Directed Algebraic Topology and Concurrency

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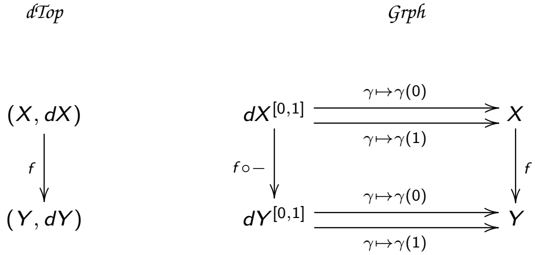
MPRI : Concurrency (2.3)

Wednesday, the 1st of February 2017

Extracting the “largest” graph from a d-space

The following functor is right adjoint to graph realization in $d\mathcal{Top}$



Compactly generated spaces

following the terminology from *the category of CGWH spaces*, N. P. Strickland, 2009.

- A *k*-probe on $X \in \mathcal{Top}$ is a continuous map from some compact Hausdorff space to X.
- $U \subseteq X$ is said to be *k*-open when $t^{-1}(U)$ is open for all k-probes t on X.
- X is a k-space when all its k-open subsets are open.
- Given $X, Y \in \mathcal{Top}$, a mapping $f : X \to Y$ is said to be *k*-continuous when for all k-probe on X, the mapping $f \circ t$ is continuous.
- X is a *k*-space iff for all $Y \in \mathcal{Top}$ and all $f \in \mathcal{Top}(X, Y)$ if f is k-continuous, then it is continuous.
- The k-spaces and the continuous functions between them form the category $\mathcal{CG}$.

Weak Hausdorff spaces

- A topological space in which distinct points admit disjoint neighborhoods is said to be Hausdorff. The category of Hausdorff spaces is denoted by $\mathcal{Haus}$.
- A topological space is said to be weak Hausdorff when the image of every k-probe on it is closed. The category of weak Hausdorff spaces is denoted by $w\mathcal{Haus}$.
- The category of compactly generated weak Hausdorff spaces is denoted by $\mathcal{CGwH}$.
- The category of compactly generated Hausdorff spaces is denoted by $\mathcal{CGH}$.

k-ification

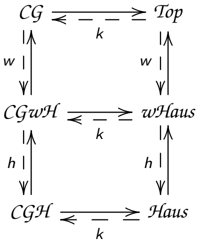
The k-ification kX of a topological space X is the underlying set of X together with all its k-open subsets.

That construction is functorial and provides the following inclusion functors with a right adjoint.

- $\mathcal{Top} \hookleftarrow \mathcal{CG}$
- $w\mathcal{Haus} \hookleftarrow \mathcal{CG}w\mathcal{H}$
- $\mathcal{Haus} \hookleftarrow \mathcal{CGH}$

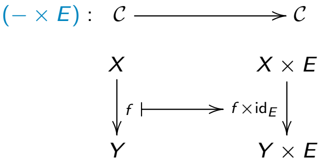
Properties

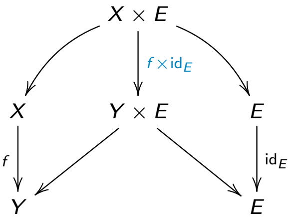
- Given $X, Y \in \mathcal{CG}$, $X \times_{\mathcal{CG}} Y \cong k(X \times Y)$
- A k-space X is weak Hausdorff iff its diagonal is a closed subset of $X \times_{\mathcal{CG}} X$
- $X \in \mathcal{CG}$ iff it is a colimit in $\mathcal{Top}$ of compact Hausdorff spaces.
- $X \in \mathcal{CGwH}$ iff it is a colimit in $w\mathcal{Haus}$ of compact Hausdorff spaces.
- $X \in \mathcal{CGH}$ iff it is a colimit in $\mathcal{Haus}$ of compact Hausdorff spaces.


