Transversal structures on triangulations, with application to straight-line drawing

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Abstract. We define and study a structure called transversal edge-partition related to triangulations without non empty triangles, which is equivalent to the regular edge labeling discovered by Kant and He. We study other properties of this structure and show that it gives rise to a new straight-line drawing algorithm for triangulations without non empty triangles, and more generally for 4-connected plane graphs with at least 4 border vertices. Taking uniformly at random such a triangulation with 4 border vertices and n vertices, the size of the grid is almost surely $\frac{n}{2} \cdot \left(1 - \frac{5}{27}\right) \times \frac{n}{2} \cdot \left(1 - \frac{5}{27}\right)$ up to fluctuations of order $\sqrt{n}$, and the half-perimeter is bounded by $n-1$. The best previously known algorithms for straight-line drawing of such triangulations only guaranteed a grid of size $(\lceil n/2 \rceil - 1) \times \lfloor n/2 \rfloor$. The reduction-factor of $\frac{5}{27}$ can be explained thanks to a new bijection between ternary trees and triangulations of the 4-gon without non empty triangles.

1 Introduction

A plane graph is a connected graph embedded in the plane so that edges do not cross each other. In many drawing algorithms of plane graphs [4, 13, 2, 8], the method consists in endowing the graph with a particular structure, from which it is possible to gives coordinates to vertices in a natural way. For example, triangulations, i.e., plane graphs with only faces of degree 3, are characterized by the fact that their inner edges can essentially be partitioned into three spanning trees, called Schnyder Woods, with specific incidence relations [13]. Using these spanning trees it is possible to associate coordinates to each vertex by counting faces on each side of particular paths passing by the vertex. Placing vertices in this way and linking adjacent vertices by segments yields a straight-line drawing algorithm, which can be refined to produce a drawing on a regular grid of size $(n-2) \times (n-2)$, see [14].

A plane graph with an outer face of degree k and inner faces of degree 3 is called a triangulation of the k-gon. If the interior of any 3-cycle of edges is a face, the triangulation is said *irreducible*. Observe that it implies $k > 3$, unless the graph is reduced to a unique triangle. There exist more compact straight-line drawing algorithms for irreducible triangulations [7, 9], the size of the grid being guaranteed to be $(\lceil n/2 \rceil - 1) \times \lfloor n/2 \rfloor$ in the worst case.

In this paper we concentrate on irreducible triangulations of the 4-gon, which carry a good level of generallity. Indeed many graphs, including 4-connected plane graphs with at least 4 border vertices, can be triangulated (after adding 4 vertices in the outer face) into an irreducible triangulation of the 4-gon, see [1]. By investigating a bijection with ternary trees, we have observed that each irreducible triangulation of the 4-gon can be endowed with a structure, called transversal edge-partition, which can be summarized as follows. Calling S_b, N_r, N_b, S_r (like south-blue, north-red, north-blue, south-red) the 4 border vertices of T in clockwise order, the inner edges of T can be oriented and partitioned into two sets: red edges that "flow" from S_r to N_r, and blue edges that "flow" from S_b to N_b. For those familiar with bipolar orientations [5], i.e. acyclic orientations with two poles, the structure can also be seen as a transversal couple of bipolar orientations, see Section 2.3. As we learned after completing a first draft of this paper, Kant and He used an equivalent structure in [8] and derived nice algorithms of rectangular-dual drawing and of visibility representation. We explore the properties of this structure and show in particular in Theorem 1 that it is of the lattice type.

We derive from the transversal structure a straight-line drawing algorithm of an irreducible triangulation T of the 4-gon. Like drawing algorithms using Schnyder Woods [13, 2], it is based on face counting operations. The first step is to endow T with a particular transversal edge-partition, said *minimal*, which is obtained by application of an iterative algorithm described in Section 2.4. Then the transversal structure is used to associate to each vertex v a path P_r of red edges and a path P_b of blue edges, both passing by v. The abscissa (resp. ordinate) of v is obtained by counting faces on each side of P_r (resp. P_b). Our algorithm outputs a straight line embedding on a regular grid of width W and height H with $W + H \leq n - 1$ if the triangulation has n vertices. This algorithm can be compared to [7] and [9], which produces straight-line drawing on a grid of size $(\lceil n/2 \rceil - 1) \times \lfloor n/2 \rfloor$. However, algorithms of [7] and [9] rely on a particular order of treatment of vertices called canonical ordering, and a step of coordinate-shifting makes them difficult to implement and to carry out by hand. As opposed to that, our algorithm can readily be performed on a piece of paper, because coordinates of vertices can be computed independently with simple face-counting operations. Finally, our algorithm has the nice feature that it respects the structure of transversal edge-partition. Indeed, Theorem 2 ensures that red edges are geometrically oriented from S_r to N_r and blue edges are geometrically oriented from S_b to N_b.

A compact version of the algorithm even ensures that, for a random triangulation with n vertices, the size of the grid is asymptotically almost surely $\frac{n}{2}(1-5/27) \times \frac{n}{2}(1-5/27)$ up to small fluctuations, of order $\sqrt{n}$. Compared to [7] and [9], we do not improve on the size of the grid in the worst case, but improve asymptotically by a reduction-factor 5/27 on the width and the height of the grid for an object of large size (see Figure 4.2 for an example with $n = 200$). The reduction factor $\frac{5}{27}$ that is obtained for the size of the grid output by the compact straight-line drawing algorithm can be explained thanks to a new bijection

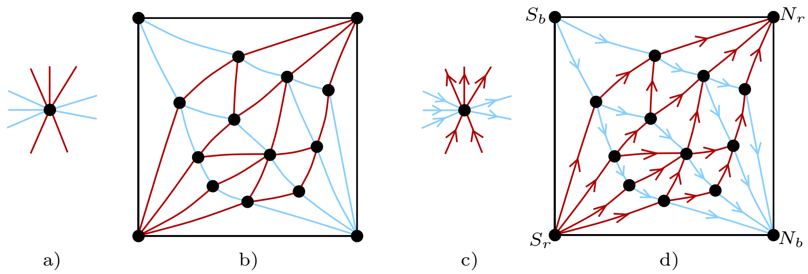

Fig. 1. The structure of transversal edge-partition: local condition (a) and a complete example (b). In parallel, the structure of transversal couple of bipolar orientations: local condition (c) and a complete example (d).

between ternary trees and irreducible triangulations of the 4-gon. This bijection is described in Section 4 and relies on "closure operations", as introduced by G. Schaeffer [11], see also [10] for a bijection with unconstrained triangulations. This bijection has, truth to tell, brought about our discovery of transversal edge-partitions. Indeed, it turns out to "transport" a so-called transversal edge-bicoloration of a ternary tree into the minimal transversal edge-partition of its associated triangulation, in the same way that bijection of [10] transports the structure of Schnyder woods. In addition, the bijection gives a combinatorial way to enumerate rooted 4-connected triangulations, which were already counted in [15] using algebraic methods.

2 Definition of transversal structures

2.1 Transversal edge-partition

Let T be an irreducible triangulation of the 4-gon. Edges and vertices of T are said *inner* or *outer* whether they belong to the outer face or not. A transversal edge-partition of T is a partition of the inner edges of T into two sets, say in blue and in red edge, such that the following conditions are satisfied.

- C1 (Inner vertices): In clockwise order around each inner vertex, its incident edges form: a non empty interval of red edges, a non empty interval of blue edges, a non empty interval of red edges, and a non empty interval of blue edges, see Figure 1a.
- C2 (Border vertices): Writing a_1, a_2, a_3, a_4 for the border vertices of T in clockwise order, all inner edges incident to a_1 and to a_3 are of one color and all inner edges incident to a_2 and to a_4 are of the other color.

Figure 1b gives an example of transversal edge-partition, where we use DARK RED for red edges and LIGHT BLUE for blue edges (the same convention will be used for all figures).

2.2 Lattice structure

As it is the case with Schnyder Woods and bipolar orientations, the set of transversal edge-partitions of a fixed irreducible triangulation of the 4-gon has a lattice structure. In addition, the "flip" operation has a nice geometric interpretation. To describe it, we have to introduce some terminology. Given T an irreducible triangulation of the 4-gon endowed with a transversal edge-partition X, we define an *alternating 4-cycle* as a 4-cycle $\mathcal{C}$ of inner edges (e_1, e_2, e_3, e_4) of T which are color-alternating (i.e. two adjacent edges of $\mathcal{C}$ have different colors). Given a border vertex v of $\mathcal{C}$, we call left-edge (resp. right-edge) of v the edge of $\mathcal{C}$ starting from v and having the exterior of $\mathcal{C}$ on its left (resp. on its right). It can easily be proven that two cases can occur for $\mathcal{C}$: either all edges interior to $\mathcal{C}$ and incident to a border vertex v of $\mathcal{C}$ have the color of the left-edge of v, then $\mathcal{C}$ is called a left alternating 4-cycle; or all edges interior to $\mathcal{C}$ and incident to a border vertex v of $\mathcal{C}$ have the color of the right-edge of v, then $\mathcal{C}$ is called a right alternating 4-cycle.

