked directed edges: the merged edge receives the labels of the first marked edge, and e receives the labels of the second marked edge.

We now claim that the image of $\mathcal{E}_g$ is included in $\mathcal{F}_g + \mathcal{H}_{g-1}$, i.e., that the connected components of the graph associated with $G \in \mathcal{E}_g$ are 2-connected of face-width at least 2. This is best seen by going to quadrangulations. Take the case where C is not surface-separating (the other case works similarly). With the notations above, let Q be the quadrangulation associated with M, and Q' the quadrangulation associated with M'. The effect of cutting along C is the same as cutting along a 4-cycle c of Q, so Q' is obtained from Q by cutting along c. Recall that simple bipartite quadrangulations of genus g exactly correspond to 2-connected maps of face-width at least 2. Since M is 2-connected of face-width at least 2, Q is a simple quadrangulation, so Q' is also a simple quadrangulation, so M' is a 2-connected map of face-width at least 2 (and genus $g-1$). Since M' has face-width at least 2 and genus $g-1$, G' has face-width at least 2 and genus at most $g-1$.

Thanks to the injective mapping we have constructed it is an easy matter to conclude the proof by induction. By injectivity we have

$$x^2 y B_g^{\mathrm{fw}=2}(x,y) = E_g(x,y) \preceq F_g(x,y) + H_{g-1}(x,y).$$

By induction, each series $B_h^{\mathrm{fw} \ge 2}(x,y)$ for $h < g$ is of strong approximate-singular order $5/2(1-h)$ at every singular point (x_0, y_0) of $D(x,y)$. Hence the partial derivative according to y of such a series is of order $3/2 - 5/2h$. Hence, for $g_1 + g_2 = g$, the product of such two series is of order $3 - 5/2g$, so $F_g(x,y)$ is also of order $3 - 5/2g = \alpha + 1/2$. Similarly, $H_{g-1}(x,y)$ is of order $\alpha + 1/2$, so the sum of F_g and H_{g-1}, which dominates $B_g^{\mathrm{fw}=2}$, is of order $\alpha + 1/2$. Finally, the series $B_g^{\mathrm{fw} \ge 2} = B_g^{\mathrm{fw} \ge 3} + B_g^{\mathrm{fw}=2}$ is the sum of two series respectively of strong approximate-singular orders α (according to Lemma 4.3) and $\alpha + 1/2$ (as just shown), so the second term is negligible and $B_g^{\mathrm{fw} \ge 2}$ is thus of strong approximate-singular order α. □

[2] The edge-component is only used to make the resulting graph well-labelled.

Even if only Lemma 4.4 is needed to do the asymptotics of connected graphs of genus g, we add here for the sake of completeness a lemma (bounding the series of 2-connected graphs of face-width 1) in order to have a theorem on the asymptotic enumeration of 2-connected graphs of genus g (Theorem 4.6 at the end of this section).

Lemma 4.5 (2-connected graphs of genus g)**.** *For every singular pair (x_0, y_0) of $D(x, y)$, the series $B_g^{\mathrm{fw}=1}(x, y)$ is bounded coefficient-wise by a series of strong singular order $\alpha + 1/2$ at (x_0, y_0).*

And the series $B_g^{\mathrm{fw}\geq 1}(x, y) = B_g(x, y)$ is of strong approximate-singular order α around (x_0, y_0).

Proof. Similarly as in Lemma 4.4 we reason by induction on g. The stated assertions are true for $g = 0$ (the first assertion is trivial, the second assertion is proved in [20]).

Let $g > 0$, and assume the stated properties are true up to genus $g - 1$. Given a 2-connected graph G of genus g and face-width 1, let M be an embedding of G on $\mathbb{S}_g$ with face-width 1, and let C be a non-contractible closed curve intersecting M at a single vertex, call it v. Cutting along C splits v into two "marked" vertices and yields a connected graph M' embedded on $\mathbb{S}_{g-1}$ and with two marked vertices (the case of C surface-separating is excluded, since it would contradict the 2-connectivity of G). Let G' be the underlying connected graph, so G' has genus at most $g-1$. Note that even after forgetting the embedding, G can be recovered from G' by merging the 2 marked vertices of G'. So the mapping from G to G' and its two marked vertices is injective (upon using a similar artifact as in Lemma 4.4 to make G' well-labelled). The graph G' is not necessarily 2-connected, but close to. We use the following easy claim:

Claim A. Let G be a 2-connected graph, let v be a vertex of G and (E_1, E_2) a partition of the edges incident to v into 2 non-empty parts. Let G' be the graph obtained from G by splitting v into 2 vertices v_1 and v_2, v_i keeping as incident edges the edges in E_i for $i \in \{1, 2\}$. Then the block-structure of G' is a chain of 2-connected components $B_1, \ldots, B_k$, i.e., each B_i has two distinct marked vertices w_i, w_i' such that $w_1 = v_1$, $w_k = v_2$, and $w_i' = w_{i+1}$ for $i \in [1..k-1]$.