CG
Top
k
w
w
CGwH
wHaus
k
h
h
CGH
Haus
k

The product functor

Let E be an object of $\mathcal{C}$ such that for all objects X of $\mathcal{C}$ the Cartesian product $X \times E$ exists.





with $f \times \mathrm{id}_E$ defined by right hand side diagram
(the unlabelled arrows being the projection morphism)

Exponentiable object

- An object E is said to be exponentiable when the functor $E \times -$ (is well defined and) admits a right adjoint often denoted by $(-)^E$
  In particular for all objects X, Y of $\mathcal{C}$ we have:

$$\mathcal{C}(E \times X, Y) \cong \mathcal{C}(X, Y^E)$$

- Any compact Hausdorff space is exponentiable (in $\mathcal{Top}$ and $\mathcal{Haus}$)
- A category with all binary cartesian products is said to be cartesian.
- A cartesian category whose objects are all exponentiable is said to be closed.
- $\mathcal{Top}$, $w\mathcal{Haus}$ and $\mathcal{Haus}$ are cartesian but not closed.
- The categories $\mathcal{CG}$, $\mathcal{CGwH}$, and $\mathcal{CGH}$ are cartesian closed. For $X, E \in \mathcal{CG}$ ($\mathcal{CGwH}$, $\mathcal{CGH}$), X^E is the k-ification of the compact-open topology on $\mathcal{Top}(E, X)$, and $X \times E$ is the k-ification of the product $X \times_{\mathcal{Top}} E$.
- By restricting the class of topological spaces allowed in the definition of d-spaces to ((weak) Hausdorff) compactly generated spaces, one obtains Cartesian closed categories.
- The categories $\mathcal{Set}$ and $\mathcal{Grph}$ are cartesian closed (they are actually toposes).

Streams

S. Krishnan, 2006

A stream is a Hausdorff space X together with a mapping that associates each open subset U of X with a preorder $\preccurlyeq_U$ on U such that for all open covering $\mathcal{V}$ of U, the preorder $\preccurlyeq_U$ is the least preorder containing $\preccurlyeq_V$ for all $V \in \mathcal{V}$.

A stream morphism from X to Y is a mapping from the underlying set of X to that of Y such that for all $x \in X$, there exist U and V, open neighbourhoods of x and $f(x)$ such that f induces a preorder morphism from U to V.

The category of streams is denoted by $\mathcal{S}tr$, it is complete and cocomplete.

By restricting the class of topological spaces allowed in the definition of streams to ((weak) Hausdorff) compactly generated spaces, one obtains Cartesian closed categories.

Directed paths on a stream

For all U open subsets of an interval of $\mathbb{R}$, write $u \preccurlyeq_U v$ when $u \leqslant v$ and $[u, v] \subseteq U$. Then any interval of $\mathbb{R}$ is a stream. In particular $[0, r]$ is a stream.

A directed path on a stream X is a stream morphism from some stream $[0, r]$ to X.

Adjunction

For any open subset U of d-space X, the preorder on U is given by $u \preccurlyeq_U v$ when there exists a directed path from u to v whose image is contained in U.

The collection of directed paths on stream gives rise to a d-space.

We have an adjunction

$$\mathcal{S}tr \underset{S}{\overset{D}{\rightleftarrows}} d\mathcal{T}op$$

That adjunction induces an isomorphism between the full subcategories $\mathcal{S}tr_{\mathsf{d}} = \{SX \mid X \in d\mathcal{T}op\}$ and $d\mathcal{T}op_{\mathsf{f}} = \{DX \mid X \in \mathcal{S}tr\}$ of $\mathcal{S}tr$ and $d\mathcal{T}op$.

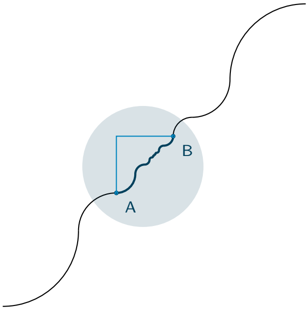
Characterization of $\mathcal{dTop}_{\mathsf{f}}$ and $\mathcal{Str}_{\mathsf{d}}$

A stream X belongs to $\mathcal{Str}_{\mathsf{d}}$ when for all open subset U of X, if $u \preccurlyeq_U v$ then there exists a directed path on X from u to v whose image is contained in U.

