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BAXTER PERMUTATIONS AND PLANE BIPOLAR ORIENTATIONS

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Abstract. We present a simple bijection between Baxter permutations of size n and plane bipolar orientations with n edges. This bijection translates several classical parameters of permutations (number of ascents, right-to-left maxima, left-to-right minima ...) into natural parameters of plane bipolar orientations (number of vertices, degree of the sink, degree of the source ...), and has remarkable symmetry properties. By specializing it to Baxter permutations avoiding the pattern 2413, we obtain a bijection with non-separable planar maps. A further specialization yields a bijection between permutations avoiding 2413 and 3142 and series-parallel maps.

1. Introduction

In 1964, Glen Baxter, in an analysis context, introduced a class of permutations that now bear his name [2]. A permutation π of the symmetric group $\mathfrak{S}_n$ is *Baxter* if one cannot find $i < j < k$ such that $\pi(j+1) < \pi(i) < \pi(k) < \pi(j)$ or $\pi(j) < \pi(k) < \pi(i) < \pi(j+1)$. These permutations were first enumerated around 1980 [9, 28, 33, 11]. More recently, they have been studied in a slightly different perspective, in the general and very active framework of *pattern avoiding* permutations [7, 15, 22, 25]. In particular, the number of Baxter permutations of $\mathfrak{S}_n$ having m ascents, i left-to-right maxima and j right-to-left maxima is known to be [28]:

$$\frac{ij}{n(n+1)}\binom{n+1}{m+1}\left[\binom{n-i-1}{n-m-2}\binom{n-j-1}{m-1}-\binom{n-i-1}{n-m-1}\binom{n-j-1}{m}\right]. \qquad (1)$$

A few years ago, another Baxter, the physicist *Rodney* Baxter, studied the sum of the Tutte polynomials $T_M(x,y)$ of non-separable planar maps M having a fixed size [3]. He proved that the coefficient of x^1y^0 in $T_M(x,y)$, summed over all rooted non-separable planar maps M having $n+1$ edges, $m+2$ vertices, root-face of degree $i+1$ and a root-vertex of degree $j+1$, was given by (1). He was unaware that these numbers had been met before (and bear "his" name), and that the coefficient of x^1y^0 in $T_M(x,y)$ is the number of *bipolar orientations* of M [24, 20]. This number is also, up to a sign, the derivative of the chromatic polynomial of M, evaluated at 1 [26].

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This is an amusing coincidence — are all Baxters doomed to invent independently objects that are counted by the same numbers? — which was first noticed in [7]. It is the aim of this paper to explain it via a direct bijection between Baxter permutations and plane bipolar orientations. Our bijection is non-recursive, simple to implement, and has a lot of structure: it translates many natural statistics of permutations into natural statistics of maps, and behaves well with respect to symmetries. When restricted to Baxter permutations avoiding the pattern 2413, it specializes to a bijection between these permutations and rooted non-separable planar maps, which is related to a recursive description of these maps by Dulucq *et al.* [14]. The key of the proofs, and (probably) the reason why this bijection has so much structure, is the existence of *two isomorphic generating trees* for Baxter permutations and plane orientations.

It is not the first time that intriguing connections arise between pattern avoiding permutations and planar maps[1]. For instance, several families of permutations, including two-stack sortable permutations, are equinumerous with non-separable maps [34, 36, 14, 10]. Connected 1342-avoiding permutations are equinumerous with bicubic planar maps [6], and 54321-avoiding involutions are equinumerous with tree-rooted maps [29, 23, 4, 7]. Finally, a bijection that sends plane bipolar orientations to 3-tuples of non-intersecting paths appears in [19]. These configurations of paths are known to be in one-to-one correspondence with Baxter permutations [15]. More bijections between these three families of objects are presented in [17]. However, our construction is the first direct bijection between Baxter permutations and plane bipolar orientations. Moreover, it has interesting properties (symmetries, specializations . . .) that the bijections one may obtain by composing the bijections of [17] do not have.

Let us finish this introduction with the outline of the paper. After some preliminaries in Section 2, we present our main results in Section 3: we describe the bijection Φ, its inverse Φ^{-1}, and state some of its properties. In particular, we explain how it transforms statistics on permutations into statistics on orientations (Theorem 2), and how it behaves with respect to symmetry (Proposition 4). In Section 4 we introduce two *generating trees*, which respectively describe a recursive construction of Baxter permutations and plane bipolar orientations. We observe that these trees are isomorphic, which means that Baxter permutations and orientations have a closely related recursive structure. In particular, this isomorphism implies the existence of a (recursively defined) canonical bijection Λ between these two classes of objects, the properties of which reflect the properties of the trees (Theorem 12). We prove in Section 5 that our construction Φ coincides with the canonical bijection Λ, and then (Section 6) that our description of Φ^{-1} is correct. Section 7 is devoted to the study of two interesting specializations of Φ, and the final section presents possible further developments of this work.

[1] Nor is it the first time that connections between Baxter permutations and planar objects appear: several authors have discussed bijections between these permutations and certain *floor-plans*, which have similarities with the present paper. See [1] and references therein.

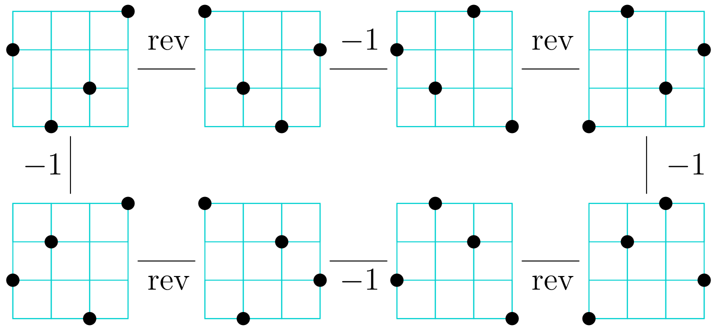


FIGURE 1. The symmetries of the square act on Baxter permutations.

2. Preliminaries

2.1. Baxter permutations

A permutation $\pi = \pi(1) \cdots \pi(n)$ will be represented by its *diagram*, that is, the set of points $(i, \pi(i))$. Hence the group of symmetries of the square, of order 8, acts on permutations of $\mathfrak{S}_n$. This group is generated (for instance) by the following two involutions:

– the inversion $\pi \mapsto \pi^{-1}$, which amounts to a reflection in the first diagonal,
– the reversion of permutations, which maps $\pi = \pi(1) \cdots \pi(n)$ to $\text{rev}(\pi) = \pi(n) \cdots \pi(1)$.

It is clear from the definition given at the beginning of the introduction that the reversion leaves the set $\mathcal{B}_n$ of Baxter permutations of size n invariant. It is also true that the inversion leaves $\mathcal{B}_n$ invariant, so that $\mathcal{B}_n$ is invariant under the 8 symmetries of the square (Fig. 1). The invariance of $\mathcal{B}_n$ by inversion follows from an alternative description of Baxter permutations in terms of *barred* patterns [22], which we now describe. Given two permutations π and $\tau = \tau_1 \cdots \tau_k$, an *occurrence of the pattern* τ in π is a subsequence $\pi(i_1), \ldots, \pi(i_k)$, with $i_1 < \cdots < i_k$, which is order-isomorphic to τ. If no such occurrence exists, then π *avoids* τ. Now, π avoids the *barred* pattern $\tau = \tau_1 \cdots \tau_{i-1} \overline{\tau_i} \tau_{i+1} \cdots \tau_k$ if every occurrence of $\tau_1 \cdots \tau_{i-1} \tau_{i+1} \cdots \tau_k$ in π is a sub-sequence of an occurrence of $\tau_1 \cdots \tau_{i-1} \tau_i \tau_{i+1} \cdots \tau_k$. Then Baxter permutations are exactly the $25\bar{3}14$- and $41\bar{3}52$-avoiding permutations.

We equip $\mathbb{R}^2$ with the natural product order: $v \leq w$ if $x(v) \leq x(w)$ and $y(v) \leq y(w)$, that is, if w lies to the North-East of v. Recall that for two elements v and w of poset $\mathcal{P}$, w *covers* v if $v < w$ and there is no u such that $v < u < w$. The *Hasse diagram* of $\mathcal{P}$ is the digraph having vertex set $\mathcal{P}$ and edges corresponding to the covering relation. We orient these edges from the smaller to the larger element.

Given a permutation π, a *left-to-right maximum* (or: *lr-maximum*) is a value $\pi(i)$ such that $\pi(i) > \pi(j)$ for all $j < i$. One defines similarly rl-maxima, lr-minima and rl-minima.

2.2. Plane bipolar orientations

A *planar map* is a connected graph embedded in the plane with no edge-crossings, considered up to continuous deformation. A map has vertices, edges, and *faces*, which are the connected components of $\mathbb{R}^2$ remaining after deleting the edges. The *outer face* is unbounded, the *inner faces* are bounded. The map is *separable* if there exists a vertex whose deletion disconnects the map. A *plane bipolar orientation* O is an acyclic orientation of a planar map M with a unique *source* s (vertex with no ingoing edge) and a unique *sink* t (vertex with no outgoing edge), both on the outer face (Fig. 2). The *poles* of O are s and t. One of the oriented paths going from s to t has the outer face on its right: this path is the *right border* of O, and its length is the *right outer degree* of O. The *left outer degree* is defined similarly.

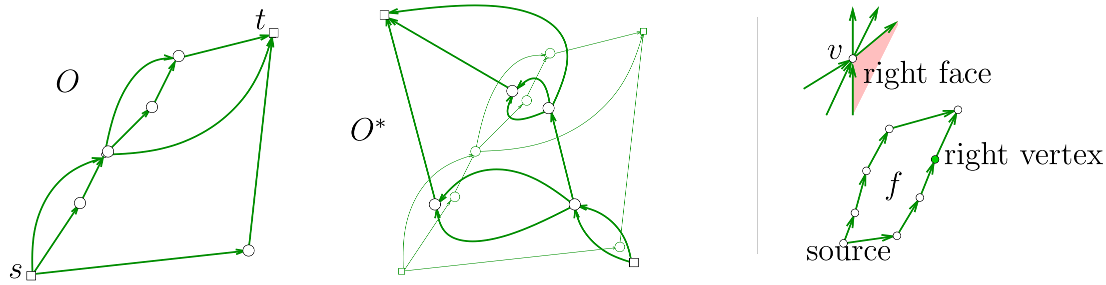


FIGURE 2. Left: A plane bipolar orientation, having right (left) outer degree 2 (3), and its dual. Right: Local properties of plane bipolar orientations.

