Directed Algebraic Topology and Concurrency

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Some standard topological spaces

- compact unit segment, circle, cylinder, and torus
- cube, sphere
- Mobius band, Klein bottle
- projective plane

The singular precubical set of an object X

- We are given $C : \square^+ \to \mathcal{C}$, the singular cubical set functor $Sing : \mathcal{C} \to \mathit{pcSet}$ is defined as follows:
  - $Sing(X) := \left(\cdots \ \mathcal{C}(C(n-1), X) \xleftarrow{C(\delta^{\varepsilon}_{k,n}) \circ -} \mathcal{C}(C(n), X) \ \cdots \right)$
  - $Sing(\ X \xrightarrow{f} Y\ ) := \left(\cdots \ \mathcal{C}(C(n), X) \xrightarrow{- \circ f} \mathcal{C}(C(n), Y) \ \cdots \right)$
- Then we have $|\_| \dashv Sing$

Conjectures

Any precubical set K can be realized in $\mathcal{Lpo}$, and

- $U(\rceil K \lfloor_{\mathcal{Lpo}}) \quad = \quad |K|_{\mathcal{Top}}$
- $I(\rceil K \lfloor_{\mathcal{Lpo}}) \quad \cong \quad \rceil K \lfloor_{d\mathcal{Top}}$
- $\rceil K \lfloor_{d\mathcal{Top}} \quad \cong \quad \rceil K \lfloor_{d\mathcal{Top}_{\mathsf{f}}}$
- There exists a precubical set K such that $\rceil K \lfloor \quad \cong \quad \vec{\mathbb{S}}^3$

The category $\square$ of face inclusions and projections

- For $n \in \mathbb{N}$, $k \in \{0, \dots, n-1\}$, $\varepsilon \in \{+, -\}$, we have the face inclusion map

$$\begin{array}{rcl} [0,1]^n & \to & [0,1]^{n+1} \\ (t_0, \dots, t_{n-1}) & \mapsto & (t_0, \dots, t_{k-1}, \varepsilon, t_k, \dots, t_{n-1}) \end{array}$$

- For $n \in \mathbb{N}$, $k \in \{0, \dots, n\}$, we have the projection map

$$\begin{array}{rcl} [0,1]^{n+1} & \to & [0,1]^n \\ (t_0, \dots, t_n) & \mapsto & (t_0, \dots, t_{k-1}, t_{k+1}, \dots, t_n) \end{array}$$

- Then $\square$ is the subcategory of $\mathcal{S}et$ generated by all the face inclusions and projections.
- Cubical sets are presheaves over $\square$ i.e. $pc\mathcal{S}et = \mathcal{S}et^{\square^{op}}$

A description of the $\square$ category

- $\{\text{Objects of } \square\} = \mathbb{N}$
- $\square[n, m]$ is the set of ordered pairs (n, w) such that $n \in \mathbb{N}$ and w is a word of length m on $\{0, 1, x_0, \ldots, x_{n-1}\}$ such that:
  - every variable occurs at most once in w, and
  - if $i < j$ and both x_i and x_j occur in w, then x_i occurs before x_j.
- $\mathrm{id}_n = \big(n, (x_0, \ldots, x_{n-1})\big)$
- $\delta^{\varepsilon}_{i,n} = \big(n, (x_0 \cdots x_{i-1} \varepsilon x_i \cdots x_{n-1})\big)$ and $\sigma_{i,n} = \big(n+1, (x_0 \cdots x_{i-1} x_{i+1} \cdots x_n)\big)$
- if $w : a \to b$ and $w' : b \to c$ then $w'w$ is obtained by replacing, for $k \in \{0, \ldots, b-1\}$, the occurrence of x_k in w' (if any) by the k^{th} letter of w.

The diagram from which the $\mathcal{C}$-realization is built

K: cubical set.

Consider the diagram made of the arrows

$$\begin{array}{rcl} \{\partial^{\varepsilon}_{k,n}(x)\} \times [0,1]^n & \to & \{x\} \times [0,1]^{n+1} \\ \Big(\partial^{\varepsilon}_{k,n}(x), (t_0, \ldots, t_{n-1})\Big) & \mapsto & \Big(x, (t_0, \ldots, t_{k-1}, \varepsilon, t_k, \ldots, t_{n-1})\Big) \end{array}$$

for $n \in \mathbb{N}$, $x \in K_{n+1}$, $k \in \{0, \ldots, n-1\}$, $\varepsilon \in \{+,-\}$ and

$$\begin{array}{rcl} \{\sigma_{k,n}(x)\} \times [0,1]^{n+1} & \to & \{x\} \times [0,1]^n \\ \Big(\sigma_{k,n}(x), (t_0, \ldots, t_n)\Big) & \mapsto & \Big(x, (t_0, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)\Big) \end{array}$$

for $n \in \mathbb{N}$, $x \in K_n$, $k \in \{0, \ldots, n\}$.

The realization of K in $\mathcal{C}$ is the colimit of the above diagram.

That construction defines a functor $|\_| : c\mathcal{S}et \to \mathcal{C}$.

Effect of projections on the "bricks" of the realization

$$\begin{array}{ccc} & |K| & \\ {}^{\phi_{\sigma_{k,n}(x)}}\nearrow & & \nwarrow^{\phi_x} \\ \{\sigma_{k,n}(x)\} \times [0,1]^{n+1} & \xrightarrow[\text{proj}]{} & \{x\} \times [0,1]^n \end{array}$$

Hence $\phi_{\sigma_{k,n}(x)}(t_0, \ldots, t_n)$ does not depend on t_k.

