# COUNTING UNROOTED MAPS USING TREE-DECOMPOSITION

ÉRIC FUSY

Abstract. We present a new method to count unrooted maps on the sphere up to orientation-preserving homeomorphism. It is based on tree-decomposition and turns out to be very efficient to enumerate unrooted 2-connected and unrooted 3-connected maps. In particular, our method improves significantly on the best-known complexity to enumerate unrooted 3-connected maps, also called oriented convex polyedra.

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Résumé. Nous présentons une nouvelle méthode pour compter les cartes non-enracinées sur la sphère orientée. La méthode est basée sur la notion de décomposition en arbre et s'avère très efficace pour énumérer les cartes 2-connexes et 3-connexes non-enracinées. En particulier, notre méthode améliore significativement les meilleurs résultats de complexité pour énumérer les cartes 3-connexes non enracinées, aussi appelées polyèdres convexes orientés.

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## Introduction

The enumeration of unrooted maps has been a well-studied problem for more than 20 years. Liskovets [4] was the first one to develop a general method for the enumeration of unrooted maps on the sphere up to orientation-preserving homeomorphism. It is based on two main tools: Burnside formula and study of the quotient maps.

With an adaptation of Burnside (orbit counting) lemma, the enumeration of unrooted maps comes down to enumerating rooted maps with a symmetry (rotation) of order $k$: for a family of maps enumerated according to the number $n$ of edges, we write respectively $c_n$, $c'_n$ and $c_n^{(k)}$ for the number of unrooted maps, rooted maps and rooted maps with a symmetry of order $k$; then $c_n$ can be computed with the formula:

$$c_n = \frac{1}{2n}\left(c'_n + \sum_{k=2}^{n} \phi(k) c_n^{(k)}\right) \tag{1}$$

and a similar formula exists for the enumeration according to the number of vertices and faces, see Section 1. We represent rooted maps with a symmetry of order $k$ as $k$-rooted maps, which are maps with $k$ undistinguishable roots. Then, the *quotient map* of such a symmetric map is essentially a rooted map with two marked cells (a vertex, or the middle of a face or of an edge). The enumeration of such maps is easy to handle for the family of unconstrained maps [4], and we use these results in our article . Their approach can also be used for families of constrained maps, such as loopless maps [7], eulerian and unicursal maps [6] and 2-connected maps [5], but their treatment is less easy for these cases.

For enumeration of unrooted maps of a constrained family, instead of using the method of quotient map, we use tree-decomposition. In this article, using this method, we carry out the enumeration of unrooted 2-connected and, above all, of unrooted 3-connected maps (also done by Walsh [13]). A first tree-decomposition "by multiple edges", allows (basically) to repercute a symmetry of order $k$ of a $k$-rooted map on a symmetry of order $k$ of a $k$-rooted 2-connected map. Hence it allows to find equations linking generating functions of $k$-rooted 2-connected maps and generating functions of $k$-rooted maps, which are easy to obtain from the method of quotient map. Then a second tree-decomposition "by separating 4-cycles" allows to find equations linking generating functions of $k$-rooted 3-connected maps, and generating functions of $k$-rooted 2-connected maps, which have already been obtained thanks to the first tree-decomposition. Finally, using Equation 1, we can enumerate unrooted 2-connected and unrooted 3-connected maps.

**Main results** Two main results are obtained: a theorem about the *algebraic structure* of $k$-rooted maps and a theorem giving the complexity of enumeration of unrooted 2-connected and unrooted 3-connected maps. First we need a few notations.

Given a series $\alpha(t)$, a series $f(t)$ is said $\alpha$-rational if there exists a rational function $\mathrm{R}(\mathrm{T})$ such that $f(t) = \mathrm{R}(\alpha(t))$. Given two series in two variables $\alpha_1(t_\bullet, t_\circ)$ and $\alpha_2(t_\bullet, t_\circ)$, a series in two variables $f(t_\bullet, t_\circ)$ is said $(\alpha_1, \alpha_2)$-rational if there exists a rational expression $\mathrm{R}(\mathrm{T}_1, \mathrm{T}_2)$ in two variables such that $f(t_\bullet, t_\circ) = \mathrm{R}(\alpha_1(t_\bullet, t_\circ), \alpha_2(t_\bullet, t_\circ))$.

Now we introduce the three "easily" algebraic series in one variable (they correspond to families of trees) $\beta(x)$, $\eta(y)$ and $\gamma(z)$ [1] given by

$$\beta(x) = x + 3\beta(x)^2, \qquad \eta(y) = \frac{y}{(1-\eta(y))^2}, \qquad \gamma(z) = z(1+\gamma(z))^2$$

and their versions in two variables $\beta_{1,2}(x_\bullet, x_\circ)$, $\eta_{1,2}(y_\bullet, y_\circ)$, and $\gamma_{1,2}(z_\bullet, z_\circ)$ (corresponding to bicolored trees of the respective families) given by

$$\left\{ \begin{array}{lcl} \beta_1 & = & x_\bullet + \beta_1^2 + 2\beta_1\beta_2 \\ \beta_2 & = & x_\circ + \beta_2^2 + 2\beta_1\beta_2 \end{array} \right., \qquad \left\{ \begin{array}{lcl} \eta_1 & = & \frac{y_\bullet}{(1-\eta_2)^2} \\ \eta_2 & = & \frac{y_\circ}{(1-\eta_1)^2} \end{array} \right., \qquad \left\{ \begin{array}{lcl} \gamma_1 & = & z_\bullet(1+\gamma_2)^2 \\ \gamma_2 & = & z_\circ(1+\gamma_1)^2 \end{array} \right. .$$

**Theorem 1.**
- *All series of $k$-rooted maps, $k$-rooted 2-connected maps and $k$-rooted 3-connected maps counted according to the number of edges of their quotient map are respectively $\beta$-rational, $\eta$-rational, and $\gamma$-rational.*
- *All series of $k$-rooted maps, $k$-rooted 2-connected maps and $k$-rooted 3-connected maps counted according to the number of vertices and faces (two parameters) of their quotient map are respectively $(\beta_1, \beta_2)$-rational, $(\eta_1, \eta_2)$-rational and $(\gamma_1, \gamma_2)$-rational.*

*In particular, all these series are algebraic.*

Using algebraicity of the series of $k$-rooted maps, methods of computer algebra can be used to quickly extract their initial coefficients. Using Equation 1 (and its version in two variables if counting is done according to the num 5.3661 0 5 0 Td (er)Tj 11.7552 0 TdR34 5. 77.j 11d88 T 5.036 (378 030 0 5 0 Td (es)Tj Tj j R25 8. (an