Theorem 1. *Let T be an irreducible triangulation of the 4-gon. Then the set $\mathcal{E}$ of transversal edge-partitions of T is non empty and has a lattice structure. Given $X \in \mathcal{E}$, the flip operation consists in finding a right alternating 4-cycle $\mathcal{C}$ of X and then switching the colors of all edges interior to $\mathcal{C}$, making $\mathcal{C}$ a left alternating 4-cycle. The (unique) transversal edge-partition of T without right alternating 4-cycle is said minimal.*

Proof. The non emptiness of $\mathcal{E}$ will be proven constructively in Section 2.4 by providing an algorithm computing the minimal transversal edge-partition of T.Concerning the lattice structure of $\mathcal{E}$, see Section A.

2.3 Transversal couple of bipolar orientations

Given a plane graph G and two vertices S (like South) and N (like North) of G incident to the outer face of G, a *bipolar orientation* of G with poles S and N is an acyclic orientation of the edges of G such that, for each vertex v different from S and N, there exists an oriented path from S to N passing by v, see [5] for a detailed decription.

Let T be an irreducible triangulation of the 4-gon. Call N_r, N_b, S_r and S_b the 4 border vertices of T in clockwise order around the outer face of T. A *transversal couple of bipolar orientations* is an orientation and a partition of the inner edges of T into red and blue edges such that the following two conditions are satisfied (see Figure 1d for an example):

- C1' (Inner vertices): In clockwise order around each inner vertex of T, its incident edges form: a non empty interval of outgoing red edges, a non empty interval of outgoing blue edges, a non empty interval of ingoing red edges, and a non empty interval of ingoing blue edges, see Figure 1c.
- C2' (Border vertices): All inner edges incident to N_b, N_r, S_b and S_r are respectively ingoing blue, ingoing red, outgoing blue, and outgoing red.

This structure is also defined in [8] under the name of regular edge labeling. The following proposition explains the name of transversal couple of bipolar orientations and is also stated in [8]:

Proposition 1. *Let T be an irreducible triangulation of the 4-gon. Given a transversal couple of bipolar orientations of T, the (oriented) red edges induce a bipolar orientation of the plane graph obtained from T by removing S_b, N_b, and all non red edges. Similarly, the blue edges induce a bipolar orientation of T deprived from S_r, N_r and all non blue edges.*

Proof. Ommited, see Section B.

Proposition 2. *To each transversal couple of bipolar orientations of T corresponds a transversal edge-partition of T, obtained by removing the orientation of the edges (Compare Figure 1d and Figure 1b). This correspondence is a bijection.*

Proof. Ommited, see Section A.

Proposition 2 allows us to manipulate equivalently transversal edge-partitions or transversal couples of bipolar orientations. The first point of view is more convenient to describe the lattice structure, the second one will be more convenient to describe the drawing algorithm in Section 3.

2.4 Algorithm computing the minimal transversal edge-partition

Let us now describe a simple iterative algorithm to compute transversal edge-partitions. Two different algorithms computing such transversal structures were already presented in [8]. However we need to compute the minimal transversal edge-partition, to be used later in the straight-line drawing algorithm. During the execution, we also orient the edges, so that we compute in fact the underlying transversal couple of bipolar orientations. The algorithm consists in maintaining and iteratively shrinking a cycle $\mathcal{C}$ of edges of T such that, in particular (we do not detail all invariants here):

- The cycle $\mathcal{C}$ contains the two edges (S_r, S_b) and (S_r, N_b).
- No edge interior to $\mathcal{C}$ connects two vertices of $\mathcal{C}\backslash\{S_r\}$
- All inner edges of T outside of $\mathcal{C}$ are colored and oriented such that Inner-vertex Condition C1' is satisfied for each inner vertex of T outside of $\mathcal{C}$.

We initialize the cycle $\mathcal{C}$ with vertices S_r, S_b, N_b and all interior neighbours of N_r, color in red all inner edges incident to N_r and orient them toward N_r, see Figure 2b. Observe also that vertices of $\mathcal{C}$ different from S_r can be ordered from left to right with S_b as leftmost and N_b as rightmost vertex. For two vertices v and v' of $\mathcal{C}\backslash\{S_r\}$ with v on the left of v', we write $[v, v']$ for the path on $\mathcal{C}$ going from v to v' without passing by S_r.

To explain how to update (shrink) $\mathcal{C}$ at each step, we need a few definitions. An *internal path* of $\mathcal{C}$ is a path $\mathcal{P}$ of edges interior to $\mathcal{C}$ and connecting two vertices v and v' of $\mathcal{C}$. We write $\mathcal{C}_\mathcal{P}$ for the cycle constituted by the concatenation of $\mathcal{P}$ and $[v, v']$. The path $\mathcal{P}$ is said *eligible* if the following conditions are satisfied:

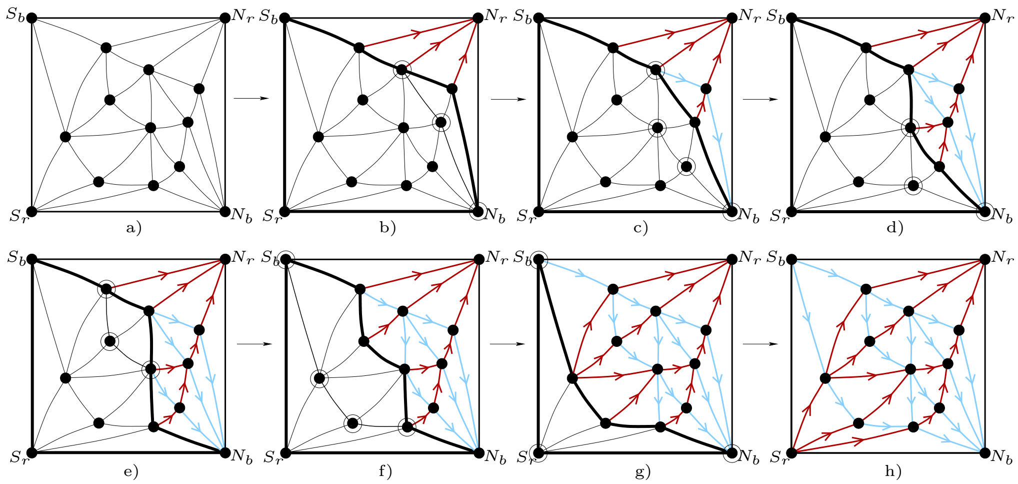

Fig. 2. An example of execution of the algorithm computing the minimal transversal couple of bipolar orientations. Vertices of the rightmost eligible path are surrounded.

- The paths $\mathcal{P}$ and $[v, v']$ both contain at least one vertex different from v and v'.
- Each edge interior to $\mathcal{C}_{\mathcal{P}}$ connects a vertex of $\mathcal{P}\backslash\{v, v'\}$ to a vertex of $[v, v']\backslash\{v, v'\}$. In particular, the interior of $\mathcal{C}_{\mathcal{P}}$ contains no vertex.
- The cycle $\mathcal{C}'$ obtained from $\mathcal{C}$ by replacing $[v, v']$ by $\mathcal{P}$ has no interior edge connecting two vertices of $\mathcal{C}\backslash\{S_r\}$.

Now we can simply explain the update operation: find an eligible internal path $\mathcal{P}$ of $\mathcal{C}$ and write v and v' for its extremities with v on the left of v' (so that v and v' are called respectively left and right extremity of $\mathcal{P}$). Then, color each internal edge of $\mathcal{C}_{\mathcal{P}}$ in red and orient it toward $[v, v']\backslash\{v, v'\}$. Color all edges of $[v, v']$ in blue and orient them from v to v'. Finally update $\mathcal{C}$ by replacing in $\mathcal{C}$ the path $[v, v']$ by the path $\mathcal{P}$.

It can easily be shown that the absence of non empty triangle on T ensures that the algorithm terminates, i.e. that at each step the cycle $\mathcal{C}$ has an eligible internal path and can be updated (shrinked). After the last update operation, $\mathcal{C}$ is empty. Using all invariants of color and orientation of edges satisfied by $\mathcal{C}$, it can be shown that the obtained orientation and coloration of inner edges of T is a transversal couple of bipolar orientations. Figure 2 illustrates the complete execution of the algorithm on an example.