It can be seen that around each non-polar vertex v of a plane bipolar orientation, the edges are *sorted* into two blocks, one block of ingoing edges and one block of outgoing edges: around v, one finds a sequence of outgoing edges, then a sequence of ingoing edges, and that's it (Fig. 2, right). The face that is incident to the last outgoing edge and the first ingoing edge (taken in clockwise order) is called the *right face* of v. Analogously, the face that is incident to the last ingoing edge and the first outgoing edge (still in clockwise order) is called the *left face* of v. Dually, the border of every finite face f contains exactly two maximal oriented paths (Fig. 2, right), forming a small bipolar orientation. Its source (resp. sink) is called the *source* (resp. *sink*) of f. The other vertices of the face are respectively called *right* and *left* vertices of f: if v is a right vertex of f, then f is the left face of v.

A map M is *rooted* if one of the edges adjacent to the outer face is oriented, in such a way the outer face lies on its right. In this case, a bipolar orientation of M is required to have source s and sink t, where s and t are the endpoints of the root edge. As recalled above, the number of such orientations of M is the coefficient of x^1y^0 in the Tutte polynomial $T_M(x, y)$ (this is actually true of any graph G with an oriented edge). This number is non-zero if and only if M is non-separable [27].

A map M is *bipolar* if two of its vertices, lying on the outer face and respectively called the *source* (s) and the *sink* (t), are distinguished. In this case, a bipolar orientation of M is required to have source s and sink t. There is of course a simple one-to-one correspondence between rooted maps and bipolar maps, obtained by deleting the root edge and taking its starting point (resp. ending point) as the source (resp. sink). It will be convenient (and, we hope, intuitive) to use the following notation: if M is rooted, $\check{M}$ is the bipolar map obtained by deleting the root edge. If M is bipolar, $\hat{M}$ is the rooted map obtained by adding a root-edge. In this case, M admits a bipolar orientation if and only if $\hat{M}$ is non-separable.

Two natural transformations act on the set $\mathcal{O}_n$ of (unrooted) plane bipolar orientations having n edges. For a plane bipolar orientation O, we define $\mathrm{mir}(O)$ to be the *mirror image* of O, that is, the orientation obtained by flipping O around any line (Fig. 3). Clearly, mir is an involution.

The other transformation is *duality.* The dual plane orientation O^* of O is constructed as shown in Fig. 2. There is a vertex of O^* in each bounded face of O, and *two* vertices of O^* (its poles) in the outer face of O. The edges of O^* connect the vertices corresponding to faces of O that are adjacent to a common edge, and are oriented using the convention shown in the figure (that is, from the face located to the right of the edge to the face located to the left of the edge). Note that $(O^*)^*$ is obtained by changing all edge directions in O, so that duality is a transformation of order 4. The transformations mir and $O \mapsto O^*$ generate a group of order 8 (Fig. 3). We shall see that our bijection Φ allows us to superimpose Figs. 1 and 3. That is, if $\Phi(\pi) = O$, then $\Phi(\pi^{-1})$ is $\mathrm{mir}(O)$ and $\Phi(\mathrm{rev}(\pi))$ is $\mathrm{mir}(O^*)$.

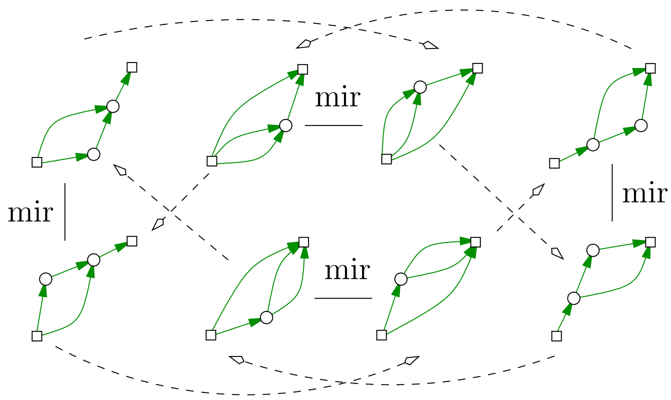


FIGURE 3. A group of order 8 acts on plane bipolar orientations. The dashed edges join an orientation to its dual, the others join mirror images.

3. Main results

3.1. From Baxter permutations to plane bipolar orientations

Let π be a Baxter permutation. We first construct an embedded digraph $\phi(\pi)$ with straight edges having black and white vertices. The black vertices are the points $b_i = (i, \pi(i))$. The white vertices correspond to the *ascents* of π. More precisely, for each ascent a (i.e., $\pi(a) < \pi(a+1)$), let $\ell_a = \max\{\pi(i) : i < a+1 \text{ and } \pi(i) < \pi(a+1)\}$. The Baxter property implies that for all $i > a$ such that $\pi(i) > \pi(a)$, one has actually $\pi(i) > \ell_a$ (see Fig. 4). Create a white vertex $w_a = (a + 1/2, \ell_a + 1/2)$. Finally, add two more white vertices $w_0 = (1/2, 1/2)$ and $w_n = (n + 1/2, n + 1/2)$. (In other words, we consider that $\pi(0) = 0$ and $\pi(n+1) = n+1$.) We define the embedded digraph $\phi(\pi)$ to be the Hasse diagram of this collection of black and white vertices for the product order on $\mathbb{R}^2$, drawn with straight edges (Fig. 5, centre). Of course, all edges point to the North-East. We will prove in Section 5 the following proposition.

Proposition 1. *For all Baxter permutation π, the embedded graph $\phi(\pi)$ is planar (no edges cross), bicolored (every edge joins a black vertex and a white one), and every black vertex has indegree and outdegree* 1.

These properties can be observed in the example of Fig. 5. Erasing all black vertices yields a plane bipolar orientation with source w_0 and sink w_n, which we define to be $\Phi(\pi)$ (see Fig. 5, right). Observe that every point of the permutation gives rise to an edge of $\Phi(\pi)$: we will say that the point *corresponds* to the edge, and vice-versa. We draw the attention of the reader to the fact that the slightly vague word "correspond" has now a very precise meaning in this paper.

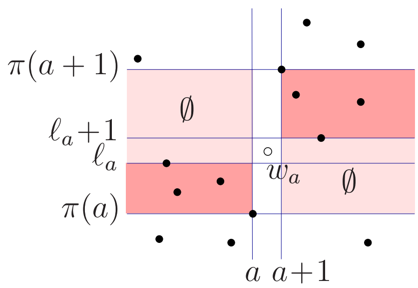


FIGURE 4. The insertion of white vertices in ascents.

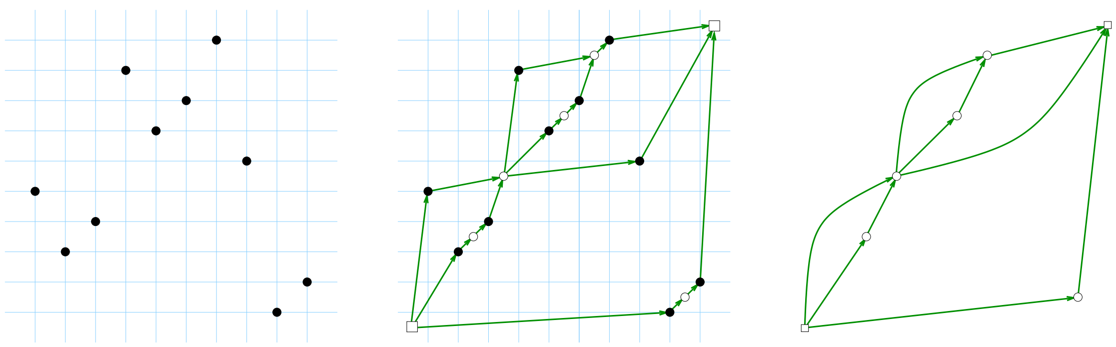


FIGURE 5. The Baxter permutation $\pi = 5\,3\,4\,9\,7\,8\,10\,6\,1\,2$, the Hasse diagram $\phi(\pi)$ and the plane bipolar orientation $\Phi(\pi)$.

Theorem 2. *The map Φ is a bijection between Baxter permutations and plane bipolar orientations, which transforms standard parameters as follows:*

size $\leftrightarrow$	*# edges,*	*# ascents* $\leftrightarrow$	*# non-polar vertices,*
# lr-maxima $\leftrightarrow$	*left outer degree,*	*# rl-minima* $\leftrightarrow$	*right outer degree,*
# rl-maxima $\leftrightarrow$	*degree of the sink,*	*# lr-minima* $\leftrightarrow$	*degree of the source.*

By Euler's formula, the number of inner faces of $\Phi(\pi)$ is the number of descents of π. Theorem 2 is proved in Section 5. It explains why the numbers (1) appear both in the enumeration of Baxter permutations [28] and in the enumeration of bipolar orientations [3].

3.2. **The inverse bijection**

Let O be a plane bipolar orientation, with vertices colored white. Let O' be the bicolored oriented map obtained by inserting a black vertex in the middle of each edge of O (Fig. 6, left). Recall how the in- and out-going edges are organized around the vertices of O (Fig. 2, right). Let T_x be obtained from O' by retaining, at each white vertex, only the first incoming edge in clockwise order (at the sink, we only retain the incoming edge lying on the right border). All vertices of T_x, except from the source of O, have now indegree one, and can be reached from the source. Hence T_x is a plane tree rooted at the source.

Similarly, let T_y be obtained from O' by retaining, at each white vertex, the last incoming edge in clockwise order (at the sink, we only retain the incoming edge lying on the left border). Then T_y is also a spanning tree of O', rooted at the source of O.

Label the black vertices of T_x in prefix order, walking around T_x clockwise. Label the black vertices of T_y in prefix order, but walking around T_y counterclockwise. For every black vertex v of O', create a point at coordinates $(x(v), y(v))$, where $x(v)$ and $y(v)$ are the labels of v in T_x and T_y respectively. Let $\Psi(O)$ be the collection of points thus obtained (one per edge of O).

Theorem 3. *For every plane bipolar orientation O, the set of points $\Psi(O)$ is the diagram of the Baxter permutation $\Phi^{-1}(O)$.*

This can be checked in the examples of Figs. 5–6, and is proved in Section 6. The construction Ψ is closely related to an algorithm introduced in [13] to draw planar orientations on a grid.

3.3. **Symmetries and specializations**

3.3.1. ***Symmetries.*** We now describe how the symmetries of the square, which act in a natural way on Baxter permutations, are transformed through our bijection.