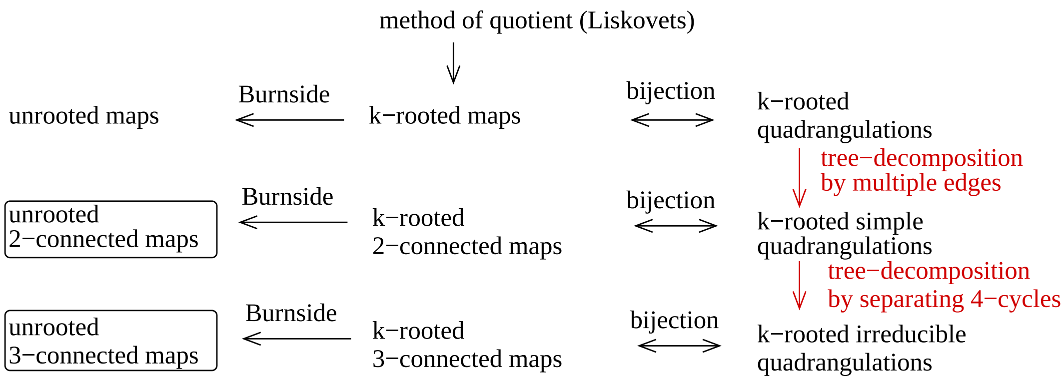


FIGURE 1. The scheme of the method to enumerate unrooted 2-connected and unrooted 3-connected maps

A map is said *2-connected* (or non-separable) if it has no loops and at least 2 of its vertices have to be removed to disconnect the map. A map is said *3-connected* if it has no loops nor multiple edges and at least 3 of its vertices have to be removed to disconnect the map.

A map is *rooted* by marking and orienting one of its edges. This operation suffices to eliminate all non trivial homeomorphism of the map. Hence, enumeration of rooted maps is more easy as we can use the root to start a recursive decomposition.

A *k-rooted map* (with $k \geqslant 2$) is a map with $k$ **undistinguishable** roots. This means that the $k$ objects obtained by marking differently (say, in blue) one of the $k$ roots are equal. Rooted maps endowed with an automorphism of order $k$ are in bijection with $k$-rooted maps (see [4] for more details). As $k$-rooted maps are easier to handle for our purpose, we will manipulate them rather than rooted maps with an automorphism of order $k$.

1.2. **Quadrangulations.** A *quadrangulation* is a map whose all faces have degree 4. A quadrangulation is said *simple* if it has no multiple edge. A quadrangulation is said *irreducible* if each 4-cycle of edges of the quadrangulation is the contour of one of its faces.

For each quadrangulation, its vertices can be colored in black and white such that each edge connects a black and a white vertex. Such a bicoloration is unique up to the choice of the colors. A quadrangulation endowed with such a bicoloration is said *bicolored*.

1.3. **Structure of $k$-rooted maps and method of quotient maps.** It was observed by Liskovets [4] that a $k$-rooted map can be realized by an embedding on the geometrical sphere which is invariant by a certain rotation of angle $2\pi/k$ of the sphere [2]. In addition, the points of the sphere crossed by the rotation-axis are either a vertex or the centre of a face, and can also be the middle of an edge if $k = 2$. These points are called the *poles* of the $k$-rooted map. The *type* of a $k$-rooted map is the type of its two poles. For example, if the two poles are a vertex and a face, then the $k$-rooted map is said to have type face-vertex.

Then, if we cut the sphere of the symmetrical embedding along two meridians forming a dihedral angle of $2\pi/k$, we extract a sector of the map borded by these two meridians. By pasting together the two meridians, the sector becomes a map on the sphere. The symmetry of order $k$ of the initial geometrical embedding ensures that this map is independant of the choice of the two meridians. We call this map the *quotient-map* of the $k$-rooted map. Remark that this quotient map has one root and two marked cells (the poles of the $k$-rooted map). The method of quotient maps developed by Liskovets consists in counting $k$-rooted maps of a family by studying the structure of their quotient map. In the case of unconstrained maps, it works very well, as quotient maps are essentially rooted maps with two marked cells.

[2]This point of view is not topologically relevant but it helps to have a geometrical intuition and it allows to define nicely the quotient of a $k$-rooted map.

**1.4. Burnside formula adapted to unrooted maps.** Consider a family of maps on the sphere (for example the family of 2-connected maps). Let $c_n$, $c'_n$ and $c_n^{(k)}$ denote respectively the number of unrooted, rooted and $k$-rooted maps of the family with $n$ edges. Let $c_{ij}$, $c'_{ij}$ and $c_{ij}^{(k)}$ denote respectively the number of unrooted, rooted and $k$-rooted maps of the family with $i+1$ vertices and $j+1$ faces. Then, Burnside (orbit counting) formula was adapted by Liskovets [4] to give the two following enumerative formulas for unrooted maps, where $\phi()$ is Euler totient function.

(2) $$2nc_n = c'_n + \sum_k \phi(k)c_n^{(k)} \qquad\qquad 2(i+j)c_{ij} = c'_{ij} + \sum_k \phi(k)c_{ij}^{(k)}$$

As a consequence, enumeration of unrooted maps in one parameter (resp. two parameters) comes down to the enumeration of rooted maps (already done for 2-connected and 3-connected maps, see [8]) and of $k$-rooted maps of the family with one parameter (resp. two parameters).

**1.5. Bijection between maps and quadrangulations.** A classical result in map theory is a bijection between maps and bicolored quadrangulations, that we shall refer to as Tutte's application. We just detail its properties here. Tutte's application is a bijection between maps with $n$ edges (resp. with $i$ vertices and $j$ faces) and bicolored quadrangulations with $n$ faces (resp. with $i$ black and $j$ white vertices). Indeed, by this bijection, vertices, faces and edges of a map correspond respectively to black vertices, white vertices and faces of the bicolored quadrangulation.

In addition, under Tutte's application, rooted maps are in bijection with rooted quadrangulations and $k$-rooted maps are in bijection with $k$-rooted bicolored quadrangulations, which can also be seen as $k$-rooted quadrangulations such that the origins of the $k$ roots have the same color when the quadrangulation is bicolored. We will only deal with such $k$-rooted quadrangulations and will shortly call them $k$-rooted quadrangulations. Remark that the type of a $k$-rooted map and the type of its associated $k$-rooted quadrangulation are linked by the above-mentioned correspondance (for example 2-rooted maps with type edge-face are in bijection with 2-rooted quadrangulations with type face-white vertex), so that a $k$-rooted quadrangulation can only have type vertex-vertex if $k>2$, and can also have type face-face and type face-vertex if $k=2$.

