Directed Algebraic Topology and Concurrency

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Goal

Let $\mathcal{C}$ be a one-way category:

- Define a class Σ of morphisms of $\mathcal{C}$ so we can keep one representative in each class of Σ-related objects without loss of information
- To do so, we are in search for a class that behaves much like the one of isomorphisms
- From now on $\mathcal{C}$ denotes a one-way category

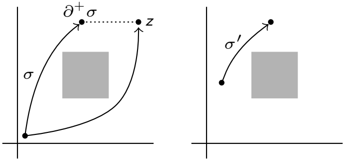
Potential weak isomorphisms

Let $\mathcal{C}$ is one-way

- For all morphisms σ and all objects z define
  - the σ, z-precomposition as $\gamma \in \mathcal{C}(\partial^{+}\sigma, z) \;\to\; \gamma \circ \sigma \in \mathcal{C}(\partial^{-}\sigma, z)$
  - the z, σ-postcomposition as $\delta \in \mathcal{C}(z, \partial^{-}\sigma) \;\mapsto\; \sigma \circ \delta \in \mathcal{C}(z, \partial^{+}\sigma)$
- One may have $\mathcal{C}(\partial^{+}\sigma, z) = \emptyset$ or $\mathcal{C}(z, \partial^{-}\sigma) = \emptyset$
- Note that σ is an isomorphism iff for all z both precomposition and postcomposition are bijective.
- The latter condition is weakened: σ is said to preserve the future cones (resp. past cones) when for all z if $\mathcal{C}(\partial^{+}\sigma, z) \neq \emptyset$ (resp. $\mathcal{C}(z, \partial^{-}\sigma) \neq \emptyset$) then the precomposition (resp. postcomposition) is bijective.
- Then σ is a potential weak isomorphism when it preserves both future cones and past cones. Potential weak isomorphisms compose.
- If $\mathcal{C}(x, y)$ contains a potential weak isomorphism, then it is a singleton
  Requires the assumption that $\mathcal{C}$ is one-way

An example

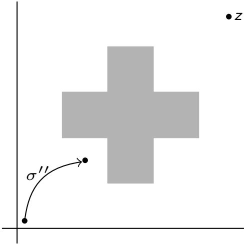
of potential weak isomorphism



Due to the lower dipath, the σ, z-precomposition is not bijective; yet σ' is a potential weak isomorphism.

An unwanted example

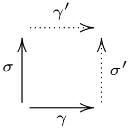
of potential weak isomorphism



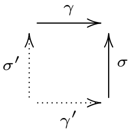
Note that σ'' is a potential weak isomorphism though there exists a morphism from $\partial^{-}\sigma''$ to z but none from $\partial^{+}\sigma''$ to z.

Stability under pushout and pullback

- A collection of morphisms Σ is said to be stable under pushout when for all $\sigma \in \Sigma$, for all γ with $\partial^{-}\gamma = \partial^{-}\sigma$, the pushout of σ along γ exists and belongs to Σ



- A collection of morphisms Σ is said to be stable under pullback when for all $\sigma \in \Sigma$, for all γ with $\partial^{+}\gamma = \partial^{+}\sigma$, the pullback of σ along γ exists and belongs to Σ



Greatest inner collection

stable under pushout and pullback

- Any collection Σ of morphisms of a category $\mathcal{C}$ admits a greatest subcollection that is stable under pushout and pullback
- Construction:
  - Start with $\Sigma_0 = \Sigma$
  - For $n \in \mathbb{N}$ define Σ_{n+1} as the collection of morphisms $\sigma \in \Sigma_n$ s.t. the pushout and the pullback of σ along with all morphisms exist (when sources or targets match) and belong to Σ_n

$$\Sigma_0 \supseteq \cdots \Sigma_1 \supseteq \cdots \supseteq \Sigma_n \supseteq \Sigma_{n+1} \supseteq \cdots$$

  - The expected subcollection is the decreasing intersection

$$\Sigma_\infty := \bigcap_{n \in \mathbb{N}}^{\downarrow} \Sigma_n$$

- The collection Σ_∞ is stable under the action of $\mathrm{Aut}(\mathcal{C})$

Systems of weak isomorphisms

- The class of isomorphisms of any category is stable under pushout and pullback
- A system of weak isomorphisms is a collection of potential weak isomorphisms that is stable under pushout and pullback
- The class of all isomorphisms of any category is a system of weak isomorphisms
- If Σ is a system of weak isomorphisms, then so is its closure under composition
- Hence we suppose the systems of weak isomorphisms are closed under composition

Examples

of systems of weak ismorphisms

- Given a partition $\mathcal{P}$ of $\mathbb{R}$ into intervals, the following collection is a system of weak isomorphisms
$$\big\{(x, y) \mid x \leqslant y;\ \exists \boldsymbol{I} \in \mathcal{P},\ [x, y] \subseteq \boldsymbol{I}\big\}$$
- In the preceding example, $\mathbb{R}$ can be replaced by any totally ordered set
- Let $\Sigma_i \subseteq \mathcal{C}_i$ be a family of collections of morphisms, then

$$\boxed{\prod_i \Sigma_i \text{ is a swi of } \prod_i \mathcal{C}_i \text{ iff each } \Sigma_i \text{ is a swi of } \mathcal{C}_i}$$

- The inverse image (resp. the direct image) of a system of weak isomorphisms by an equivalence of categories is a system of weak isomorphisms.