A pseudo-directed path on a d-space X is a continuous path γ such that for all subpaths γ' of γ and all neighbourhood U of $\mathrm{im}(\gamma')$, there exists a directed path on U from $\partial^{-}\gamma'$ to $\partial^{+}\gamma'$.

A d-space D belongs to $\mathcal{dTop}_{\mathsf{f}}$ when for every pseudo-directed path on X is directed. Such d-spaces are said to be filled.

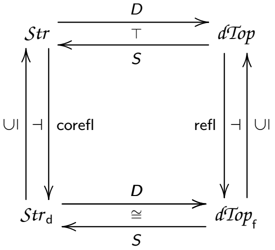
Infinite rounded staircase



Filled directions on a space X

- The collection of constant paths on X is a filled direction.
- The collection of all paths on X is a filled direction.
- The forgetful functor $U : \mathit{dTop}_{\mathrm{f}} \to \mathit{Top}$ has both a left and a right adjoint.
- An intersection of filled directions on X is still a filled direction on X.
- The collection of filled directions on X is thus a complete lattice.
- Hence a Galois connection $\{\text{filled directions on } X\} \rightleftarrows \{\text{directions on } X\}$.
- The inclusion functor $\mathit{dTop}_{\mathrm{f}} \hookrightarrow \mathit{dTop}$ has a left adjoint

Relating streams and d-spaces



Properties of filled d-spaces

- The functor $I : \mathcal{Lpo} \to \mathcal{dTop}$ actually takes its values in $\mathcal{dTop}_{\mathrm{f}}$.
- Let F be the operator sending any direction on X to the least filled direction containing it,

> Conjecture: given $V_1, \ldots, V_n$ vector fields on $\mathcal{M}$,
>
> $$F\big(d_{V_1}\mathcal{M} \vee \cdots \vee d_{V_n}\mathcal{M}\big) = F\big(d_{V_1,\ldots,V_n}\mathcal{M}\big)$$

- The full subcategory $\mathcal{dTop}_{\mathrm{f}}$ of filled d-spaces is complete and cocomplete.

Fundamental categories

On defines a fundamental category functor from $\mathcal{S}tr$ to $\mathcal{C}at$ in the same way as we define the fundamental category functors over $\mathcal{L}po$ and $d\mathcal{T}op$.

For all streams X, $\overrightarrow{\pi_1}X \cong \overrightarrow{\pi_1}DX$

For all filled d-spaces X, $\overrightarrow{\pi_1}X \cong \overrightarrow{\pi_1}SX$

The category $\square^+$ of face inclusions

- For $n \in \mathbb{N}$, $k \in \{0, \ldots, n-1\}$, $\varepsilon \in \{+, -\}$, we have the face inclusion map

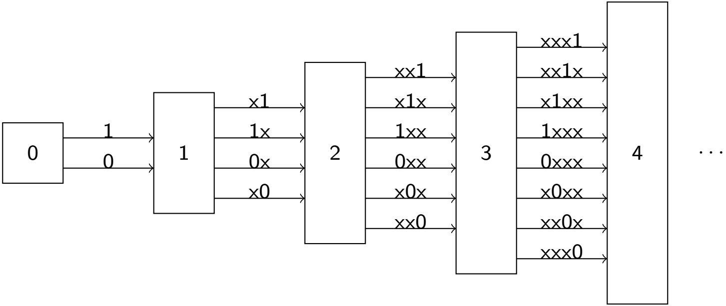
$$\begin{array}{rcl} [0,1]^n & \to & [0,1]^{n+1} \\ (t_0, \ldots, t_{n-1}) & \mapsto & (t_0, \ldots, t_{k-1}, \varepsilon, t_k, \ldots, t_{n-1}) \end{array}$$

- Then $\square^+$ is the subcategory of $\mathcal{S}et$ generated by all the face inclusions