Moreover, Tutte's application has the nice property that 2-connected maps are in bijection with bicolored simple quadrangulations and 3-connected maps are in bijection with bicolored irreducible quadrangulations. As a consequence, thanks to Tutte's bijection, the enumeration of $k$-rooted 2-connected maps by number of edges (resp. by numbers of vertices and faces) comes down to the enumeration of $k$-rooted simple quadrangulations by number of faces (resp. by numbers of black vertices and white vertices). The situation is the same for 3-connected maps, but with irreducible quadrangulations instead of simple quadrangulations, see Figure 1.

**1.6. Notations.** We will use the letters F, $g$ and $q$ to denote respectively generating functions of $k$-rooted, $k$-rooted simple and $k$-rooted irreducible quadrangulations. We will use the subscripts $f$, $v$, $b$ and $w$ to denote respectively a pole which is a face, a vertex, a black vertex and a white vertex (the subscripts $b$ and $w$ are only used for generating functions with two parameters, where we have to take the bicoloration into account). For example, $g_{vv}^{(k)}(y)$ is the series counting $k$-rooted simple quadrangulations of type vertex-vertex by the number of faces in their quotient map, and $q_{bw}^{(k)}(z_\bullet, z_\circ)$ is the series counting $k$-rooted irreducible quadrangulations, whose poles are a black and a white vertex, by the number of black and white vertices in their quotient map and without counting the two axial vertices.

**Lemma 3.** *All generating functions of $k$-rooted quadrangulations in one (resp. two) variable are $\beta$-rational (resp. $(\beta_1, \beta_2)$-rational).*

*Proof.* From the method of quotient-map of Liskovets, the quotient-map of a $k$-rooted quadrangulation is essentially a quadrangulation with two marked cells (these cells can be a vertex or also a face if $k=2$). Hence the series counting these objects involve the first and second derivatives (or partial derivatives for two variables) of the series F counting rooted quadrangulations. This series is well-known to be $\beta$-rational in one variable [2] and $(\beta_1, \beta_2)$-rational in two variables [1] (see [10] for a combinatorial explanation). In

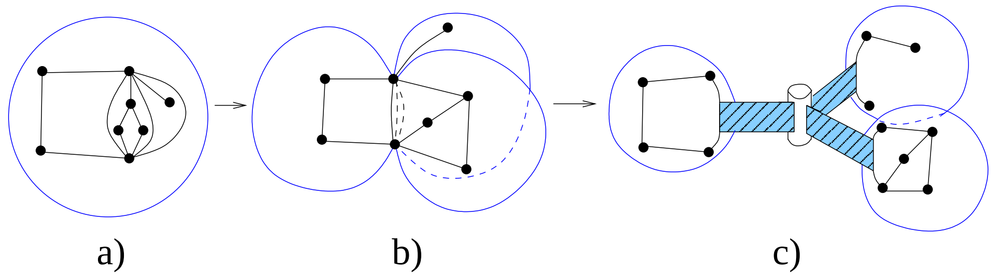


FIGURE 2. The tree-decomposition by multiple edges of a quadrangulation.

addition, the fact of being $\beta$-rational (resp. $(\beta_1, \beta_2)$-rational) can easily be proved to be stable under derivation. Indeed, $d\mathrm{F}/dx = (d\mathrm{F}/d\beta)/(dx/d\beta)$ is the quotient of two $\beta$-rational expressions, and we can proceed similarly for two variables. The result follows. □

## 2. Tree-decompositions

**2.1. Tree-decomposition by multiple edges.** We explain here how to transform an unrooted quadrangulation Q (that may have multiple edges) in a tree with two kinds of nodes: nodes representing multiple edges and nodes representing simple quadrangulations.

One way to see this decomposition is as follows. Take a multiple edge of Q of multiplicity $d$. Cut the sphere along each of the $d$ edges forming the multiple edge. In this way we obtain $d$ sectors, each sector being delimited by two consecutive edges of the multiple edge. Now, for each sector, identify the two meridians corresponding to the two edges delimiting the sector by pasting them together. Thus we make out of each sector a map on the sphere and we can link these $d$ maps, at their edge corresponding to the initial multiple edge, around a new node: this will be the node of the tree corresponding to the multiple edge. Now we can carry on recursively the tree-decomposition for each of the $d$ maps, until all multiple edges have been split into nodes of the tree.

Another way to see this decomposition is to imagine that we do not cut along the edges of the multiple edge, but that we "blow" equally, from the interior of the sphere, each of the $d$ sectors delimited by the multiple edge. We obtain thus $d$ components drawn each on a sphere, where the $d$ spheres are connected (glued) at the multiple edge, see Figure 2b. We can then represent this multiple edge as a rigid link (see Figure 2c) around which the $d$ components are linked via their unique edge belonging to the multiple edge. We can also here carry on the decomposition for each of the $d$ components.

**2.2. Tree-decomposition by separating 4-cycles.** In this section we transform a simple quadrangulation with at least 3 faces into a tree with two kinds of nodes: so-called axis-nodes and nodes corresponding to irreducible quadrangulations. The description of this tree-decomposition can also be found in [3]. We describe first the tree-decomposition for rooted objects and we will see then that we can also see the tree-decomposition on unrooted objects.

Let us first define the axis-map with $k$ faces ($k \geqslant 3$) as the simple quadrangulation consisting of two pole-vertices linked by $k$ parallel chains of 2 edges, each couple of two consecutive paths forming one of the $k$ faces of the axis-map, see Figure 3a.

Now we state the following lemma of decomposition of a rooted simple quadrangulation Q with at least 3 faces:

**Lemma 4.** *There exists a unique rooted quadrangulation* $\mathrm{Q}_0$*, with maximal possible number* $k+1$ *of faces such that:*

- $\mathrm{Q}_0$ *is an axis-map or an irreducible quadrangulation.*
- *There are* $k$ *rooted simple quadrangulations* $\mathrm{Q}_1, \ldots, \mathrm{Q}_k$ *with at least 2 faces such that* Q *can be seen as the quadrangulation* $\mathrm{Q}_0$ *where each of the* $k$ *non root faces* $f_i$ *of* $\mathrm{Q}_0$ *is substituted in a canonical*

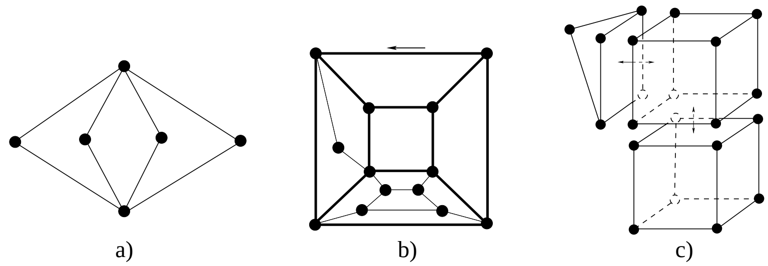


FIGURE 3. An axis-map with 4 faces (a). The tree-decomposition of a quadrangulation by separating 4-cycles, performed with a root (b) or without a root (c).