This algorithm can easily be adapted to give an algorithm, called ComputeMinimal(T), which computes the transversal couple of bipolar orientations associated (by removing orientation of edges) to the minimal transversal edge-partition of T, as defined in Theorem 1. Observe that, at each step of the algorithm, eligible paths of $\mathcal{C}$ can be (partially) ordered from left to right, by saying that $\mathcal{P}_1 \geq \mathcal{P}_2$ if the left extremity and the right extremity of $\mathcal{P}_1$ are (weakly) on the left respectively of the left extremity and of the right extremity of $\mathcal{P}_2$. Although this order is only partial, it can easily be shown that it admits a unique minimum, called rightmost eligible path of $\mathcal{C}$. Algorithm ComputeMinimal(T) consists in choosing the rightmost eligible path at each step of the iterative algorithm described above, see also Figure 2, where the execution respects this choice.

Proposition 3. *Given an irreducible triangulation T of the 4-gon, Algorithm* ComputeMinimal*(T) outputs the transversal couple of bipolar orientations of T associated to the minimal transversal edge-partition of T (by removing edge orientations). In addition,* ComputeMinimal*(T) can be implemented to run in linear time.*

Proof. Ommited.

3 Application to straight-line drawing

We recall that a straight line drawing of a plane graph G consists in placing all points of G on a regular grid of size $[0, W] \times [0, H]$, so that for two edges e and e' of G, the two segments on the grid corresponding to e and e' can only meet at their endpoint (which happens if e and e' share an extremity). The integers W and H are called the width and the height of the grid.

The structure of transversal edge-partition can be used to derive a simple algorithm, called TransversalDraw, to perform straight line drawing of an irreducible triangulation T of the 4-gon. First we have to give a few definitions. The plane graph obtained from T by removing all blue (resp. red) edges is called the red-map (resp. blue-map) of T and is denoted by T_r (resp. T_b). We write f_r and f_b for the number of inner faces of T_r and T_b. Given an inner vertex v of T, we define the leftmost outgoing red path of v as the oriented path starting from v and such that each edge of the path is the leftmost outgoing red edge at its origin. As the orientation of red edges is bipolar, this path has no cycle and ends at N_r. We also define the rightmost ingoing red path of v as the path starting from v and such that each edge of the path is the rightmost ingoing red edge at its extremity. This path is also acyclic and ends at S_r. We call separating red path of v the concatenation of these two paths and denote it by $\mathcal{P}_r(v)$. The path $\mathcal{P}_r(v)$ goes from S_r to N_r passing by v, and separates inner faces of T_r into two sets: those on the left of $\mathcal{P}_r(v)$ and those on the right of $\mathcal{P}_r(v)$. Similarly, we define the leftmost outgoing blue path, the rightmost ingoing blue path, and write $\mathcal{P}_b(v)$ for their concatenation, called separating blue path of v.

Now we can describe Algorithm TransversalDraw, which runs as follows (see Figure 3 for a complete execution):

- Perform ComputeMinimal(T) to endow T with its minimal transversal couple of bipolar orientations.
- For each inner vertex v of T, place v on the grid in the following way:

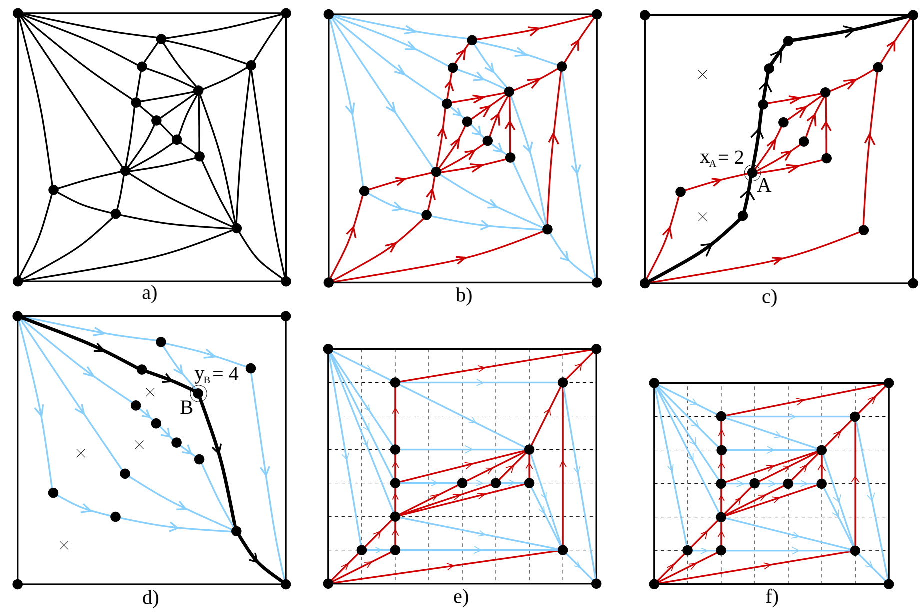

Fig. 3. The execution of Algorithm TransversalDraw (a-to-e) and of Algorithm CompactTransversalDraw (a-to-f) on an example.

- The abscissa of v is the number of inner faces of T_r on the left of $\mathcal{P}_r(v)$, see Figure 3c.
- The ordinate of v is the number of inner faces of T_b on the right of $\mathcal{P}_b(v)$, see Figure 3d.

– Place the border vertices S_r, S_b, N_b, N_r respectively at coordinates $(0,0)$, $(0,f_b)$, $(f_r,0)$ and (f_r,f_b).

A further step can be added to Algorithm TransversalDraw to have an algorithm, called CompactTransversalDraw, giving a more compact drawing. The further step consists in deleting the unused abscissas and ordinates of the drawing computed by TransversalDraw. An example is given on Figure 3d, obtained from Figure 3c after having deleted the unused abscissa 3 and the unused ordinate 5.

Theorem 2. *Given an irreducible triangulation T of the 4-gon with n vertices, Algorithm* TransversalDraw *and Algorithm* CompactTransversalDraw *can be implemented to run in linear time and compute a straight line drawing of T such that:*

– *All red edges are oriented from bottom to top and weakly oriented from left to right.*

– *All blue edges are oriented from left to right and weakly oriented from top to bottom.*

– *The width W and height H of the grid of the drawing given by* TransversalDraw*(T) verify $W + H = n - 1$.*
– *Let T be taken uniformly at random among irreducible triangulations of the 4-gon with n vertices. The width W_c and the height H_c of the grid of the drawing given by* CompactTransversalDraw*(T) are asymptotically almost surely equal to $\frac{n}{2}(1 - 5/27)$, up to fluctuations ϵ_{W_c} and ϵ_{H_c} of order $\sqrt{n}$.*

Proof. Ommited. Proof of the the third point relies on the fact that $W = f_r$ and $H = f_b$ and then on Euler relation.

In fact the transversal structure used to give coordinates to vertices need not to be the minimal one. Using any other transversal couple of bipolar orientations, the three first points of Theorem 2 remain true. However the analysis of the reduction-factor $\frac{5}{27}$ with CompactTransversalDraw(T) crucially requires that the transversal structure is the minimal one, see Section 4.2 and Section C.

Corollary 1. *Each 4-connected plane graph G with n vertices and at least 4 vertices on the outer face can be embedded on a grid $W \times H$ with $W + H \leq n - 1$.*

Proof. It relies on the fact that an irreducible triangulation T of the 4-gon with $n + 4$ vertices can be associated to G: triangulate G, then draw a quadrangle outside of G and finally link outer vertices of G to vertices of the quadrangle by new edges, so as to produce a triangulation of the 4-gon (see Figure 4c-d for a similar operation). This triangulation is easily proven to be irreducible by 4-connectivity of G.

4 Bijection with ternary trees and applications

We present a bijection which associates to a ternary tree A an irreducible triangulation T of the 4-gon. In addition, A can be naturally endowed with a structure of transversal edge-bicoloration, which is transported by the bijection into the minimal transversal edge-partition of T. In fact, this bijection was the starting point of our discovery of the structure of transversal edge-partition.

4.1 Description of the bijection

A *ternary tree* A is a tree embedded in the plane with nodes of degree 4, called *inner nodes* and nodes of degree 1, called *leaves*. Edges of A connecting two inner nodes are called *inner edges* and edges incident to a leaf are called *stems* (these are “pending” edges). A ternary tree can be rooted by marking one of its leaves, and such rooted ternary trees correspond to the classical definition of ternary trees (i.e. all nodes have either 0 or 3 children).