A description of the $\square^+$ category

- $\{\text{Objects of } \square^+\} = \mathbb{N}$
- $\square^+[n, m] =$
  - $\{$words of length m on $\{0, 1, x\}$ with n occurences of $x\}$
  - empty when $n > m$
- $\mathrm{id}_n = x^n \quad \in \quad \square^+(n, n)$
- $\delta^-_{k,n} \cong (x \cdots x \underbrace{0}_{k^{th}} x \cdots x)$ and $\delta^+_{k,n} \cong (x \cdots x \underbrace{1}_{k^{th}} x \cdots x) \quad \in \quad \square^+(n-1, n)$
- if $w : a \to b$ and $w' : b \to c$ then $w'w$ is obtained by
  replacing the k^{th} occurrence of x in w' by the k^{th} letter of w.

Presentation of $\square^+$ with faces as generators



Presentation of $\square^+$: the (co)precubical relations

The category $\square^+$ is generated by the morphisms $\delta^{\varepsilon}_{k,n}$ for $n \in \mathbb{N}$, $k \in \{0, \dots, n-1\}$, and $\varepsilon \in \{+, -\}$ together with the following relations for all $n \in \mathbb{N}$, $i \in [n]$, $j \in [n+1]$ and $\alpha, \beta \in \{0, 1\}$

$$\delta^{\beta}_{j,n+1} \circ \delta^{\alpha}_{i,n} = \begin{cases} \delta^{\alpha}_{i,n+1} \circ \delta^{\beta}_{j-1,n} & \text{if } i < j \\ \delta^{\alpha}_{i+1,n+1} \circ \delta^{\beta}_{j,n} & \text{if } i \geqslant j \end{cases}$$

Precubical sets are presheaves over $\square^+$

- The category of precubical sets, denoted by pcSet, can be defined as $\mathrm{Fun}[{\square^+}^{\mathrm{op}}, \mathcal{S}et]$
- If K is such a functor then $\partial^\varepsilon_{k,n} = K(\delta^\varepsilon_{k,n})$

Higher dimensional automata

labelled precubical sets

Modeling Concurrency with Geometry. Pratt, V. PoPL 1991.

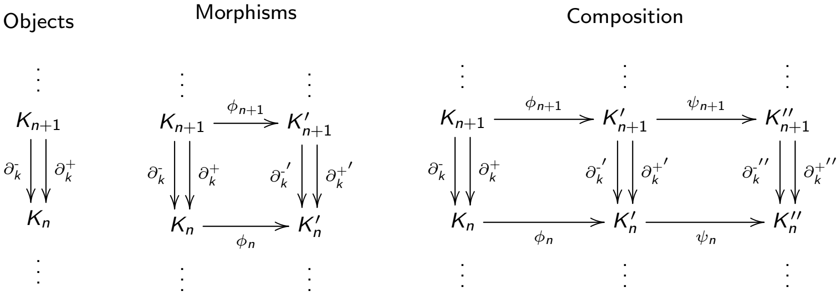
Bisimulations for Higher Dimensional Automata. van Glabbeek, R.J. Manuscript 1991.
`http://theory.stanford.edu/~rvg/hda`

Higher dimensional automata revisited. Pratt, V. Mathematical Structures in Computer Science 10(4):525-548, 2000.

Erratum to "On the Expressiveness of Higher Dimensional Automata". van Glabbeek, R.J. Theoretical Computer Science 368(1-2):168-194. 2006.

Morphisms of precubical sets from K to K'

- An $\mathbb{N}$ indexed family of mapping $\phi_n : K_n \to K'_n$ such that
- for all $n \in \mathbb{N}$, for all $k \in \{0, \dots, n-1\}$, for all $\varepsilon \in \{+, -\}$, $\partial'^{\varepsilon}_k \circ \phi_{n+1} = \phi_n \circ \partial^{\varepsilon}_k$