Proposition 4. *Let π be a Baxter permutation and $O = \Phi(\pi)$. Then*

$$\Phi(\pi^{-1}) = \mathrm{mir}(O) \quad \textit{and} \quad \Phi(\mathrm{rev}(\pi)) = \mathrm{mir}(O^*).$$

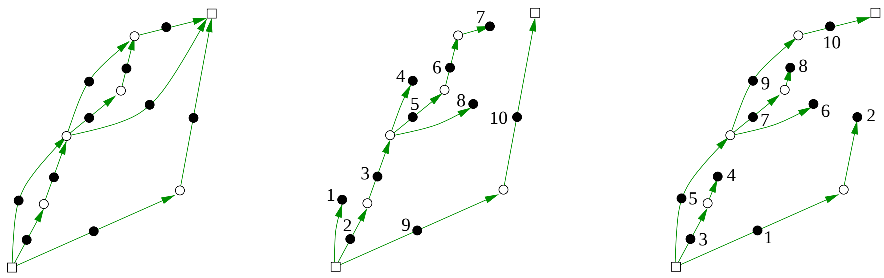


FIGURE 6. A bipolar orientation O, and the trees T_x and T_y used to compute the coordinates of the points of $\Psi(O)$.

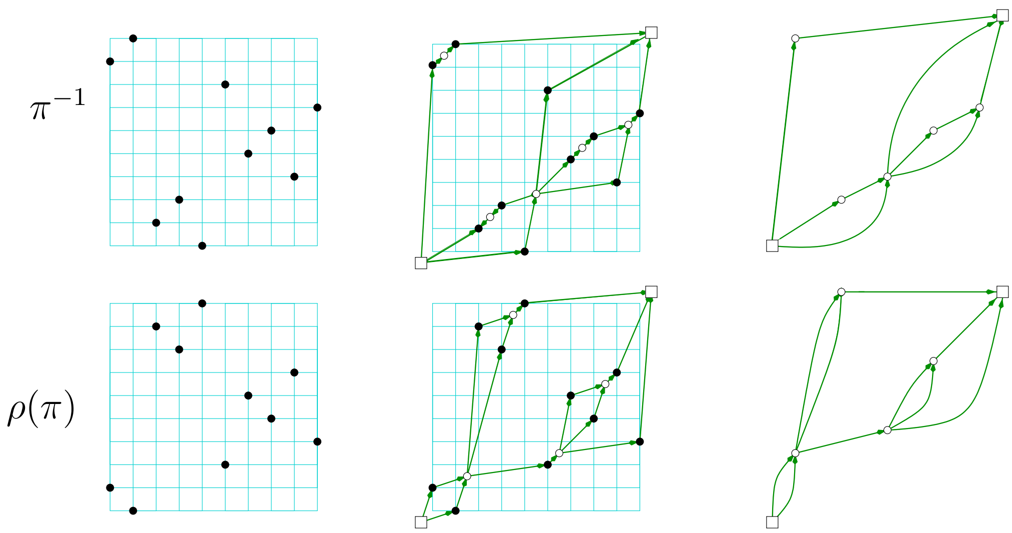


FIGURE 7. The Hasse diagram and the bipolar orientations associated to π^{-1} and $\rho(\pi)$, where $\pi = 5\,3\,4\,9\,7\,8\,10\,6\,1\,2$ is the Baxter permutation of Fig. 5.

By combining both properties, this implies

$$O^* = \Phi(\rho(\pi)),$$

where ρ *is the clockwise rotation by* 90 *degrees.*

Moreover, if the point $p = (i, \pi(i))$ *of* π *corresponds (via the bijection* Φ*) to the edge* e *of* O*, then the point* $(\pi(i), i)$ *of* π^{-1} *corresponds to the edge* $\text{mir}(e)$ *in* $\text{mir}(O)$*, and the point* $\rho(p)$ *in* $\rho(\pi)$ *corresponds to the dual edge of* e *in* O^*.

The symmetry properties dealing with the inverse and the rotation are illustrated in Fig. 7. The first one is easily proved from the definition of ϕ and Φ. Indeed, it is clear from Fig. 4 that π has an ascent at a if and only if π^{-1} has a ascent at ℓ_a, and that the white dots of the embedded graph $\phi(\pi^{-1})$ are the mirror images of the white dots of $\phi(\pi)$ across the first diagonal. Of course this holds for black vertices as well. Hence the Hasse diagram $\phi(\pi^{-1})$ is obtained by flipping the diagram $\phi(\pi)$ about the first diagonal. The first property follows, as well as the correspondence between the point $(\pi(i), i)$ and the edge $\text{mir}(e)$.

The second property is non-trivial, and will be proved in Section 5.

3.3.2. ***Specializations.*** As recalled in Section 2, a bipolar map M has a bipolar orientation if and only if the rooted map $\hat{M}$ is non-separable.

Take a Baxter permutation π and the bipolar orientation $\Phi(\pi)$, with poles s and t. Let M be the underlying bipolar map, and define $\widehat{\Phi}(\pi)$ to be the rooted non-separable $\hat{M}$. We still call s and t the source and the sink of the rooted map. It is not hard to see that different permutations may give the same map. However, the following is true.

Proposition 5. *The restriction of $\widehat{\Phi}$ to Baxter permutations avoiding the pattern* 2413 *(that is, to permutations avoiding* 2413 *and* $41\bar{3}52$*) is a bijection between these permutations and rooted non-separable planar maps, which transforms standard parameters as follows:*

– *if π has length n, m ascents, i lr-maxima, j rl-maxima, k lr-minima and ℓ rl-minima,*
– *then $\widehat{\Phi}(\pi)$ has $n+1$ edges, m non-polar vertices, a sink of degree $j+1$, a source of degree $k+1$ and the face that lies to the right (resp. left) of the root edge has degree $i+1$ (resp. $\ell+1$).*

This proposition is proved in Section 7.1. The fact that permutations avoiding 2413 and $41\bar{3}52$ are equinumerous with non-separable planar maps was already proved in [14], by exhibiting isomorphic *generating trees* for these two classes. This isomorphism could be used to describe a recursive bijection between permutations and maps. It turns out that, up to simple symmetries, our direct, non-recursive bijection, is equivalent to the one that is implicit in [14]. This is explained in Section 7.3.

Observe that, if π is a Baxter permutation avoiding 2413, then π^{-1} is a Baxter permutation avoiding 3142. As $\Phi(\pi^{-1}) = \mathrm{mir}(\Phi(\pi))$, the restriction of $\widehat{\Phi}$ to Baxter permutations avoiding 3142 is also a bijection with non-separable planar maps. We now describe what happens when we restrict $\widehat{\Phi}$ to permutations avoiding both 2413 and 3142 (such permutations are always Baxter as they obviously avoid the barred patterns $25\bar{3}14$ and $41\bar{3}52$).

Proposition 6. *The restriction of $\widehat{\Phi}$ to permutations avoiding the patterns* 2413 *and* 3142 *is a bijection between these permutations and rooted* series-parallel maps*, which transforms the standard parameters in the same way as in Proposition 5.*

We say that a rooted non-separable map M is *series-parallel* if it does not contain the complete graph K_4 as a minor. The terminology is slightly misleading: the map that can be constructed recursively using the classical series and parallel constructions (Fig. 17) is not M itself, but the bipolar map $\check{M}$. This is the case for the bipolar map of Fig. 5, and one can check that the associated permutation avoids both 2413 and 3142. Details are given in Section 7.2, with the proof of Proposition 6.

It is easy to count series-parallel maps, using their recursive description (we are in the simple framework of *decomposable structures* [18]). The series that counts them by edges is (up to a factor t accounting for the root edge) the generating function of large Schröder numbers [31, Exercise 6.39],

$$S(t) = t\,\frac{1-t-\sqrt{1-6t+t^2}}{2} = \sum_{n\geq 1} t^{n+1} \sum_{k=0}^{n} \frac{(n+k)!}{k!(k+1)!(n-k)!}.$$

Thus the number of permutations of size n avoiding 2413 and 3142 is the nth Schröder number, as was first proved in [35]. See also [22], where this result is proved via a bijection with certain trees.

4. Generating trees

A *generating tree* is a rooted plane tree with labelled nodes satisfying the following property: if two nodes have the same label, the lists of labels of their children are the same. In other words, the (ordered) list of labels of the children is completely determined by the label of the parent. The rule that tells how to label the children of a node, given its label, is called the *rewriting rule* of the tree. In particular, the tree $\mathcal{T}$ with root $(1,1)$ and rewriting rule:

$$(i,j) \leadsto \left\{ \begin{array}{llll} (1,j+1), & (2,j+1), & \dots & (i,j+1), \\ (i+1,j), & \dots & (i+1,2), & (i+1,1), \end{array} \right. \tag{2}$$

will be central in the proofs of our results. The first three levels of this tree are shown in Fig. 8, left. In this section, we describe a generating tree for Baxter permutations, and another one for plane orientations, which are both isomorphic to $\mathcal{T}$. The properties of these trees, combined with this isomorphism, imply the existence of a canonical bijection between Baxter permutations and plane orientations satisfying the conditions stated in Theorem 2. In the next section, we will prove that this bijection coincides with our map Φ.

4.1. A generating tree for Baxter permutations

Take a Baxter permutation π of length $n+1$, and remove the value $n+1$: this gives another Baxter permutation, denoted $\overline{\pi}$, of length n. Conversely, given $\sigma \in \mathcal{B}_n$, it is well known, and easy to see, that the permutations π such that $\overline{\pi} = \sigma$ are obtained by inserting the value $n+1$:

- either just before an lr-maximum of σ,
- or just after an rl-maximum of σ.

This observation was used already in the first paper where Baxter permutations were counted [9]. We write $\pi = L_k(\sigma)$ if π is obtain by inserting $n+1$ just before the kth lr-maximum of σ, and $\pi = R_k(\sigma)$ if π is obtain by inserting $n+1$ just after the kth rl-maximum of σ (with the convention that the first lr-maximum is $\sigma(n)$). We henceforth distinguish *left* and *right* insertions, or L- and R-insertions for short. This is refined by talking about (L,k)-insertions (and (R,k)-insertions) when we need to specify the position of the insertion.

This construction allows us to display Baxter permutations as the nodes of a generating tree $\mathcal{T}_b$. The root is the unique permutation of size 1, and the children of a node σ having i lr-maxima and j rl-maxima are, from left to right,

$$L_1(\sigma),\quad L_2(\sigma),\quad \ldots,\quad L_i(\sigma),\quad R_j(\sigma),\quad \ldots,\quad R_2(\sigma),\quad R_1(\sigma).$$

Hence we find at level n the permutations of $\mathcal{B}_n$. The first layers of $\mathcal{T}_b$ are shown on the right of Fig. 8. Observe that, if σ has i lr-maxima and j rl-maxima, then $L_k(\sigma)$ has k lr-maxima and $j+1$ rl-maxima, while $R_k(\sigma)$ has $i+1$ lr-maxima and k rl-maxima. In other words, if we replace in the tree $\mathcal{T}_b$ every permutation by the pair (i,j) giving the number of lr-maxima and rl-maxima, we obtain the generating tree $\mathcal{T}$ defined by (2). That is, the following holds true.

