BIJECTIVE COUNTING OF PLANE BIPOLAR ORIENTATIONS

ÉRIC FUSY, DOMINIQUE POULALHON, AND GILLES SCHAEFFER

Abstract. We introduce a bijection between plane bipolar orientations with fixed numbers of vertices and faces, and non-intersecting triples of upright lattice paths with some specific extremities. Writing ϑ_{ij} for the number of plane bipolar orientations with $(i+1)$ vertices and $(j+1)$ faces, our bijection provides a combinatorial proof of the following formula due to Baxter:

$$\vartheta_{ij} = 2\,\frac{(i+j-2)!\,(i+j-1)!\,(i+j)!}{(i-1)!\,i!\,(i+1)!\,(j-1)!\,j!\,(j+1)!}. \quad (1)$$

1. Introduction

A *bipolar orientation* of a graph $G = (V, E)$ is an acyclic orientation of G such that the induced partial order on the vertex set has a unique minimum s called the *source*, and a unique maximum t called the *sink*; s and t are the two *poles*. Bipolar orientations are a powerful combinatorial structure and prove insightful to solve many algorithmic problems such as planar graph embedding [11, 4] and geometric representations of graphs in various flavours (visibility [12], floor planning [10], straight-line drawing [13, 7]). As a consequence, it is an interesting issue to have a better understanding of their combinatorial properties.

This article focuses on the enumeration of bipolar orientations in the planar case. We consider bipolar orientations on rooted planar maps, where a *planar map* is a connected planar graph embedded in the plane without edge-crossings and up to continuous deformation, and *rooted* means with a marked oriented edge (called the *root*) having the outer face on its left. A *plane bipolar orientation* is a pair (M, X), where X is a bipolar orientation of a rooted planar map M such that the poles are the extremities of the root. It is well known that the graphs admitting such a bipolar orientation with adjacent poles are exactly 2-connected graphs, *i.e.,* graphs with no separating vertex. Given M a rooted 2-connected map, let $\vartheta(M)$ be the number of (rooted) bipolar orientations of M. As shown in [5], $\vartheta(M)$ is equal to the coefficient $[x]T_M(x, y)$ in the Tutte polynomial of M, but no explicit formula is known for $\vartheta(M)$. The purpose of this article is to show an explicit formula for the quantities $\vartheta(M)$ when added up over maps with same size parameters. Precisely, let $\mathcal{M}_{ij}$ be the set of rooted 2-connected maps with $(i+1)$ vertices and $(j+1)$ faces (including the outer one); then the quantity $\vartheta_{ij} := \sum_{M\in\mathcal{M}_{ij}} \vartheta(M)$ satisfies Formula 1 (given in the abstract), as was guessed and checked by Baxter [1, Eq 5.3] using algebraic manipulations on generating functions of maps weighted by their Tutte polynomials. The main result of this article is a direct bijective proof of Formula (1), exemplified in Figure 1:

Theorem 1. *There is a bijection between plane bipolar orientations with $(i+2)$ vertices and $(j+2)$ faces, and non-intersecting triples of upright lattice paths on the grid $\mathbb{Z}^2$ with respective origins $(-1, 1)$, $(0, 0)$, $(1, -1)$, and respective endpoints $(i-1, j+1)$, (i, j), $(i+1, j-1)$.*

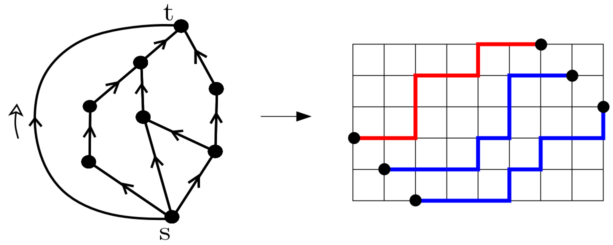


Figure 1. A plane bipolar orientation and the associated triple of non-intersecting upright lattice paths.

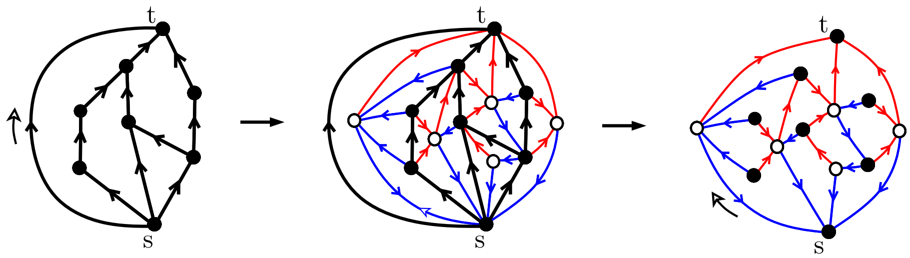


FIGURE 2. From a rooted map endowed with a bipolar orientation to a rooted bicoloriented quadrangulation.