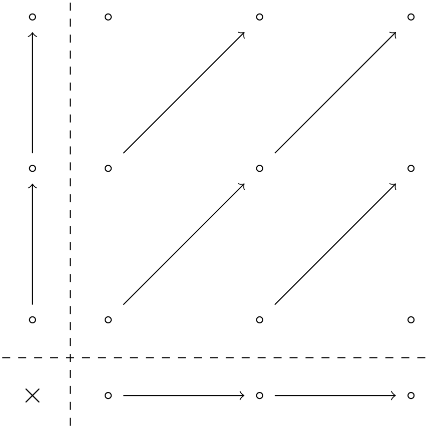


Cartesian product of precubical sets

$$\begin{pmatrix} \vdots \\ K_{n+1} \\ \partial_k^+ \downarrow\downarrow \partial_k^- \\ K_n \\ \vdots \end{pmatrix} \times \begin{pmatrix} \vdots \\ K'_{n+1} \\ \partial_k^{+\prime} \downarrow\downarrow \partial_k^{-\prime} \\ K'_n \\ \vdots \end{pmatrix} \cong \begin{pmatrix} \vdots \\ K_{n+1} \times K'_{n+1} \\ \partial_k^+ \times \partial_k^{+\prime} \downarrow\downarrow \partial_k^- \times \partial_k^{-\prime} \\ K_n \times K'_n \\ \vdots \end{pmatrix}$$

The Cartesian product in $\mathcal{pcSet}$ is deduced form the Cartesian product in $\mathcal{Set}$

Example of Cartesian product



Tensor product

Given precubical sets K and K' of dimension p and q, the set of n-cubes for $0 \leqslant n \leqslant p+q$ is

$$(K \otimes K')_n = \bigsqcup_{i+j=n} K_i \times K'_j$$

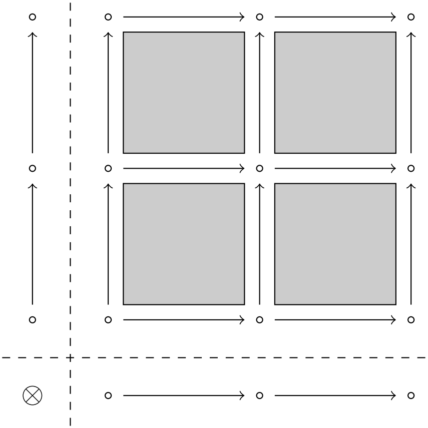
For $x \otimes y \in K_i \times K'_j$ with $i+j=n$ the k^{th} face map, with $0 \leqslant k < n$, is given by

$$\partial_k^{\pm}(x \otimes y) = \begin{cases} \partial_k^{\pm}(x) \otimes y & \text{if } 0 \leqslant k < i \\ x \otimes \partial_{k-p}^{\pm}(y) & \text{if } i \leqslant k < n \end{cases}$$

Remark that $\square^+(-,1) = \{\cdot \to \cdot\}$ and defines the standard n-cube as $\square_n^+ := \square^+(-,n)$. Then

$$\square_n^+ = \bigotimes_{i=1}^{n} \square_1^+$$

Example of tensor product



The diagram from which the $\mathcal{C}$-realization is built

K: precubical set.

Consider the diagram made of the arrows

$$\begin{array}{ccc} \{\partial_k^\varepsilon(x)\} \times [0,1]^n & \to & \{x\} \times [0,1]^{n+1} \\ \Big(\partial_k^\varepsilon(x), (t_0, \ldots, t_{n-1})\Big) & \mapsto & \Big(x, (t_0, \ldots, t_{k-1}, \varepsilon, t_k, \ldots, t_{n-1})\Big) \end{array}$$

for $n \in \mathbb{N}$, $x \in K_{n+1}$, $k \in \{0, \ldots, n-1\}$, $\varepsilon \in \{+, -\}$.

The realization of K in $\mathcal{C}$ is the colimit of the above diagram.

That construction defines a functor $\rceil\_\lfloor : \mathit{pcSet} \to \mathcal{C}$.