Proposition 7. *The generating tree $\mathcal{T}_b$ of Baxter permutations is isomorphic to the tree $\mathcal{T}$ defined by* (2).

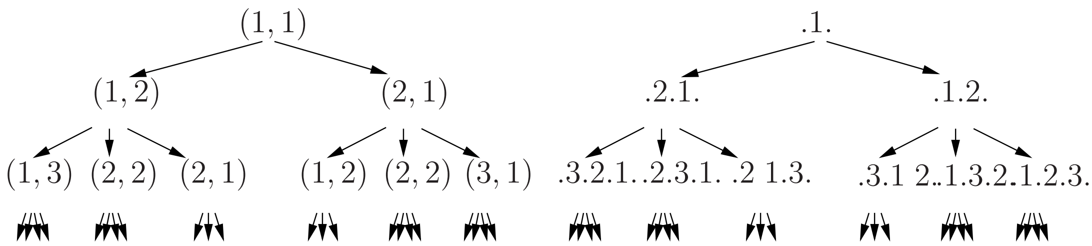


FIGURE 8. The generating tree $\mathcal{T}$ with rewriting rule (2), and the generating tree $\mathcal{T}_b$ of Baxter permutations. The dots represent the possible insertion positions for the new maximal element.

We now observe a property of the tree $\mathcal{T}_b$ that will be crucial to prove one of the symmetry properties of our bijection.

Proposition 8. *The tree obtained from $\mathcal{T}_b$ by replacing each permutation π by* $\mathrm{rev}(\pi)$ *coincides with the tree obtained by reflecting $\mathcal{T}_b$ in a (vertical) mirror.*

In other words, if the sequence of insertions $(S_1,k_1),(S_2,k_2),\ldots,(S_{n-1},k_{n-1})$, where $S_i \in \{L,R\}$ and $k_i \in \mathbb{N}$, leads from the permutation 1 to the permutation $\pi \in \mathcal{B}_n$, then the sequence $(\bar{S}_1,k_1),(\bar{S}_2,k_1),\ldots,(\bar{S}_{n-1},k_{n-1})$ obtained by replacing all L's by R's and vice-versa leads from 1 to $\mathrm{rev}(\pi)$.

Proof of Proposition 8. It suffices to observe that

$$\mathrm{rev}(L_k(\pi)) = R_k(\mathrm{rev}(\pi)) \quad \text{and} \quad \mathrm{rev}(R_k(\pi)) = L_k(\mathrm{rev}(\pi)).$$

∎

4.2. **A generating tree for plane bipolar orientations**

Let O be a plane bipolar orientation with $n+1$ edges. Let e be the last edge of the left border of O, and let v be its starting point. The endpoint of e is the sink t. Perform the following transformation: if v has outdegree 1, contract e, otherwise delete e. This gives a new plane bipolar orientation, denoted $\overline{O}$, having n edges. Indeed, the degree condition on v guarantees that the contraction never creates a cycle and that the deletion never creates a sink.

Conversely, let P be an orientation with n edges. We want to describe the orientations O such that $\overline{O} = P$. Our discussion is illustrated in Fig. 9. Let i be the left outer degree of P, and j the (in)degree of the sink t of P. Let $v_1 = s, v_2, \ldots, v_i, v_{i+1} = t$ be the vertices of the left border, visited from s to t. Denote $e_1, e_2, \ldots, e_j$ the edges incident to t, from right to left (the infinite face is to the right of e_1 and to the left of e_j). The orientations O such that $\overline{O} = P$ are obtained by adding an edge e whose contraction or deletion gives P. This results in two types of edge-insertion:

- **Type L.** For $k \in [\![1, i]\!] = \{1, 2, \ldots, i\}$, the orientation $L_k(P)$ is obtained by adding an edge from $v := v_k$ to t, having the infinite face on its left.
- **Type R.** For $k \in [\![1, j]\!]$, the orientation $R_k(P)$ is constructed as follows. Split the vertex t into two neighbour vertices t and v, and re-distribute the edges adjacent to t: the edges $e_1, e_2, \ldots, e_{k-1}$ remain connected to t, while $e_k, e_{k+1}, \ldots, e_j$ are connected to v. Add an edge from v to t.

In both cases, the last edge of the left border of the resulting orientation O joins v to t. After an L-insertion, v has outdegree 2 or more, so that e will be deleted in the construction of $\overline{O}$, giving the orientation P. After an R-insertion, v has outdegree 1, so that e will be contracted in the construction of $\overline{O}$, giving the orientation P. The fact that we use, as in the construction of Baxter permutations, the notation L_k and R_k, is of course not an accident.

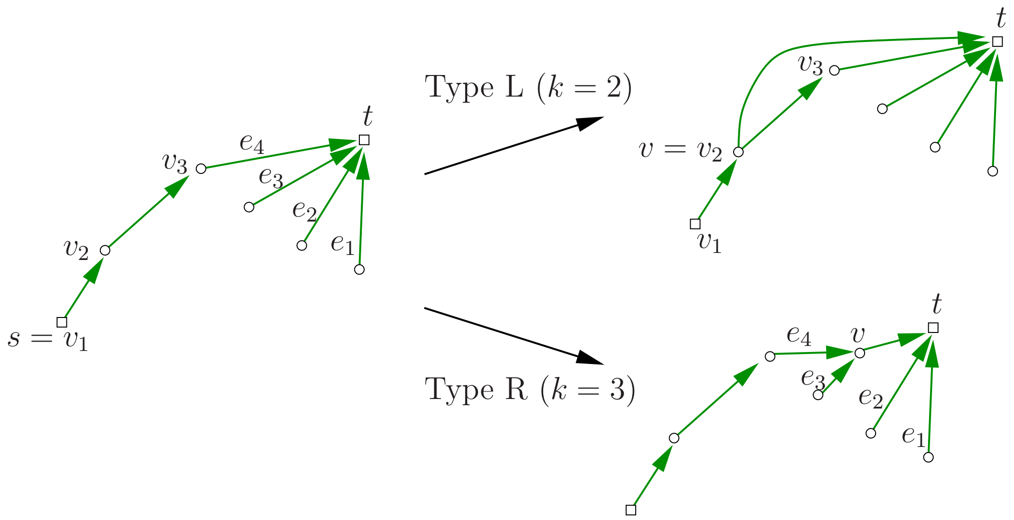


FIGURE 9. Inserting a new edge in a plane bipolar orientation ($i = 3, j = 4$).

We can now define the generating tree $\mathcal{T}_o$ of plane bipolar orientations: the root is the unique orientation with one edge, and the children of a node P having left outer degree i and sink-degree j are, from left to right,

$$L_1(P), \quad L_2(P), \quad \ldots, \quad L_i(P), \quad R_j(P), \quad \ldots, \quad R_2(P), \quad R_1(P).$$

Hence we find at level n the orientations with n edges. The first levels of this tree are shown in Fig. 10. Observe that, if P has left outer degree i and sink-degree

j, then $L_k(P)$ has left outer degree k and sink-degree $i+1$, while $R_k(P)$ has left outer degree $i+1$ and sink-degree k. In other words, if we replace in the tree $\mathcal{T}_o$ every orientation by the pair (i,j) giving the left outer degree and the sink-degree, we obtain the generating tree $\mathcal{T}$ defined by (2).

Proposition 9. *The generating tree $\mathcal{T}_o$ of plane bipolar orientations is isomorphic to the tree $\mathcal{T}$ defined by* (2).

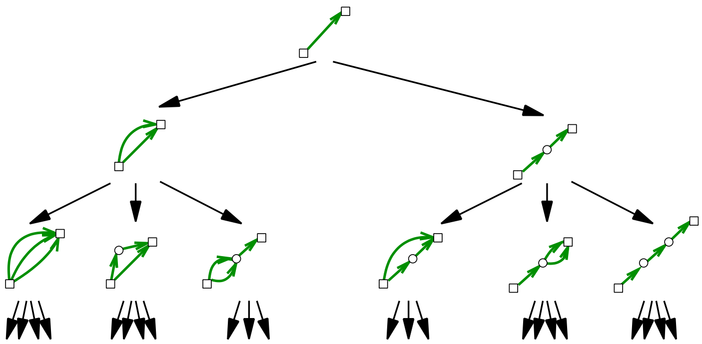


FIGURE 10. The generating tree $\mathcal{T}_o$ of plane bipolar orientations.

We finally observe a property of the tree $\mathcal{T}_o$ that is the counterpart of Proposition 8. Recall the definitions of the dual and mirror orientations, given in Section 2.2.

Proposition 10. *The tree obtained from $\mathcal{T}_o$ by replacing each orientation O by* $\mathrm{mir}(O^*)$ *coincides with the tree obtained by reflecting $\mathcal{T}_o$ in a (vertical) mirror.*

In other words, if the sequence of insertions $(S_1,k_1),(S_2,k_2),\dots,(S_{n-1},k_{n-1})$, where $S_i \in \{L,R\}$ and $k_i \in \mathbb{N}$, leads from the root of $\mathcal{T}_o$ to the orientation $O \in \mathcal{O}_n$, then the sequence $(\bar{S}_1,k_1),(\bar{S}_2,k_1),\dots,(\bar{S}_{n-1},k_{n-1})$ obtained by replacing all L's by R's and vice-versa leads from the root to $\mathrm{mir}(O^*)$.

Proof of Proposition 10. It suffices to observe that

$$\mathrm{mir}((L_k(O))^*) = R_k(\mathrm{mir}(O^*)) \quad \text{and} \quad \mathrm{mir}((R_k(O))^*) = L_k(\mathrm{mir}(O^*)).$$

This should be clear from Fig. 11, which shows that applying L_k to O boils down to applying R_k to $\mathrm{mir}(O^*)$, and that conversely, applying R_k to O boils down to applying L_k to $\mathrm{mir}(O^*)$. Note that this figure only shows O^*, and not its mirror image $\mathrm{mir}(O^*)$. ∎

We conclude with an observation that will be useful in the proof of Proposition 4.

Remark 11. *Take a plane bipolar orientation O having n edges, and label these edges with $1,2,\dots,n$ in the order where they were created in the generating tree. Fig.* 11 *shows that when the edge e is added to O in the generating tree, the edge* $\mathrm{mir}(e^*)$ *is added to* $\mathrm{mir}(O^*)$ *(we denote by e^* the dual edge of e). Consequently, for all edges e of O, the label of e in O coincides with the label of* $\mathrm{mir}(e^*)$ *in* $\mathrm{mir}(O^*)$.