Pureness

- A collection Σ of morphisms is said to be pure when

$$\gamma \circ \delta \in \mathbf{\Sigma} \ \Rightarrow \ \gamma, \delta \in \mathbf{\Sigma}$$

- Given a one-way category $\mathcal{C}$ we have:

All the systems of weak isomorphisms of $\mathcal{C}$ are pure

The locale of systems of weak isomorphisms

- A locale is a complete lattice whose binary meet distributes over arbitrary join i.e.

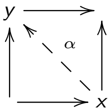
$$x \wedge \left(\bigvee_i y_i \right) = \bigvee_i (x \wedge y_i)$$

- The collection ΩX open subsets of a topological space X form a locale and we have the functor $L : \mathcal{Top} \to \mathcal{Loc}$ (that admits a left adjoint) defined by
  - $L(X) = \Omega X$
  - $L(f)(W) = f^{-1}(W)$ for all $f : X \to Y$ and $W \in \Omega Y$
- The collection of systems of weak isomorphisms of a category has a greatest element
- Given a one-way category $\mathcal{C}$ we have:
  - The collection of systems of weak isomorphisms of $\mathcal{C}$ forms a locale
  - The greatest swi is invariant under the action of $\mathrm{Aut}(\mathcal{C})$

The filling square property

of a category $\mathcal{C}$

- By definition, a filling square category $\mathcal{C}$ is such that for all commutative squares which are both pushout and pullack (see below), if $\mathcal{C}(x, y) \neq \emptyset$ then there exists $\alpha \in \mathcal{C}(x, y)$ that makes both triangles commute.



- If $\mathcal{C}$ satisfies the filling square property, then any collection of morphisms of $\mathcal{C}$ that is stable under pushout and pullback is a system of weak isomorphisms.
- A conjecture:

> For all loop-free isothetic region X, $\overrightarrow{\pi_1}X$ satisfies the square filling property

Components

of a one-way category $\mathcal{C}$

- From now on $\mathcal{C}$ is a one-way category and Σ is a system of weak isomorphisms on it
- Recall that is $\mathcal{C}(x, y)$ meets Σ, then $\mathcal{C}(x, y)$ is a singleton, a fact that we represent on diagrams by: $x \xrightarrow{\Sigma} y$
- Given two objects x and y of $\mathcal{C}$ t.f.a.e.:
  - there exists a Σ-zigzag between x and y
  - there exists z such that $x \xleftarrow{\Sigma} z \xrightarrow{\Sigma} y$
  - there exists z such that $x \xrightarrow{\Sigma} z \xleftarrow{\Sigma} y$
- When any of the following property is satisfied x and y are said to be Σ-connected
- Σ-connectedness is an equivalence relation on the objects of $\mathcal{C}$
- The equivalence classes are called a Σ-components

Structure of the Σ-components

Σ system of weak isomorphisms of $\mathcal{C}$ one-way category

A prelattice is a preordered set in which $x \wedge y$ and $x \vee y$ exist for all x and y.
However they are defined only up to isomorphism

Let K be a Σ-component of $\mathcal{C}$ and $\mathcal{K}$ be the full subcategory of $\mathcal{C}$ whose objects are the elements of K. The following properties are satisfied:

- [1.] The category $\mathcal{K}$ is isomorphic with the preorder $(K, \preccurlyeq)$ where $x \preccurlyeq y$ stands for $\mathcal{C}[x, y] \neq \emptyset$. In particular, every diagram in $\mathcal{K}$ commutes.
- [2.] The preordered set $(K, \preccurlyeq)$ is a prelattice.
- [3.] If d and u are respectively a greatest lower bound and a least upper bound of the pair $\{x, y\}$, then Diagram 1 is both a pullback and a pushout in $\mathcal{C}$, and all the arrows appearing on the diagram belong to Σ.
- [4.] $\mathcal{C} = \mathcal{K}$ iff $\mathcal{C}$ is a prelattice, and Σ is the greatest system of weak isomorphisms of $\mathcal{C}$ i.e. all the morphisms in this case.