Describing the realization in “concrete” categories

- Start with the disjoint union (assuming that $[0,1]^0$ is a singleton)

$$\bigcup_{n \in \mathbb{N}} K_n \times [0,1]^n$$

- then for all $n \in \mathbb{N}$, all $x \in K_{n+1}$, all $k \in \{0, \ldots, n-1\}$, and $\varepsilon \in \{+,-\}$, identify

$$(\partial_k^\varepsilon x, t_0, \ldots, t_{n-1}) \sim (x, t_0, \ldots, t_{k-1}, \varepsilon, t_k, \ldots, t_{n-1})$$

  and for all $n \in \mathbb{N}$, all $x \in K_n$ and all $k \in \{0, \ldots, n\}$, identify

$$(\sigma_k x, t_0, \ldots, t_n) \sim (x, t_0, \ldots, t_{k-1}, t_{k-1}, \ldots, t_n)$$

About directed realization of cubical sets

- Some cubical set K cannot be realized in $\mathcal{Lpo}$,
- $D(\rceil K \lfloor_{\mathcal{Str}}) \quad \cong \quad \rceil K \lfloor_{d\mathcal{Top}_{\mathsf{f}}}$
- $S(\rceil K \lfloor_{d\mathcal{Top}}) \quad \cong \quad \rceil K \lfloor_{\mathcal{Str}_{\mathsf{d}}}$
- $\rceil K \lfloor_{\mathcal{Str}} \quad \cong \quad \rceil K \lfloor_{\mathcal{Str}_{\mathsf{d}}}$
- $\rceil K \lfloor_{d\mathcal{Top}}$ and $\rceil K \lfloor_{d\mathcal{Top}_{\mathsf{f}}}$ may differ.

Cartesian product of cubical sets

$$\left(\begin{array}{c} \vdots \\ K_{n+1} \\ \partial_k^+ \downarrow \; \uparrow \sigma_k \; \downarrow \partial_k^- \\ K_n \\ \vdots \end{array}\right) \times \left(\begin{array}{c} \vdots \\ K'_{n+1} \\ \partial_k^{+\prime} \downarrow \; \uparrow \sigma'_k \; \downarrow \partial_k^{-\prime} \\ K'_n \\ \vdots \end{array}\right) \cong \left(\begin{array}{c} \vdots \\ K_{n+1} \times K'_{n+1} \\ \partial_k^+ \times \partial_k^{+\prime} \downarrow \; \uparrow \sigma_k \times \sigma'_k \; \downarrow \partial_k^- \times \partial_k^{-\prime} \\ K_n \times K'_n \\ \vdots \end{array}\right)$$

The Cartesian product in $\mathcal{pcSet}$ is deduced form the Cartesian product in $\mathcal{Set}$

Cartesian product of two segments in $c\mathcal{S}et$

Compute the product $\square_1 \times \square_1$

Tensor product

Given the cubical sets K and K', the set of n-cubes is

$$(K \otimes K')_n = \left(\bigsqcup_{i+j=n} K_i \times K'_j \right) / \sim$$

For $x \otimes y \in K_i \times K'_j$ with $i + j = n$ the k^{th} face map, with $0 \leqslant k < n$, is given by

$$\partial_k^{\pm}(x \otimes y) = \begin{cases} \partial_k^{\pm}(x) \otimes y & \text{if } 0 \leqslant k < i \\ x \otimes \partial_{k-i}^{\pm}(y) & \text{if } i \leqslant k < n \end{cases}$$

For $x \otimes y \in K_i \times K'_j$ with $i + j = n$ the k^{th} degeneracy map, with $0 \leqslant k \leqslant n$, is given by

$$\varepsilon_k(x \otimes y) = \begin{cases} \varepsilon_k(x) \otimes y & \text{if } 0 \leqslant k \leqslant i \\ x \otimes \varepsilon_{k-i}(y) & \text{if } i \leqslant k < n \end{cases}$$

with $\sim$ generated by $\varepsilon_i(x) \otimes y \quad \sim \quad x \otimes \varepsilon_0(y)$.
The "segment" is $\square(-, 1)$ and the standard n-cube is $\square_n := \square(-, n)$. We have

$$\square_n = \bigotimes_{i=1}^{n} \square_1$$

The singular cubical set of an object X

- We are given $C : \square \to \mathcal{C}$, the singular cubical set functor $Sing : \mathcal{C} \to c\mathcal{S}et$ is defined as follows:
  - $Sing(X) := \left(\cdots \ \mathcal{C}(C(n), X) \underset{C(\sigma_k)\circ -}{\overset{C(\delta_k^\varepsilon)\circ -}{\leftrightarrows}} \mathcal{C}(C(n+1), X) \ \cdots \right)$
  - $Sing(\ X \xrightarrow{f} Y\ ) := \left(\cdots \ \mathcal{C}(C(n), X) \xrightarrow{-\circ f} \mathcal{C}(C(n), Y) \ \cdots \right)$
- Then we have $|\_| \dashv Sing$

Theorem

Nonabelian Algebraic Topology, Brown, R., Higgins, P. J., and Sivera R., EMS, 2011. Proposition 11.1.17, p.372

For any "topological space" X, the counit at X

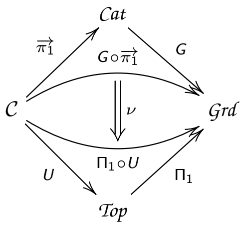
$$\varepsilon_X : |Sing\ X| \to X$$

of the adjunction $\text{“}\mathcal{Top}\text{”} \underset{|\_|}{\overset{\text{Sing}}{\rightleftarrows}} c\mathcal{S}et$ is a weak homotopy equivalence.

("topological space" maybe mean compactly generated space here.)

The canonical natural transformation $G \circ \overrightarrow{\pi_1} \to \Pi_1 \circ U$

- If X is undirected (i.e. all its paths are directed) then $\overrightarrow{\pi_1}X$ is actually $\Pi_1 \circ U(X)$ the fundamental groupoid of UX
- Denote by $G : \mathcal{Cat} \to \mathcal{Grd}$ the left adjoint to the inclusion functor $\mathcal{Grd} \hookrightarrow \mathcal{Cat}$
- Let X be an object of $\mathcal{C}$
- $\Pi_1 \circ U(X)$ is the fundamental groupoid of the underlying space of X
- $G \circ \overrightarrow{\pi_1}X$ is the enveloping groupoid of the fundamental category of X
- There exists a unique functor $\nu_X : G \circ \overrightarrow{\pi_1}(X) \to \Pi_1 \circ U(X)$,



we would like to known when it is an isomorphism.