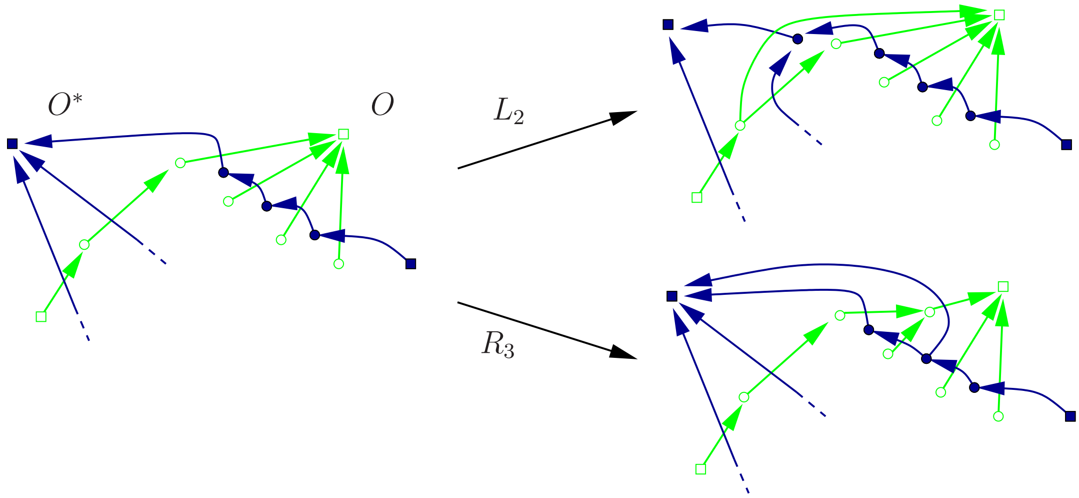


FIGURE 11. How the orientation O^* is changed when L_k or R_k is applied to O.

4.3. The canonical bijection

We have seen that the generating trees $\mathcal{T}_b$ and $\mathcal{T}_o$ are both isomorphic to the tree (2) labelled by pairs (i, j). This gives immediately a canonical bijection Λ between Baxter permutations and plane bipolar orientations: this bijection maps the permutation 1 to the one-edge orientation, and is then defined recursively by

$$\Lambda(L_k(\pi)) = L_k(\Lambda(\pi)) \quad \text{and} \quad \Lambda(R_k(\pi)) = R_k(\Lambda(\pi)).$$

In other words, if π is obtained in the Baxter tree by the sequence of insertions $(S_1, k_1), (S_2, k_2), \ldots, (S_{n-1}, k_{n-1})$, then $\Lambda(\pi)$ is the orientation obtained by *the same sequence* of insertions in the tree of orientations.

Theorem 12. *The map Λ is a bijection between Baxter permutations and plane bipolar orientations, which transforms standard parameters as follows:*

size $\leftrightarrow$	*# edges,*	*# ascents* $\leftrightarrow$	*# non-polar vertices,*
# lr-maxima $\leftrightarrow$	*left outer degree,*	*# rl-minima* $\leftrightarrow$	*right outer degree,*
# rl-maxima $\leftrightarrow$	*degree of the sink,*	*# lr-minima* $\leftrightarrow$	*degree of the source.*

Moreover, if π is a Baxter permutation and $O = \Lambda(\pi)$, then

$$\Lambda(\text{rev}(\pi)) = \text{mir}(0^*).$$

Proof. The properties of Λ dealing with the size, lr-maxima and rl-maxima follow directly from the isomorphism of the trees $\mathcal{T}_b$, $\mathcal{T}_o$ and $\mathcal{T}$. The next three properties are proved by observing that the relevant parameters evolve in the same way in the recursive construction of Baxter permutations and plane bipolar orientations. Indeed, if we replace every node π of $\mathcal{T}_b$ by $(i, j; m, k, \ell)$, where i, j, m, k, ℓ are respectively the number of lr-maxima, rl-maxima, ascents, lr-minima and rl-minima of π, we obtain the generating tree with root $(1,1;0;1,1)$ and rewriting rule:

$$(i,j;m,k,\ell) \rightsquigarrow \begin{cases} (1,j+1;m,k+1,\ell),(2,j+1;m,k,\ell),\dots,(i,j+1;m,k,\ell), \\ (i+1,j;m+1,k,\ell),\dots,(i+1,2;m+1,k,\ell), \\ (i+1,1;m+1,k,\ell+1). \end{cases} \tag{3}$$

Phrased in words, the number of ascents increases by 1 in an R-insertion, and is unchanged otherwise. The number of lr-minima is only changed if we perform an $(L,1)$-insertion (and then it increases by 1), and the number of rl-minima is only changed if we perform an $(R,1)$-insertion (and then it increases by 1).

It is not hard to see that one obtains the same tree by replacing every orientation O of $\mathcal{T}_o$ by $(i,j;m,k,\ell)$, where i,j,m,k,ℓ are respectively the left outer degree, the sink-degree, the number of non-polar vertices, the source-degree and the right outer degree. That is, the number of vertices only increases (by 1) in an R-insertion, the degree of the source only increases (by 1) in an $(L,1)$-insertion and the right outer degree only increases (by 1) in an $(R,1)$-insertion.

Let us finally prove the symmetry property. Let (S_1,k_1), (S_2,k_2), $\dots$, (S_{n-1},k_{n-1}) be the sequence of insertions that leads to π in the Baxter tree. By definition of Λ, this sequence leads to $O=\Lambda(\pi)$ in the tree of orientations. By Propositions 8 and 10, the sequence $(\bar{S}_1,k_1),(\bar{S}_2,k_1),\dots,(\bar{S}_{n-1},k_{n-1})$ obtained by swapping the L's and the R's leads respectively to $\mathrm{rev}(\pi)$ and $\mathrm{mir}(O^*)$ in the trees $\mathcal{T}_b$ and $\mathcal{T}_o$. By definition of Λ, this means that $\mathrm{mir}(O^*)=\Lambda(\mathrm{rev}(\pi))$. ∎

4.4. **A lattice path interpretation of the generating tree**

Baxter permutations have been shown to be in bijection with certain 3-tuples of non-intersecting lattice paths [33, 16]. We describe here a simple generating tree for these paths, which is again isomorphic to the tree $\mathcal{T}$ defined by (2).

Let $n \geq 0$, and consider 3-tuples of non-intersecting paths consisting of north and east steps, starting from the points $(-1,1)$, $(0,0)$ and $(1,-1)$ and ending at 3 consecutive points on the diagonal $x+y=n$, say $(m-1,n-m+1)$, $(m,n-m)$ and $(m+1,n-m-1)$. These path configurations can be displayed as the nodes of a generating tree as follows. The root of the tree is labelled by the configuration formed of 3 paths of length 0. If C is a configuration of paths of length $n>0$, the parent of C in the tree is obtained by contracting the last north (resp. east) step in every path if the central path ends with a north (resp. east) step. The first levels of the tree are shown of Fig. 12.

Conversely, let C be a configuration of length $n-1$. Assume the top path ends with (exactly) $i-1$ east steps, and the bottom path with (exactly) $j-1$ north steps, with $i,j \geq 1$. For $1 \leq k \leq i$, let $L_k(C)$ be obtained by inserting 3 north steps in C as follows:

- one north step is inserted in the last horizontal run of the top path, at distance $k-1$ from the endpoint,
- one north step is added at the end of the central path, and one at the end of the bottom path.

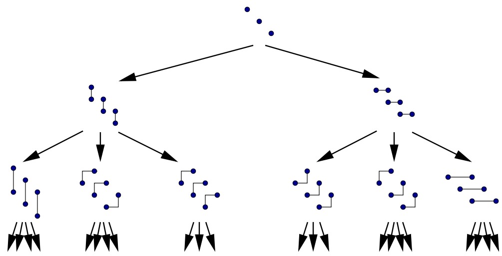


FIGURE 12. The generating tree of path configurations.

One defines analogously $R_k(C)$, for $1 \leq k \leq j$, by inserting 3 east steps in the configuration (at the end of the top and central paths, and in the last vertical run of the bottom path). Then the $L_k(C)$ and $R_k(C)$ are the children of the node C in the tree. We order them from left to right as follows: $L_1(C), \ldots, L_i(C), R_j(C), \ldots, R_1(C)$. It is not hard to see that, if we replace every configuration C by the associated pair (i, j), we obtain once again the tree $\mathcal{T}$.

This construction induces a canonical bijection between Baxter permutations of size n and configurations of length $n-1$. We have not checked whether it is related to one of the already published ones [33, 16]. Of course, the number of lr-maxima (resp. rl-maxima) of the permutation becomes the parameter i (resp. j) in the path configuration. Moreover, the number of ascents of the permutation is the number m of east steps in each path of the configuration. Indeed, it is easy to check that this number evolves as described in (3). (The remaining parameters, k and ℓ, count certain "bottlenecks" in the path configuration.)

From these observations, it follows that the formula (1) giving the number of Baxter permutations of length n, having m ascents, i lr-maxima and j rl-maxima results now from a simple application of the Gessel–Viennot determinantal formula for configurations of non-intersecting paths [21]. This was already used in [16].

5. **The mapping Φ is the canonical bijection**

In the previous section, we have described recursively a bijection Λ that implements the isomorphism between the generating trees $\mathcal{T}_b$ and $\mathcal{T}_o$, and we have shown that it has some interesting properties (Theorem 12). We now prove that the mapping Φ defined in Section 3 coincides with this canonical bijection Λ. Simultaneously, we prove the properties of the map ϕ stated in Proposition 1.

Proposition 13. *For each Baxter permutation π, the embedded graph $\phi(\pi)$ is planar and bicolored, with black vertices of indegree and outdegree* 1. *Moreover,* $\Phi(\pi) = \Lambda(\pi)$.

Proof. The proof is by induction on the size of π. Both statements are obvious for $\pi = 1$. Now assume that the proposition holds for Baxter permutations of size n. Let $\pi \in \mathcal{B}_{n+1}$, and let σ be the parent of π in the tree $\mathcal{T}_b$. Then either $\pi = L_k(\sigma)$, or $\pi = R_k(\sigma)$ for some k. Let $O = \Phi(\sigma) = \Lambda(\sigma)$. We want to prove that $\phi(\pi)$ satisfies the required conditions and that $\Phi(\pi) = L_k(O)$ (or that $\Phi(\pi) = R_k(O)$ in the case of an R-insertion). Both statements follow from a careful observation of

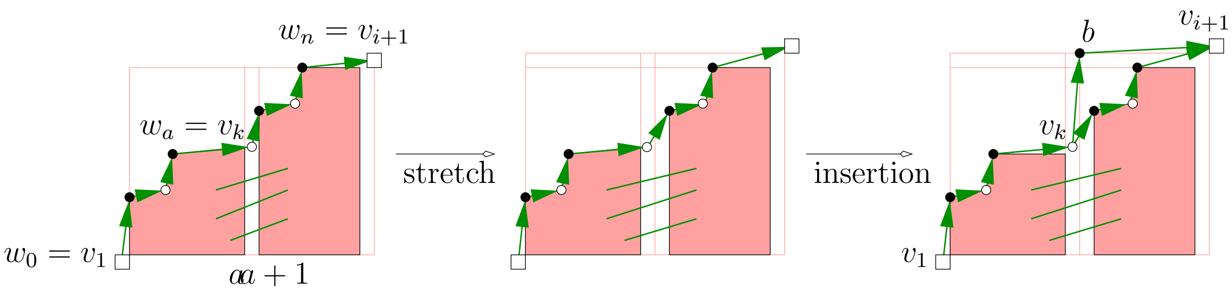


FIGURE 13. How the graph $\phi(\sigma)$ changes during an (L, k)-insertion.

how $\phi(\sigma)$ is changed into $\phi(\pi)$ as $n+1$ is inserted in σ. Some readers will think that looking for two minutes at Figs. 13 and 14 is sufficient to get convinced of the result. For the others, we describe below in greater detail what happens during the insertion of $n+1$ in σ. Essentially, we describe the generating tree whose nodes are the embedded graphs $\phi(\pi)$, for $\pi \in \mathcal{B}_n$.