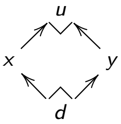


Diagram 1

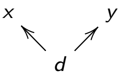


Diagram 2

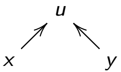
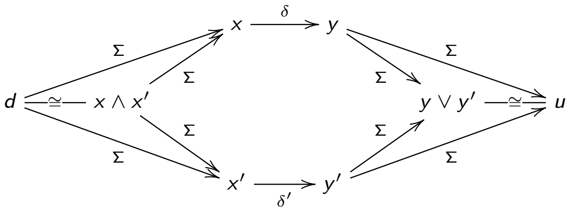


Diagram 3

Equivalent morphisms

with respect to Σ

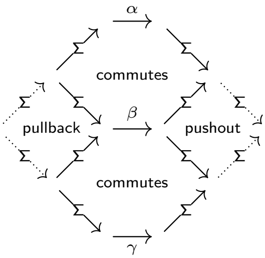
- Let $\delta \in \mathcal{C}(x, y)$ and $\delta' \in \mathcal{C}(x', y')$. Then write $\delta \sim \delta'$ when
  - $x \sim x'$ and $y \sim y'$, and
  - the inner hexagon of the next diagram commutes



  - Note that if $d \cong x \wedge x'$ and $u \cong y \vee y'$ then the outter hexagon also commutes, hence the relation $\sim$ is well defined.
- If $\gamma \sim \delta$ then $\partial^- \gamma \sim \partial^- \delta$ and $\partial^+ \gamma \sim \partial^+ \delta$

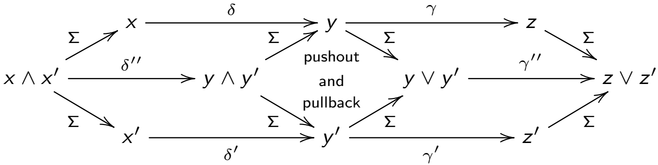
The relation $\sim$ is an equivalence

- The relation $\sim$ is:
  - reflexive since Σ contains all identities
  - symmetric by definition
  - transitive



The relation $\sim$ fits with composition

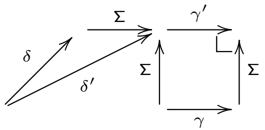
- Suppose $\partial^-\gamma = \partial^+\delta$, $\partial^-\gamma' = \partial^+\delta'$ and $\gamma \sim \gamma'$ and $\delta \sim \delta'$.
- Then we have $\gamma \circ \delta \sim \gamma' \circ \delta'$



The category of components

denoted by $\mathcal{C}/\Sigma$

- The quotient category $\mathcal{C}/\Sigma$ (obtained by turning each morphism of Σ into an identity) can be defined as follows:
  - The objects are the Σ-components
  - The morphisms are the $\sim$-equivalence classes
- If $\partial^{-}\gamma \sim \partial^{+}\delta$ then
  - there exists γ' and δ' such that $\gamma' \sim \gamma$, $\delta' \sim \delta$, and $\partial^{-}\gamma' = \partial^{+}\delta'$



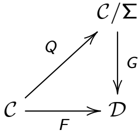
  - so we define $[\gamma] \circ [\delta] = [\gamma' \circ \delta']$
- We have the quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$
- The category of components is $\mathcal{C}/\Sigma$ with Σ being the greatest swi of $\mathcal{C}$

Characterizing the identities of $\mathcal{C}/\Sigma$

For any morphism δ of $\mathcal{C}$ t.f.a.e.

- $\delta \in \Sigma$
- $[\delta] \subseteq \Sigma$
- $[\delta]$ is an identity of $\mathcal{C}/\Sigma$

The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ satisfies the following universal property:
for all functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{isomorphisms of } \mathcal{D}\}$
there exists a unique $G : \mathcal{C}/\Sigma \to \mathcal{D}$ s.t. $F = G \circ Q$



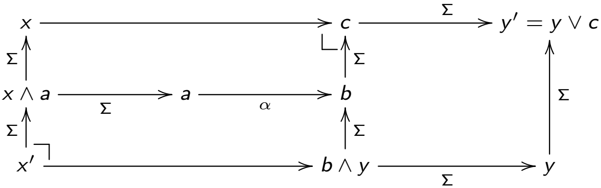
The fundamental properties of $\mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ is surjective on morphisms
- The quotient category $\mathcal{C}/\Sigma$ is loop-free
- If $\mathcal{C}(x, y) \neq \emptyset$ then the following map is a bijection.

$$\delta \in \mathcal{C}(x, y) \;\mapsto\; Q(\delta) \in \mathcal{C}/\Sigma\big(Q(x), Q(y)\big)$$

- If $\mathcal{C}/\Sigma\big(Q(x), Q(y)\big) \neq \emptyset$ then there exist x' and y' such that $\Sigma(x', x)$, $\Sigma(y, y')$, $\mathcal{C}(x', y)$, and $\mathcal{C}(x, y')$ are nonempty.