Examples

The functor ν_X is an isomorphism when

- all the paths on UX are directed since $\overrightarrow{\pi_1}(X) = \Pi_1(X)$
- XU is totally disconnected since both $G \circ \overrightarrow{\pi_1}(X)$ and $\Pi_1 \circ U(X)$ are discrete
- X is the directed circle: just replace $\mathbb{N}$ by $\mathbb{Z}$ in the description of $\overrightarrow{\pi_1}\mathbb{S}^1$
- X is the directed complex plane or Riemann sphere: precompose by α_x

The functor ν_X is an not isomorphism when

- the direction on X is discrete while UX has a non constant path

Conjecture: η_X is an isomorphism when $X \cong \rceil K \lfloor_{\mathcal{L}po}$ for some precubical set K.

Conjecture: η_X is an isomorphism when X is an isothetic region.

Skeleta and equivalences of categories

- A skeleton of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ whose class of objects meets every isomorphism class of $\mathcal{C}$ exactly once.
- The skeleton of $\mathcal{C}$ is unique up to isomorphism, it is denoted by sk$\mathcal{C}$.
- Two categories are equivalent (i.e. there exists an equivalence of categories between them) iff their skeleta are isomorphic.
- The skeleton of the category of finite sets is the full subcategory whose objects are $\{0, \ldots, n-1\}$ for $n \in \mathbb{N}$.
- The skeleton of the fundamental groupoid of a path-connected space is the fundamental group of this space
- Problem: The fundamental category of a local pospace has no isomorphisms but its identities, hence it is its own skeleton

The categories $\mathcal{LfCat}$ and $\mathcal{OwCat}$

- A category $\mathcal{C}$ is said to be one-way when all its endomorphisms are identities i.e. $\mathcal{C}(x,x) = \{\mathrm{id}_x\}$ for all x

  Every Grothendieck topos has a one-way site. C. MacLarty. Theor. Appl. of Cat. 16(5) pp 123-126 (2006).

- A one-way category $\mathcal{C}$ is said to be loop-free when for all x, y

$$\mathcal{C}(x,y) \neq \emptyset \text{ and } \mathcal{C}(y,x) \neq \emptyset \text{ implies } x = y$$

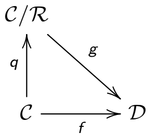
  Complexes of groups and orbihedra *in* Group theory from a geometrical viewpoint.
  A. Haefliger. World Scientific (1991).

- A loop-free category is its own skeleton
- A category is one-way iff its skeleton is loop-free

Generalized congruences

M. A. Bednarczyk, A. M. Borzyszkowski, W. Pawlowski. Theor. Appl. Cat. 5(11). 1999

- Given a binary relation $\mathcal{R}$ on the set of morphisms of a category $\mathcal{C}$, there is a unique category $\mathcal{C}/\mathcal{R}$ and a unique functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{R}$ such that for all functors $f : \mathcal{C} \to \mathcal{D}$, if $\alpha \mathcal{R} \beta \Rightarrow f(\alpha) = f(\beta)$, then there is a unique functor $g : \mathcal{C}/\mathcal{R} \to \mathcal{D}$ such that $f = g \circ q$



- Examples
  - any congruence is a generalized congruence.
  - $\mathcal{C}$ freely generated by $x \xrightarrow{\alpha} y$ with $\mathrm{id}_x \mathcal{R} \mathrm{id}_y$ (resp. with $\alpha \mathcal{R} \mathrm{id}_x$).
  - $(\mathbb{N}, +, 0)$ with $0 \mathcal{R} n$ for some $n \in \mathbb{N}$.

Reflections

- One has the full inclusions $\mathcal{LfCat} \subseteq \mathcal{OwCat} \subseteq \mathcal{Cat}$ each of which having a left adjoint
- The left adjoint to $\mathcal{OwCat} \subseteq \mathcal{Cat}$ is obtained by identifing every endomorphism with the identity of its homset
- The left adjoint to $\mathcal{LfCat} \subseteq \mathcal{OwCat}$ identifying every isomorphism with the identify of its extremities
- The left adjoint R to a full inclusion functor is called (when it exists) a reflection. By extension $R(x)$ is called the reflect of x
- The skeleton of a one-way category is its reflect in $\mathcal{LfCat}$.
- Any poset can be seen as a loop-free category whose homsets contains at most on element $\mathcal{Pos} \hookrightarrow \mathcal{LfCat}$
- Any preordered set can be seen as a one-way category whose homsets contains at most on element $\mathcal{Pre} \hookrightarrow \mathcal{LfCat}$
- In both cases the left adjoint is obtained by identifying two morphisms with the same sources and targets
- Any equivalence relation can be see as a one-way groupoid

Objective

Let $\mathcal{C}$ be a one-way category

- Define a class Σ of morphisms of $\mathcal{C}$ so we can keep one representative in each class of Σ-related objects without loss of information
- To do so, we are in search for a class that behaves much like the one of isomorphisms
- From now on $\mathcal{C}$ denotes a one-way category

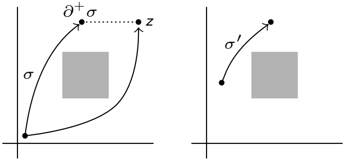
Potential weak isomorphisms

Let $\mathcal{C}$ is one-way

- For all morphisms σ and all objects z define
  - the σ, z-precomposition as $\gamma \in \mathcal{C}(\partial^+\sigma, z) \rightarrow \gamma\circ\sigma \in \mathcal{C}(\partial^-\sigma, z)$
  - the z, σ-postcomposition as $\delta \in \mathcal{C}(z, \partial^-\sigma) \mapsto \sigma\circ\delta \in \mathcal{C}(z, \partial^+\sigma)$
- One may have $\mathcal{C}(\partial^+\sigma, z) = \emptyset$ or $\mathcal{C}(z, \partial^-\sigma) = \emptyset$
- Note that σ is an isomorphism iff for all z both precomposition and postcomposition are bijective.
- The latter condition is weakened: σ is said to preserve the future cones (resp. past cones) when for all z if $\mathcal{C}(\partial^+\sigma, z) \neq \emptyset$ (resp. $\mathcal{C}(z, \partial^-\sigma) \neq \emptyset$) then the precomposition (resp. postcomposition) is bijective.
- Then σ is a potential weak isomorphism when it preserves both future cones and past cones. Potential weak isomorphisms compose.
- If $\mathcal{C}(x, y)$ contains a potential weak isomorphism, then it is a singleton
  Requires the assumption that $\mathcal{C}$ is one-way

An example

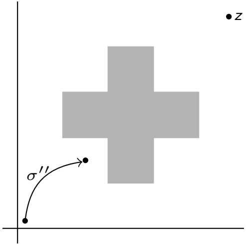
of potential weak isomorphism



Due to the lower dipath, the σ, z-precomposition is not bijective; yet σ' is a potential weak isomorphism.