We describe briefly the bijection (see [6, 10] for detailed descriptions of similar bijections), consisting of three main steps: local closure, partial closure and complete closure. Perform a counterclockwise traversal of A (imagine an ant

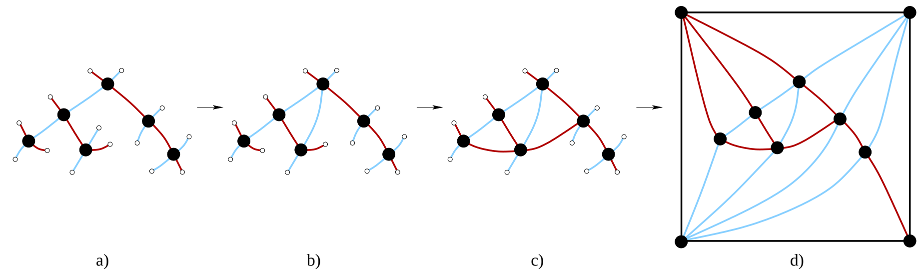

Fig. 4. The execution of the closure on an example.

walking around A with the infinite face on its right). If a stem s and then two inner edges e_1 and e_2 are successively encountered during the traversal, merge the extremity of s with the extremity of e_2, so as to *close* a triangular face. This operation is called *local closure*, see Figure 4b. Now we can restart a counter-clockwise traversal around the new Figure F, which is identical to A, except that it contains a triangular face and, more important, the stem s has become an inner edge. Each time we find a succession (stem, edge, edge), we perform a local closure, update the figure, and restart, until no local closure is possible. This greedy execution of local closures is called the *partial closure* of A, see Figure 4c. It can easily be shown that the figure F obtained by partial closure of A does not depend of the order of execution of the local closures. Finally, the last step, called *complete closure* (see Figure 4d), consists in drawing a 4-gon, and then merging the extremity of each unmatched stem with a border vertex, so as to create only triangular inner faces. It can be shown that the choice of an outer 4-gon is the good one so that this last operation works without conflict.

Now we explain how the closure "transports" the structure of transversal edge-partition. The edges of a ternary tree A can be bicolored in blue and red edges so that two successive edges incident to an inner node of A have always different color, see Figure 4a. This bicoloration, unique up to the choice of the colors, is called the transversal edge-bicoloration of A. Observe that Inner Vertex Condition C1 is satisfied on A and remains satisfied throughout the closure. The following theorem states the bijection and describes more precisely the transportation of the transversal structures:

Theorem 3. *The closure is a bijection between the set of ternary trees with n inner nodes and the set of irreducible triangulations of the 4-gon with n inner vertices.*

The closure transports the transversal edge-bicoloration of a ternary tree into the minimal transversal edge-partition of its image.

Proof. Ommited. Injectivity can easily be proven by uniqueness of the transversal edge-partition without right alternating 4-cycle. The inverse of the closure consists in computing the minimal transversal edge-partition of T and using the colors to disconnect certain edges at their extremity, so as to leave a ternary tree.

An irreducible triangulation of the 4-gon is *rooted* by choosing one of its 4 border edges and orienting this edge with the infinite face on its right. This well known operation eliminates symmetries of the triangulation.

Corollary 2. *The closure induces a 4-to-*$(2n{+}2)$ *correspondence between the set* $\mathcal{A}_n$ *of rooted ternary trees with* n *inner nodes and the set* $\mathcal{T}_n$ *of rooted irreducible triangulations of the 4-gon with* n *inner vertices. In other words,* $\mathcal{A}_n \times \{1, \ldots, 4\}$ *is in bijection with* $\mathcal{T}_n \times \{1, \ldots, 2n+2\}$*.*

As an enumerative consequence, $|\mathcal{T}_n| = \frac{4}{2n+2}|\mathcal{A}_n| = \frac{4(3n)!}{(2n+2)!n!}$.

Proof. The proof follows easily from the bijection stated in Theorem 3 and from the fact that a ternary tree with n inner nodes has $2n+2$ leaves and an object of $\mathcal{T}_n$ has 4 edges (the 4 border edges) to carry the root.

4.2 Applications

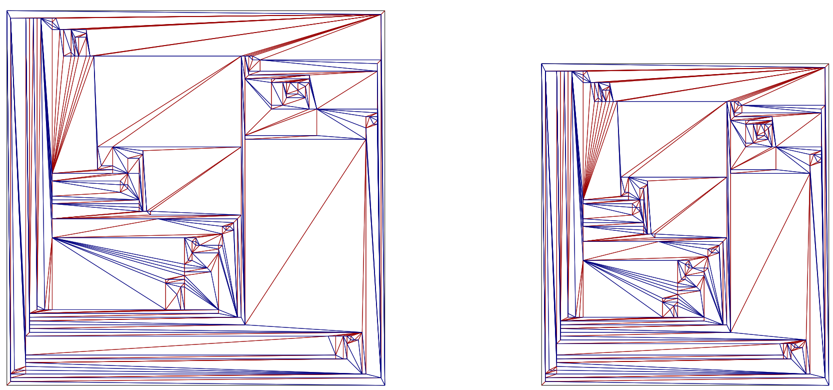

Fig. 5. A triangulation with 200 vertices embedded with Algorithms TransversalDraw and CompactTransversalDraw.

The closure-bijection has several applications. A first one is a linear time algorithm to perform uniform random sampling of objects of $\mathcal{T}_n$, using the fact that rooted ternary trees with n inner nodes can readily be uniformly sampled using parenthesis words. A thorough study of such sampling algorithms is given in [12]. In addition, sampled objects of $\mathcal{T}_n$ are naturally endowed, through the closure, with their minimal transversal edge-partition. Hence, we can easily run face-counting algorithms TransversalDraw and CompactTransversalDraw on the sampled objects. Performing simulations on objects of large size ($n \approx 50000$), it was observed by the author that the size of the grid is always approximately $\frac{n}{2} \times \frac{n}{2}$ with TransversalDraw and $\frac{n}{2}(1-\alpha) \times \frac{n}{2}(1-\alpha)$ with CompactTransversalDraw, where $\alpha \approx 0.18$. It turns out that the size of the grid can be readily analyzed thanks to our closure-bijection (see Section C), in the same way that bijection of [10] allowed to analyze parameters of Schnyder woods in [3]. Indeed, the number of unused abscissas and ordinates of TransversalDraw correspond to certain inner edges of the ternary tree, whose number can be proven to be asymptotically almost surely $\frac{5n}{27}$ up to fluctuations of order $\sqrt{n}$.

A second application is counting rooted 4-connected triangulations with n vertices, whose set is denoted by $\mathcal{C}_n$. It is well known that a 4-connected triangulation is a triangulation where each 3-cycle delimits a face. Hence, the operation of removing the root edge of an object of $\mathcal{C}_n$ and carrying the root on the counterclockwise-consecutive edge is an (injective) mapping from $\mathcal{C}_n$ to $\mathcal{T}_{n-4}$. However, given $T \in \mathcal{T}_{n-4}$, the inverse edge-adding-operation can create a separating 3-cycle if there exists an internal path of length 2 connecting the origin of the root of T to the vertex diametrically opposed in the outer face of T. Objects of $\mathcal{T}_{n-4}$ having no such internal path are said *undecomposable* and their set is denoted by $\mathcal{U}_{n-4}$. The above discussion ensures that they are in bijection with $\mathcal{C}_n$. A maximal decomposition of an object T of $\mathcal{T}$ along the above mentioned interior paths of length 2 ensures that T is a sequence of objects of $\mathcal{U}$. After a few simple manipulations and using Corollary 2 (see Section B), we obtain the following enumerative result for rooted 4-connected triangulations:

Proposition 4. *The series $C(z)$ counting rooted 4-connected triangulations by their number of inner vertices has the following expression:*

$$C(z) = \frac{z(A(z) - A(z)^2 + 1)}{1 + z(A(z) - A(z)^2 + 1)}$$

where $A(z) = z(1 + A(z))^3$ is the series counting rooted ternary trees by their number of inner nodes

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A Proof of Proposition 2 and of the lattice structure of transversal edge-partitions

A.1 Lattice structure of α-orientations

Let T be an irreducible triangulation of the 4-gon. We write $\mathcal{E}$ for the set of transversal edge-partitions of T. To prove that $\mathcal{E}$ is a lattice, we rely on a well known theorem on the lattice structure of so-called *α-orientations*. Given a plane graph $G = (V, E)$, and a function $\alpha : V \rightarrow \mathbf{N}$, an *$\alpha$-orientation* of G is an orientation of the edges of G such that each vertex v of G has outdegree $\alpha(v)$.