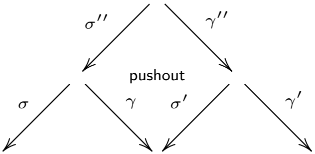


- The quotient functor Q preserves and reflects potential weak isomorphisms
- If $\mathcal{C}$ is finite then so is the quotient $\mathcal{C}/\Sigma$
- $\mathcal{C}$ is a preorder iff $\mathcal{C}/\Sigma$ is a poset

Describing the localization of $\mathcal{C}$ by Σ

with Σ system of weak isomoprphisms of $\mathcal{C}$

- The objects of $\mathcal{C}[\Sigma^{-1}]$ are the objects of $\mathcal{C}$
- The morphisms are the equivalence classes of ordered pairs of coinitial morphisms (γ, σ) with $\sigma \in \Sigma$,
  - Two pairs (γ, σ) and (γ', σ') being equivalent when $\partial^{-}\sigma = \partial^{-}\sigma'$, $\partial^{-}\gamma = \partial^{-}\gamma'$, and $Q(\gamma) = Q(\gamma')$
  - In the diagram below we have $Q(\gamma' \circ \gamma'') = Q(\gamma') \circ Q(\gamma'') = Q(\gamma') \circ Q(\gamma)$ hence the composite $(\gamma' \circ \gamma'', \sigma \circ \sigma'')$ neither depend on the choice of the pushout nor on the representatives (γ, σ) and (γ', σ').



The canonical inclusion $I : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$

with Σ system of weak isomoprphisms of $\mathcal{C}$

- Define I by $I(\gamma) := (\gamma, \mathrm{id}_{\partial^- \gamma})$ and the identity on objects
- Given a functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{$iomorphisms of $\mathcal{D}\}$ define
  - $G(x) := F(x)$ for all objects x of $\mathcal{C}[\Sigma^{-1}]$ and
  - $G(\gamma, \sigma) := F(\gamma) \circ (F(\sigma))^{-1}$ for any representative (γ, σ) of a morphism of $\mathcal{C}[\Sigma^{-1}]$
- The functor $I : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ then satisfies the universal property: for all functor $F : \mathcal{C} \to \mathcal{D}$ there exists a unique $G : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ s.t. $F = G \circ I$
- In particular there is a unique functor P s.t. $Q = P \circ I$ with $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ and we have
-

The functor P is an equivalence of categories

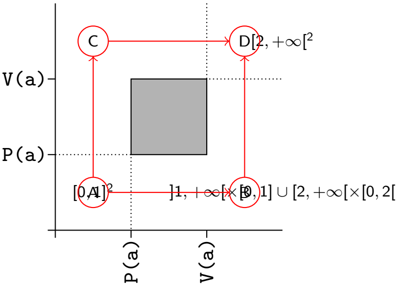
- The skeleton of $\mathcal{C}[\Sigma^{-1}]$ is $\mathcal{C}/\Sigma$ and $\mathcal{C}[\Sigma^{-1}]$ is one-way.

Embeding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

- Let ϕ : Σ-components of $\mathcal{C} \to \mathrm{Ob}(\mathcal{C})$ such that
  - for all Σ-components K, K', if there exists $x \in K$ and $x' \in K'$ such that $\mathcal{C}(x, x') \neq \emptyset$, then $\mathcal{C}(\phi(K), \phi(K')) \neq \emptyset$
  - in this case $\mathcal{C}/\Sigma$ is isomorphic with the full subcategory of $\mathcal{C}$ whose set of objects is $\mathrm{im}(\phi)$.
  - the mapping ϕ is called an admissible choice (of canonical objects)
- Write $\phi \preccurlyeq \phi'$ when $\mathcal{C}(\phi(K), \phi'(K)) \neq \emptyset$ for all Σ-components K
  - The collection of admissible choice then forms a (pre)lattice
  - If $\mathcal{C}/\Sigma$ is finite then there exists an admissible choice
  - If $\mathcal{C}/\Sigma$ is infinite the existence of an admissible choice is a open question

Plane without a square

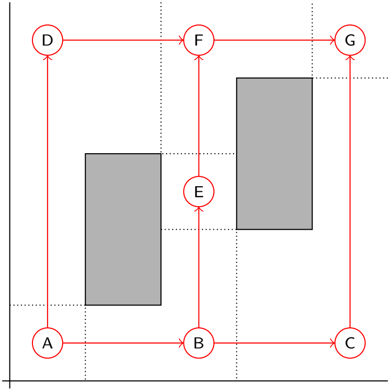
$x = \mathbb{R}_+^2 \setminus]0,1[^2$



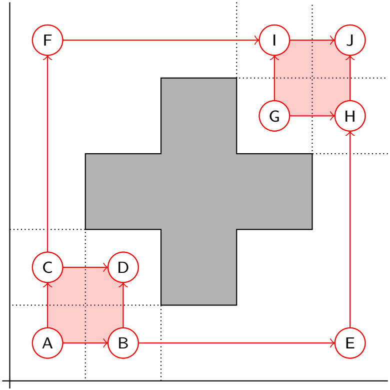
Let x, y such that $x \leqslant^2 y$, then $\overrightarrow{\pi_1} X(x,y)$ only depends on which elements of the partition x and y belong to

$\rightarrow$	A	B	C	D
A	σ	β	γ	$\beta' \circ \beta$ $\alpha' \circ \alpha$
B		σ		β'
C			σ	γ'
D				σ