*way by one of the* $Q_i$, $1 \leqslant i \leqslant k$, *the contour of* $f_i$ *being replaced by the contour of the root face of* $Q_i$.

*Proof.* If there exists an internal chain of length 2 between two opposite vertices of the outer face of Q, take the sequence of all chains of length 2 (including the 2 outer ones) between these two vertices. Forgetting all other edges, we get an axis-map. Hence Q can be seen as this axis-map where each non root face is substituted by a quadrangulation.

Otherwise, define a proper 4-cycle of Q as a 4-cycle of edges different from the contour of the root face of Q. Here we have to see Q as drawn on the plane with its root face as infinite face, so that we can distinguish interior and exterior. A proper 4-cycle is said maximal if it is not strictly included in the interior of any other proper 4-cycle. It can easily be shown (see [8]) that the interiors of maximal proper 4-cycles partition the interior of Q. Let $Q_0$ be the rooted quadrangulation obtained from Q by keeping the contour of the root face and of the maximal proper 4-cycles of Q. The quadrangulation $Q_0$ is trivially irreducible by maximallity of the 4-cycles of which we have kept the contour. Hence we are in the case where Q can be seen as a rooted irreducible quadrangulation where each inner face is substituted by a rooted quadrangulation. □

The first (resp. second) case of Lemma 4 correspond to the case where the root node of the (rooted) decomposition-tree is an axis-node (resp. a node which is an irreducible quadrangulation). For example, on Figure 3b, the rooted quadrangulation can be seen as a (rooted) cube where two faces are substituted by another cube and an axis-map with 3 faces.

**Remark** We make the following distinction for the case of an axis-node: if the parallels chains of length 2 are incident to the origin of the root, the root node of the tree is said a *vertical* axis-node, otherwise, it is said an *horizontal* axis-node.

Now we can carry on the tree-decomposition for each rooted quadrangulation $Q_i$ and get finally a (rooted) decomposition-tree with axis nodes and nodes which are irreducible quadrangulations. Remark that, if $Q_0$ and the root node of one of the $Q_i$ are simultaneously axis-nodes, then they are stretched in perpendicular directions by maximallity of the number of faces of $Q_0$.

Observe that the preceding decomposition on rooted objects ensures that, as in Section 2.1, we can "blow" from the interior of the sphere to "sculpt" the quadrangulation Q in a tree with nodes which are irreducible quadrangulations and axis-nodes, these nodes being connected (glued) at so-called *interconnection-faces*, see Figure 3c. Hence we can say that an unrooted simple quadrangulation "is" its tree-decomposition (after a judicious deformation of the sphere). We see thus that the geometrical shape of the tree in the space does not depend on the face of the quadrangulation where we choose to place the root to start the tree-decomposition.

2.3. **Centre of a tree.** The *centre* of a tree T is defined in the following recursive way. If T is reduced to an edge or a node, then the centre of T is this edge (resp. this node). Otherwise, remove all leaves of T to obtain a (shrinked) tree $\widetilde{T}$. Then the centre of T is defined to be the centre of $\widetilde{T}$.

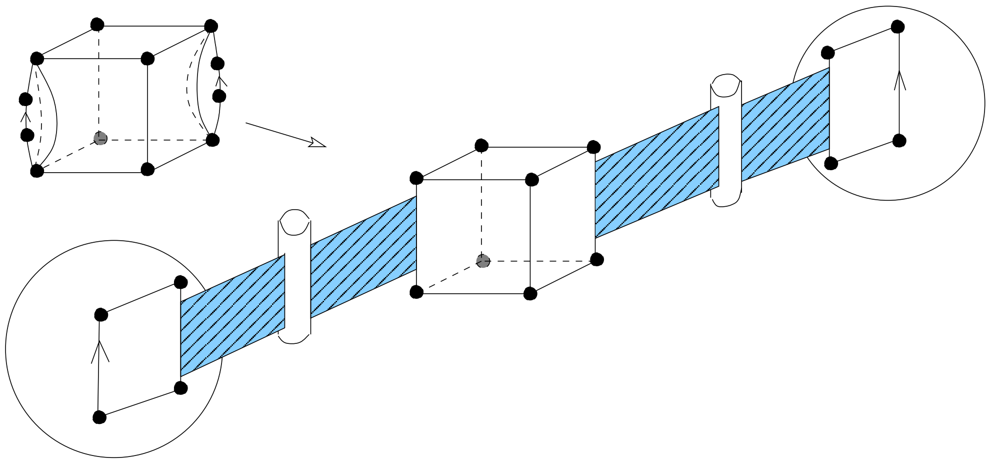

FIGURE 4. Repercussion of the symmetry of a $k$-rooted quadrangulation on its decomposition-tree.

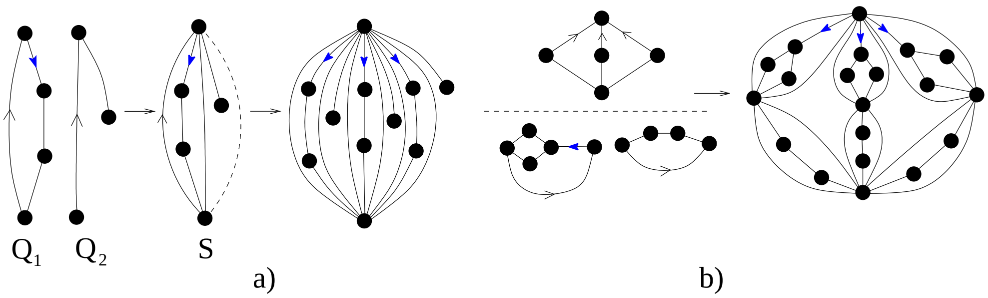


FIGURE 5. Construction of a $k$-rooted quadrangulation of type $a$ (Figure a), and of type $b$ (Figure b).

The important point is that the definition does not need that T is rooted, hence the centre is invariant under any symmetry of T.

## 3. Using the tree-decomposition by multiple edges to enumerate unrooted 2-connected maps

**3.1. Repercussion of the symmetry of a $k$-rooted quadrangulation on its decomposition-tree.** As we have seen, the tree-decomposition by multiple edges of a quadrangulation Q can be seen as a deformation of the sphere on which Q is drawn and by splitting multiple edges into links so as to form a decomposition-tree "living" in the 3D-space. In addition, if Q is $k$-rooted, then its decomposition-tree is invariant under the symmetry (rotation) of order $k$ induced by its $k$-root. Hence, the centre of the tree is fixed by the symmetry, see Figure 4. This centre can be a node or an edge of the tree. However, the case of an edge is excluded because an edge of the tree always links a node of type "multiple edge" and a node of type "simple quadrangulation", hence an edge of the tree can not be invariant under a non-trivial symmetry of the tree. As a consequence, the centre is a node and there are two cases: either it is a node of type multiple edge -we say that Q has type $a$- or it is a node of type simple quadrangulation -we say that Q has type $b$-.