Hence the graph G' obtained from G is a chain of bi-pointed 2-connected graphs with sum of genera strictly smaller than g (because the genus of a graph is the sum of genera of its blocks). For each composition κ of $k \leq g - 1$ as a sum of positive integers $i_1, \ldots, i_j$, let $\mathcal{S}^{(\kappa)}$ be the family of connected graphs G' made of a chain of blocks, so that the blocks of positive genera in the chain have genera $i_1, \ldots, i_j$ and there are two unlabelled marked vertices in G' contained respectively in the first and in the last block of the chain. Note that (with $\mathcal{Z}$ marking a labelled vertex, Seq denoting the sequence construction, and $\star$ denoting the labelled product in the sense of [16])

$$\mathcal{S}^{(\kappa)} = {\mathcal{B}_{i_1}}'' \star \cdots \star {\mathcal{B}_{i_j}}'' \star \mathrm{Seq}(\mathcal{Z} \star {\mathcal{B}_0}'')^{j+1}.$$

Claim B. Let $\kappa = i_1 + \cdots + i_j$ be a composition of $k < g$ by positive integers. Then the generating series $S^{(\kappa)}(x, y)$ of $\mathcal{S}^{(\kappa)}$ is of strong approximate-singular order $j/2 - 5/2k$ at every singular point (x_0, y_0) of $D(x, y)$.

Proof of the claim. By induction, for $h < g$, each series ${B_h}''(x, y)$ is of strong approximate-singular order $1/2 - 5/2h$ at (x_0, y_0). Hence, by Lemma 2.2, ${B_{i_1}}''(x, y) \cdots {B_{i_j}}''(x, y)$ is of strong approximate-singular order $j/2 - 5/2k$. To conclude the proof of the claim we just have to show that the series $S(x, y)$ of $\mathrm{Seq}(\mathcal{Z} \star \mathcal{B}_0)''$ is of (strong) positive approximate-singular order; we show it is of order $1/2$. Since $B_0(x, y)$ is of order $5/2$ at (x_0, y_0), $B_0''(x, y)$ is of order $1/2$, so $1/(1 - xB_0''(x, y))$ is either of order $1/2$ (if $x_0B_0''(x_0,y_0)=1$) or is of order $-1/2$ (if $x_0B_0''(x_0,y_0)<1$), so proving that $S(x,y)$ is of order $1/2$ reduces to proving that $1/(1-x_0B_0''(x_0,y_0))$ is finite. Recall the equation

$$F(z,y) = z\exp(B_0'(F(z,y),y))$$

relating the series $F(z,y)$ of pointed connected planar graphs and the series $B_0(x,y)$ of 2-connected planar graphs. Differentiating this equation with respect to z yields

$$F'(z,y) = F(z,y)F'(z,y)B_0''(F(z,y),y) + F(z,y)/z,$$

so

$$F'(z,y) = \frac{x}{z}\frac{1}{1-xB_0''(x,y)}, \quad \text{where } x = F(z,y).$$

Let z_0 be the positive value such that (z_0,y_0) is a singular point of $F(z,y)$. As proved in [20], the composition scheme from 2-connected to connected planar graphs is critical, i.e., $x_0 = F(z_0,y_0)$. Moreover, it is shown in [20] that $F(z,y)$ is of singular order $3/2$ at (x_0,y_0), so $F'(z,y)$ is of order $1/2$. Hence $F'(z,y)$ converges at (z_0,y_0), which implies that $1/(1-xB_0''(x,y))$ converges at (x_0,y_0), and thus is of singular order $1/2$. Observe that the same argument shows that the denominator $1-xB_0''(x,y)$ does not vanish before we reach the singular point (x_0,y_0). This concludes the proof of the claim. △

The injection presented in the beginning of the proof of the lemma ensures that $B_g^{\mathrm{fw}=1}(x,y)$ is bounded coefficientwise by the sum of series $B^{(\kappa)}(x,y)$, where κ runs over compositions of integers strictly smaller than g. By Claim B, each series $B^{(\kappa)}(x,y)$ is of strong approximate-singular order at least $1/2-5/2(g-1)=\alpha+1/2$ (the least singular order of $B^{(\kappa)}(x,y)$ is attained for κ the composition of $g-1$ with a single part). This completes the proof of the first assertion of the lemma.

The second assertion follows straightforwardly. The series $B_g^{\mathrm{fw}\geq 1}(x,y)=B_g(x,y)$ is the sum of $B_g^{\mathrm{fw}\geq 2}(x,y)$ and $B_g^{\mathrm{fw}=1}$, the first term being of strong approximate-singular order α by Lemma 4.4, and the second term being bounded coefficientwise by a series of strong approximate-singular order $\alpha+1/2$, hence of negligible contribution. Hence $B_g(x,y)$ is also of strong approximate-singular order α at (x_0,y_0). □

Applying the transfer theorems stated in Section 2 to the series counting 2-connected graphs of genus g, we obtain:

Theorem 4.6. *For $g \geq 0$, the number $b_n^{(g)}$ of 2-connected graphs with n vertices that can be embedded in the orientable surface $\mathbb{S}_g$ of genus g satisfies*

$$b_n^{(g)} \sim d^{(g)} n^{5(g-1)/2-1}\gamma^n n! \tag{24}$$

where $d^{(g)}$ is a positive constant and γ is the exponential growth constant of 2-connected planar graphs.