An unwanted example

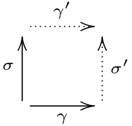
of potential weak isomorphism



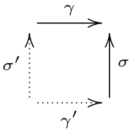
Note that σ'' is a potential weak isomorphism though there exists a morphism from $\partial^- \sigma''$ to z but none from $\partial^+ \sigma''$ to z.

Stability under pushout and pullback

- A collection of morphisms Σ is said to be stable under pushout when for all $\sigma \in \Sigma$, for all γ with $\partial^-\gamma = \partial^-\sigma$, the pushout of σ along γ exists and belongs to Σ



- A collection of morphisms Σ is said to be stable under pullback when for all $\sigma \in \Sigma$, for all γ with $\partial^+\gamma = \partial^+\sigma$, the pullback of σ along γ exists and belongs to Σ



Greatest inner collection

stable under pushout and pullback

- Any collection Σ of morphisms of a category $\mathcal{C}$ admits a greatest subcollection that is stable under pushout and pullback
- Construction:
  - Start with $\Sigma_0 = \Sigma$
  - For $n \in \mathbb{N}$ define Σ_{n+1} as the collection of morphisms $\sigma \in \Sigma_n$ s.t. the pushout and the pullback of σ along with all morphisms exist (when sources or targets match) and belong to Σ_n

$$\Sigma_0 \supseteq \cdots \Sigma_1 \supseteq \cdots \supseteq \Sigma_n \supseteq \Sigma_{n+1} \supseteq \cdots$$

  - The expected subcollection is the decreasing intersection

$$\Sigma_\infty := \bigcap_{n \in \mathbb{N}}^{\downarrow} \Sigma_n$$

- The collection Σ_∞ is stable under the action of $\mathrm{Aut}(\mathcal{C})$

Systems of weak isomorphisms

- The class of isomorphisms of any category is stable under pushout and pullback
- A system of weak isomorphisms is a collection of potential weak isomorphisms that is stable under pushout and pullback
- The class of all isomorphisms of any category is a system of weak isomorphisms
- If Σ is a system of weak isomorphisms, then so is its closure under composition
- Hence we suppose the systems of weak isomorphisms are closed under composition

Examples

of systems of weak ismorphisms

- Given a partition $\mathcal{P}$ of $\mathbb{R}$ into intervals, the following collection is a system of weak isomorphisms
$$\{(x, y) \mid x \leqslant y;\ \exists I \in \mathcal{P},\ [x, y] \subseteq I\}$$
- In the preceding example, $\mathbb{R}$ can be replaced by any totally ordered set
- Let $\Sigma_i \subseteq \mathcal{C}_i$ be a family of collections of morphisms, then

$$\boxed{\prod_i \Sigma_i \text{ is a swi of } \prod_i \mathcal{C}_i \text{ iff each } \Sigma_i \text{ is a swi of } \mathcal{C}_i}$$

- The inverse image (resp. the direct image) of a system of weak isomorphisms by an equivalence of categories is a system of weak isomorphisms.

Pureness

- A collection Σ of morphisms is said to be pure when

$$\gamma \circ \delta \in \boldsymbol{\Sigma} \;\Rightarrow\; \gamma, \delta \in \boldsymbol{\Sigma}$$

- Given a one-way category $\mathcal{C}$ we have:

All the systems of weak isomorphisms of $\mathcal{C}$ are pure

The locale of systems of weak isomorphisms

- A locale is a complete lattice whose binary meet distributes over arbitrary join i.e.

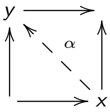
$$x \wedge \left(\bigvee_i y_i \right) = \bigvee_i (x \wedge y_i)$$

- The collection ΩX open subsets of a topological space X form a locale and we have the functor $L : \mathcal{Top} \to \mathcal{Loc}$ (that admits a left adjoint) defined by
    - $L(X) = \Omega X$
    - $L(f)(W) = f^{-1}(W)$ for all $f : X \to Y$ and $W \in \Omega Y$
- The collection of systems of weak isomorphisms of a category has a greatest element
- Given a one-way category $\mathcal{C}$ we have:
    - The collection of systems of weak isomorphisms of $\mathcal{C}$ forms a locale
    - The greatest swi is invariant under the action of $\mathrm{Aut}(\mathcal{C})$

The filling square property

of a category $\mathcal{C}$

- By definition, a filling square category $\mathcal{C}$ is such that for all commutative squares which are both pushout and pullack (see below), if $\mathcal{C}(x, y) \neq \emptyset$ then there exists $\alpha \in \mathcal{C}(x, y)$ that makes both triangles commute.