Formula (1) is easily derived from this theorem using Gessel-Viennot Lemma [8, 9] (classical determinant-type formula to enumerate non-intersecting paths).

Overview. Our bijection relies on several steps. Counting plane bipolar orientations is first reduced to counting quadrangulations endowed with specific edge-bicolorations. Then these edge-bicolored quadrangulations are bijectively encoded by triples of words with some conditions on the prefixes. The encoding draws its inspiration from a nice bijection found by Bonichon [3], between Schnyder Woods of triangulations and pairs of non-crossing Dyck words. The final step of our bijection is to translate each binary word of the triple to an upright lattice path; the prefix conditions of the words are equivalent to the property that the three paths are non-intersecting.

2. Reduction to counting edge-bicolored quadrangulations

Let M be a rooted map; its *quadrangulation* Q is the bipartite map with vertex set consisting of vertices and faces of M, and edges corresponding to incidences between these vertices and faces. Q is naturally rooted with the same root vertex as M, as shown in Figure 2. From now on, rooted quadrangulations are endowed with their unique bicoloration of vertices in black and white such that the root vertex, called s, is black; the other outer black vertex is denoted by t.

If M is endowed with a bipolar orientation, this classical construction can be enriched in order to transfer the orientation on Q; a rooted quadrangulation is said to be *bicoloriented* if the edges are oriented and partitioned into red and blue edges such that the following conditions are satisfied, see Figure 3:

- each inner vertex has exactly two outgoing edges, a red one and a blue one;
- around each inner black (*resp.* white) vertex, ingoing edges in each color follow the outgoing one in clockwise (*resp.* counterclockwise) order;
- all edges incident to s are ingoing blue, and all edges incident to t are ingoing red.

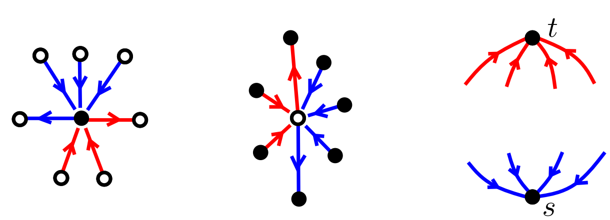


FIGURE 3. Rules of bicolorientation around inner vertices and the two poles.

Any face f of a plane bipolar orientation (M, X) has two poles [5]; let us orient the two corresponding edges of Q from the white vertex f to the two black ones, and color the up-edge red and the down-edge blue. Other edges incident to f are oriented and colored so as to satisfy the circular order condition around f. This defines actually a bicolorientation of Q, and this mapping from plane bipolar orientations to rooted bicoloriented quadrangulations is one-to-one, as proved by an easy extension of [5, Theo 5.3]:

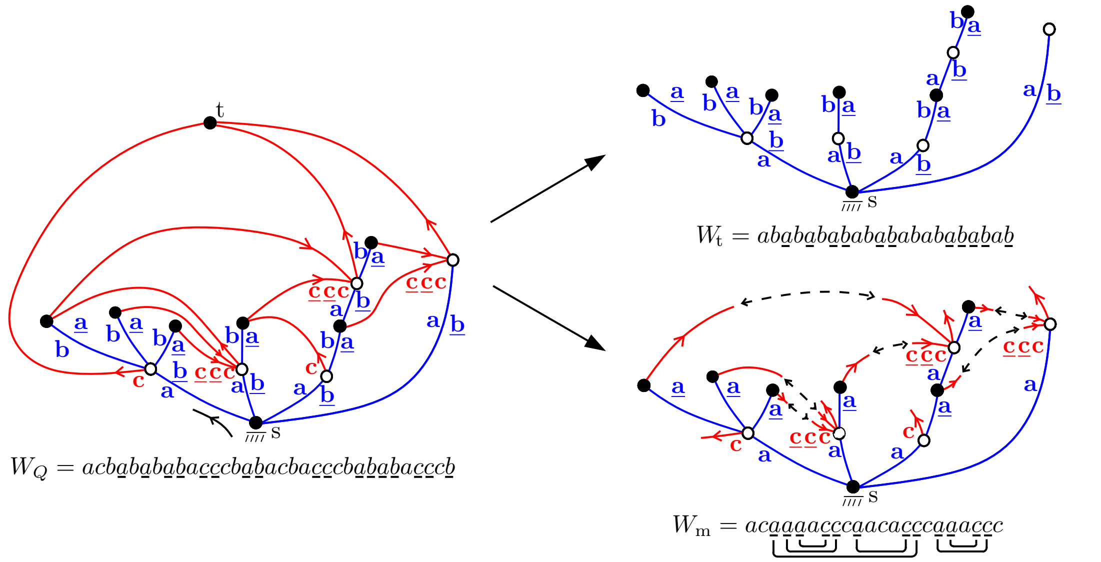


FIGURE 4. Contour words of a bicoloriented quadrangulation.