For $\mu \in (1,3)$, the number $b_{n,m}^{(g)}$ of graphs with n vertices and $m=\lfloor \mu n\rfloor$ edges that can be embedded in $\mathbb{S}_g$ satisfies

$$b_{n,m}^{(g)} \sim d_\mu^{(g)} n^{5g/2-4}(\gamma_\mu)^n n! \quad \text{when } n\to\infty,$$

where $d_\mu^{(g)}$ is a positive constant and γ_μ is the exponential growth constant of planar graphs with ratio edges/vertices tending to μ (γ and $\gamma(\mu)$ are characterized analytically in [20]).

4.3.3 Connected graphs of genus g

Lemma 4.7 (Connected graphs of genus g and fw ≥ 2)**.** *For $g \geq 0$, the series $C_g^{\mathrm{fw}\geq 2}(x,y)$ is of strong approximate-singular order α around every singular pair (x_0, y_0) of the series $G_0(x,y)$ counting planar graphs.*

Proof. In this proof the first variable of the series of 2-connected graph families is denoted by z to avoid confusion. Recall Equation (21): $C_g^{\mathrm{fw}\geq 2}(x,y) = B_g^{\mathrm{fw}\geq 2}(F(x,y),y)$. Giménez and Noy [20] have shown that, if (x_0, y_0) is a singular point of $G_0(x,y)$, then (z_0, y_0) —where $z_0 = F(x_0, y_0)$— is a singular point of $D(z,y)$. Hence, according to Lemma 4.4, (z_0, y_0) is also a singular point of $B_g^{\mathrm{fw}\geq 2}(z,y)$ (precisely, is a singular point of the two series bounding $B_g^{\mathrm{fw}\geq 2}(z,y)$ from above and below). Hence the composition scheme is critical. In addition, as proved in [20], $F(x,y)$ admits a log-free singular expansion of order 3/2 around (x_0, y_0). Hence by Corollary 2.8, $C_g^{\mathrm{fw}\geq 2}(x,y)$ is of strong approximate-singular order α around every singular point (x_0, y_0) of $F(x,y)$. ☐

Lemma 4.8 (Connected graphs of genus g, and with fixed face-width)**.** *For $g \geq 0$ and $k \geq 0$, the series $C_g^{\mathrm{fw}=k}(x,y)$ is bounded coefficientwise by a series that is of strong approximate-singular order $\alpha + 1/2$ around every singular point (x_0, y_0) of the series $G_0(x,y)$ counting planar graphs.*

And the series $C_g(x,y)$ is of strong approximate-singular order α around (x_0, y_0).

Proof. As in Lemma 4.4 (and using similar notations), the proof works by induction on g. The stated properties are true when $g = 0$ since by convention the face-width of any planar graph is infinite. Assume the property is true up to $g-1$, for $g > 0$. Let $\mathcal{E}_g^{(k)}$ be the family of graphs (as usual well-labelled, the vertices have distinct labels in $[1..n]$) made of two connected components, the first one in $\mathcal{C}_g^{\mathrm{fw}=k}$, the second one an oriented path of k vertices (again the second component is just an artifact to have a reserve of k labelled vertices). Note that the series of $\mathcal{E}_g^{(k)}$ is $x^k C_g^{\mathrm{fw}=k}(x,y)$. Let $\mathcal{F}_g$ be the family of graphs with two connected components of positive genera (g_1, g_2) adding up to g, each connected component having a marked vertex that is unlabelled; so the series of $\mathcal{F}_g$ satisfies

$$F_g(x,y) = \sum_{\substack{g_1+g_2=g \\ g_1>0, g_2>0}} C_{g_1}{}'(x,y) C_{g_2}{}'(x,y).$$

And let $\mathcal{H}_g$ be the doubly derived family of connected graphs of genus at most g (i.e., with two marked unlabelled vertices): $\mathcal{H}_g = \cup_{h \leq g} \mathcal{C}_h{}''$, so the series of $\mathcal{H}_g$ satisfies

$$H_g(x,y) = \sum_{h \leq g} C_h{}''(x,y).$$

Denote by $\mathfrak{S}_k$ the group of permutations of k elements. We are going to define an injective mapping from $\mathcal{E}_g^{(k)}$ to $(\mathcal{F}_g + \mathcal{H}_{g-1}) \times \mathfrak{S}_k^2$. (Note that $\mathcal{F}_g$ and $\mathcal{H}_{g-1}$ only involve families of connected graphs of genus smaller than g, which will allow us to use induction to conclude the proof).

Let $(G,p) \in \mathcal{E}_g^k$. Associate with G an embedding M (map of genus g) of face-width k. Let C be an oriented non-contractible curve intersecting M at k vertices, one of which is marked as the "starting vertex" of C. Cut M along C, thus creating two cylindric ends, as in Figure 1. When doing this, each vertex along C is split into two vertices, one in each cylindric end; to make the resulting graph well-labelled, the vertices $(v_1, \ldots, v_k)$ of the contour of C (with v_1 the starting vertex) on the right tip are labelled according to p, i.e., v_i receives the label of the ith vertex on the path p.