- If $\mathcal{C}$ satisfies the filling square property, then any collection of morphisms of $\mathcal{C}$ that is stable under pushout and pullback is a system of weak isomorphisms.
- A conjecture:

> For all loop-free isothetic region X, $\overrightarrow{\pi_1}X$ satisfies the square filling property

Components

of a one-way category $\mathcal{C}$

- From now on $\mathcal{C}$ is a one-way category and Σ is a system of weak isomorphisms on it
- Recall that is $\mathcal{C}(x, y)$ meets Σ, then $\mathcal{C}(x, y)$ is a singleton, a fact that we represent on diagrams by: $x \xrightarrow{\Sigma} y$
- Given two objects x and y of $\mathcal{C}$ t.f.a.e.:
  - there exists a Σ-zigzag between x and y
  - there exists z such that $x \xleftarrow{\Sigma} z \xrightarrow{\Sigma} y$
  - there exists z such that $x \xrightarrow{\Sigma} z \xleftarrow{\Sigma} y$
- When any of the following property is satisfied x and y are said to be Σ-connected
- Σ-connectedness is an equivalence relation on the objects of $\mathcal{C}$
- The equivalence classes are called a Σ-components

Structure of the Σ-components

Σ system of weak isomorphisms of $\mathcal{C}$ one-way category

A prelattice is a preordered set in which $x \wedge y$ and $x \vee y$ exist for all x and y.
However they are defined only up to isomorphism

Let K be a Σ-component of $\mathcal{C}$ and $\mathcal{K}$ be the full subcategory of $\mathcal{C}$ whose objects are the elements of K. The following properties are satisfied:

- [1.] The category $\mathcal{K}$ is isomorphic with the preorder $(K, \preccurlyeq)$ where $x \preccurlyeq y$ stands for $\mathcal{C}[x, y] \neq \emptyset$. In particular, every diagram in $\mathcal{K}$ commutes.
- [2.] The preordered set $(K, \preccurlyeq)$ is a prelattice.
- [3.] If d and u are respectively a greatest lower bound and a least upper bound of the pair $\{x, y\}$, then Diagram 1 is both a pullback and a pushout in $\mathcal{C}$, and all the arrows appearing on the diagram belong to Σ.
- [4.] $\mathcal{C} = \mathcal{K}$ iff $\mathcal{C}$ is a prelattice, and Σ is the greatest system of weak isomorphisms of $\mathcal{C}$ i.e. all the morphisms in this case.

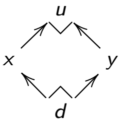


Diagram 1

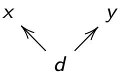


Diagram 2

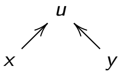
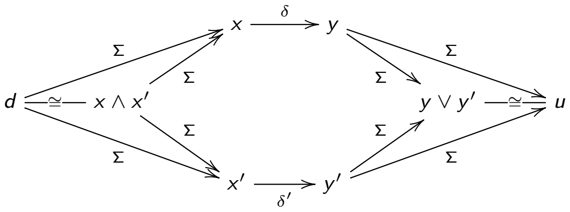


Diagram 3

Equivalent morphisms

with respect to Σ

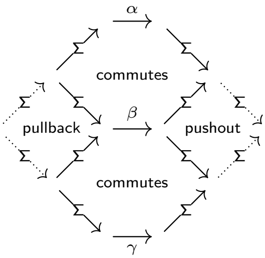
- Let $\delta \in \mathcal{C}(x, y)$ and $\delta' \in \mathcal{C}(x', y')$. Then write $\delta \sim \delta'$ when
  - $x \sim x'$ and $y \sim y'$, and
  - the inner hexagon of the next diagram commutes



  - Note that if $d \cong x \wedge x'$ and $u \cong y \vee y'$ then the outter hexagon also commutes, hence the relation $\sim$ is well defined.
- If $\gamma \sim \delta$ then $\partial^- \gamma \sim \partial^- \delta$ and $\partial^+ \gamma \sim \partial^+ \delta$

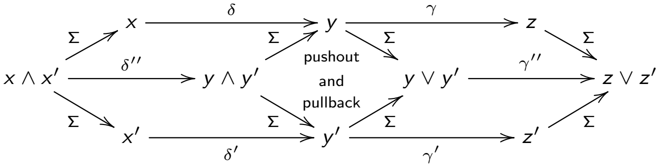
The relation $\sim$ is an equivalence

- The relation $\sim$ is:
  - reflexive since Σ contains all identities
  - symmetric by definition
  - transitive



The relation $\sim$ fits with composition

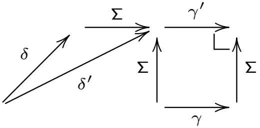
- Suppose $\partial^- \gamma = \partial^+ \delta$, $\partial^- \gamma' = \partial^+ \delta'$ and $\gamma \sim \gamma'$ and $\delta \sim \delta'$.
- Then we have $\gamma \circ \delta \sim \gamma' \circ \delta'$



The category of components

denoted by $\mathcal{C}/\Sigma$

- The quotient category $\mathcal{C}/\Sigma$ (obtained by turning each morphism of Σ into an identity) can be defined as follows:
  - The objects are the Σ-components
  - The morphisms are the $\sim$-equivalence classes
- If $\partial^{-}\gamma \sim \partial^{+}\delta$ then
  - there exists γ' and δ' such that $\gamma' \sim \gamma$, $\delta' \sim \delta$, and $\partial^{-}\gamma' = \partial^{+}\delta'$



  - so we define $[\gamma] \circ [\delta] = [\gamma' \circ \delta']$
- We have the quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$
- The category of components is $\mathcal{C}/\Sigma$ with Σ being the greatest swi of $\mathcal{C}$

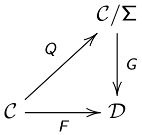
Characterizing

the identities of $\mathcal{C}/\Sigma$

For any morphism δ of $\mathcal{C}$ t.f.a.e.

- $\delta \in \Sigma$
- $[\delta] \subseteq \Sigma$
- $[\delta]$ is an identity of $\mathcal{C}/\Sigma$

The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ satisfies the following universal property:
for all functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{isomorphisms of } \mathcal{D}\}$
there exists a unique $G : \mathcal{C}/\Sigma \to \mathcal{D}$ s.t. $F = G \circ Q$



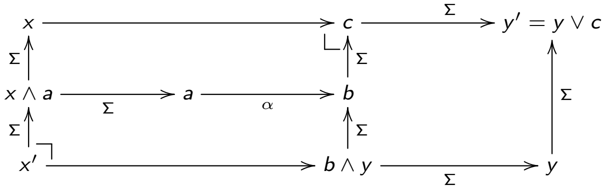
The fundamental properties of $\mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ is surjective on morphisms
- The quotient category $\mathcal{C}/\Sigma$ is loop-free
- If $\mathcal{C}(x, y) \neq \emptyset$ then the following map is a bijection.

$$\delta \in \mathcal{C}(x, y) \ \mapsto \ Q(\delta) \in \mathcal{C}/\Sigma\big(Q(x), Q(y)\big)$$

- If $\mathcal{C}/\Sigma\big(Q(x), Q(y)\big) \neq \emptyset$ then there exist x' and y' such that $\Sigma(x', x)$, $\Sigma(y, y')$, $\mathcal{C}(x', y)$, and $\mathcal{C}(x, y')$ are nonempty.