Theorem 2. *Plane bipolar orientations with i vertices and j faces are in bijection with rooted bicoloriented quadrangulations with i black vertices and j white vertices.*

This reduces the bijective encoding of plane bipolar orientations *w.r.t.* the numbers of vertices and faces to the bijective encoding of rooted bicoloriented quadrangulations *w.r.t.* the numbers of black vertices and white vertices.

3. Encoding a bicoloriented quadrangulation by a triple of words

Bicoloriented quadrangulations have an interesting property: as shown in [2, 6], blue (*resp.* red) edges form two trees spanning all vertices except t (*resp.* s) and oriented toward s (*resp.* t). Let Q be a rooted bicoloriented quadrangulation with $(i+2)$ black vertices and j white vertices, and let T_{blue} be its blue tree. We define the *contour word* W_Q of Q as the word on the alphabet $\{a, \underline{a}, b, \underline{b}, c, \underline{c}\}$ obtained as follows (see Figure 4). Perform a clockwise traversal of the contour of T_{blue} starting at the root edge, and write a letter a (*resp.* b) each time an edge e of T_{blue} is traversed from a black to a white vertex (*resp.* from a white to a black one). Underline the letter if this is the second traversal of e. Write a letter c each time a red edge is crossed at a white vertex, and underline it if the edge is ingoing.

We shall consider three subwords of W_Q: for any l in $\{a, b, c\}$, let W_l denote the word obtained by keeping only the letters in the alphabet $\{l, \underline{l}\}$. In order to describe the properties of these words, we also introduce the *tree-word* W_{t} and the *matching word* W_{m}, that are respectively obtained by keeping the letters in $\{a, \underline{a}, b, \underline{b}\}$, and in $\{a, \underline{a}, c, \underline{c}\}$.

3.1. **The tree-word encodes the blue tree.** Observe that W_{t} corresponds to a classical Dyck encoding of T_{blue}, in which the two alphabets $\{a, \underline{a}\}$ and $\{b, \underline{b}\}$ are used alternatively to encode the bicoloration of vertices. Hence W_{t} is just the shuffle of W_a and W_b at even and odd positions respectively, and each prefix of W_{t} has at least as many non-underlined letters as underlined letters.

It is easily seen that W_a has j occurences of a and i occurences of $\underline{a}$, shortly written $W_a \in \mathfrak{S}(a^j \underline{a}^i)$, with the additionnal property that its first and last letters are a (because of the rightmost branch of T_{blue} being reduced to an edge, see Figure 4). Similarly, $W_b \in \mathfrak{S}(b^i \underline{b}^j)$, and its two last letters are $\underline{b}$.

Last, it is an easy exercise to show that the prefix condition on W_{t} translates to the following property for the pair (W_a, W_b):

Property 1. *For $1 \le k \le i$, the number of a's on the left of the kth occurence of $\underline{a}$ in W_a is strictly larger than the number of $\underline{b}$'s on the left of the kth occurence of b in W_b.*

3.2. **The matching word encodes the red edges.** Let us now focus on W_c and on the matching word W_{m}. Clearly, any occurence of a c (*resp.* $\underline{c}$) in W_Q corresponds to a red edge with white (*resp.* black) origin, see Figure 4. Hence $W_c \in \mathfrak{S}(c^j \underline{c}^i)$. Moreover W_c starts and ends with a letter c, corresponding to the two outer red edges.

Observe also that any occurence of a in W_{m}, which corresponds to the first visit to a white vertex v, is immediately followed by a pattern $\underline{c}^l c$, with l the number of ingoing red edges at v. Hence W_{m} satisfies the regular expression $ac(\underline{a}^* a \underline{c}^* c)^*$, which uniquely defines W_{m} as a shuffle of W_a and W_c.