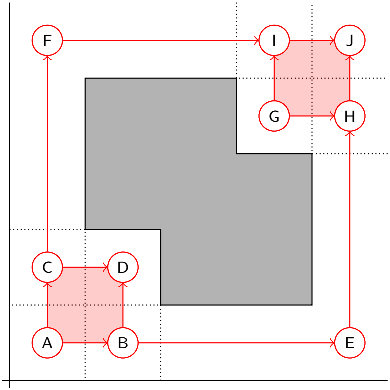
Two rectangles



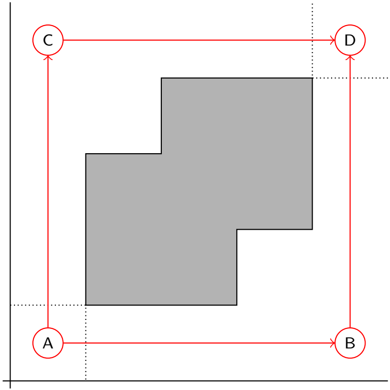
Swiss Flag



Achronal overlaping square

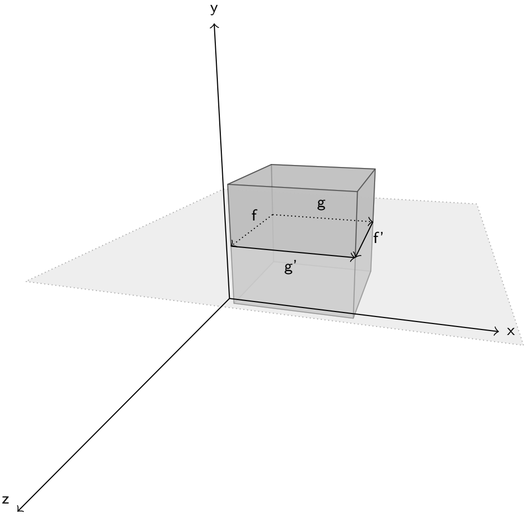


Diagonal overlaping squares



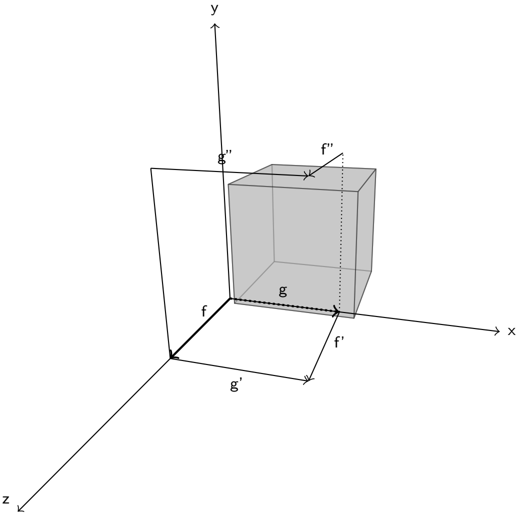
The floating cube

Non potential weak isomorphisms



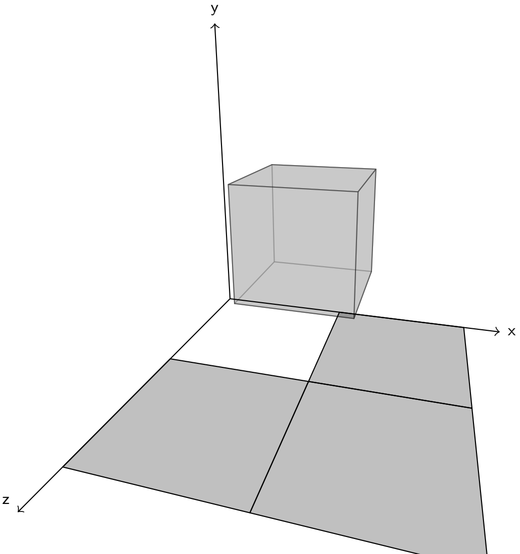
The floating cube

A "vee" that does not extend to a pushout



The floating cube

Some pushouts squares



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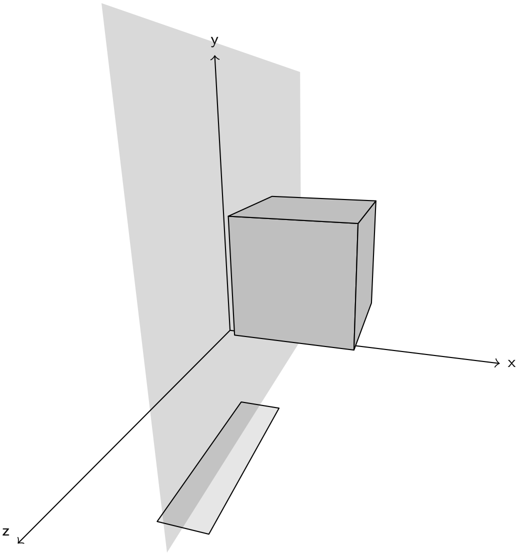
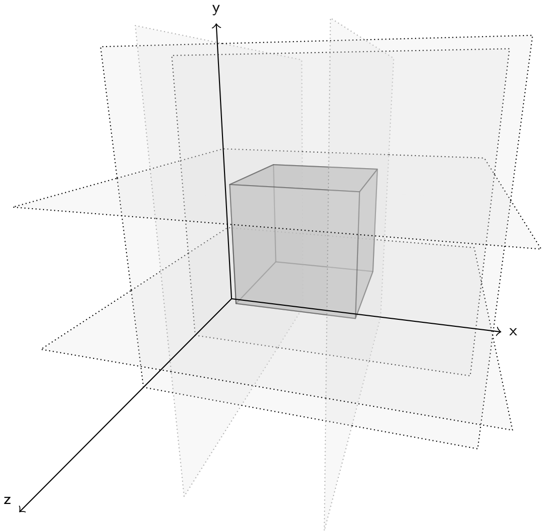