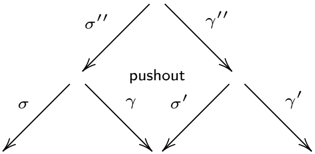


- The quotient functor Q preserves and reflects potential weak isomorphisms
- If $\mathcal{C}$ is finite then so is the quotient $\mathcal{C}/\Sigma$
- $\mathcal{C}$ is a preorder iff $\mathcal{C}/\Sigma$ is a poset

Describing the localization of $\mathcal{C}$ by Σ

with Σ system of weak isomoprphisms of $\mathcal{C}$

- The objects of $\mathcal{C}[\Sigma^{-1}]$ are the objects of $\mathcal{C}$
- The morphisms are the equivalence classes of ordered pairs of coinitial morphisms (γ, σ) with $\sigma \in \Sigma$,
  - Two pairs (γ, σ) and (γ', σ') being equivalent when $\partial^{-}\sigma = \partial^{-}\sigma'$, $\partial^{-}\gamma = \partial^{-}\gamma'$, and $Q(\gamma) = Q(\gamma')$
  - In the diagram below we have $Q(\gamma' \circ \gamma'') = Q(\gamma') \circ Q(\gamma'') = Q(\gamma') \circ Q(\gamma)$ hence the composite $(\gamma' \circ \gamma'', \sigma \circ \sigma'')$ neither depend on the choice of the pushout nor on the representatives (γ, σ) and (γ', σ').



The canonical inclusion $I : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$

with Σ system of weak isomoprphisms of $\mathcal{C}$

- Define I by $I(\gamma) := (\gamma, \mathrm{id}_{\partial^- \gamma})$ and the identity on objects
- Given a functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{iomorphisms of } \mathcal{D}\}$ define
  - $G(x) := F(x)$ for all objects x of $\mathcal{C}[\Sigma^{-1}]$ and
  - $G(\gamma, \sigma) := F(\gamma) \circ (F(\sigma))^{-1}$ for any representative (γ, σ) of a morphism of $\mathcal{C}[\Sigma^{-1}]$
- The functor $I : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ then satisfies the universal property: for all functor $F : \mathcal{C} \to \mathcal{D}$ there exists a unique $G : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ s.t. $F = G \circ I$
- In particular there is a unique functor P s.t. $Q = P \circ I$ with $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ and we have
-

The functor P is an equivalence of categories

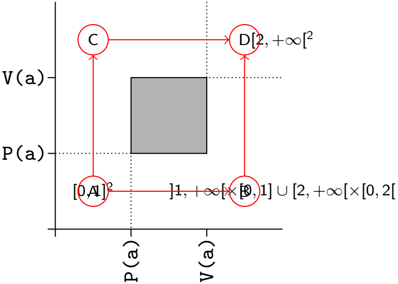
- The skeleton of $\mathcal{C}[\Sigma^{-1}]$ is $\mathcal{C}/\Sigma$ and $\mathcal{C}[\Sigma^{-1}]$ is one-way.

Embeding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

- Let ϕ : Σ-components of $\mathcal{C} \to \mathrm{Ob}(\mathcal{C})$ such that
  - for all Σ-components K, K', if there exists $x \in K$ and $x' \in K'$ such that $\mathcal{C}(x, x') \neq \emptyset$, then $\mathcal{C}(\phi(K), \phi(K')) \neq \emptyset$
  - in this case $\mathcal{C}/\Sigma$ is isomorphic with the full subcategory of $\mathcal{C}$ whose set of objects is $\mathrm{im}(\phi)$.
  - the mapping ϕ is called an admissible choice (of canonical objects)
- Write $\phi \preccurlyeq \phi'$ when $\mathcal{C}(\phi(K), \phi'(K)) \neq \emptyset$ for all Σ-components K
  - The collection of admissible choice then forms a (pre)lattice
  - If $\mathcal{C}/\Sigma$ is finite then there exists an admissible choice
  - If $\mathcal{C}/\Sigma$ is infinite the existence of an admissible choice is a open question

Plane without a square

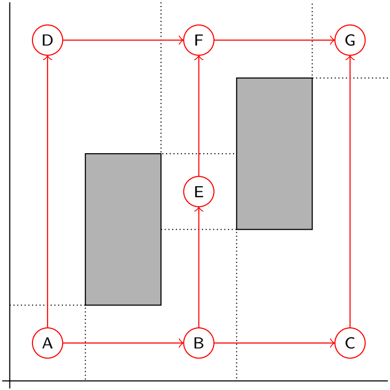
$x = \mathbb{R}_+^2 \setminus]0,1[^2$



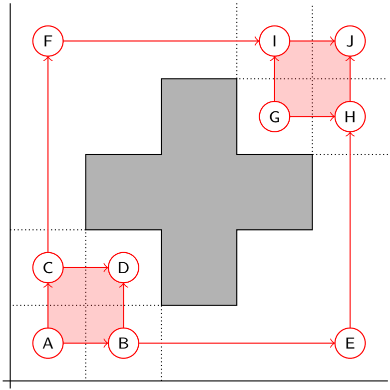
Let x, y such that $x \leqslant^2 y$, then $\overrightarrow{\pi_1}X(x,y)$ only depends on which elements of the partition x and y belong to

$\rightarrow$	A	B	C	D
A	σ	β	γ	$\beta' \circ \beta$ $\alpha' \circ \alpha$
B		σ		β'
C			σ	γ'
D				σ