Case 1. The permutation π is obtain by left insertion, that is, $\pi = L_k(\sigma)$. Let $a+1$ be the abscissa of the kth lr-maximum of σ. Let i be the number of lr-maxima in σ. Then the left border of O has $i+1$ vertices, ranging from $v_1 = w_0$ to $v_{i+1} = w_n$. The vertex w_a is the kth vertex v_k of the left border of O. As $n+1$ is inserted in σ, the ascents of σ become ascents of π, and no new ascent is created. All the vertices that occur in $\phi(\sigma)$ occur in $\phi(\pi)$, but some of them are translated: the white vertex w_n moves from $(n+1/2, n+1/2)$ to $(n+3/2, n+3/2)$, and all vertices located at abscissa $x \geq a+1$ move one unit to the right. These translations, illustrated by the first two pictures of Fig. 13, stretch some edges but do not affect the covering relations among vertices. We claim that they also do not affect planarity. Indeed, it is easy to see that the Hasse diagram of a set $\mathcal{S}$ of points in the plane having distinct abscissas and ordinates, has no crossing if and only if $\mathcal{S}$ avoids the pattern $21\bar{3}54$ [8]. Clearly, this property is not affected by translating to the right the rightmost points of $\mathcal{S}$.

After the translation operations, one new vertex is created: a black vertex $b = (a+1, n+1)$ corresponding to the new value $n+1$ in π. We need to study how this affects the covering relations. That is, which vertices cover b, and which vertices are covered by b?

As b lies at ordinate $n+1$, it is only covered by w_n. This results in a new bicolored edge from b to w_n, which lies sufficiently high not to affect the planarity. Hence b has outdegree 1.

It remains to see which vertices b covers. Clearly it covers w_a. But then all the vertices lying to the South-West of b are actually smaller than w_a for our ordering (because $\sigma(a)$ was an lr-maximum), so that b covers no vertex other than w_a. This means it has indegree 1.

To summarize, one goes from $\phi(\sigma)$ to $\phi(\pi)$, where $\pi = L_k(\sigma)$, by

- stretching some edges by a translation of certain vertices,
- inserting one new vertex, $b = (a+1, n+1)$,

- adding an edge from w_a to b, and another one from b to w_n.

The resulting graph is still bicolored, with black vertices of indegree and outdegree 1. The planarity is preserved as all the changes occur to the North-West of the lr-maxima of σ. Finally, recall that the orientation $O = \Phi(\sigma)$ and $\Phi(\pi)$ are respectively obtained by erasing the black vertices in $\phi(\sigma)$ and $\phi(\pi)$. It should now be clear that $\Phi(\pi)$ is exactly the result of an (L, k)-insertion in O: denoting $v_1, \ldots, v_i, v_{i+1} = w_n$ the vertices of the left border of $\Phi(\sigma)$, one has simply added a new edge from $v_k = w_a$ to the sink w_n in the outer face of $\Phi(\sigma)$.

Case 2. The permutation π is obtain by right insertion, that is, $\pi = R_k(\sigma)$. Let $m_1, \ldots, m_j$ be the points of the diagram of σ corresponding to its rl-maxima, from right to left. Then the sink $t = w_n$ of O has degree j, and for all ℓ, the ℓth edge that arrives at t (from right to left) corresponds to the point m_ℓ of σ via the correspondence Φ. As noticed in the proof of Theorem 12, the ascents of σ become ascents of π, and a new ascent occurs at position a, if the kth rl-maximum of σ is $\sigma(a)$. In particular, all the vertices that occur in $\phi(\sigma)$ occur in $\phi(\pi)$. However, some of them are translated: the white vertex w_n moves from $(n+1/2, n+1/2)$ to $(n + 3/2, n + 3/2)$, while the vertices located at abscissa $x \geq a + 1$ move one unit to the right. These translations, illustrated by the first two pictures of Fig. 14, do not affect the planarity nor the covering relations.

Then two new vertices are created: a black vertex $b = (a+1, n+1)$ corresponding to the new value $n+1$ in π, and a white vertex $w = (a+1/2, n+1/2)$ corresponding to the new ascent. We need to study how they affect the covering relations. That is, which vertices cover w or b, and which vertices are covered by w or b?

As w and b lie respectively at ordinate $n + 1/2$ and $n + 1$, it is easily seen that w is only covered by b, which is only covered by w_n. This results in two new bicolored edges, from w to b, and from b to w_n, which lie sufficiently high not to affect the planarity. Moreover, b does not cover any vertex other than w. We have thus proved that b has indegree and outdegree 1.

It remains to see which vertices w covers. These vertices were covered in $\phi(\sigma)$ by vertices that are now larger than w. But w_n is the only vertex larger than w that was already in $\phi(\sigma)$. Since by assumption, $\phi(\sigma)$ is bicolored, the vertices covered by w are black, and hence were rl-maxima in σ. The vertices $m_1, \ldots, m_{k-1}$ are still covered by w_n (they lie to the right of w), but $m_k, \ldots, m_j$ are covered by w (Fig. 14, right).

To summarize, one goes from $\phi(\sigma)$ to $\phi(\pi)$, where $\pi = R_k(\sigma)$, by

- stretching some edges by a translation of certain vertices,
- inserting two new vertices, $w = (a + 1/2, n + 1/2)$ and $b = (a + 1, n + 1)$,
- adding an edge from w to b, and another one from b to w_n,
- re-directing the edge that starts from m_r so that it points to w rather than w_n, for $r \geq k$.

The resulting graph is still bicolored, with black vertices of indegree and outdegree 1. The planarity is preserved as all the changes occur to the North-East of the rl-maxima of σ. Finally, recall that the orientations $O = \Phi(\sigma)$ and $\Phi(\pi)$ are respectively obtained by erasing the black vertices in $\phi(\sigma)$ and $\phi(\pi)$. It should now

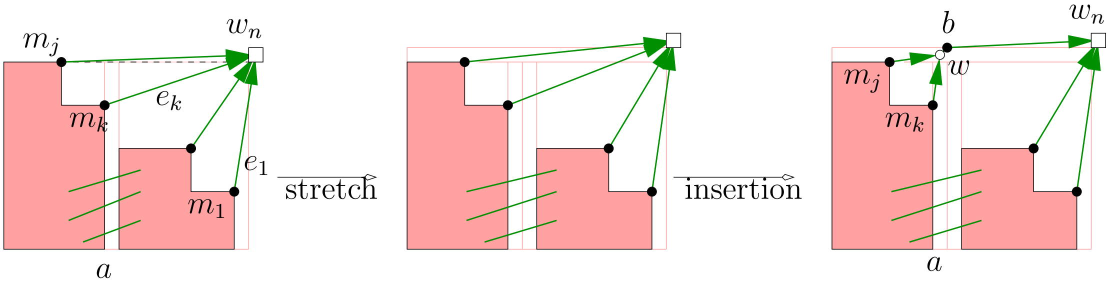


FIGURE 14. How the graph $\phi(\sigma)$ changes during an (R, k)-insertion.

be clear that $\Phi(\pi)$ is exactly the result of an (R, k)-insertion in O: the leftmost $j - k + 1$ edges that were pointing to w_n are now pointing to the new white vertex w. ■

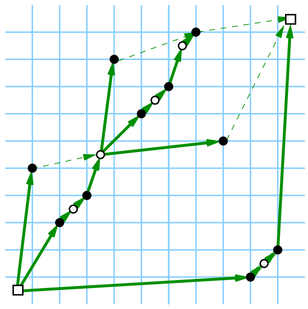
We now know that Φ coincides with the canonical bijection Λ, whose properties were stated in Theorem 12. This implies Theorem 2, and the second symmetry property of Proposition 4. The first property was proved just after the statement of this proposition, as well as the correspondence between the point $(\pi(i), i)$ and the edge $\text{mir}(e)$. To conclude the proof of Proposition 4, it remains to prove that $\rho(p)$ corresponds to the edge e^*.

Observe that, as $n+1$ is inserted in σ to form the permutation π, the edge e that is added to $O = \Phi(\sigma)$ to form $\Phi(\pi)$ corresponds to the point of ordinate $n + 1$ in the diagram of π. Hence, the edge labelling introduced in Remark 11 boils down to labelling every edge of $\Phi(\pi)$ by the ordinate of the corresponding point of π. Consequently, Remark 11 can be reformulated as follows: if the point $p = (i, \pi(i))$ corresponds to the edge e, then the point $\text{rev}(p)$ (which is the point of $\text{rev}(\pi)$ with the same ordinate as p) corresponds to $\text{mir}(e^*)$. Combining this correspondence with the first one (which deals with π^{-1} and $\text{mir}(O)$) gives the final statement of Proposition 4.

6. **The inverse bijection**

It is now an easy task to prove that the map Ψ described in Section 3.2 is indeed the inverse of the bijection Φ. As we already know that Φ is bijective, it suffices to prove that $\Psi(\Phi(\pi)) = \pi$ for all Baxter permutation π. For a Baxter permutation π, we denote by O the orientation $\Phi(\pi)$ and by O' the bicolored oriented map obtained from O by adding a black vertex in the middle of each edge. Recall that $\phi(\pi)$ is an embedding of O'. This allows us to consider the trees T_x and T_y as embedded in $\mathbb{R}^2$. In particular, the edge of T_x joining a white vertex v to its parent is the steepest edge ending at v in $\phi(\pi)$. See Fig. 15, left.