3.2. **Case where the centre is a multiple edge (type $a$).** First we need to define a *simply rooted* quadrangulation as a quadrangulation whose root edge does not belong to a multiple edge. We also define a bi-rooted quadrangulation as a quadrangulation having a secondary root which is differently marked (say in blue).

Now we explain how to construct a $k$-rooted quadrangulation whose centre of the decomposition-tree is a multiple edge with multiplicity $k\cdot d$ ($d\geqslant 1$), see Figure 5a. Take a bi-rooted, simply-rooted (i.e. whose primary root is a simple edge) quadrangulation $\mathrm{Q}_1$. Cut it along its primary root-edge, thus transforming $\mathrm{Q}_1$ into a sector with two bording meridians. Among these two meridians, we call root-meridian the one corresponding to the right part of the cutted edge (we imagine that the edge we have cut along has a "width").

Now take $d-1$ simply rooted quadrangulations $\mathrm{Q}_2,\dots,\mathrm{Q}_d$ and perform the same cutting operation as for $\mathrm{Q}_1$. Then paste the root meridian of $\mathrm{Q}_2$ with the non-root meridian of $\mathrm{Q}_1$, the pasting operation being such that the orientations of the roots of the two sectors coincide. Then, iteratively, for each $i\leqslant d$, paste the root meridian of $\mathrm{Q}_i$ with the non-root meridian of $\mathrm{Q}_{i-1}$.

We obtain finally a big sector S whose root meridian is the root meridian of $\mathrm{Q}_1$. Now make $k$ copies $\mathrm{S}_1,\dots,\mathrm{S}_k$ of S and, for each $1\leqslant i\leqslant k$, paste the root meridian of $\mathrm{S}_i$ with the non-root meridian of $\mathrm{S}_{i-1}$. In this way we obtain finally a quadrangulation consisting of $k$ identical sectors, each carrying a blue root (the secondary root of $\mathrm{Q}_1$). By erasing the mark of the primary root of $\mathrm{Q}_1$ and of the roots of $\mathrm{Q}_2\dots\mathrm{Q}_d$ in each sector, we obtain a $k$-rooted quadrangulation of type $a$. Remark that each $k$-rooted quadrangulation of type $a$ is obtained exactly twice by this construction. Indeed, the inverse operation consists in choosing an extremity $v$ (two possibilities) of the central multiple edge and then orienting all edges of the multiple edge toward $v$.

Writing $f(x)$ for the series counting simply rooted quadrangulations by their number of faces, this construction gives the series counting $k$-rooted quadrangulations of type $a$:

$$\frac{1}{2}(4xf'(x)).\frac{1}{1-f(x)}$$

In addition, all objects constructed in this way have clearly type vertex-vertex.

3.3. **Case where the centre is a simple quadrangulation (type $b$).** Here we give a construction of $k$-rooted quadrangulations of type $b$ as composed objects, see Figure 5b. Take a $k$-rooted simple quadrangulation $\mathrm{Q}_s$. For the $k$-orbite of root edges, either leave its $k$ edges untouched or perform the following operation. Take a bi-rooted quadrangulation $\widetilde{\mathrm{Q}}$. Then cut $\mathrm{Q}_s$ along each of its $k$ root edges and cut $\widetilde{\mathrm{Q}}$ along its primary root edge, transforming $\widetilde{\mathrm{Q}}$ into a sector bordered by two meridians. Take $k$ copies of this sector and for each (cutted) root-edge $e$ of $\mathrm{Q}_s$, place a copy of the sector in the empty sector of $\mathrm{Q}_s$ leaved by the cutting of $e$, pasting the two meridians of the sector with the two border-edges of $\mathrm{Q}_s$ created by the cutting of $e$, and making the orientation of $e$ and of the primary root edge of $\widetilde{\mathrm{Q}}$ coincide.

Proceed similarly for each $k$-orbite of non-root edges of $\mathrm{Q}_s$, with the only difference that the quadrangulation $\widetilde{\mathrm{Q}}$ used for the substitution is not bi-rooted but just rooted. Finally, by removing all marks of the primary root of the substituted quadrangulations, we obtain a $k$-rooted quadrangulation Q of type $b$.

Remark that $k$-rooted quadrangulations of type $b$ obtained by this construction are such that their $k$ root edges are simple i.e their incident face (the face on their right) belongs to the central simple quadrangulation. The missing $k$-rooted quadrangulations of type $b$ are obtained by the same construction, with the difference that we always cut the $k$ root edges of $\mathrm{Q}_s$, splitting each of them into two edges, and we carry the root on the one having the empty sector on its left. Then the other difference is that the first substituted quadrangulation $\widetilde{\mathrm{Q}}$ is not bi-rooted but just rooted. At the end of this construction, we only keep the mark of the $k$ roots of $\mathrm{Q}_s$

Similarly as in Section 3.2, these two complementary constructions allow to obtain all $k$-rooted quadrangulations of type $b$ exactly twice. Writing $\mathrm{F}(x)$ for the series counting rooted quadrangulations by their number of faces and $\mathrm{E}(x)=2x\mathrm{F}'(x)+\mathrm{F}(x)+1$, this construction gives the following three series, depending on the type of $\mathrm{Q}_s$

$$\frac{\mathrm{E}(x)}{1+\mathrm{F}(x)} g_{vv}^{(k)}\left((1+\mathrm{F}(x))^2\right), \quad \mathrm{E}(x) g_{fv}^{(2)}\left((1+\mathrm{F}(x))^2\right), \quad \mathrm{E}(x)(1+\mathrm{F}(x)) g_{ff}^{(2)}\left((1+\mathrm{F}(x))^2\right).$$

3.4. **Obtaining the equations.** As $k$-rooted quadrangulations are partitioned in two sets whether the centre of their decomposition-tree is a multiple edge or a simple quadrangulation, taking the sum of the series obtained in Section 3.2 and Section 3.3, we obtain the following equations:

$$\mathrm{F}_{vv}^{(k)}(x) = 2\frac{xf'(x)}{1-f(x)} + \frac{\mathrm{E}(x)}{1+\mathrm{F}(x)} g_{vv}^{(k)}\left((1+\mathrm{F}(x))^2\right) \tag{3}$$

$$\mathrm{F}_{fv}^{(2)}(x) = \mathrm{E}(x) g_{fv}^{(2)}\left((1+\mathrm{F}(x))^2\right) \tag{4}$$

$$\mathrm{F}_{ff}^{(2)}(x) = \mathrm{E}(x)(1+\mathrm{F}(x)) g_{ff}^{(2)}\left((1+\mathrm{F}(x))^2\right) \tag{5}$$

where the only unknown series are $g_{vv}^{(k)}$, $g_{fv}^{(2)}$ and $g_{ff}^{(2)}$.