This process either yields two maps whose genera add up to g (case where C is surface-separating) or cuts a handle and yields a map of genus $g-1$. Add a special vertex v_{left} (v_{right}, resp.) in the tip-area of the left (right, resp.) cylindric end, and connect the special vertex to the k vertices on the contour of the tip-area (in Figure 1 a similar operation occurs, but the special vertex is only connected to the white vertices of the contour). Let G' be the graph (with two components if C is surface-separating, with one component otherwise) obtained after this process and forgetting the embedding. Note that the k neighbours of v_{left} and the k neighbours of v_{right} were matched in G along C; record this matching by two permutations $(\sigma, \sigma') \in \mathfrak{S}_k$ defined as follows: if $(v_1, \ldots, v_k)$ is the occurrence of C on the left (right, resp.) tip, then $\sigma(i) = j$ ($\sigma'(i) = j$, resp.) means that v_i has the jth smallest label among $v_1, \ldots, v_k$.

We claim that the mapping from (G, p) to (G', σ, σ') is injective. Indeed (G, p) can be recovered from (G', σ, σ') as follows:

- For $i \in [1..k]$, let u_i be the neighbour of v_{left} in G' with $\sigma(i)$th smallest label, and let u'_i be the neighbour of v_{right} in G' with $\sigma'(i)$th smallest label. Then G is recovered from G' by merging u_i with u'_i for $i \in [1..k]$ (the merged vertex keeps the label of u_i) and then by removing the special vertices v_{left} and v_{right} and their incident edges.

- And p is recovered as the oriented path of k vertices, where for $i \in [1..k]$ the ith vertex of p receives the label of u'_i.

Since the construction is injective, we have

$$x^k C_g^{\text{fw}=k}(x,y) = E_g^{(k)}(x,y) \preceq (F_g(x,y) + H_{g-1}(x,y)) \cdot k!^2.$$

By induction, for $h < g$, $C_h(x,y)$ is of strong approximate-singular order $5/2(1-h)$ at any singular point (x_0, y_0) of $G_0(x,y)$ (series counting connected planar graphs), hence $C_h{}'(x,y)$ ($C_h{}''(x,y)$, resp.) is of strong approximate-singular order $3/2 - 5/2h$ ($1/2 - 5/2h$, resp.). Hence $F_g(x,y)$ (by Lemma 2.2) and $H_g(x,y)$ are both of strong approximate-singular order $3 - 5/2g = \alpha + 1/2$ at (x_0, y_0). This proves the first assertion.

The second assertion follows easily. Indeed $C_g = C_g^{\text{fw}\geq 2} + C_g^{\text{fw}=1}$. By Lemma 4.7, the first term is of strong approximate-singular order α. And the second term is negligible —bounded by a series of strong approximate-singular order $\alpha + 1/2$ at (x_0, y_0)— according to the first assertion applied to $k = 1$. ☐

4.3.4 Graphs of genus g

Lemma 4.9 (Graphs of genus g)**.** *For $g \geq 0$, the series $G_g(x,y)$ counting graphs of genus g is of strong approximate-singular order α around every singular point (x_0, y_0) of the series $G_0(x,y)$ counting planar graphs.*

Proof. Again the proof works by induction on g. The result has been proved by Giménez and Noy [20] for $g = 0$. Let $g > 0$, and assume the stated property is true up to genus $g-1$. The genera of the connected components of a genus g graph G add up to g. If two connected components are non-planar then there is a partition of G into two non-planar graphs G_1, G_2 whose genera add up to g

$$G_g(x,y) = C_g(x,y) G_0(x,y) + K_g(x,y), \tag{25}$$

where

$$K_g(x,y) \preceq \sum_{\substack{g_1+g_2=g \\ g_1>0, g_2>0}} G_{g_1}(x,y)G_{g_2}(x,y).$$

By induction, for $h < g$, $G_h(x,y)$ is of strong approximate-singular order $5(1-h)/2$ around (x_0, y_0). Hence, for $g_1 > 0$, $g_2 > 0$ such that $g_1+g_2 = g$, Lemma 2.2 ensures that the series $G_{g_1}(x,y)G_{g_2}(x,y)$ is of strong approximate-singular order $5/2(1-g_1) + 5/2(1-g_2) = \alpha + 5/2$ around (x_0, y_0). Thus the dominating part of $G_g(x,y)$ is $C_g(x,y)G_0(x,y)$, which by Lemma 2.2 is of strong approximate-singular order α around (x_0, y_0) (indeed $C_g(x,y)$ is of approximate-singular order α by Lemma 4.8 and $G_0(x,y)$ is of approximate-singular order $5/2$, as proved in [20]). □

Lemma 4.10 (Graphs embeddable on $\mathbb{S}_g$)**.** *For $g \geq 0$, the series F_g counting graphs embeddable on the genus g surface is of strong approximate-singular order α around every singular point (x_0, y_0) of the series $C(x,y)$ counting connected planar graphs.*

Proof. Note that a graph embeddable on $\mathbb{S}_g$ has genus at most g, hence

$$F_g(x,y) = \sum_{i=0}^{g} G_i(x,y).$$

By Lemma 4.9, each series $G_i(x,y)$ in the sum is of strong approximate-singular order $5(1-i)/2$, hence the dominating series in the sum is $G_g(x,y)$, which is of strong approximate-singular order $5(1-g)/2 = \alpha$. □

To conclude, applying the transfer theorems (Corollary 2.7) to the singular expansions of the series counting graphs embeddable on $\mathbb{S}_g$ (Lemma 4.10), we obtain the asymptotic enumeration results stated in Theorem 1.1.