Describing the realization in "concrete" categories

- Start with the disjoint union (assuming that $[0,1]^0$ is a singleton)

$$\bigcup_{n\in\mathbb{N}} K_n \times [0,1]^n$$

- then for all $n \in \mathbb{N}$, all $x \in K_{n+1}$, all $k \in \{0,\dots,n-1\}$, and $\varepsilon \in \{+,-\}$, identify

$$(\partial_k^\varepsilon x, t_0, \dots, t_{n-1}) \sim (x, t_0, \dots, t_{k-1}, \varepsilon, t_k, \dots, t_{n-1})$$

- in particular for $x, y \in K_{n+1}$, the equality $\partial_k^\varepsilon x = \partial_{k'}^{\varepsilon'} y$ makes
  the k^{th} ε-face of $\{x\} \times [0,1]^{n+1}$ be identified with
  the k'^{th} ε'-face of $\{y\} \times [0,1]^{n+1}$.
  This is the way cubes are "glued" with each other.

Abstract realization in a cocomplete category $\mathcal{C}$

- Any functor $C : \square^+ \to \mathcal{C}$ can be understood as an interpretation of all the standard cubes and their face inclusions. If $\mathcal{C}$ is cocomplete one has the corresponding realization.
- One has $|K \otimes K'| \cong |K| \times |K'|$ when the following are satisfied
  - for all $n \in \mathbb{N}$, $|\square_n^+| \cong |\square_1^+|^n$
  - for all $n \in \mathbb{N}$, $|\square_n^+|$ is exponentiable
  - at least one of the realizations $|K|$ and $|K'|$ is exponentiable

Examples of realizations

- $C(n) = \{n\}$ in $\mathcal{Set}$, $\mathcal{Top}$, $\mathcal{Haus}$, or $\mathcal{Ke}$: connected components
- $C(n) = \square_n^+$ in $\mathcal{pcSet}$: $|K| = K$
- $C(n)$ the underlying graph of $\square_n^+$ in $\mathcal{Grph}$: the underlying graph of K. NB: $|\square_n^+|_{\mathcal{Grph}} \ncong |\square_1^+|^n_{\mathcal{Grph}}$
- $C(n) = [0,1]^n$ in $\mathcal{Set}$, $\mathcal{Top}$, $\mathcal{Haus}$, and $\mathcal{Ke}$
    - $|K \otimes K'| \cong |K| \times |K'|$ in $\mathcal{Set}$ and $\mathcal{Ke}$ for all K and K'
    - $|K \otimes K'| \cong |K| \times |K'|$ in $\mathcal{Top}$ and $\mathcal{Haus}$ for all K and K' with K or K' finite. The finiteness of K or K' implies that the corresponding realization is compact, hence exponentiable

Some standard topological spaces

- compact unit segment, circle, cylinder, and torus
- cube, sphere
- Mobius band, Klein bottle
- projective plane

The singular precubical set of an object X

- We are given $C : \square^+ \to \mathcal{C}$, the singular cubical set functor $Sing : \mathcal{C} \to \mathit{pcSet}$ is defined as follows:
  - $Sing(X) := \left(\cdots \ \mathcal{C}(C(n), X) \xleftarrow{C(\delta_k^\varepsilon)\circ -} \mathcal{C}(C(n+1), X) \ \cdots \right)$
  - $Sing(\ X \xrightarrow{f} Y\ ) := \left(\cdots \ \mathcal{C}(C(n), X) \xrightarrow{-\circ f} \mathcal{C}(C(n), Y) \ \cdots \right)$
- Then we have $|\_| \dashv Sing$

Conjectures

Any precubical set K can be realized in $\mathcal{Lpo}$, and

- $U(\rceil K \lfloor_{\mathcal{Lpo}}) \quad = \quad |K|_{\mathcal{Top}}$
- $I(\rceil K \lfloor_{\mathcal{Lpo}}) \quad \cong \quad \rceil K \lfloor_{d\mathcal{Top}}$
- $\rceil K \lfloor_{d\mathcal{Top}} \quad \cong \quad \rceil K \lfloor_{d\mathcal{Top}_{\mathsf{f}}}$