Theorem 4. *Given $G = (V, E)$ a plane graph and $\alpha : V \rightarrow \mathbf{N}$ a function, the set of α-orientations of G is either empty or it carries a lattice structure. The flip operation consists in reversing the orientation of "minimal" oriented cycles.*

Of course, for two different transversal edge-partitions X and X' of T, the outdegree of a vertex v in X and X' are not always the same. Hence, Theorem 4 does not apply directly to prove the lattice structure of $\mathcal{E}$. However we will see that transversal edge-partitions of T are in bijection with α-orientations of a plane graph $Q(T)$ associated to T.

A.2 Angular graph of T

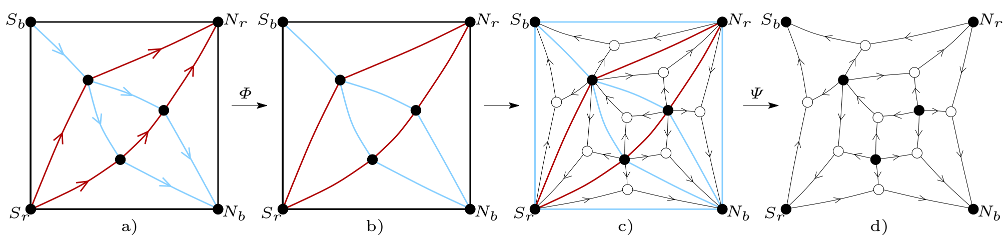

Fig. 6. Given an irreducible triangulation T of the 4-gon endowed with a transversal edge-partition Z (Figure a), construction of $Q(T)$ and of the α_0-orientation of $Q(T)$ image of Z by $\Psi \circ \Phi$.

We define an angle of T as a couple (v, f) composed of an inner face f of T and a vertex v of T incident to f. The *angular map* $Q(T)$ of T is constructed as follows: the vertices of $Q(T)$ are composed of two kinds of vertices: the vertices of T, colored in black; and so-called *face-vertices* colored in white, a face-vertex v_f being placed in the center of each inner face f of T. Then, for each angle (v, f) of T, we construct an edge of $Q(T)$ by connecting v to the white vertex v_f associated to f. Figure 6 illustrates the construction of $Q(T)$ from T (ignore here the orientation of the edges of $Q(T)$).

A.3 Statement of the theorem

We denote by V the set of vertices of $Q(T)$ and write $\alpha_0 : V \to \mathbf{N}$ for the function such that:

- For each black vertex of $Q(T)$ corresponding to an inner vertex of $Q(T)$, $\alpha_0(v) = 4$.
- For the two red poles N_r and S_r, $\alpha_0(v) = 2$.
- For the two blue poles N_b and S_b, $\alpha_0(v) = 0$.
- For each white vertex of T, $\alpha_0(v) = 1$.

We are going to show that transversal edge-partitions of T are in bijection with α_0-orientations of $Q(T)$. More precisely, we state the following theorem, which also yields the bijection stated in Proposition 2:

Theorem 5. *Given T an irreducible triangulation of the 4-gon, the set $\mathcal{B}$ of transversal couples of bipolar orientations of T is in bijection with the set $\mathcal{E}$ of transversal edge-partitions of T. The set $\mathcal{E}$ is itself in bijection with the set $\mathcal{O}$ of α_0-orientations of $Q(T)$. Hence Theorem 4 ensures that $\mathcal{B}$ and $\mathcal{E}$ have a lattice structure.*

To prove Theorem 5, we introduce two functions Φ and Ψ respectively from $\mathcal{B}$ to $\mathcal{E}$ and from $\mathcal{E}$ to $\mathcal{O}$.

Given $Z \in \mathcal{B}$, $\Phi(Z)$ is obtained from Z by removing the orientations of edges. It is straightforward that $\Phi(Z) \in \mathcal{E}$. In addition, Φ is clearly injective.

Now, given $X \in \mathcal{E}$, we define $\Psi(X)$ as the following orientation of edges of $Q(T)$. First, color in blue the 4 border edges of T. Then, for each angle (v, f) of T, let e be the edge of $Q(T)$ associated to (v, f) and write v_f for the white vertex of $Q(T)$ associated to f. We say that (v, f) is *bicolored* if it is delimited by two edges of T having different colors. Two cases can arise: either (v, f) is bicolored, in which case we orient e from v to v_f; or it is not bicolored, in which case we orient e from v_f to v. Observe that Inner-vertex Condition C1 satisfied by X implies that all inner black vertices of $Q(T)$ have outdegree 4. In addition Border-vertex Condition C2 and the fact that the 4 border edges of T have been colored in blue imply that N_b and S_b have outdegree 0 and that N_r and S_r have outdegree 2.

To prove that $\Psi(X)$ is an α_0-orientation, it remains to prove that all white vertices have outdegree 1, which relies on the following lemma:

Lemma 1. *Consider a transversal edge-partition X of an irreducible triangulations T of the 4-gon and color all border edges of T in blue. Then each inner face of T has two sides of one color and one side of the other color. Hence exactly one of the three angles of the face is delimited by two edges having the same color.*

Proof. Let A_b be the number of bicolored angles of T and let n be the number of inner vertices of T. Inner-vertex Condition C1 implies that there are $4n$ bicolored angle incident to an inner vertex of T. Border-vertex Condition C2 and the fact that all border edges are colored in blue imply that 2 angles incident to N_r and two angles incident to S_r are bicolored. Hence, we have $A_b = 4n + 4$.

Moreover Euler relation, together with the fact that T is outer-quadrangular and has triangular inner faces, ensure that T has $2n + 2$ inner faces. For each inner face, two cases can arise: either its three sides have the same color or it has two sides of one color and one side of the other color. In the first (resp. second) case, the face has 0 (resp. 2) bicolored angles. Finally, the fact that $A_b = 4n + 4$ and the pigeonhole principle imply that all inner faces have a contribution of 2 in the number of bicolored angles, which concludes the proof.

Hence Ψ is a mapping from $\mathcal{E}$ to $\mathcal{O}$ and it is clear that Ψ is injective. To prove Theorem 5, it remains to prove that Φ and Ψ are surjective, which comes down to proving surjectivity of $\Psi \circ \Phi$. Thus, given $O \in \mathcal{O}$, we have to find $B \in \mathcal{B}$ such that $\Psi \circ \Phi(B) = O$. To do that, we could simply greedily traverse the triangulation and propagate a coloration and orientation of its edges that is locally (around each vertex) compatible with the given α-orientation O. However we have no guarantee that the same edge can not receive two different colors (thus in conflict) during such a greedy traversal. It turns out that no such conflict can arise and we can get sure of that by traversing vertices of the triangulation in a certain order, as explained in the next section.

A.4 Iterative algorithm to find the preimage of an α_0-orientation

As in Section 2.4, the algorithm consists in maintaining and iteratively shrinking a cycle $\mathcal{C}$ of edges of T such that the following invariants hold:

- The cycle $\mathcal{C}$ contains the edges (S_r, S_b) and (S_r, N_b).
- All edges of T outside of $\mathcal{C}$ are colored in such a way that:
  - They satisfy the transformation $\Psi \circ \Phi$, i.e. for two edges (e_1, e_2) outside of $\mathcal{C}$ delimiting an angle (v, f) of T, the associated edge of $Q(T)$ is directed toward v iff e_1 and e_2 have the same color.
  - They satisfy Inner-vertex Condition C1' for each inner vertex v of T outside of $\mathcal{C}$.
  - For each inner vertex v of T on $\mathcal{C}$, a partial version of C1' holds: in the clockwise order around v, edges incident to v and exterior to $\mathcal{C}$ form: a (possibly empty) interval of ingoing blue edges, a non empty interval of outgoing red edges, and a (possibly empty) interval of outgoing blue edges.
- Each inner vertex of T on $\mathcal{C}$ has 2 of its 4 outgoing edges of $Q(T)$ inside of $\mathcal{C}$ and the 2 other ones outside of $\mathcal{C}$.

To explain how to update (shrink) $\mathcal{C}$ at each step, we have to give a few definitions. First we recall that the vertices of $\mathcal{C} \backslash \{S_r\}$ can be ordered from left to right, with S_b as leftmost and N_b as rightmost vertex. Given v and v' on $\mathcal{C}$, we also use the notation $[v, v']$ for the path going from v to v' on $\mathcal{C}$ without passing by S_r.