4.4 The non-orientable case

The proof of Theorem 1.2 follows the same lines as the proof of Theorem 1.1. Here, we explain the mild differences and how to adapt the proof at those points. Let $h > 0$ and let $\mathbb{N}_h$ be the non-orientable surface with h crosscaps. In this section, we use the same notations as before for the generating series of maps and graphs, but with a different meaning: all series will refer to non-orientable surfaces, and the subscript in the notation will refer to the non-orientable genus h (whereas in previous sections it referred to the oriented genus). For example, the generating series of near-irreducible quadrangulations on the surface $\mathbb{N}_h$ will be denoted by $S_h(x,w)$.

Before giving the outline of the proof of Theorem 1.2, let us give some definitions. The *non-orientable genus* of a graph G is the smallest h such that G can be embedded in the non-orientable surface $\mathbb{N}_h$, and the *non-orientable face-width* of G is the largest face-width among all the embeddings of G on $\mathbb{N}_h$. Since we will have to consider at the same time orientable and non-orientable surfaces, it makes sense to introduce the *Euler genus* κ of a surface S, which is defined as $\kappa = 2-\chi(S)$, where $\chi(S)$ is the Euler characteristic of S. Therefore $\kappa = 2g$ for the orientable surface $\mathbb{S}_g$, and $\kappa = h$ for the non-orientable surface $\mathbb{N}_h$. The *Euler genus* of a graph is the minimal Euler genus of a surface (orientable or non-orientable) in which G can be embedded.

We now define $\alpha = \frac{5}{2}(1-h/2)$ (as we will see, the singular exponent for a given surface depends only on its Euler genus, and h plays the same role here as $2g$ in the orientable case). The bijection between maps and bipartite quadrangulations also works on the non-orientable surface $\mathbb{N}_h$, and the extraction of a near-irreducible core works in the same way, so the equation relating rooted maps and near-simple quadrangulations on $\mathbb{N}_h$ is the same as for $\mathbb{S}_g$ (Equation (13)). The only difference is in the expression of the series $\overrightarrow{M}_h(x,u)$ of rooted maps on $\mathbb{N}_h$; Arquès and Giorgetti have shown in [3] that there exists a trivariate polynomial $P_h(X,Y,Z)$ such that

$$\overrightarrow{M}_h(x,u) = \frac{P_h(p,q,\sqrt{\Delta})}{\Delta^{5h/2-3}},$$

i.e., the numerator-polynomial has $\sqrt{\Delta}$ as additional parameter, where Δ is as before the Jacobian of (9). Recall that around a singular point of (9), $\Delta(p(x,u),q(x,u))$ is of the order of $\sqrt{1-x/\rho(u)}$, so $\sqrt{\Delta}$ is of the order of $(1-x/\rho(u))^{1/4}$. Hence, in the non-orientable case singular expansions are of the same form as in the orientable case, but in terms of $\widetilde{X} = (1-x/\rho(u))^{1/4}$ instead of $X = (1-x/\rho(u))^{1/2}$, which is a mild extension of the definition given in Section 2. The definition of the order of a singular expansion is also readily extended.

Similarly as in the orientable case, the sketch of proof in [6] ensures that P_h is non-zero at a singular point (x_0,u_0) for (p,q), so $\overrightarrow{M}_h(x,u)$ admits a singular expansion of order α at (x_0,u_0). Adapting Equation (17), the expression of the series S_h' of rooted near-simple quadrangulations on $\mathbb{N}_h$ is of the form

$$S_h'(x,w) = \frac{1}{\widetilde{A}(r,s)} \frac{\widetilde{P}_g\left(r,s,\sqrt{\widetilde{\Delta}(r,s)}\right)}{(\widetilde{\Delta}(r,s))^{5g-3}}.$$