Let us now consider a red edge with black origin; its origin (encoded by a letter $\underline{a}$) has to be encountered before its endpoint (encoded by a letter $\underline{c}$). Hence planarity ensures that the restriction of W_{m} to the alphabet $\{\underline{a}, \underline{c}\}$ is a parenthesis word, which translates in the following way:

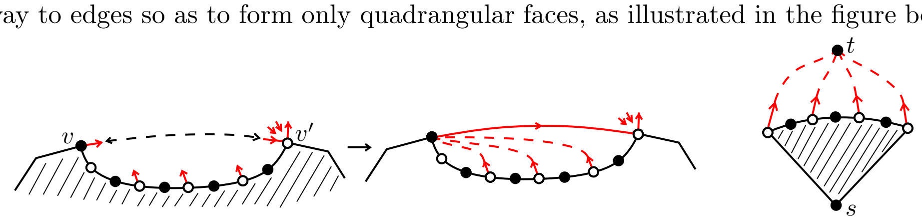
Property 2. *For $1 \leq k \leq j$, the number of $\underline{a}$'s on the left of the kth occurence of a in W_a is at least as large as the number of $\underline{c}$'s on the left of the kth occurence of c in W_c.*

Definition. A triple of words (W_a, W_b, W_c) in $\mathfrak{S}(a^j \underline{a}^i) \times \mathfrak{S}(b^i \underline{b}^j) \times \mathfrak{S}(c^j \underline{c}^i)$ is said to be *admissible of type* (i,j) if W_a (*resp.* W_c) ends with a letter a (*resp.* c) and the properties 1 and 2 are satisfied.

Observe that this implies in particular that W_a (*resp.* W_c) starts with a letter a (*resp.* c), and that W_b ends with two letters $\underline{b}$. To sum up, we have described a mapping Φ from rooted bicoloriented quadrangulations with $(i+2)$ black and j white vertices to admissible triples of words of type (i,j). This mapping is proved to be bijective by defining an inverse mapping in a way that naturally reverses the operations performed by Φ.

3.3. **Inverse mapping (sketch).** Starting from an admissible triple (W_a, W_b, W_c) of type (i,j), form a so-called *tree-word* W_{t} as the shuffle of W_a and W_b at even and odd positions. Property 1 ensures that each prefix of W_{t} has not more underlined than non-underlined letters. Hence there exists a (unique) tree, called T_{blue}, whose classical Dyck encoding is W_{t}. In addition, T_{blue} has $(i+2)$ (black) vertices at even depth and j (white) vertices at odd depth.

The next step is to insert the red edges. Precisely we first insert red half-edges and then merge them into complete red edges. Form a so-called *matching word* W_{m} as the unique shuffle of W_a and W_c that satisfies the regular expression $ac(\underline{a}^* a \underline{c}^* c)^*$. For $1 \leq k \leq j$, consider the kth white vertex v in T_{blue} (with vertices ordered *w.r.t.* first visit), and let $l \geq 0$ be the number of consecutive $\underline{c}$'s that follow the kth occurence of a in W_{m}. Insert l ingoing and one outgoing red half-edges (in clockwise order) in the angle of T_{blue} traversed during the first visit to v. Then, add an outgoing red half-edge to each black vertex v, in the angle traversed during the last visit to v. Next, we match outgoing red half-edges at black vertices and ingoing red half-edges at white vertices following Property 2. Finally it is easily shown that the red half-edges going out of white vertices can be completed in a unique way to edges so as to form only quadrangular faces, as illustrated in the figure below.



a) b)

By construction, the obtained figure is a quadrangulation endowed with a bicolorientation, and the construction is inverse to Φ.

4. Representation as a triple of non-intersecting paths

Given Theorem 2 and the bijection presented in Section 3, proving Theorem 1 reduces to showing that admissible triples of words of type (i,j) are in bijection with non-intersecting triples of upright lattice paths with origins $(-1,1)$, $(0,0)$, $(1,-1)$ and endpoints $(i-1,j-1)$, $(i,j-2)$, $(i+1,j-3)$. This section describes the correspondence, illustrated in Figure 5.

Consider an admissible triple of words (W_a, W_b, W_c) of type (i,j), and represent each word as an upright lattice path starting at the origin, the binary word being read from left to right, and