The category $\square$ of face inclusions and projections

- For $n \in \mathbb{N}$, $k \in \{0, \ldots, n-1\}$, $\varepsilon \in \{+, -\}$, we have the face inclusion map

$$\begin{array}{rcl} [0,1]^n & \to & [0,1]^{n+1} \\ (t_0, \ldots, t_{n-1}) & \mapsto & (t_0, \ldots, t_{k-1}, \varepsilon, t_k, \ldots, t_{n-1}) \end{array}$$

- For $n \in \mathbb{N}$, $k \in \{0, \ldots, n\}$, we have the projection map

$$\begin{array}{rcl} [0,1]^{n+1} & \to & [0,1]^n \\ (t_0, \ldots, t_n) & \mapsto & (t_0, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n) \end{array}$$

- Then $\square$ is the subcategory of $\mathcal{S}et$ generated by all the face inclusions and projections.
- Cubical sets are presheaves over $\square$ i.e. $pc\mathcal{S}et = \mathcal{S}et^{\square^{\mathrm{op}}}$

A description of the $\square$ category

- $\{\text{Objects of } \square\} = \mathbb{N}$
- $\square[n, m]$ is the set of ordered pairs (n, w) such that $n \in \mathbb{N}$ and w is a word of length m on $\{0, 1, x_0, \ldots, x_{n-1}\}$ such that:
  - every variable occurs at most once in w, and
  - if $i < j$ and both x_i and x_j occur in w, then x_i occurs before x_j.
- $\mathrm{id}_n = \big(n, (x_0, \ldots, x_{n-1})\big)$
- $\delta^{\varepsilon}_{i,n} = \big(n, (x_0 \cdots x_{i-1} \varepsilon x_i \cdots x_{n-1})\big)$ and $\sigma_{i,n} = \big(n+1, (x_0 \cdots x_{i-1} x_{i+1} \cdots x_n)\big)$
- if $w : a \to b$ and $w' : b \to c$ then $w'w$ is obtained by replacing, for $k \in \{0, \ldots, b-1\}$, the occurrence of x_k in w' (if any) by the k^{th} letter of w.

The diagram from which the $\mathcal{C}$-realization is built

K: cubical set.

Consider the diagram made of the arrows

$$\begin{array}{rcl} \{\partial^{\varepsilon}_{k,n}(x)\} \times [0,1]^n & \to & \{x\} \times [0,1]^{n+1} \\ \big(\partial^{\varepsilon}_{k,n}(x), (t_0, \ldots, t_{n-1})\big) & \mapsto & \big(x, (t_0, \ldots, t_{k-1}, \varepsilon, t_k, \ldots, t_{n-1})\big) \end{array}$$

for $n \in \mathbb{N}$, $x \in K_{n+1}$, $k \in \{0, \ldots, n-1\}$, $\varepsilon \in \{+,-\}$ and

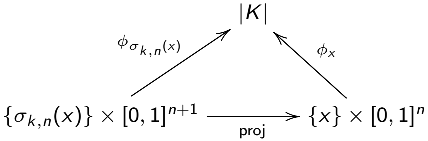
$$\begin{array}{rcl} \{\sigma_{k,n}(x)\} \times [0,1]^{n+1} & \to & \{x\} \times [0,1]^n \\ \big(\sigma_{k,n}(x), (t_0, \ldots, t_n)\big) & \mapsto & \big(x, (t_0, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)\big) \end{array}$$

for $n \in \mathbb{N}$, $x \in K_n$, $k \in \{0, \ldots, n\}$.

The realization of K in $\mathcal{C}$ is the colimit of the above diagram.

That construction defines a functor $\lceil \_ \rfloor : c\mathcal{S}et \to \mathcal{C}$.

Effect of projections on the “bricks” of the realization



Hence $\phi_{\sigma_{k,n}(x)}(t_0, \ldots, t_n)$ does not depend on t_k.