Similar equations can be easily obtained in two variables by taking the bicoloration of vertices into account. Writing $df(x_\bullet, x_\circ) = \frac{d}{dt} f(tx_\bullet, tx_\circ)_{t=1}$ and adapting E in two variables as $\mathrm{E}(x_\bullet, x_\circ) = 2\frac{d}{dt}\mathrm{F}(tx_\bullet, tx_\circ)_{t=1} + \mathrm{F}(x_\bullet, x_\circ) + 1$, Equation 3 becomes for example:

$$\begin{cases} \mathrm{F}_{bw}^{(k)}(x_\bullet, x_\circ) = 2\frac{df}{1-f} + \frac{\mathrm{E}}{1+\mathrm{F}} g_{bw}^{(k)}\left(x_\bullet(1+\mathrm{F})^2, x_\circ(1+\mathrm{F})^2\right) \\ \mathrm{F}_{bb}^{(k)}(x_\bullet, x_\circ) = \frac{\mathrm{E}}{1+\mathrm{F}} g_{bb}^{(k)}\left(x_\bullet(1+\mathrm{F})^2, x_\circ(1+\mathrm{F})^2\right) \\ \mathrm{F}_{ww}^{(k)}(x_\bullet, x_\circ) = \frac{\mathrm{E}}{1+\mathrm{F}} g_{ww}^{(k)}\left(x_\bullet(1+\mathrm{F})^2, x_\circ(1+\mathrm{F})^2\right) \end{cases} \tag{6}$$

where all series (including $f$ and F) have two variables, one for the number of black vertices, the other one for the number of white vertices.

Remark that, as the series $\mathrm{F}_{vv}^{(k)}$ (in one or two variables) does not depend on $k$ as was observed in Lemma 3, it follows from the form of Equation 3 and 6 that the series $g_{vv}^{(k)}$ does not depend on $k$, hence the exponent $(k)$ can be ommited.

**Lemma 5.** *The series $g$ counting rooted simple quadrangulations and all series of $k$-rooted simple quadrangulations in one variable (resp. two variables) are $\eta$-rational (resp. $(\eta_1, \eta_2)$-rational).*

*Proof.* Using Lemma 3, we know that $\mathrm{F}(x)$, $\mathrm{F}_{vv}(x)$, $\mathrm{F}_{fv}(x)$ and $\mathrm{F}_{ff}(x)$ are $\beta$-rational, and so are $x$ (because $x = \beta - 3\beta^2$), $f(x)$ (because $\mathrm{F} = f/(1-f)$), and $\mathrm{E}(x)$. Hence it follows from Equations 3, 4 and 5 that $g_{vv}\left(x(1+\mathrm{F})^2\right)$, $g_{fv}\left(x(1+\mathrm{F})^2\right)$ and $g_{ff}\left(x(1+\mathrm{F})^2\right)$ are $\beta$-rational. Now we have to make the change of variable $y = x(1+\mathrm{F})^2$. It can easily be proved (or found in [2]) that $\beta(x) = \eta(y)/(1+3\eta(y))$ when $y$ and $x$ are linked by the change of variable $y = x(1+\mathrm{F})^2$. Hence, replacing $\beta(x)$ by $\eta(y)/(1+3\eta(y))$ in the $\beta$-rational expression of $g_{vv}\left(x(1+\mathrm{F})^2\right)$, $g_{fv}\left(x(1+\mathrm{F})^2\right)$ and $g_{ff}\left(x(1+\mathrm{F})^2\right)$, we obtain $\eta$-rational expressions for $g_{vv}(y)$, $g_{fv}(y)$ and $g_{ff}(y)$. Finally, $g(y)$ is $\eta$-rational from [2].

We can proceed similarly in two variables, using the fact that $\beta_1(x_\bullet, x_\circ)$ and $\beta_2(x_\bullet, x_\circ)$ have a rational expression in terms of $\eta_1(y_\bullet, y_\circ)$ and $\eta_2(y_\bullet, y_\circ)$ when $(y_\bullet, y_\circ)$ and $(x_\bullet, x_\circ)$ are linked by the change of variable $(y_\bullet, y_\circ) = (x_\bullet(1+\mathrm{F})^2, x_\circ(1+\mathrm{F})^2)$. □

**Lemma 6.** *The* N *initial coefficients counting unrooted 2-connected maps according to their number of edges can be computed with $\mathcal{O}(\mathrm{N}\log(\mathrm{N}))$ operations.*

*The table of initial coefficients with indices $(i, j)$ and $i + j \leqslant \mathrm{N}$ counting unrooted 2-connected maps according to their number of vertices and faces can be computed with $\mathcal{O}(\mathrm{N}^2)$ operations.*

*Proof.* First, we use the following notation. For a series $f$ in one variable (resp. two variables), we denote by $\mathcal{C}_\mathrm{N}(f)$ the number of operations necessary to extract its N initial coefficients (resp. its coefficients with indices $(i, j)$ and $i + j \leqslant \mathrm{N}$). Writing $c_n$ (resp. $c_{ij}$) for the number of unrooted 2-connected maps with $n$ edges (resp. $i+1$ vertices and $j+1$ faces), Equation 2 (Burnside formula) can be easily transposed in the following equations on series:

$$\sum_n 2nc_n y^n = g(y) + y g_{fv}(y^2) + y^2 g_{ff}(y^2) + \sum_{k\geqslant 2} \phi(k) g_{vv}(y^k)$$

$$\sum_{i,j} 2(i+j)c_{ij} y_\bullet^i y_\circ^j = g(y_\bullet, y_\circ) + y_\bullet g_{fb}(y_\bullet^2, y_\circ^2) + y_\circ g_{fw}(y_\bullet^2, y_\circ^2) + y_\bullet y_\circ g_{ff}(y_\bullet^2, y_\circ^2)$$
$$+ \sum_{k\geqslant 2} \phi(k) \left( y_\bullet/y_\circ g_{bb}(y_\bullet^k, y_\circ^k) + g_{bw}(y_\bullet^k, y_\circ^k) + y_\circ/y_\bullet g_{ww}(y_\bullet^k, y_\circ^k) \right)$$