Every point of the diagram of π corresponds to an edge of O. Thus the black vertices that we have added to form O' are in one-to-one correspondence with the points of π. This allows us to identify the black vertices of O' with the points of π. We want to check that the order induced on these vertices by the



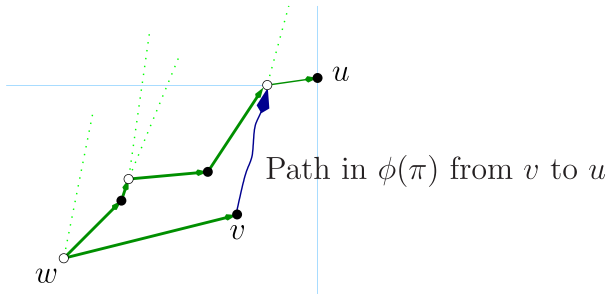


FIGURE 15. Left: The tree T_x obtained from the Baxter permutation $\pi = 5\,3\,4\,9\,7\,8\,10\,6\,1\,2$ Right: Why the prefix order and the abscissa order coincide.

clockwise prefix order of T_x coincides with the order induced by the abscissas of the points. Similarly, we want to check that the order induced on the vertices by the counterclockwise prefix order of T_y coincides with the order induced by the ordinates of the points. As the constructions of ϕ and Ψ are symmetric with respect to the reflection in the first diagonal, it suffices to prove the statement for the tree T_x. Since we are comparing two total orders, it suffices to prove that, if the vertex v comes just after the vertex u in the prefix order of T_x, then v lies to the right of u in $\phi(\pi)$. Two cases occur.

If u is not a leaf of T_x, it has a (unique, white) child, the first child of which is v. As all edges in $\phi(\pi)$ point North-East, v is to the right of u.

If u is a leaf (Fig. 15, right), let w be its closest ancestor (necessarily white) that has at least one child to the right of the branch leading to u. By definition of the prefix order, the first of these children is v. Observe that, by definition of Ψ, the path of T_x joining u to its ancestor w is the *steepest* (unoriented) path of $\phi(\pi)$ leading from u to w. More precisely, it is obtained by starting at u, and choosing at each time the steepest down edge, until w is reached. Assume v is to the left of u. Then it is also below u, and, as $\phi(\pi)$ is a Hasse diagram, there exist paths in $\phi(\pi)$ from v to u. Take the steepest of these: that is, start from u, take the steepest down edge that can be extended into a down path ending at v, and iterate until v is reached. Finally, add the edge (w, v): this gives a path joining w and u that is steeper than the one in T_x, a contradiction. Hence v lies to the right of u. ∎

7. Specializations

7.1. Baxter permutations avoiding 2413 and rooted non-separable maps

The aim of this subsection is to prove Proposition 5: if we restrict Φ to Baxter permutations avoiding 2413, add a root-edge from the source to the sink, and forget the orientations of all (non-root) edges, we obtain a bijection with rooted non-separable planar maps, which transforms standard parameters of permutations into standard parameters of maps. Our first objective will be to describe the orientations corresponding via Φ to 2413-avoiding Baxter permutations. Recall that the faces of a bipolar orientation have *left* and *right* vertices (Fig. 2). The following definition is illustrated in Fig. 16(a).

Definition 14. *Given a plane bipolar orientation O, a* right-oriented piece *(ROP) is a 4-tuple (v_1, v_2, f_1, f_2) formed of two vertices v_1, v_2 and two inner faces f_1, f_2 of O such that:*

- *v_1 is the source of f_1 and is a left vertex of f_2,*
- *v_2 is the sink of f_2 and is a right vertex of f_1.*

A left-oriented piece *(LOP) is defined similarly by swapping 'left' and 'right' in the definition of a ROP. Consequently, a ROP in O becomes a LOP in* mir(O) *and vice-versa.*

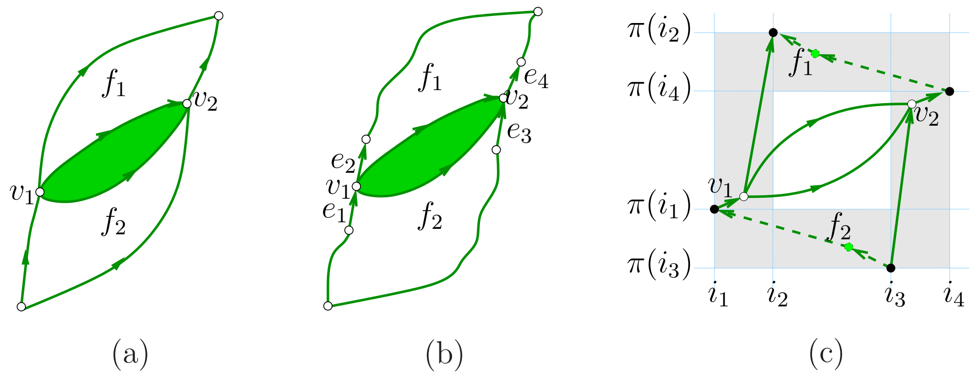


FIGURE 16. (a) A right-oriented piece (ROP) in a plane bipolar orientation. (b) The four distinguished edges of a ROP. (c) A minimal pattern 2413 in a Baxter permutation yields a ROP in the associated plane bipolar orientation. The dashed edges come from $\phi(\rho(\pi))$.

Proposition 15 ([30]). *Every bipolar planar map admits a unique bipolar orientation with no ROP, and a unique plane bipolar orientation with no LOP.*

The set of bipolar orientations of a fixed bipolar map can actually be equipped with the structure of a distributive lattice, the minimum (resp. maximum) of which is the unique orientation with no ROP (resp. LOP) [30].

We can now characterize the image by Φ of 2413-avoiding Baxter permutations.

Proposition 16. *A Baxter permutation π contains the pattern 2413 if and only if the bipolar orientation $O = \Phi(\pi)$ contains a ROP. Analogously, π contains the pattern 3142 if and only if O contains a LOP.*

Simple examples are provided by $\pi = 25314$ and $\pi = 41352$. The corresponding orientations (with a root edge added) are those of Fig. 18.

Proof. Assume O contains a ROP (v_1, v_2, f_1, f_2). Denote e_1, e_2, e_3, e_4 the four edges shown in Fig. 16(b). To each of them corresponds a point p_i of the diagram of π, with $1 \le i \le 4$. We will prove that these points form an occurrence of 2413.

Recall that the order of the abscissas and ordinates of these points is obtained from the trees T_x and T_y defined in Section 3.2. Denote by $P(e)$ the path of T_x that joins (the middle of) the edge $e \in O$ to the source. By definition of T_x, the point of π corresponding to e occurs in π to the left of the point corresponding to e' if and only if either e lies on the path $P(e')$, or $P(e)$ is on the left of $P(e')$ when the two paths meet. From this observation and the configuration of a ROP, it is clear that the x-order of the points p_i is p_1, p_2, p_3, p_4. By similar arguments, the y-order of these points is p_3, p_1, p_4, p_2. Hence they form an occurrence of 2413.

Conversely, let π be a Baxter permutation containing an occurrence p_1, p_2, p_3, p_4 of the pattern 2413. This means that $p_j = (i_j, \pi(i_j))$ with $i_1 < i_2 < i_3 < i_4$ and $\pi(i_3) < \pi(i_1) < \pi(i_4) < \pi(i_2)$. We can assume that the p_i form a *minimal* occurrence of 2413: that is, the bounding rectangle R of the p_i does not contain any other occurrence of the pattern. Then it is easy to see that no point of R lies between columns i_1 and i_2 (this would create a smaller pattern). Similarly, no point of R lies between columns i_3 and i_4, or between rows $\pi(i_3)$ and $\pi(i_1)$, or between rows $\pi(i_4)$ and $\pi(i_2)$. The empty areas are shaded in Fig. 16(c).

We now want to prove that O contains a ROP. Consider an oriented path going from p_1 to p_2 in the embedded Hasse diagram $\phi(\pi)$. As $\phi(\pi)$ is bipartite and there is no point in R between columns i_1 and i_2, this path has length 2, and goes from p_1 to p_2 via a unique white vertex, which we denote by v_1. As the p_i have in- and out-degree 1, there is no other path between p_1 and p_2. Similarly, there is a unique oriented path going from p_3 to p_4, which has length 2. Denote by v_2 the (unique) white vertex of this path.

We will exhibit a ROP whose vertices are v_1 and v_2. To find the faces of this ROP, we consider the Baxter permutation $\pi^* = \rho(\pi)$ obtained by a clockwise rotation of π by 90 degrees. After this rotation, the points $\rho(p_i)$ still form a minimal occurrence of 2413. The above arguments imply that there exists in $\phi(\pi^*)$ a unique path from $\rho(p_3)$ to $\rho(p_1)$, which has length 2 and contains only one white vertex w_2. Similarly, there exists in $\phi(\pi^*)$ a unique path from $\rho(p_4)$ to $\rho(p_2)$, which has length 2 and contains only one white vertex w_1. But $\Phi(\pi^*)$ is the dual orientation O^* (Proposition 4). Let f_1 and f_2 be the faces of O that are the duals of w_1 and w_2. We claim that (v_1, v_2, f_1, f_2) is a ROP of O.

Consider the superposition of the edges of $\phi(\pi)$ and of the edges of $\phi(\pi^*)$, rotated by 90 degrees counterclockwise (dashed lines in Fig. 16(c)). Let e_i be the edge of O corresponding to the point p_i, for $1 \le i \le 4$. By Proposition 4, the dual edge e_i^* corresponds to the point $\rho(p_i)$. As e_1^* starts from w_2, the edge e_1 lies on the left border of the face f_2 in O. By considering the points p_2, p_3 and p_4, one proves similarly that e_2 lies on the left border of f_1, that e_3 lies on the right border of f_2 and e_4 on the right border of f_1. In particular, both v_1 and v_2 are adjacent to f_1 and f_2.

As p_4 lies North-East of p_1, there is an oriented path from p_1 to p_4 in $\phi(\pi)$. As p_1 has outdegree 1, the second vertex on this path is v_1, and similarly, its next-to-last vertex is v_2. Thus the edge (v_1, p_2) of $\phi(\pi)$ is followed, in clockwise order around v_1, by another outgoing edge: this implies that v_1 is the source of the face

f_1. At the other end of the path, we observe that the edge (p_3, v_2) is followed, in clockwise order around v_2, by another ingoing edge: thus v_2 is the sink of f_2.

The edge e_1 ends at v_1 and is on the left border of f_2: as the sink of f_2 is v_2 (and $v_2 \neq v_1$), v_1 is a left vertex of f_2. Symmetrically, v_2 is a right vertex of f_1.

Hence (v_1, v_2, f_1, f_2) is a ROP.

It remains to prove the statement on 3142-avoiding Baxter permutations. Recall that $\Phi(\pi^{-1}) = \text{mir}(O)$ and observe that $2413^{-1} = 3142$. Note also that the map mir transforms right-oriented pieces on left-oriented pieces. Consequently, a Baxter permutation π avoids 3142 if and only if the orientation $\Phi(\pi)$ has no LOP.