Given two vertices v and v' on $\mathcal{C}$, we define the *internal path* of the couple $\{v, v'\}$, writen $\mathcal{P}_{\{v,v'\}}$, as the unique path satisfying the following conditions:

– $\mathcal{P}_{\{v,v'\}}$ connects v to v' using edges of T in the interior of $\mathcal{C}$.
– If we write $\mathcal{C}_{\mathcal{P}_{\{v,v'\}}}$ for the cycle made of the concatenation of $\mathcal{P}_{\{v,v'\}}$ and $[v, v']$, all edges of T interior to $\mathcal{C}_{\mathcal{P}_{\{v,v'\}}}$ connect a vertex of $\mathcal{P}_{\{v,v'\}} \backslash \{v, v'\}$ to a vertex of $[v, v'] \backslash \{v, v'\}$.

A couple $\{v, v'\}$ with v on the left of v' is said *admissible* if the following two conditions hold: the edge e_v of $Q(T)$ corresponding to the angle formed by the first edge of $\mathcal{P}_{\{v,v'\}}$ and the first edge of $[v, v']$ is directed toward v; and the edge $e_{v'}$ of $Q(T)$ corresponding to the angle formed by the last edge of $\mathcal{P}_{\{v,v'\}}$ and the last edge of $[v, v']$ is directed toward v'. Observe that the set of admissible couples is non empty: indeed the couple (S_b, N_b) is always admissible because these two vertices have only ingoing edges in $Q(T)$. An admissible couple $\{v, v'\}$ is said minimal if there exists no other admissible couple $\{v_1, v'_1\}$ such that $[v_1, v'_1]$ is properly included in $[v, v']$. Then, minimallity of $\{v, v'\}$ implies the following important remark: for each vertex in $[v, v'] \backslash \{v, v'\}$, if we consider the bunch of its incident edges of $Q(T)$ in the interior of $\mathcal{C}$, then the leftmost and the rightmost are outgoing and all other are ingoing.

Now we can explain the update operation shrinking the cycle $\mathcal{C}$, see Figure 7:

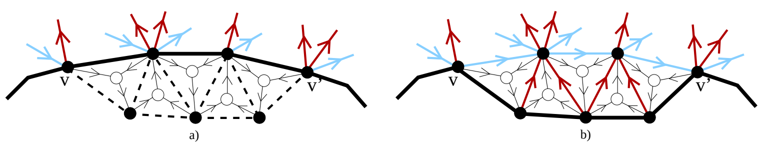

Fig. 7. The update step of the iterative algorithm finding the preimage of an α_0-orientation.

1. Choose a minimal admissible couple $\{v, v'\}$ on $\mathcal{C}$.
2. Color in blue all edges of $[v, v']$ and orient them from v to v'.
3. Color in red all edges of T interior to $\mathcal{C}_{\mathcal{P}_{\{v,v'\}}}$ and orient them from $\mathcal{P}_{\{v,v'\}}$ to $[v, v']$.
4. Replace $[v, v']$ by $\mathcal{P}_{\{v,v'\}}$ in $\mathcal{C}$, so as to update (shrink) $\mathcal{C}$.

It can easily be proven (see Figure 7) that the invariants on $\mathcal{C}$ stated above are still satisfied after these operations. As we have said, as long as $\mathcal{C}$ is not empty, there exists an admissible couple on $\mathcal{C}$ (hence there is also a minimal one), so that the algorithm terminates. At the end, $\mathcal{C}$ is empty. Then the invariants satisfied by $\mathcal{C}$ exactly imply that the coloration and orientation of inner edges of T is a transversal couple of bipolar orientation Z such that $\Psi \circ \Phi(Z) = O$, see also Figure 8 for a complete execution of the algorithm. This concludes the proof of Theorem 5.

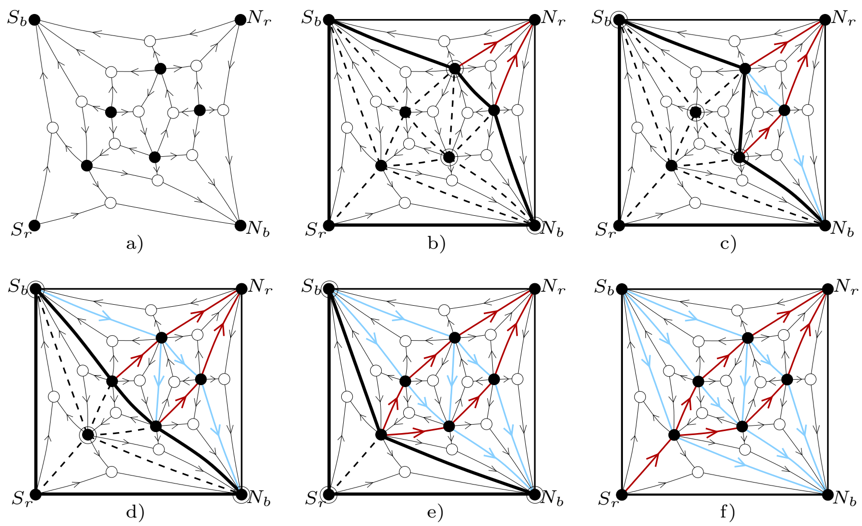

Fig. 8. The complete execution of the algorithm calculating the preimage of an α_0-orientation O by $\Psi \circ \Phi$. At each step, the vertices of the chosen internal path $\mathcal{P}_{\{v,v'\}}$ are surrounded.

B Other proofs

Proof of Proposition 1. First we prove that the orientation induced by the red edges is acyclic. Suppose there is an oriented cycle $\mathcal{C}$ of red edges of T. Let k and n be respectively the number of vertices on $\mathcal{C}$ and in the interior of $\mathcal{C}$. Then Inner-vertex Condition C1' implies that the number A_b of bicolored angles in the interior of $\mathcal{C}$ is $4n + 2k$. Moreover Euler relation and the fact that all faces in the interior of $\mathcal{C}$ are triangular imply that there are $2n + k - 2$ faces interior to $\mathcal{C}$. Then Lemma 1 implies that $A_b = 2(2n + k - 2) = 4n + 2k - 4$, which gives a contradiction.

Hence the orientation induced by the red edges is acyclic. In addition, Inner-vertex Condition C1' implies that inner vertices of T can not be extrema for the underlying partial order. Finally, Border-vertex Condition C2' ensures that S_r and N_r are respectivelly the minimum and the maximum of the partial order (on the vertices of T different from S_b and N_b) induced by the orientation of the red edges. The same holds similarly for the orientation of the blue edges.

Partial proof of Theorem 2. Let e be a red edge of T and let v_1 and v_2 be its origin and extremity. To show that e is oriented from bottom to top in the embedding computed by CompactDraw(T), we have to show that the ordinate of v_2 is greater than the ordinate of v_1, which comes down to proving that $\mathcal{P}_b(v_1)$ is on the right of $\mathcal{P}_b(v_2)$. This last fact is trivial because $\mathcal{P}_b(v_1)$ and $\mathcal{P}_b(v_2)$ can not cross (we do not provide a proof here but it is easy to check), and Inner-vertex

Condition C1' ensures that v_1 is on the right of v_2 in the orientation induced by the blue edges of T. Now we prove that e is weakly oriented from left to right. We have here to prove that the abscissa of v_1 is not greater than the abscissa of v_2, which comes down to proving that $\mathcal{P}_r(v_1)$ is not on the right of $\mathcal{P}_r(v_2)$. This last fact follows from the two easy observations: the leftmost outgoing red path of v_1 is (weakly) on the left of the leftmost outgoing red path of v_2; and the rightmost ingoing red path of v_2 is (weakly) on the right of the rightmost ingoing red path of v_1. Similary the blue edges are directed from left to right and weakly directed from top to bottom.

Now the proof that the embedding is a straight line drawing (i.e. that two edges can not cross) relies on an iterative algorithm very similar to the ones presented in Section 2.4 and Section A.4. The idea consists in maintaining and iteratively shrinking a cycle $\mathcal{C}$ of edges of T such that two edges outside of $\mathcal{C}$ can not cross in the embedding. Using the property of orientations of red edges and of blue edges stated above, it can easily be established that the cycle $\mathcal{C}$ can be shrinked so that the the embedding of the edges outside of $\mathcal{C}$ remains planar.