According to Lemma 5, $g(y)$, $g_{fv}(y)$, $g_{ff}(y)$ and $g_{vv}(y)$ are $\eta$-rational, hence they are algebraic (because they live in the algebraic extension of the algebraic series $\eta(y)$). As a consequence, they are differentiably finite (see [11]), i.e. solution of a linear differential equation with polynomial coefficients. Taking coefficient $[y^n]$ in this differential equation yields that the coefficients of these series verify a linear recurrence with polynomial coefficients. As a consequence, the N initial coefficients of these series can be computed with $\mathcal{O}(\mathrm{N}\log(\mathrm{N}))$ "arithmetical" operations, which are the multiplication of a "small" integer with $\mathcal{O}(\log(\mathrm{N}))$ bits and of a "large" integer with $\mathcal{O}(\mathrm{N})$ bits (same operations as in [14]). Hence, $\mathcal{C}_\mathrm{N}(\sum 2nc_n) = \mathcal{C}_\mathrm{N}(g) + \mathcal{C}_{\mathrm{N}/2}(g_{fv}+g_{ff}) + \sum_{k=2}^{\mathrm{N}} \mathcal{C}_{\mathrm{N}/k}(g_{vv}) = \mathcal{O}(\mathrm{N}) + \mathcal{O}(\mathrm{N}/2) + \sum_{k=2}^{\mathrm{N}} \mathcal{O}(\mathrm{N}/k) = \mathcal{O}(\mathrm{N}\log(\mathrm{N}))$.

Similarly, the coefficients of an algebraic series in two variables "essentially" verify a linear recurrence, this time with two indices. As a consequence, if $f(y_\bullet, y_\circ)$ is algebraic, then $\mathcal{C}_\mathrm{N}(f) = \mathcal{O}(\mathrm{N}^2)$. As series of $k$-rooted simple quadrangulations in two variables are $(\eta_1, \eta_2)$-rational, they are algebraic. Hence, $\mathcal{C}_\mathrm{N}\left(\sum_{i,j} 2(i+j)c_{ij}\right) = \mathcal{C}_\mathrm{N}(g) + \mathcal{C}_{\mathrm{N}/2}(g_{ff}+g_{fb}+g_{fw}) + \sum_{k=2}^{\mathrm{N}} \mathcal{C}_{\mathrm{N}/k}(g_{bb}+g_{bw}+g_{ww}) = \mathcal{O}(\mathrm{N}) + \mathcal{O}((\mathrm{N}/2)^2) + \sum_{k=2}^{\mathrm{N}} \mathcal{O}((\mathrm{N}/k)^2) = \mathcal{O}(\mathrm{N}^2)$ where we use the fact that $\sum_k 1/k^2$ converges. □

## 4. Using the tree-decomposition by separating 4-cycles to enumerate unrooted 3-connected maps

**4.1. Repercussion of the symmetry of a $k$-rooted simple quadrangulation on its decomposition-tree.** First we introduce the families $\mathcal{W}$ of rooted simple quadrangulations with at least two faces and the family $\mathcal{G}$ consisting of the objects of $\mathcal{W}$ whose root node of the decomposition tree is not an horizontal axis-node. We write $\mathrm{W}(y)$ and $\mathrm{G}(y)$ for the series counting these families by their number of faces (notations of [3]). Remark that $\mathrm{W}(y) = g(y) - 2y$ and $\mathrm{W}(y)/y = \frac{\mathrm{G}(y)/y}{1-\mathrm{G}(y)/y}$. We define also the families $\mathcal{W}'$ and $\mathcal{G}'$ of objects of $\mathcal{W}$ and $\mathcal{G}$ having a secondary root incident to a face different from the root face. The series counting objects of $\mathcal{W}'$ and $\mathcal{G}'$ by their number of faces are respectively $4\mathrm{C}(y)$ and $4\mathrm{B}(y)$ where $\mathrm{C}(y) = y^2 \frac{d}{dy}(\mathrm{W}(y)/y)$ and $\mathrm{B}(y) = y^2 \frac{d}{dy}(\mathrm{G}(y)/y)$.

Let Q be a simple $k$-rooted quadrangulation with at least 3 faces. Here we work with $k \geqslant 3$. The case $k=2$ is more difficult (for example a symmetry of order 2 of an axis-map can exchange its poles), but can also be thoroughly treated, see the full version. As in Section 3.1, the decomposition tree of Q is invariant under the symmetry of order $k$ induced by the $k$-root of Q. Hence, the centre of the tree (which is a node because $k > 2$) is invariant by the symmetry. Also here two cases arise: either the centre is an axis-node -we say that Q has type $\mathfrak{a}$- or it is an irreducible quadrangulation -we say that Q has type $\mathfrak{b}$-.

**4.2. Construction of $k$-rooted simple quadrangulations of type $\mathfrak{a}$.** Similarly as in Section 3.2, we construct a $k$-rooted simple quadrangulation, whose centre of the decomposition tree is an axis-map with $k \cdot d$ faces, as a composed object. Take a $k$-rooted axis-map with $k \cdot d$ faces and whose all roots point toward a pole of the axis-map, that we call the *north pole*. Then take $k$ copies of an object $\mathrm{Q}_1$ of $\mathcal{G}'$ and substitute each root face of the axis-map by one of these copies, making the primary root of the copies of $\mathrm{Q}_1$ be oriented toward the north pole of the axis-map. Proceed similarly for each $k$-orbit of non-root faces of the axis-map, with the only difference that the substituted objects are $k$ copies of an object of $\mathcal{G}$ instead of $\mathcal{G}'$. Finally keep only the marks of the secondary root of the $k$ copies of $\mathrm{Q}_1$.

As in Section 3.2, each $k$-rooted simple quadrangulation of type $\mathfrak{a}$ is obtained exactly twice by this construction. The series counting $k$-rooted simple quadrangulations of type $\mathfrak{a}$ is:

$$2\frac{\mathrm{B}(y)}{y}\frac{1}{1-\mathrm{G}(y)/y}$$

and all these objects have type vertex-vertex.

4.3. **Construction of $k$-rooted simple quadrangulations of type $\mathfrak{b}$.** As precedently, we give a construction of $k$-rooted simple quadrangulations of type $\mathfrak{b}$ as composed objects. Take a $k$-rooted irreducible quadrangulation $\mathrm{Q}_{irr}$ (remark that $\mathrm{Q}_{irr}$ has type vertex-vertex because $k>2$). Take $k$ copies of an object $\mathrm{Q}_1$ of $\mathcal{W}'$ and substitute each face of $\mathrm{Q}_{irr}$ by one of the copies of $\mathrm{Q}_1$ in a "canonical" way, e.g. by superposing the primary root edge of $\mathrm{Q}_1$ with the root edge of the face where the substitution takes place. Then proceed similarly for each $k$-orbite of non-root faces of $\mathrm{Q}_{irr}$, except that the substituted objects are $k$ copies of an object of $\mathcal{W}$ instead of $\mathcal{W}'$. Finally keep only the marks of the secondary root of the $k$ copies of $\mathrm{Q}_1$.