All the arguments of Lemma 3.3 still apply, even with the extra parameter $\sqrt{\widetilde{\Delta}}$ in the numerator-polynomial, so S_h has a singular expansion of order α at every singular point of (14). In the statement of Lemma 3.5, $\alpha+1/2$ has to be changed to $\alpha+1/4$ (this is no problem, the contribution of near-irreducible quadrangulations of a fixed edge-width is still negligible since the singular order is larger than the one of near-irreducible quadrangulations). This is due to the fact that there is one more case to consider in the proof: that the marked cycle C of length $2i$ crosses a crosscap. In that case, cutting along C removes the crosscap, and yields a surface of Euler genus $h-1$. So we obtain either a quadrangulation on $\mathbb{N}_{h-1}$ or a quadrangulation on the orientable surface $\mathbb{S}_g$, where $g = \frac{h-1}{2}$ (the latter case may only appear if h is odd). In both cases, C yields a boundary of length $4i$ delimiting a "hole" in the surface. Filling this hole with a disk and placing a marked black vertex connected to all white vertices on the contour, we obtain a quadrangulation on $\mathbb{N}_{h-1}$ or $\mathbb{S}_g$ with a marked black vertex. This quadrangulation may not be near-irreducible. However, any contractible 2-cycle or 4-cycle bounds a disk that contains the marked black vertex. These cycles are therefore nested around the marked vertex. Hence by similar arguments as in Lemma 3.5, the contribution of "C crossing a crosscap" is bounded by a series of the same singular order as the series counting near-irreducible quadrangulations on $\mathbb{N}_{h-1}$ with a marked vertex, plus those on $\mathbb{S}_g$ (when h is odd). By induction, the latter series with no marked vertex is of singular order $\alpha+5/4$ (since $\alpha+5/4 = \frac{5}{2}(1-(h-1)/2) = \frac{5}{2}(1-g)$). The effect of marking a vertex is to decrease the order by 1, so the bounding series is actually of order $\alpha+1/4$. The other two cases (C being surface-separating and C cutting a handle) still yield contributions bounded by a series of singular order $\alpha+1/2$. Let us remark that in these cases we may also end up with an orientable surface $\mathbb{S}_{g'}$ but since for each g' the singular orders for $\mathbb{S}_{g'}$ and $\mathbb{N}_{2g'}$ are the same, the orientability of the surface obtained after cutting along C has no effect on the singular order of the corresponding bounding series.

The next step is to go to 3-connected graphs of non-orientable genus h via 3-connected maps on $\mathbb{N}_h$. By Proposition 5.5.12 in [25], a map on $\mathbb{N}_h$ is 3-connected if the associated quadrangulation is irreducible (no 2-cycles and no non-facial 4-cycles at all). Conversely, the quadrangulation associated with a 3-connected map of face-width ≥ 2 is near-irreducible (no 2-cycles and no contractible non-facial 4-cycles). So 3-connected maps on $\mathbb{N}_h$ have the same approximate-singular expansion as near-irreducible quadrangulations. Since a 3-connected graph of non-orientable genus h has a unique embedding if the face width is at least $2h+3$ [28], one shows that the corresponding series is of strong approximate-singular order α similarly as in Lemma 4.2. Finally, one proceeds with 2-connected graphs of non-orientable genus h and then with connected graphs and arbitrary graphs similarly as in the orientable case, since the statements in Theorem 4.1 also hold with the non-orientable genus. Again in Lemmas 3.5, 4.4, 4.5, and 4.8, the bounding series for graphs with fixed face-width is of strong approximate-singular order $\alpha + 1/4$ instead of $\alpha + 1/2$ (which is still fine as long as the order is larger than α), since there is a third case where the distinguished non-contractible curve crosses a crosscap.

Euler genus. Our results also show that, for each integer $\kappa \geq 0$ the number $e_n^{(\kappa)}$ of graphs with n vertices and Euler genus κ satisfies:

$$e_n^{(\kappa)} \sim \hat{c}^{(\kappa)} n^{5(\kappa-2)/4-1} \gamma^n n! \tag{26}$$

where $\hat{c}^{(\kappa)} = \tilde{c}^{(\kappa)}$ if κ is odd, and $\hat{c}^{(\kappa)} = c^{(\kappa/2)} + \tilde{c}^{(\kappa)}$ if κ is even. This result follows from Lemma 4.8, from the analogous result for non-orientable surfaces, and from the fact (shown in [28]) that for each κ every graph of Euler genus κ and face-width more than $2\kappa+3$ can be embedded either in the non-orientable surface $\mathbb{N}_\kappa$ or in the orientable surface $\mathbb{S}_{\kappa/2}$, but not both.

5 Random graphs of genus g

In this section we analyze several fundamental parameters of graphs of genus g and derive limit distribution laws for them. In all cases the limit laws do not depend on the genus, a phenomenon that has been observed previously for *maps* on surfaces (see, for instance, [18]). When we say that an event holds *with high probability* we mean that the probability of the event tends to 1 as n tends to infinity. The variance of a random variable X is denoted by $\sigma^2(X)$, and a sequence of random variables $(X_n)_{n\geq 1}$ is called *asymptotically normal* if $(X_n - \mathbf{E}[X_n])/\sigma^2(X_n)$ converges in distribution to a standard Gaussian random variable $\mathcal{N}(0,1)$ (see [16, Part C]).

We start with two basic parameters in order to motivate the general analysis. We show that, as for planar graphs, the number of edges is asymptotically normal and the number of connected components is asymptotically Poisson distributed.

Theorem 5.1. *The number of edges X_n in a random graph of fixed genus g with n vertices is asymptotically normal and*

$$\mathbf{E}(X_n) \sim \kappa n, \qquad \sigma^2(X_n) \sim \lambda n$$

where $\kappa \approx 2.21326$ and $\lambda \approx 0.43034$ are the same constants as for planar graphs.