Finally we show that the width W and height H of the grid of the drawing computed by TransversalDraw(T) are such that $W + H = n - 1$ if T has n vertices. We recall that we define T_r (resp. T_b) as T deprived from its blue (resp. red) edges and we write respectively f_r and f_b for the number of inner faces of T_r and T_b. By construction of TransversalDraw, $W = f_r$ and $H = f_b$. Hence we have to prove that $f_r + f_b = n - 1$. We write respectively e_r and e_b for the number of edges of T_r and T_b. Euler relation ensures that the total number e of edges of T is $3n-7$. Hence, $e_b + e_r = e + 4 = 3n - 3$. In addition Euler relation, taken respectively in T_r and T_b, ensures that $n + f_r = e_r + 1$ and $n + f_b = e_b + 1$. Finally, $f_r + f_b = e_r + e_b - 2n + 2 = n - 1$.

Proof of Proposition 4 First we provide an expression of the series $T(z) = \sum_n T_n z^n$ counting irreducible triangulations of the 4-gon by their number of inner vertices. Let $A(z) = \sum_n A_n z^n$ be the series counting rooted ternary trees by their number of inner nodes, so that we have $A(z) = z(1 + A(z))^3$. We define respectively the sets $\overline{\mathcal{A}_n}$ and $\widehat{\mathcal{A}_n}$ of ternary trees with n inner nodes and having the following marks: an inner edge is marked and oriented for objects of $\overline{A}_n$; an inner edge is marked and oriented and a leaf is marked for objects of $\widehat{A}_n$. Observe that a ternary tree with n inner nodes has $n-1$ inner edges and $2n + 2$ leaves (this last fact can be proven recursively). Hence we have $A_n \cdot 2(n-1) = |\widehat{\mathcal{A}_n}| = |\overline{\mathcal{A}_n}|(2n+2)$, so that $|\overline{A_n}| = 2\frac{n-1}{2n+2}A_n$. In addition, an object of $\overline{A} := \cup_n \overline{A}_n$ can be seen as a couple of rooted ternary trees whose root leaves have been merged into a marked (and oriented) inner edge. Hence the series counting objects of $\overline{A}$ by their number of inner vertices is $A(z)^2$. Finally, we know from Proposition 4 that $T_n = \frac{4}{2n+2}A_n$. Hence, we also have $T_n = \frac{2n+2}{2n+2}A_n - 2\frac{n-1}{2n+2}A_n$, from which we conclude that $T(z) = A(z) - A(z)^2$.

Now, writing $U(z)$ for the series counting undecomposable irreducible triangulations of the 4-gon, we have easily from the discussion of Section 4.2:

$T(z)+1 = \frac{U(z)+1}{1-z(U(z)+1)}$. Hence $U(z)+1 = \frac{T(z)+1}{1+z(T(z)+1)} = \frac{A(z)-A(z)^2+1}{1+z(A(z)-A(z)^2+1)}$. Finally, writing $C(z)$ for the series counting rooted 4-connected triangulations by their number of inner vertices, we also know from the discussion of Section 4.2 that $C(z) = zU(z)$, which gives the expression for $C(z)$ in terms of $A(z)$ stated in Proposition 4.

C Analysis of the reduction of the grid from TransversalDraw to CompactTransversalDraw

This section gives a precise analysis of the probability distribution of the grid size of TransversalDraw and CompactTransversal. It yields the following theorem:

Theorem 6. *Let T be taken uniformly at random among rooted irreducible triangulations of the 4-gon with n vertices. Let $W \times H$ and $W_c \times H_c$ be the dimensions of the grid obtained respectively with* TransversalDraw(T) *and* CompactTransversal(T)*. Then W and H are asymptotically almost surely equal to $\frac{n}{2}$ up to fluctuations of order $\sqrt{n}$; and W_c and H_c are asymptotically almost surely equal to $\frac{n}{2} \cdot \left(1 - \frac{5}{27}\right)$ up to fluctuations of order $\sqrt{n}$.*

C.1 Characterization of unused abscissas as particular edges of T

We analyze the number of unused abscissas of TransversalDraw(T), i.e., the number of vertical lines of the grid that bear no vertex of T. Recall that the width of the grid of TransversalDraw(T) is the number of inner faces of the red-map T_r of T, where T_r is obtained by computing the minimal transversal couple of bipolar orientations of T and then removing all blue edges. The abscissa $Abs(v)$ of a vertex v is obtained by associating to v an oriented path $\mathcal{P}_r(v)$ of red edges called separating path of v, as defined in Section 3 and then counting the number of inner faces of T_r on the left of $\mathcal{P}_r(v)$. In addition, we orient the 4 border edges of T_r from S_r to N_r. By construction of TransversalDraw(T), abscissas 0 and f_r are occupied respectively by S_r and N_r. Then, an abscissa i with $1 \leq i \leq f_r - 1$ is not used if there exists no inner vertex v of T such that $Abs(v) = i$. In what follows we show that each abscissa-candidate $1 \leq i \leq f_r$ can be associated to a unique inner face of T_r, for which we write $Abs(f) = i$. Then we show that the existence of an inner vertex v of T with $Abs(v) = i$ only depends on the configuration of the edge at the bottom right corner of f.

First we give a few definitions. Let $e = (v_1, v_2)$ be an inner edge of T_r, oriented from v_1 to v_2. The *sepatating path* of e, denoted by $\mathcal{P}_r(e)$, is defined as the concatenation of the leftmost outgoing red path of v_2, of the edge e, and of the rightmost ingoing red path of v_1. Let f be an inner face of T_r. As the oriented edges of T_r form a bipolar orientation form a bipolar orientation, it is well known that there exist two vertices S_f and N_f of f, called south pole and north pole of f, such that the contour of f consists of two "lateral" oriented paths P_1 and P_2, the first one going from S_f to N_f with f on its right, the second one going from S_f to N_f with f on its left. The *separating path* of f, denoted by $\mathcal{P}_r(f)$, is defined as the concatenation of the leftmost outgoing red path of N_f, of the path P_2, and of the rightmost ingoing red path of v_1. Observe that the first edge e_f of P_2, called bottom right edge of f, verifies $\mathcal{P}_r(f) = \mathcal{P}_r(e_f)$. The following lemma states properties of paths that follow easily from their definition. It will prove useful to show that separating paths do not cross each other.

Lemma 2. *Let v and v' be two different inner vertices of T. Let $\mathcal{P}_r^{left}(v)$, $\mathcal{P}_r^{left}(v')$, $\mathcal{P}_r^{right}(v)$ and $\mathcal{P}_r^{right}(v')$ be respectively the leftmost outgoing red path and rightmost ingoing red path of v and v'. Then $\mathcal{P}_r^{left}(v)$ and $\mathcal{P}_r^{left}(v')$ do not cross each other, they join at a vertex v'' and then are equal between v'' and N_r. Similarly $\mathcal{P}_r^{right}(v)$ and $\mathcal{P}_r^{right}(v')$ do not cross each other, join at a vertex v'' annd then are equal between v'' and S_r. Concerning the intersection I of $\mathcal{P}_r^{left}(v)$ and $\mathcal{P}_r^{right}(v')$, either v' is on $\mathcal{P}_r^{left}(v)$, in which case I is empty (i.e. $\mathcal{P}_r^{left}(v)$ and $\mathcal{P}_r^{right}(v')$ have no common vertex); or v' is on $\mathcal{P}_r^{left}(v)$, in which case I is the part of $\mathcal{P}_r^{left}(v)$ between v and v'.*

Lemma 3. *Let e and e' be two different red edges of T. Then the separating paths $\mathcal{P}_r(e)$ and $\mathcal{P}_r(e')$ do not cross each other. If $\mathcal{P}_r(e) = \mathcal{P}_r(e')$, then e and e' have the same face of T_r on their left.*

Proof. A first observation, following from the presence of the transversal blue edges, is that an edge e' connecting two vertices of an oriented red path is on the path. Hence only three cases can arise: either e' and $\mathcal{P}_r(e)$ do not intersect, or they intersect at a unique extermity of e' or e' is on $\mathcal{P}_r(e)$. Lemma 2 allows us to carry out a examination of each of the three cases and to obtain the stated result.

Recall that the separating path of a face f is the separating path of its bottom-right edge e_f. Thus, Lemma 3 ensures that, for two different inner faces of T_r, the separating paths $\mathcal{P}_r(f)$ and $\mathcal{P}_r(f')$ do not cross each other and are different. Hence the number of inner faces of T_r on the left of each of the two paths are different, i.e., $Abs(f) \neq Abs(f')$. There are f_r inner faces in T_r and each inner face f of T_r clearly verifies $1 \leq Abs(f) \leq f_r$. Hence the pigeonhole principle yields:

Lemma 4. *For each i with $1 \leq i \leq f_r$, there exists a unique inner face f of T such that $Abs(f) = i$.*

As a consequence, inner faces of T_r can be strictly ordered from left to right according to their associated abscissa.