By this construction, all $k$-rooted simple quadrangulations of type $\mathfrak{b}$ are obtained exactly 4 times. Indeed, as a quadrangular face has 4 sides, there are 4 possibilities to guess the primary root edge of the $k$ copies of $\mathrm{Q}_1$. We obtain the following series counting $k$-rooted simple quadrangulations of type $\mathfrak{b}$:

$$\frac{\mathrm{C}(y)}{\mathrm{W}(y)}q_{vv}^{(k)}(\mathrm{W}(y)/y)$$

4.4. **Obtaining the equations.** As $k$-rooted simple quadrangulations are partitioned in two sets whether the center of their decomposition tree is an axis-node or an irreducible quadrangulation, summing the series obtained in Section 4.2 and Section 4.3, we obtain the following equation linking series of $k$-rooted simple quadrangulations with series of $k$-rooted irreducible quadrangulations:

$$g_{vv}^{(k)}(y)=2\frac{\mathrm{B}(y)}{y}\frac{1}{1-\mathrm{G}(y)/y}+\frac{\mathrm{C}(y)}{\mathrm{W}(y)}q_{vv}^{(k)}(\mathrm{W}(y)/y) \tag{7}$$

Similar equations can be easily obtained in two variables by taking the bicoloration of Q into account. Writing $\mathrm{C}(y_\bullet,y_\circ)=y_\bullet\frac{\partial \mathrm{W}}{\partial y_\bullet}+y_\circ\frac{\partial \mathrm{W}}{\partial y_\circ}-\mathrm{W}$ and $\mathrm{B}(y_\bullet,y_\circ)=y_\bullet\frac{\partial \mathrm{G}}{\partial y_\bullet}+y_\circ\frac{\partial \mathrm{G}}{\partial y_\circ}-\mathrm{G}$ for the versions in two variables of $\mathrm{C}(y)$ and $\mathrm{B}(y)$, the version in two variables of Equation 7 becomes:

$$\begin{aligned}
g_{bb}^{(k)}(y_\bullet,y_\circ) &= \frac{\mathrm{B}}{y_\bullet}\frac{1}{1-\mathrm{G}/y_\bullet}+\frac{\mathrm{C}}{\mathrm{W}}q_{bb}^{(k)}(\mathrm{W}/y_\circ,\mathrm{W}/y_\bullet)\\
g_{ww}^{(k)}(y_\bullet,y_\circ) &= \frac{\mathrm{B}}{y_\circ}\frac{1}{1-\mathrm{G}/y_\circ}+\frac{\mathrm{C}}{\mathrm{W}}q_{ww}^{(k)}(\mathrm{W}/y_\circ,\mathrm{W}/y_\bullet)\\
g_{bw}^{(k)}(y_\bullet,y_\circ) &= \frac{\mathrm{C}}{\mathrm{W}}q_{bw}^{(k)}(\mathrm{W}/y_\circ,\mathrm{W}/y_\bullet)
\end{aligned}$$

Remark that these equations are the same for all values of $k$. As we have already seen that $g_{vv}^{(k)}(y)$ does not depend on $k$, then $q_{vv}^{(k)}(z)$ does not depend on $k$ so that exponent $(k)$ can be ommited.

**Lemma 7.** *All series of $k$-rooted irreducible quadrangulation in one variable (resp. two variables) are $\gamma$-rational (resp. $(\gamma_1,\gamma_2)$-rational).*

*Proof.* Similar to the proof of Lemma 5. In one variable, we use the form of Equation 7 to see that $q_{vv}^{(k)}(\mathrm{W}(y)/y)$ is $\eta$-rational. Then we use the fact [8] that $\eta(y)$ has a rational expression in terms of $\gamma(z)$ when $z$ and $y$ are linked by the change of variable $z=\mathrm{W}(y)/y$. Substituting $\eta$ by this expression in the $\eta$-rational expression of $q_{vv}^{(k)}(\mathrm{W}(y)/y)$, we obtain a $\gamma$-rational expression for $q_{vv}^{(k)}(z)$.

The proof for two variables is similar, using in particular the fact that $\eta_1(y_\bullet,y_\circ)$ and $\eta_2(y_\bullet,y_\circ)$ have a rational expression in terms of $\gamma_1(z_\bullet,z_\circ)$ and $\gamma_2(z_\bullet,z_\circ)$ when $(z_\bullet,z_\circ)$ and $(y_\bullet,y_\circ)$ are linked by the change of variable $(z_\bullet,z_\circ)=(\mathrm{W}/y_\circ,\mathrm{W}/y_\bullet)$. □

**Lemma 8.** *The* N *initial coefficients counting unrooted 3-connected maps according to their number of edges can be computed with* $\mathcal{O}(\mathrm{N}\log(\mathrm{N}))$ *operations.*

*The table of initial coefficients with indices* $(i, j)$ *and* $i + j \leqslant \mathrm{N}$ *counting unrooted 3-connected maps according to their number of vertices and faces can be computed with* $\mathcal{O}(\mathrm{N}^2)$ *operations.*

*Proof.* Using the algebraicity of the generating function of $k$-rooted irreducible quadrangulations, we can perform the same treatment as in the proof of Lemma 6. ☐

Finally, Lemma 6 and 8 yield Theorem 2. Using Tutte's bijection between $k$-rooted objects (see also Figure 1), Lemma 3, 5 and 7 yield Theorem 1.

## 5. Conclusion

We have proposed an original and efficient method to enumerate unrooted maps. In particular, we have improved significantly on the complexity of counting oriented convex polyedra (unrooted 3-connected maps).

Our method is flexible and can be adapted to enumerate other families of unrooted maps. For example, a similar scheme can be used to count unrooted loopless and then unrooted maps without loops and multiple edges. This time, a first tree decomposition, said "by loops" allows to obtain enumeration of $k$-rooted loopless maps from $k$-rooted maps. Then the tree decomposition by multiple edges (this time on $k$-rooted maps instead of $k$-rooted quadrangulations as in this article) allows to enumerate $k$-rooted maps without loop and multiple edge from loopless $k$-rooted maps.

Another very interesting problem is the enumeration of unrooted 3-connected maps on the sphere up to all homeomorphisms (including orientation-reversing). Indeed these objects correspond exactly to 3-connected maps, by Whitney's Theorem. In this case, a Burnside formula is also available, letting the problem come down to the enumeration of oriented $k$-rooted 3-connected maps, but also orientation-reversing ones such as 2-rooted 3-connected maps representing a reflexion. The tree-decomposition by separating 4-cycles can be used to obtain an equation linking 2-rooted 2-connected maps and 2-rooted 3-connected maps of type reflexion. Hence, the method of tree decomposition is also here promising.

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Éric Fusy, INRIA Rocquencourt, Projet ALGO BP 105, 78153 Le Chesnay, and LIX (Ecole Polytechnique)
*E-mail address*: Eric.Fusy@inria.fr