Proof. By Lemma 4.9, the generating function $G_g(x,y) = \sum g_{n,k} y^k x^n/n!$ of graphs of genus g counted according to the number of vertices and edges is of approximate-singular order α. This means that there exist sequences $f_{n,k}$ and $h_{n,k}$ with

$$f_{n,k} \le g_{n,k} \le h_{n,k}$$

such that $f(x,y) = \sum f_{n,k} y^k x^n/n!$ and $h(x,y) = \sum h_{n,k} y^k x^n/n!$ have singular expansions of order α with same singularity function and the same leading coefficients. By the Quasi-powers Theorem [16], the random variables with probability generating functions

$$\frac{[x^n]f(x,y)}{[x^n]f(x,1)}, \qquad \frac{[x^n]h(x,y)}{[x^n]h(x,1)}.$$

are asymptotically normal. It follows that X_n, whose distribution is given by $[x^n]G(x,y)/[x^n]G(x,1)$, also converges to a normal law. The expectation and variance are determined by the singularity function $\rho(y)$ as

$$\mu = -\frac{\rho'(1)}{\rho(1)}, \qquad \sigma^2 = -\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \left(\frac{\rho'(1)}{\rho(1)}\right)^2.$$

By Lemma 4.9, $\rho(y)$ is independent of the genus and we are done. □

Following the lines of the former proof, one can also show that several basic parameters, such as the number of blocks, the number of cut vertices, and the number of *appearances* of a fixed planar graph (see [20] for a precise definition), follow a normal law with the same moments as for planar graphs [22]. In order to avoid repetition we omit the corresponding proofs. We remark in particular that, given a fixed planar graph H, a random graph of genus g contains a subgraph isomorphic to H with high probability; in fact, it contains a linear number of disjoint copies of H.

Theorem 5.2. *The number of connected components in a random graph of fixed genus g is distributed asymptotically as $1 + X$, where X is a Poisson law with parameter $\nu \approx 0.037439$, same as for planar graphs. In particular, the probability that a random graph of genus g is connected is asymptotically $e^{-\nu}$.*

Proof. As shown in the proof of Theorem 4.9, a graph of genus g has a unique connected component of genus g with high probability, and the remaining components are planar. Hence, in order to study the number of components it is enough to work with the generating function

$$uC_g(x)e^{uC_0(x)},$$

the first two factors encoding the component of genus g, the exponential term encoding the planar components. The generating function of graphs with exactly $k+1$ components is then $C_g(x)C_0(x)^k/k!$. By Lemma 4.8, the series $C_g(x)$ is dominated coefficientwise by series having a singular expansion of the form

$$aX^\alpha + O(X^{\alpha+1}),$$

where $X = \sqrt{1 - x/\rho}$. On the other hand, we have $C_0(x) = C_0 + O(X)$ (see [20]; because $\alpha < 0$ we only need the constant term in the singular expansion of C_0). Then the probability of a random graph having exactly $k+1$ components is asymptotically equal to

$$\frac{[x^n]C_g(x)C_0(x)^k/k!}{[x^n]C_g(x)e^{C_0(x)}} \sim \frac{aC_0^k/k!}{ae^{C_0}} = e^{-C_0}\frac{C_0^k}{k!}.$$

This is precisely a Poisson distribution with parameter C_0, as for planar graphs. □

A similar analysis as that in the previous proof shows that there is a unique giant component of genus g.

Theorem 5.3. *Let L_n denote the size of the largest connected component in a random graph of fixed genus g with n vertices, and let $M_n = L_n - n$ be the number of vertices not in the largest component. Then*

$$\mathbf{P}(M_n = k) \sim p \cdot g_k \frac{\gamma^{-k}}{k!},$$

where p is the probability of a random planar graph being connected, g_k is the number of planar graphs with k vertices, and γ is the planar growth constant.

Proof. According to the results of the previous section, the number G_n of graphs of genus g grows like

$$G_n \sim G \cdot n^{-\alpha-1} \gamma^n n!,$$

and the number C_n of connected graphs of genus g grows like

$$C_n \sim C \cdot n^{-\alpha-1} \gamma^n n!.$$

Using again the fact that there is a unique component of genus g with high probability, we find that the probability that $M_n = k$ is asymptotically equal to

$$\binom{n}{k} \frac{C_{n-k} g_k}{G_n},$$

since there are $\binom{n}{k}$ ways of choosing the labels of the vertices not in the largest component, C_{n-k} ways of choosing the largest component, and g_k ways of choosing the complement. Using the previous estimates we get

$$\mathbf{P}(M_n = k) \sim \frac{C}{G} g_k \frac{\gamma^{-k}}{k!}.$$

But C/G is the asymptotic probability of a graph of genus g being connected, and by Theorem 5.2 it is the same as for planar graphs. □

In the next two results we analyze the size of the largest block and the size of the largest 3-connected component. For the precise form of the Airy law of map type, a continuous distribution defined in terms of the Airy function, and the computation of the parameters for planar graphs, we refer to [4] and [22, Section 5].