Lemma 5. *Let $e = (v_1, v_2)$ be an inner edge of the red-map T_r oriented from v_1 to v_2. Let v be an inner vertex of T. Then $\mathcal{P}_r(v)$ and $\mathcal{P}_r(e)$ do not cross each other. In addition, there exists a vertex v such that $\mathcal{P}_r(v) = \mathcal{P}_r(e)$ if and only if either e is the rightmost ingoing red edge at v_2 or e is the leftmost outgoing red edge at v_1.*

Proof. The fact that $\mathcal{P}_r(v)$ and $\mathcal{P}_r(e)$ do not cross each other easily follows from Lemma 2. The second statement of the lemma follows from a few observations. If v is not on $\mathcal{P}_r(e)$ then clearly $\mathcal{P}_r(v)$ is not equal to $\mathcal{P}_r(e)$. If v is on $\mathcal{P}_r(e)$ between v_2 and N_r, then $\mathcal{P}_r(v) = \mathcal{P}_r(e)$ iff all edges of $\mathcal{P}_r(e)$ between v_1 and v are the rightmost ingoing red edge at their end-vertex. If v is on $\mathcal{P}_r(e)$ between S_r and v_1, then $\mathcal{P}_r(v) = \mathcal{P}_r(e)$ iff all edges of $\mathcal{P}_r(e)$ between S_r and v_2 are the leftmost outgoing red edge at their origin. It follows from these observations that $\mathcal{P}_r(v_2) = \mathcal{P}_r(e)$ if e is the rightmost ingoing red edge at its end-vertex, that $\mathcal{P}_r(v_1) = \mathcal{P}_r(e)$ if e is the leftmost outgoing red edge at its origin, and that no vertex v satisfies $\mathcal{P}_r(v) = \mathcal{P}_r(e)$ otherwise.

Definition An *internal red edge* of T is an inner edge that is neither the leftmost outgoing red edge at its origin nor the rightmost ingoing red edge at its end-vertex. An internal blue edge of T is defined similarly.

Proposition 5. *The number of unused abscissas in* TRANSVERSALDRAW(T) *is equal to the number of internal red edges of T. Similarly, the number of unused ordinates in* TRANSVERSALDRAW(T) *is equal to the number of internal blue edges of T.*

Proof. Let i with $1 \leq i \leq f_r$ be an abscissa candidate and let f be the unique inner face of the red-map T_r with $Abs(f) = i$. Recall that the separating path $\mathcal{P}_r(v)$ is equal to the separating path $\mathcal{P}_r(e_f)$ of the bottom-right edge e_f of the face f. Then Lemma 5 ensures that i is not the abscissa of any vertex v of T iff e_f is an internal red edge of T. Hence the number of unused abscissas of TRANSVERSALDRAW(T) is equal to the number of internal red edges of T which are the bottom-right edge of an inner face of T_r. But observe that an edge e is the bottom-right edge of an inner face of T_r iff e is not the leftmost outgoing red edge at its origin. In particular, any internal red edge of T is the bottom-right edge of an inner face of T_r.

C.2 Analysis of the number of internal edges of a random irreducible triangulation

According to Proposition 5, the number of coordinates of the grid that can be deleted to obtain a compact drawing of T is equal to the number of internal edges of T. Internal edges of T correspond, through the closure-mapping, to particular edges of the ternary tree A associated to T. We define an internal edge of a ternary tree A as an inner edge $e = (v_1, v_2)$ of A such that the edges following e in counterclockwise order respectively around v_1 and around v_2 are both inner edges (i.e., they are not incident to a leaf of A).

Lemma 6. *Let T be an irreducible triangulation of the 4-gon and let A be the ternary tree whose closure is T, A being endowed with its transversal edge-bicoloration and T being endowed with its minimal transversal edge-partition. Each internal red (resp. blue) edge of A corresponds, through the closure of A, to an internal red (resp. blue) edge of T.*

Proof. Ommited.

Let $\mathcal{A}_n$ be the set of rooted ternary trees with n inner nodes and $\mathcal{T}_n$ be the set of rooted irreducible triangulations of the 4-gon with n inner vertices. Proposition 5, Lemma 6 and the bijection between $\mathcal{A}_n \times \{1,2,3,4\}$ and $\mathcal{T}_n \times \{1,\ldots,2n+2\}$ imply the following statement: *The probability distribution of the number of unused coordinates of a (uniformly) random object of $\mathcal{T}_n$ is equal to the probability distribution of the number of internal edges of a (uniformly) random object of $\mathcal{A}_n$.* The following proposition, where the reduction factor already appears, is weaker than Theorem 6.

Proposition 6. *Let T be taken uniformly at random in $\mathcal{T}_n$. The mean number of deleted coordinates of* TransversalDraw(T) *is asymptotically equivalent to $\frac{5n}{27}$. As the half-perimeter of* TransversalDraw(T) *is equal to $(n-1)$, the asymptotic of the mean of the half-perimeter of* CompactTransversalDraw(T) *is $n \cdot \left(1 - \frac{5}{27}\right)$.*

Proof. Let T_n be the number of rooted ternary trees with n inner nodes. The generating function $T(z) := \sum_n T_n z^n$ verifies $T(z) = z(1+T(z))^3$. Let $\widetilde{T}_n$ be the number of rooted ternary trees with n inner nodes such that the right son of the root node is not a leaf. The associated generating function $\widetilde{T}(z)$ is clearly equal to $zT(z)(1+T(z))^2$. Let $\overline{T}_n$ be the number of ternary trees with n inner nodes and rooted at an internal edge e. By definition of an internal edge, the two subtrees hanging at each extremity have their right son different from a leaf. Hence the generating $\overline{T}(z) := \sum_n \overline{T}_n z^n$ verifies $\overline{T}(z) = \widetilde{T}(z)^2 = z^2 T(z)^2 (1+T(z))^4$. As a ternary tree with n inner nodes has $2n+2$ leaves, the quantity $\frac{1}{2}\overline{T}_n(2n+2)$ counts the number of rooted ternary trees with a marked internal edge. Hence $\frac{\overline{T}_n(n+1)}{T_n}$ is the mean number of internal edges of a rooted ternary tree with n inner nodes taken uniformly at random. Lagrange's inversion formula yields $\overline{T}_n = 2\frac{(3n-3)!(5n-10)}{(n-4)!(2n+2)!(n-2)}$ and $T_n = \frac{(3n)!}{n!(2n+1)!}$. Finally, Stirling Formula yields $\frac{\overline{T}_n(n+1)}{T_n} \sim_{n\to\infty} \frac{5n}{27}$.

To carry out the precise analysis stated in Theorem 6, we have to take bicoloration of edges into account. A rooted ternary tree is bicolored by coloring the edge connected to the root leaf in red and coloring all other edges in the unique way such that the four colors of the edges around each node alternate.

Let X_r be the random variable corresponding to the number of unused abscissas of TransversalDraw(T) where T is taken uniformly at random in $\mathcal{T}_n$ and where the ordered choice of the two colors is chosen to be red-blue or blue-red (color switch) with probability 1/2. Let Z_r be the random variable corresponding to the number of internal red edges of a tree A taken uniformly at random in $\mathcal{A}_n$. Then the bijection between $\mathcal{A}_n \times \{1,2,3,4\}$ and $\mathcal{T}_n \times \{1,\ldots,2n+2\}$, together with Proposition 5 and Lemma 6, imply that the distribution probability of X is equal to the distribution probability of Z_r.

The analysis of the distribution of Z_r is then readily carried out by introducing the generating function $A(z,u)$ of bicolored rooted ternary trees where

z marks inner nodes and u marks internal red edges. Using a decomposition grammar of ternary trees adapted to take internal red edges into account, it is possible to obtain an algebraic expression for $A(z,u)$. From this algebraic equation an analysis of the singularities of $A(z,u)$ is feasible. Then, the well-known quasi-power theorem ensures that Z_r is almost surely equal to $\frac{5n}{54}$ up to fluctuations of order $\sqrt{n}$. Similarly, the number Z_b of internal blue edges of a random irreducible triangulation of the 4-gon can be proven to be asymptotically almost surely equal to $\frac{5n}{54}$ up to fluctuations of order $\sqrt{n}$. Finally, it can be shown, using similar methods, that the variable Y_r and Y_b counting respectively inner faces of the red-map and inner faces of the blue-map of a random irreducible triangulation of the 4-gon both are almost surely equal to $\frac{n}{2}$, up to fluctuations of order $\sqrt{n}$. Theorem 6 follows easily from these distributions and from Proposition 5.