The floating cube

- Since the pushout of f (resp. g) along g (resp. f) does not exist, $f, g \notin \Sigma$
- The commutative square f, g, f', and g' is a pullback:
  - Therefore $f', g' \notin \Sigma$ (anyway they do not preserve the future cones)

The floating cube

boundaries of the components



Commutative monoid

of nonempty finite connected loop-free categories

- The Cartesian product of categories $\mathcal{A} \times \mathcal{B}$ is non-empty iff so are $\mathcal{A}$ and $\mathcal{B}$.

  If $\mathcal{A}$ and $\mathcal{B}$ are indeed nonempty then we also have
  - $\mathcal{A} \times \mathcal{B}$ finite iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ connected iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ loop-free iff so are $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A} \cong \mathcal{A}'$ and $\mathcal{B} \cong \mathcal{B}'$ implies $\mathcal{A} \times \mathcal{A}' \cong \mathcal{B} \times \mathcal{B}'$
- $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C} \cong \mathcal{A} \times (\mathcal{B} \times \mathcal{C})$
- $1 \times \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \times 1$
- $\mathcal{A} \times \mathcal{B} \cong \mathcal{B} \times \mathcal{A}$
- The collection of isomorphism classes of nonempty finite connected loop-free categories is thus a commutative monoid $\mathcal{M}$

The commutative monoid $\mathcal{M}$ is free.

Criteria for primality

- The monoid $\mathcal{M}$ is graded by the following morphisms
  - $\#Ob : \mathcal{C} \in \mathcal{M} \mapsto \mathrm{card}(\mathrm{Ob}(\mathcal{C})) \in (\mathbb{N} \setminus \{0\}, \times, 1)$
  - $\#Mo : \mathcal{C} \in \mathcal{M} \mapsto \mathrm{card}(\mathrm{Mo}(\mathcal{C})) \in (\mathbb{N} \setminus \{0\}, \times, 1)$
  - $\#Mo(\mathcal{C}) \geqslant 2 \times \#Ob(\mathcal{C}) - 1$, for all $\mathcal{C} \in \mathcal{M}$
- In particular if $\#Ob(\mathcal{C})$ or $\#Mo(\mathcal{C})$ is prime, then so is $\mathcal{C}$.
  The converse is false.
- Any element of $\mathcal{M}$ freely generated by a graph, is prime

Comparing decompositions

- The mapping $\mathcal{C} \in \mathcal{M} \mapsto \overrightarrow{\pi_0}(\mathcal{C}) \in \mathcal{M}$ is a morphism of monoids
- We would like to know which prime element of $\mathcal{M}$ are preserved by it
- We known that $\overrightarrow{\pi_0}(\mathcal{C})$ is null iff $\mathcal{C}$ is a lattices (e.g. $\overrightarrow{\pi_0}(0 < 1) = \{0\}$ though $\{0 < 1\}$ is prime in $\mathcal{M}$)
- For all d-spaces X and Y, $\overrightarrow{\pi_1}(X \times Y) \cong \overrightarrow{\pi_1}X \times \overrightarrow{\pi_1}Y$
- Hence $\mathcal{N}' := \{X \in \mathcal{H}_f\upharpoonleft G \downharpoonright \mid \overrightarrow{\pi_1}X \text{ is nonempty, connected, and loop-free}\}$ is a pure submonoid of $\mathcal{H}_f\upharpoonleft G \downharpoonright$
- Then $\mathcal{N} := \{X \in \mathcal{N}' \mid \overrightarrow{\pi_0}(\overrightarrow{\pi_1}X) \text{ is finite}\}$ is a pure submonoid of $\mathcal{N}'$
- Therefore it is free commutative and we would like to know which prime elements are preserved by $X \in \mathcal{N} \mapsto \overrightarrow{\pi_0}(\overrightarrow{\pi_1}X) \in \mathcal{M}$
- Conjecture

If $P \in \mathcal{N}$ is prime and $\overrightarrow{\pi_1}(P)$ is not a lattice, then $\overrightarrow{\pi_0}(\overrightarrow{\pi_1}(P))$ is prime