Theorem 5.4. *The size X_n of the largest block in a random connected graph of fixed genus g with n vertices follows asymptotically an Airy law of the map type, with the same parameters as for planar graphs. In particular*

$$\mathbf{E}(X_n) \sim \alpha n,$$

where $\alpha \approx 0.9598$, and the size of the second largest block is $o(n^{\frac{2}{3}+\epsilon})$, for any $\epsilon > 0$. Moreover the largest block has genus g with high probability.

Proof. By the results of the previous section, we know that with high probability a connected graph of genus g has a unique block of genus g, and the remaining blocks are planar. As we are going to show, the unique block of genus g is the largest block with high probability.

Equation (21) encodes precisely this statement. Since almost all 2-connected graphs have face-width ≥ 2, it is enough to consider the simplified composition scheme $B_g(F(x))$, where $F(x) = xC'(x)$ is the generating function of vertex-pointed connected planar graphs. In the terminology of [4] this scheme is *critical* (see Lemma 2.6), since the evaluation $F(\rho)$ of $F(x)$ at its singularity is equal to the singularity of $B_g(x)$, which is the same as the singularity of $B_0(x)$.

By general principles (see Theorem 12 and Appendix D in [4]) it follows that the size of the block of genus g follows a continuous Airy law. The parameters of the law depend only on the singular coefficients of $F(x)$, hence they are the same as for planar graphs. In particular $\mathbf{E}(X_n) \sim \alpha n$, where $\alpha = -F_0/F_2$ and $F(x) = F_0 + F_2(1-x/\rho) + O(1-x/\rho)^{3/2}$ is the singular expansion of $F(x)$ at ρ. $\square$

We can also adapt the proof in [22] for the largest 3-connected component. The key point is that now the relevant composition scheme is $T_g(x, D(x,y))$, instead of $T_0(x, D(x,y))$ as for planar graphs. Again, this is because with high probability a 2-connected graph of genus g has face-width ≥ 3, hence it has a unique 3-connected component of genus g and the remaining 3-connected components are planar. The most technical part of the proof in the planar case is to prove that two different probability distributions for 2-connected planar graphs are asymptotically equal, as the number of vertices and edges tends to infinity at a given ratio (see [22, Section 6.3]). One distribution comes from planar networks counted by the number of edges with an appropriate weight on vertices, but since we replace edges of the 3-connected component of genus g by *planar* networks, we are dealing with the same probability distribution. The second distribution comes from extracting the largest block in random planar graphs with a given number of vertices. But we have shown in the previous theorem that the distribution of the largest block is asymptotically independent of the genus.

The rest of the proof (see [22, Section 6.4]) is easily adapted. For computing the moments we need the asymptotic expected number of edges in a random 3-connected graph of genus g, but this is the same as for 3-connected planar graphs, since it depends only on the singularity function of $T_g(x,z)$, which we have proved does not depend on g. In conclusion, we obtain the following result, which is analogous to Theorem 6.1 from [22].

Theorem 5.5. *The size X_n of the largest 3-connected component in a random connected graph of fixed genus g with n vertices follows asymptotically an Airy law of the map type, with the same parameters as for planar graphs. In particular*

$$\mathbf{E}(X_n) \sim \alpha_2 n,$$

where $\alpha_2 \approx 0.7346$, and the size of the second largest 3-connected component is $o(n^{\frac{2}{3}+\epsilon})$, for any $\epsilon > 0$. Moreover the largest 3-connected component has genus g with high probability.

We conclude with the chromatic number of a random graph of genus g. According to Lemma 4.8, asymptotically almost surely a random graph of genus g has face-width greater than any fixed number $k(g)$. Taking $k(g) = 2^{14g+6}$ this implies that it is 5-colorable by a result of Thomassen [29]. Moreover, as mentioned just before Theorem 5.2, a random graph of genus g has a linear number of copies of K_4 with high probability, in particular has chromatic number at least 4. Consequently:

Theorem 5.6. *The chromatic number of a random graph of fixed genus g with n vertices is asymptotically almost surely in $\{4, 5\}$.*

Unfortunately, we do not know if both values 4 and 5 appear on a positive proportion of graphs of genus g. However, we conjecture the following:

Conjecture 5.1. *The chromatic number of a random graph of fixed genus g with n vertices is asymptotically almost surely equal to* 4.

More precise results hold for the list-chromatic number: as shown in [12], a graph of fixed genus is 5-choosable provided its face-width is large enough. Moreover, there exist planar graphs that are not 4-choosable [32]. If we fix any of them, then it is asymptotically almost surely contained in a random graph of genus g. Therefore we have:

Theorem 5.7. *The list-chromatic number of a random graph of fixed genus g with n vertices is asymptotically almost surely equal to* 5.

Remark. All these result also hold for the random graph of fixed non-orientable genus h, with exactly the same ingredients. Since a random graph with n vertices embeddable on $\mathbb{S}_g$ is asymptotically almost surely of genus g, and a random graph with n vertices embeddable on $\mathbb{N}_h$ is asymptotically almost surely of non-orientable genus h, these results hold also for the random graph embeddable on $\mathbb{S}_g$ and for the random graph embeddable on $\mathbb{N}_h$.

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