■

Proof of Proposition 5. The restriction of Φ to 2413-avoiding Baxter permutations is a bijection between these permutations and bipolar plane orientations with no ROP, which transforms standard parameters as described in Theorem 2. By Proposition 15, orientations with no ROP are in bijection with non-separable planar maps (the bijection consists in adding a root-edge from the source to the sink, and forgetting the orientation of all non-root edges). This bijection increases the edge number, the degrees of the source and of the sink by 1, and it transforms the right and left outer degrees of the orientation into the degrees of the faces lying, respectively, to the left and right of the root-edge, minus one. Proposition 5 follows. ■

7.2. **Permutations avoiding 2413 and 3142, and series-parallel maps**

We now prove Proposition 6. We have said that a rooted non-separable planar map M is *series-parallel* if it does not contain K_4 as a minor. Let $\check{M}$ be the corresponding bipolar map. We will say that $\check{M}$ itself is series-parallel. By adapting the proof given for graphs in [5], it is not hard to see that a bipolar map $\check{M}$ is series-parallel if and only if it can be constructed recursively, starting from the single-edge map, by applying a sequence of series and parallel compositions:

- the *series composition* of two series-parallel bipolar maps $\check{M}_1$ and $\check{M}_2$ is obtained by identifying the sink of $\check{M}_1$ with the source of $\check{M}_2$ (see Fig. 17 (b)),
- the *parallel composition* of $\check{M}_1$ and $\check{M}_2$ is obtained by putting $\check{M}_2$ to the right of $\check{M}_1$, and then identifying the sources of $\check{M}_1$ and $\check{M}_2$, as well as their sinks (Fig. 17 (c)).

By Proposition 16, a Baxter permutation π avoids both patterns 2413 and 3142 if and only if the corresponding plane bipolar orientation has no ROP nor LOP. We have seen in Proposition 15 that every bipolar map $\check{M}$ admits a unique orientation with no ROP and a unique orientation with no LOP, which are respectively the minimal and maximal element of the lattice of orientations of $\check{M}$. Thus $\check{M}$ admits an orientation with no ROP nor LOP if and only if it has a unique bipolar orientation. Hence, to prove Proposition 6, it suffices to establish the following lemma.

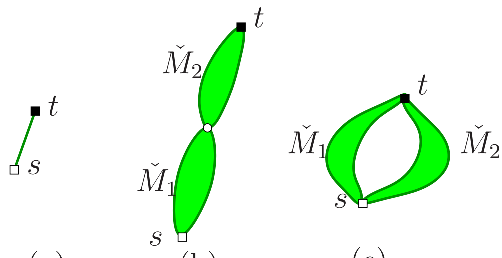


FIGURE 17. The operations that build all series-parallel bipolar maps: (a) taking a single edge, (b) a series composition, (c) a parallel composition.

Lemma 17. *A bipolar map $\check{M}$ admits a unique bipolar orientation if and only if it is series-parallel.*

Proof. From the recursive construction of series-parallel bipolar maps shown in Fig. 17, it is easily checked that such a map admits a unique bipolar orientation (see also [12, Remark 6.2]).

Conversely, assume that $\check{M}$ is not series-parallel. This means that the corresponding rooted map M contains K_4 as a minor. Observe that K_4 admits exactly two bipolar orientations, show in Fig. 18. From the *Extension Lemma* of [12], the two bipolar orientations of K_4 can be extended to two distinct bipolar orientations of M, and thus of $\check{M}$. ■

This concludes the proof of Proposition 6.

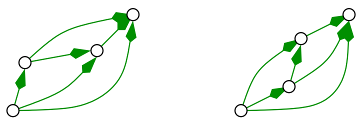


FIGURE 18. The two bipolar orientations of K_4. The first one contains a ROP, the other one a LOP.

7.3. A link with a construction of Dulucq, Gire and West

We have seen that the bijection Φ sends 3142-avoiding Baxter permutations onto orientations with no LOP. Clearly, 3142-avoiding Baxter permutations form a subtree of the generating tree $\mathcal{T}_b$ of Fig. 8: if π avoids 3142, deleting its largest entry will not create an occurrence of this pattern. As Φ is the canonical bijection between $\mathcal{T}_b$ and the tree $\mathcal{T}_o$ of orientations, this implies that orientations with no LOP form a subtree of $\mathcal{T}_o$. In this subtree, replace every orientation by the underlying bipolar map $\check{M}$, and then by the corresponding non-separable rooted map M (this means adding a root-edge to $\check{M}$). This gives a generating tree $\mathcal{T}_m$ for rooted non-separable maps.

In this section we give a description of this tree directly in terms of maps (rather than orientations). We then observe that, up to simple symmetries, this is the tree

maps given in [14, Section 2.1]. In that paper, it is shown that $\mathcal{T}_m$ is isomorphic to the generating tree of 2413-avoiding Baxter permutations. The canonical (and recursive) bijection between these permutations and non-separable planar maps that results from the existence of this isomorphism is not described explicitly in [14]. Our paper gives a non-recursive description of this bijection (up to elementary symmetries), as a special case of a more general correspondence. Note that the structure of the tree $\mathcal{T}_m$ is not as simple as that of the more general tree $\mathcal{T}_o$. In particular, the description given in [14] involves an unbounded number of labels, while $\mathcal{T}_o$ is isomorphic to a simple tree $\mathcal{T}$ with two integers labels.

We still have to prove Lemma 19.

Proof of Lemma 19. Assume the lemma is wrong, and that O is a minimal counterexample (in terms of the edge number). By assumption, s_1 is not on the left outer border of O. Thus the left face of s_1, called f_2, is finite, with source s_2 and sink t_2 (Fig. 20(a)). The left border of f_1 is an oriented path P_0 going from s_1 to t, which ends with the edge e. Let P_1 be the *leftmost* oriented path from s_1 to t: at each vertex, it takes the leftmost outgoing edge. By planarity, the last edge of P_1 is also e. The first steps of P_1 follow the right border of f_2, from s_1 to t_2. Let O_1 be the set of edges lying between $P_1 \setminus e$ and $P_0 \setminus e$ (both paths are included in O_1). Then O_1 is a bipolar orientation of source s_1 and sink v.

Observe that t_2 is not incident to f_1, otherwise (s_1, t_2, f_1, f_2) would form a LOP. Hence the right face of t_2, called f_3, is a face of O_1 (see Fig. 20(b), where only the orientation O_1 and the faces f_1 and f_2 are shown). Let s_3 be its source, and t_3 its sink. As s_1 is the source of O_1, there exists an oriented path P from s_1 to s_3. Consider the following two paths that go from s_2 to t_3: the first one follows the left border of f_2 and then the portion of the left border of f_3 from t_2 to t_3; the second follows the right border of f_2 up to s_1, then the path P, and finally the right border of f_3. Let O_2 be the plane bipolar orientation formed of the edges lying between these two paths (Fig. 20(c)).

The orientation O_2 has fewer edges than O, and we claim that it is a counterexample to the lemma. The last edge e' of its left outer border is also the last edge of the left border of f_3. Hence its starting point lies between t_2 and t_3, and has outdegree 1 in O_2. The face to the right of e' is f_3. Moreover, the source s_3 of f_3 is not on the left outer border of O_2, as it is separated from this path by the face f_2. Hence O_2 is a smaller counterexample than O, which yields a contradiction. ∎

8. **Final comments**

It is natural to study the restriction of our bijection Φ (or its inverse Φ^{-1}) to interesting subclasses of permutations (or orientations), for instance those that are counted by simple numbers. This is the case for alternating and doubly-alternating Baxter permutations [11, 15, 25], or for orientations of triangulations [32]. Recently, we have also discovered a new and intriguing result, which deals with fixed-point-free Baxter involutions. It is easy to construct for them a generating tree, analogous to the tree T_b of Baxter permutations (Fig. 8). One goes from a node to its parent by deleting the cycle containing the largest entry. Again, the

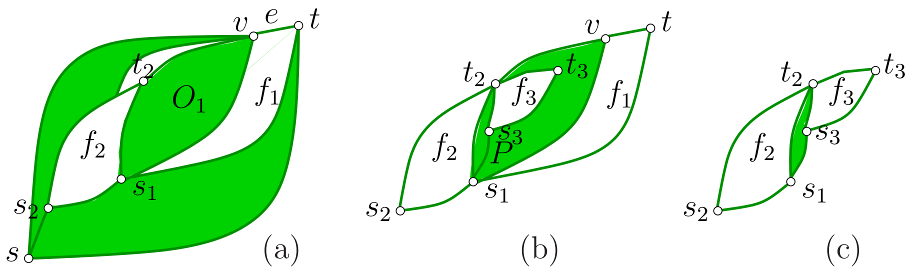


FIGURE 20. (a) A plane bipolar orientation with no LOP such that v has outdegree 1. All edges are North-East oriented. The white areas are faces. The source of f_1 has to be on the left outer border, otherwise, as shown in (b)-(c), a smaller counterexample can be produced.

tree is isomorphic to a generating tree with two labels (like (2)), that encode the number of lr-maxima and rl-maxima. Using the techniques of [7], we have proved that the number of such involutions of length $2n$ is

$$\frac{3.\ 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n},$$

which is also known to count Eulerian planar maps with n edges.

After a 90 degrees rotation, Baxter involutions having no fixed point are described by permutation diagrams that are symmetric with respect to the reflection in the second diagonal and have no point on this diagonal. Via the bijection Φ, they correspond to plane orientations having a single source, but possibly several sinks, lying in the outer face (Fig. 21). It would be interesting to have a combinatorial understanding of the above formula (and of the various refinements of it that we have obtained).

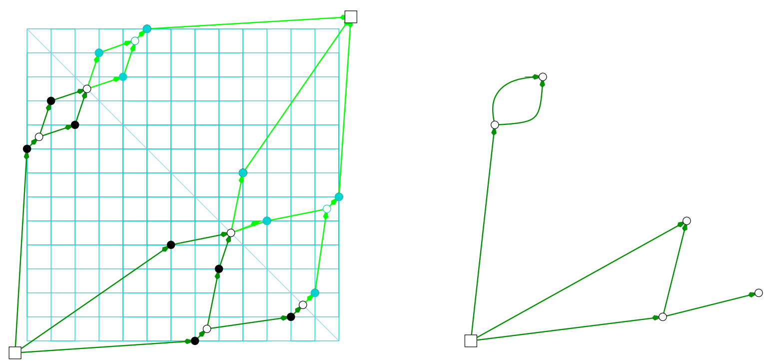


FIGURE 21. A fixed-point-free Baxter involution, rotated by 90 degrees, and the corresponding plane orientation.

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