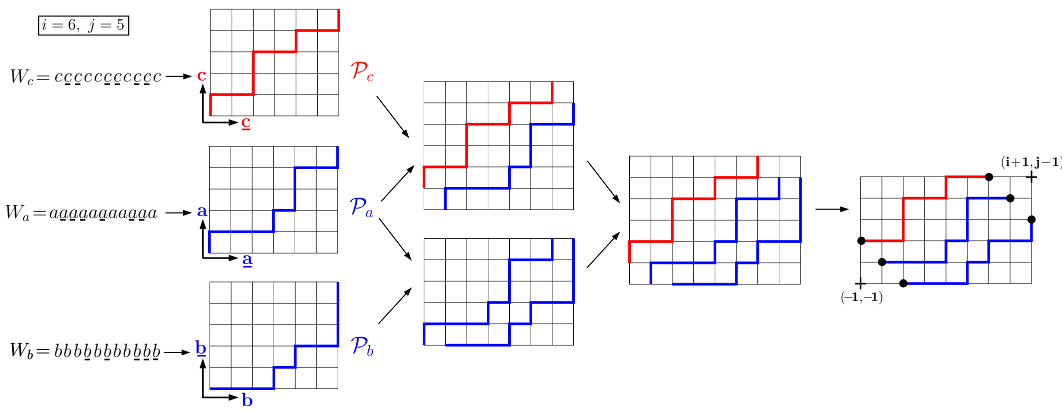


FIGURE 5. The triple of words translates to a non-intersecting triple of paths.

the associated path going up or right depending on the letter. Letters associated to up steps are a, $\underline{b}$ and c. Clearly, as $(W_a, W_b, W_c) \in \mathfrak{S}(a^j \underline{a}^i) \times \mathfrak{S}(b^i \underline{b}^j) \times \mathfrak{S}(c^j \underline{c}^i)$, the three paths end at (i, j).

Translation of Property 1 on the paths is: "for $1 \leq k \leq i$, the kth horizontal step of $\mathcal{P}_a$ (ending at abscissa k) is strictly above the kth horizontal step of $\mathcal{P}_b$". Hence, Property 1 is equivalent to the fact that $\mathcal{P}_a$ and the right-shift of $\mathcal{P}_b$ are non-intersecting. Similarly, Property 2 is translated to: "for $1 \leq k \leq j$, the kth vertical step of $\mathcal{P}_a$ is weakly on the right of the kth vertical step of $\mathcal{P}_c$". In other words, $\mathcal{P}_c$ is weakly on the left of $\mathcal{P}_a$. Hence, Property 2 is equivalent to the fact that $\mathcal{P}_a$ and the upleft-shift of $\mathcal{P}_c$ are non-intersecting. Let us now consider the redundant letters; they correspond to two vertical steps in each path, and removing them leads to a triple of non intersecting paths with origins and endpoints modified accordingly.

References

[1] R. J. Baxter. Dichromatic polynomials and Potts models summed over rooted maps. *Annals of Combinatorics*, 5:17, 2001.

[2] T. Biedl and F. J. Brandenburg. Partitions of graphs into trees. In *Proceedings of Graph Drawing'06 (Karlsruhe)*, volume 4372 of *LNCS*, pages 430–439, 2007.

[3] N. Bonichon. A bijection between realizers of maximal plane graphs and pairs of non-crossing dyck paths. In *Proceedings of FPSAC'02*, pages 123–132, 2002.

[4] N. Chiba, T. Nishizeki, S. Abe, and T. Ozawa. A linear algorithm for embedding planar graphs using pq-trees. *J. Comput. Syst. Sci.*, 30(1):54–76, 1985.

[5] H. de Fraysseix, P. Ossona de Mendez, and P. Rosenstiehl. Bipolar orientations revisited. *Discrete Appl. Math.*, 56(2-3):157–179, 1995.

[6] S. Felsner, C. Huemer, S. Kappes, and D. Orden. Binary labelings for plane quadrangulations and their relatives, 2006. Available at `http://arxiv.org/abs/math.CO/0612021`.

[7] É. Fusy. Straight-line drawing of quadrangulations. In *Proceedings of Graph Drawing (Karlsruhe)*, 2006.

[8] I. Gessel and X. Viennot. Binomial determinants, paths, and hook formulae. *Adv. Math*, 58:300–321, 1985.

[9] I. Gessel and X. Viennot. Determinants, paths, and plane partitions, 1989. Preprint.

[10] G. Kant and X. He. Regular edge labeling of 4-connected plane graphs and its applications in graph drawing problems. *Theor. Comput. Sci.*, 172(1-2):175–193, 1997.

[11] A. Lempel, S. Even, and I. Cederbaum. An algorithm for planarity testing of graphs. In *Theory of Graphs, Int. Symp (New York)*, pages 215–232, 1967.

[12] R. Tamassia and I. G. Tollis. A unified approach to visibility representations of planar graphs. *Discrete Comput. Geom.*, 1(4):321–341, 1986.

[13] R. Tamassia and I. G. Tollis. Planar grid embedding in linear time. *IEEE Trans. on Circuits and Systems*, CAS-36(9):1230–1234, 1989.

E.F. and G.S.: LIX (École Polytechnique), D.P.: LIAFA (Université Paris Diderot)