Homotopy of maps

- Let $f, g : X \to Y$ be continuous maps that agree on $A \subseteq X$
- An A-homotopy from f to g is a mapping $\eta : X \times [0, r] \to Y$, for some $r \in \mathbb{R}_+$, such that $\eta(\_, 0) = f$, $\eta(\_, r) = g$, and $t \mapsto \eta(\_, t)|_A$ is constant
- If X is exponentiable, then A-homotopies can be seen as a path on Y^X
- If η is an A-homotopy from f to g then $(x, t) \mapsto \eta(x, r - t)$ is an A-homotopy from g to f called the opposite of η
- If $\eta' : X \times [0, r']$ is an A-homotopy from g to h then the concatenation

$$\begin{array}{rcl} \eta' \cdot \eta : X \times [0, r + r'] & \to & Y \\ (x, t) & \mapsto & \begin{cases} \eta(x, t) & \text{if } t \leqslant r \\ \eta'(x, t - r) & \text{if } r \leqslant t \end{cases} \end{array}$$

  is an A-homotopy from f to g
- Writting $f \sim g$ when there is an A-homotopy from f to g, we define an equivalence relation over the mappings from X to Y.

Homotopy groups of a space Y

The n^{th} homotopy group of Y at point $p \in Y$, $\pi_n(Y, p)$, is defined as follows:

- The elements of the group are the $\partial[0,1]^n$-homotopy classes of maps from $[0,1]^n$ to Y that sends $\partial[0,1]^n$ to p
- Define, for $i \in \{1, \ldots, n\}$, $f +_i g(\ldots, t_i, \ldots)$ by $f(\ldots, 2t_i, \ldots)$ if $t_i \leqslant \frac{1}{2}$; $f(\ldots, 2t_i - 1, \ldots)$ if $\frac{1}{2} \leqslant t_i$
- One proves that
  - $[f +_i g]$ only depends on $[f]$ and $[g]$
  - $[f +_i g] = [f +_j g]$ for all i, j
  - $[f]^{-1} = [t_i \mapsto f(\ldots, 1 - t_i, \ldots)]$
  - for $n \geqslant 2$, the group $\pi_n(Y, p)$ is abelian

Some elementary facts about homotopy groups

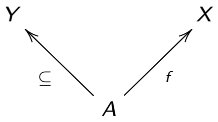
- If Y is path-connected, the homotopy groups do not depend on the base point
- for $n = 0$ the construction extends to a functor $\pi_0 : \mathcal{Top} \to \mathcal{Set}$ (the path-connected components)
- for $n = 1$ the fundamental group construction extends to a functor $\pi_1 : \mathcal{Top} \to \mathcal{Gr}$ (in general it is not abelian)
- for $n \geqslant 2$ the n^{th} homotopy group construction extends to a functor $\pi_n : \mathcal{Top} \to \mathcal{Ab}$.
  i.e. for $n \geq 2$, the n^{th} homotopy group of a space is commutative

Some advanced facts about homotopy groups

- Any group can be obtained as $\pi_1(X)$ for some polyhedron
- $\pi_n(\mathbb{S}^d) \cong \{0\}$ for $0 \leqslant n < d$
- for $n \geqslant 1$, $\pi_n(\mathbb{S}^n) \cong \mathbb{Z}$ (Hurewicz)
- for $n \in \mathbb{N}$, $\pi_n(\mathbb{S}^d)$ is finite for $n > d$ except $\pi_{4d-1}(\mathbb{S}^{2d}) \cong \mathbb{Z} \oplus F_d$ with F_d finite
  Groupes d'homotopie et classes de groupes abliens. J.-P. Serre, Ann. of Math. 58 (1953). 258-294
- $\pi_{d+k}(\mathbb{S}^d)$ does not depend on d when $d \geqslant k + 2$ (stable homotopy)
  cor. of the Freudenthal Suspension Theorem
- the function sending (n, d) to $\pi_n(\mathbb{S}^d)$ is computable
  Finite Computability of Postnikov Complexes. E. H. Brown, Jr. Ann. of Math. 65(1). 1957

Attaching spaces

- Let A be a subspace of Y and $f : A \to X$ be a continuous map.
- The resulting attaching space is the pushout of f and $A \subseteq X$ i.e. the colimit of



- As a standard example we have $Y = [0,1]^n$, A the boundary of Y i.e.

$$\big\{(x_1, \ldots, x_n) \in [0,1]^n \mid \exists k \in \{1, \ldots, n\},\ x_k \in \{0,1\}\big\}$$

- The CW-complexes arises in this way.

CW-complexes

Combinatorial homotopy I & II, J.H.C. Whitehead (1949)

- a CW-complex is the colimit in $\mathcal{CGH}$ of a (possibly infinite) sequence

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$$

  provided the spaces X_n are inductively defined as follows:
  - Define X_{-1} as the empty space $\emptyset$
  - The space X_n being given, let Y_{n+1} be a disjoint union of copies of $[0,1]^n$ i.e.

$$Y_{n+1} = \mathcal{I}_{n+1} \times [0,1]^{n+1} \cong \bigsqcup_{x \in \mathcal{I}_{n+1}} \{x\} \times [0,1]^{n+1}$$

    Let A_n be the boundary of Y_n and $\phi_n : A_n \to X_n$ be an attaching map.
    Then X_{n+1} is the attaching space

$$X_{n+1} = X_n \bigsqcup_{\phi_n} Y_{n+1}$$

    The pushout of ϕ_n is denoted by Φ_{n+1} and called the characterictic map.
- For $x \in \mathcal{I}_n$, a n-cell is the image of $\{x\} \times [0,1]^n$ under Φ_n.
- For $x \in \mathcal{I}_n$, an open n-cell is the image of $\{x\} \times]0,1[^n$ under Φ_n.
  It is a homeomorphic image.