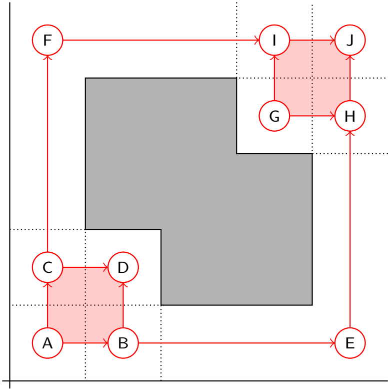
Two rectangles



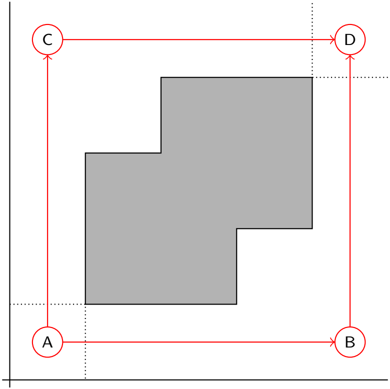
Swiss Flag



Achronal overlaping square

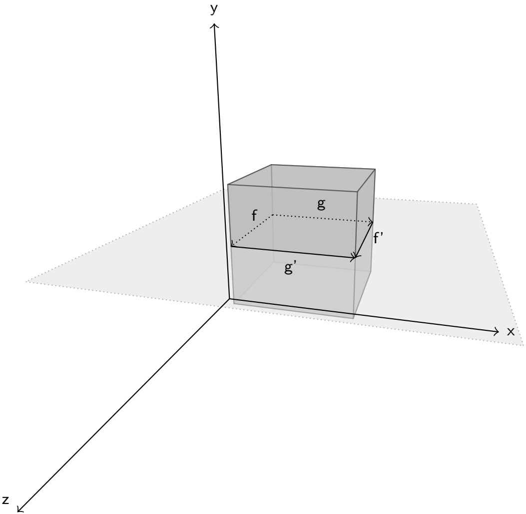


Diagonal overlaping squares



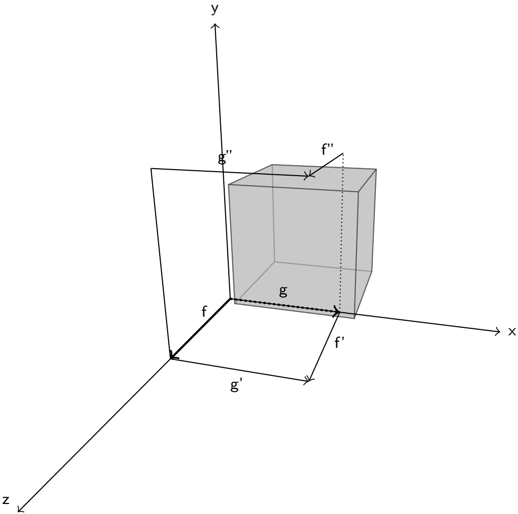
The floating cube

Non potential weak isomorphisms



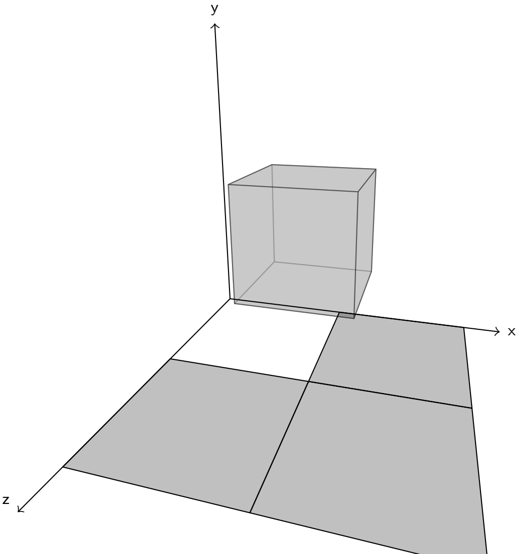
The floating cube

A “vee” that does not extend to a pushout

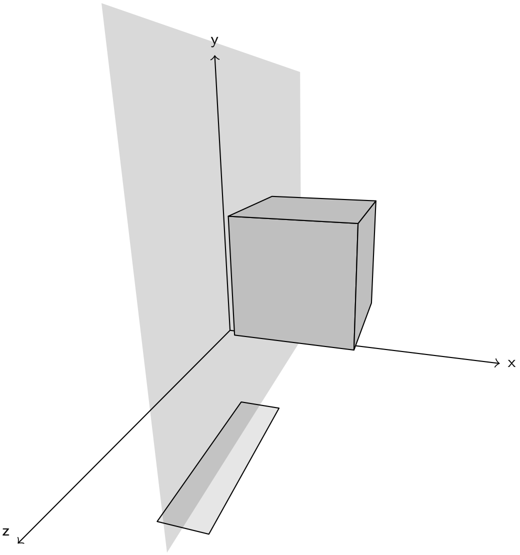


The floating cube

Some pushouts squares



The floating cube

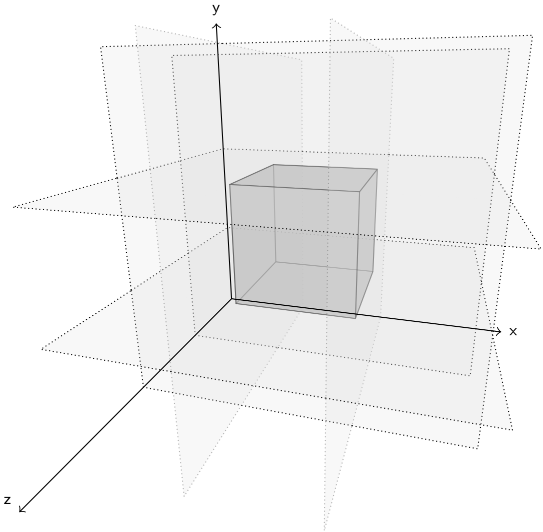


The floating cube

- Since the pushout of f (resp. g) along g (resp. f) does not exist, $f, g \notin \Sigma$
- The commutative square f, g, f', and g' is a pullback:
  - Therefore $f', g' \notin \Sigma$ (anyway they do not preserve the future cones)

The floating cube

boundaries of the components



Homotopical category and homotopical functor

Homotopy Limit Functors on Model Categories and Homotopical Categories

Dwyer, W. G., Hirschhorn, P. S., Kan, D. M., and Smith, J. H. Amer. Math. Soc. (2004)

- A category is said to be homotopical when it comes with a collection of morphisms $\mathcal{W}$ that
  - contains all the identities
  - has the two out of six property i.e. if $\gamma \circ \beta$ and $\beta \circ \alpha$ both exist and belong to $\mathcal{W}$, then α, β, γ, and $\gamma \circ \beta \circ \alpha$ belong to $\mathcal{W}$
- The second condition may be replaced by the conjonction of the following ones:
  - has the weak invertibility property i.e. if there exists γ and δ such that $\gamma \circ \omega$ and $\omega \circ \delta$ both exist and belong to $\mathcal{W}$, then $\omega \in \mathcal{W}$
  - has the two out of three property i.e. if $\gamma \circ \delta$ exist and two of γ, δ and $\gamma \circ \delta$ belong to $\mathcal{W}$, then the third also lies in $\mathcal{W}$
- The element of $\mathcal{W}$ are called the weak equivalences
- A functor between homotopical categories is said to be homotopical if it preserves weak equivalences.
- The corresponding homotopy category is defined as the localization $\mathcal{C}[\mathcal{W}^{-1}]$

Weak isomorphisms as weak equivalences

Let Σ be a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The category $\mathcal{C}$ equipped with the collection $\mathcal{W} = \Sigma$ is a homotopical category
  because Σ is pure in $\mathcal{C}$ and contains all the isomophisms of $\mathcal{C}$
- The homotopy category $Ho\mathcal{C}$ is the localization $\mathcal{C}[\mathcal{W}^{-1}]$ i.e. $\mathcal{C}/\mathcal{W}$ up to an equivalence of category
- If the collection of isomorphisms of $\mathcal{C}$ is pure then so is its greatest p.o. and p.b. stable collection of morphisms.

The canonical model category on $\mathcal{C}at$

- [reminder] $f : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories when there exists $g : \mathcal{D} \to \mathcal{C}$ s.t. $g \circ f \cong \mathrm{id}_{\mathcal{C}}$ and $f \circ g \cong \mathrm{id}_{\mathcal{D}}$
- [reminder] $f : \mathcal{C} \to \mathcal{D}$ is an equivalence of category iff it is fully faithful and every object of $\mathcal{D}$ is isomorphic with some object of the form $f(c)$
- an equivalence of category between loop-free categories is therefore an isomorphism
- The category $\mathcal{C}at$ with

  $$\mathcal{W} := \{\text{equivalences of categories}\}$$

  is also a homotopical category.
- In fact there is a unique model category structure on $\mathcal{C}at$ whose weak equivalences are the equivalences of category.

Finding a homotopical category structure on *OwCat*

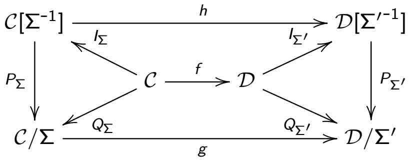
- The class of isomorphisms of a category $\mathcal{C}$ is stable under the action of $\mathrm{Aut}(\mathcal{C})$ therefore any one-way category should be equipped with its greatest system of weak isomorphisms
- The class of isomorphisms of a category $\mathcal{C}$ is preserved by any functor, so the only functors between one-way categories that should be considered are the ones that preserve the greatest systems of weak isomorphisms.
  These functors are therefore the homotopical ones.

A guiding result

for defining the weak equivalences

Given a system of weak isomorphisms Σ of a one-way category $\mathcal{C}$ we have:

> the functor $g : \mathcal{C}[\Sigma^{-1}] \to \mathcal{D}[\Sigma'^{-1}]$ is an equivalence of category
> iff the quotient functor $q : \mathcal{C}/\Sigma \to \mathcal{D}/\Sigma'$ is an isomorphism



When Σ anf Σ' are the greatest systems of weak isomorphisms of $\mathcal{C}$ and $\mathcal{D}$, then we define:

$$\vec{\pi}_0(f) = g \qquad \text{and} \qquad Ho(f) = h$$

The functors $\overrightarrow{\pi_0}$ and *Ho*

- In particular we have a functor $\overrightarrow{\pi_0} : \mathcal{O}\omega\mathcal{C}at_h \to \mathcal{L}f\mathcal{C}at$ with $\overrightarrow{\pi_0}\mathcal{C}$ being the quotient of $\mathcal{C}$ by its greatest system of weak isomorphisms

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ {\scriptstyle q_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle q_{\mathcal{D}}} \\ \overrightarrow{\pi_0}\mathcal{C} & \xrightarrow[\overrightarrow{\pi_0}f]{} & \overrightarrow{\pi_0}\mathcal{D} \end{array}$$

- We also have the functor $Ho : \mathcal{O}\omega\mathcal{C}at_h \to \mathcal{O}\omega\mathcal{C}at$ defined by

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ {\scriptstyle i_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle i_{\mathcal{D}}} \\ Ho\mathcal{C} & \xrightarrow[Ho(f)]{} & Ho\mathcal{D} \end{array}$$

The homotopical category $\mathcal{OwCat}_h$

- Then $\mathcal{OwCat}_h$ is the subcategory of $\mathcal{OwCat}$ whose morphisms are the functors that preserve the greatest systems of weak isomorphisms
- The collection of weak equivalences of $\mathcal{OwCat}_h$ is

$$\mathcal{W} := \{f \mid \overrightarrow{\pi_0} f \text{ is an isomorphism}\} = \{f \mid Ho(f) \text{ is an equivalence of category}\}$$

- Any equivalence of categories between one-way categories is a weak equivalence
- The pushout in $\mathcal{OwCat}_h$ of two copies of $\{0\} \hookrightarrow \{0 < 1\}$ is $\{0, 1\}^2$