Some properties of the CW complexes

- All CW-complexes are compactly generated Hausdorff spaces
- A CW-complex is compact iff it has finitely many cells
- The realization of a (pre)cubical set is a CW-complex
- The product in $\mathcal{CGH}$ of two CW-complexes is a CW-complex
- The following product in $\mathcal{Top}$ is not a CW-complex

$$|\mathbb{R} \rightrightarrows \{0\}| \times |\mathbb{N} \rightrightarrows \{0\}|$$

Homotopy equivalences

- If there exists $f' : Y \to X$ such that $f' \circ f \sim \mathrm{id}_X$ and $f \circ f' \sim \mathrm{id}_Y$, then f (and f') are said to be homotopy equivalences. The spaces X and Y are said to be homotopic. A space (resp. map) that is homotopic with $\{0\}$ (resp. a constant map) is said to be null homotopic
- Note the analogy with equivalences of category i.e. functors $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ such that there exists natural isomorphisms $\eta : \mathrm{id} \to FG$ and $\varepsilon : GF \to \mathrm{id}$
- Given a functor F t.f.a.e:
    - F is an equivalence of categories
    - F has a left adjoint and the unit and counit are isomorphisms
    - F is fully faithful and any object of its codomain is isomorphic with an object of its image

Basic examples

- Any homeomorphism is a homotopy equivalence
- For all $n \in \mathbb{N}$, $\mathbb{R}^n$ is null homotopic. Consider $\eta(p, t) = t \cdot p$, $p \in \mathbb{R}^n$, $t \in [0, 1]$
- In particular one has homotopy equivalences which are not homeomorphisms
- For all $n \geqslant 1$, $\mathbb{R}^{n+1} \setminus \{0\}$ is homotopically equivalent to $\mathbb{S}^n$

Whitehead theorem

Combinatorial homotopy I & II, J.H.C. Whitehead (1949)

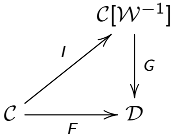
- If X and Y are homotopic, then $\pi_n(X) \cong \pi_n(Y)$ for all n
- Whitehead theorem:

> If X and Y are CW-complexes and $f : X \to Y$ induces isomorphisms of n^{th} homotopy groups for all n, then f is a homotopy equivalence

The homotopy category

Localizing with respect to homotopy equivalences

- Given a collection $\mathcal{W}$ of morphisms of a category $\mathcal{C}$
- Consider the category:
  - whose objects are functors F defined on $\mathcal{C}$ s.t. $F(\mathcal{W}) \subseteq \{\text{isomorphisms of } \mathcal{D}\}$
  - the morphisms from F to F' are the functors from $\text{cod}(F)$ to $\text{cod}(F)'$ s.t. $F' = G \circ F$
- The previous category has an initial object $I : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ i.e.
  - for all functors F sending all the elements of $\mathcal{W}$ to an isomorphism,
  - there exists a unique functor G defined over $\mathcal{C}[\mathcal{W}^{-1}]$ s.t. $F = G \circ I$



- The homotopy category is defined as the localization of $\mathcal{Top}$ (or $\mathcal{CGH}$ etc) with respect to the class of homotopy equivalences

Basic

- *Revêtements et Groupe Fondamental*, Michèle Audin, en ligne, 2004
- *Categories and Groupoids*, P. J. Higgins, Van Nostrand - Reinhold, 1971
- *Algebraic Topology*, A. Hatcher, Cambridge Univ. Press, 2002
- *Topology and Groupoids*, R. Brown, BookSurge, 2006
- *Introduction to Homotopy Theory*, M. Arkowitz, Springer, 2011
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More advanced books

- *Elements of Homotopy Theory*, G. W. Whitehead, Springer, 1978
- *Algebraic Topology: an Introduction*, Massey, Springer, 1967
- *Algebraic Topology: homotopy and homology*, R. M. Switzer, 1975
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About combinatorial homotopy and model categories

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- *Calculus of Fractions and Homotopy Theory*, P. Gabriel and M. Zisman, Springer, 1967
- *Model Categories*, M. Hovey, Amer. Math. Soc., 1999
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  P. S. Hirschhorn, Amer. Math. Soc., 2003
- *Homotopy Limit Functors on Model Categories and Homotopical Categories*, W.G. Dwyer, P.S. Hirschhorn, D.M. Kan, and J.H. Smith, Amer. Math. Soc., 2004