Describing the realization in “concrete” categories

- Start with the disjoint union (assuming that $[0,1]^0$ is a singleton)

$$\bigcup_{n\in\mathbb{N}} K_n \times [0,1]^n$$

- then for all $n \in \mathbb{N}$, all $x \in K_{n+1}$, all $k \in \{0,\dots,n-1\}$, and $\varepsilon \in \{+,-\}$, identify

$$(\partial_k^\varepsilon x, t_0, \dots, t_{n-1}) \sim (x, t_0, \dots, t_{k-1}, \varepsilon, t_k, \dots, t_{n-1})$$

  and for all $n \in \mathbb{N}$, all $x \in K_n$ and all $k \in \{0,\dots,n\}$, identify

$$(\sigma_k x, t_0, \dots, t_n) \sim (x, t_0, \dots, t_{k-1}, t_{k-1}, \dots, t_n)$$

About directed realization of cubical sets

- Some cubical set K cannot be realized in $\mathcal{L}po$,
- $D(\rceil K \lfloor_{\mathcal{S}tr}) \cong \rceil K \lfloor_{d\mathcal{T}op_\mathsf{f}}$
- $S(\rceil K \lfloor_{d\mathcal{T}op}) \cong \rceil K \lfloor_{\mathcal{S}tr_\mathsf{d}}$
- $\rceil K \lfloor_{\mathcal{S}tr} \cong \rceil K \lfloor_{\mathcal{S}tr_\mathsf{d}}$
- $\rceil K \lfloor_{d\mathcal{T}op}$ and $\rceil K \lfloor_{d\mathcal{T}op_\mathsf{f}}$ may differ.

Cartesian product of cubical sets

$$
\left(
\begin{array}{c}
\vdots \\
K_{n+1} \\
\partial_k^+ \downarrow \; \uparrow \sigma_k \; \downarrow \partial_k^- \\
K_n \\
\vdots
\end{array}
\right)
\times
\left(
\begin{array}{c}
\vdots \\
K'_{n+1} \\
\partial_k^{+\prime} \downarrow \; \uparrow \sigma'_k \; \downarrow \partial_k^{-\prime} \\
K'_n \\
\vdots
\end{array}
\right)
\cong
\left(
\begin{array}{c}
\vdots \\
K_{n+1} \times K'_{n+1} \\
\partial_k^+ \times \partial_k^{+\prime} \downarrow \; \uparrow \sigma_k \times \sigma'_k \; \downarrow \partial_k^- \times \partial_k^{-\prime} \\
K_n \times K'_n \\
\vdots
\end{array}
\right)
$$

The Cartesian product in $pc\mathcal{S}et$ is deduced form the Cartesian product in $\mathcal{S}et$

Cartesian product of two segments in *cSet*

Compute the product $\square_1 \times \square_1$

Tensor product

Given the cubical sets K and K', the set of n-cubes is

$$(K \otimes K')_n = \left(\coprod_{i+j=n} K_i \times K'_j \right) / \sim$$

For $x \otimes y \in K_i \times K'_j$ with $i + j = n$ the k^{th} face map, with $0 \leqslant k < n$, is given by

$$\partial_k^{\pm}(x \otimes y) = \begin{cases} \partial_k^{\pm}(x) \otimes y & \text{if } 0 \leqslant k < i \\ x \otimes \partial_{k-i}^{\pm}(y) & \text{if } i \leqslant k < n \end{cases}$$

For $x \otimes y \in K_i \times K'_j$ with $i + j = n$ the k^{th} degeneracy map, with $0 \leqslant k \leqslant n$, is given by

$$\varepsilon_k(x \otimes y) = \begin{cases} \varepsilon_k(x) \otimes y & \text{if } 0 \leqslant k \leqslant i \\ x \otimes \varepsilon_{k-i}(y) & \text{if } i \leqslant k < n \end{cases}$$

with $\sim$ generated by $\varepsilon_i(x) \otimes y \quad \sim \quad x \otimes \varepsilon_0(y)$.
The "segment" is $\square(-, 1)$ and the standard n-cube is $\square_n := \square(-, n)$. We have

$$\square_n = \bigotimes_{i=1}^{n} \square_1$$

The singular cubical set of an object X

- We are given $C : \square \to \mathcal{C}$, the singular cubical set functor $Sing : \mathcal{C} \to c\mathcal{S}et$ is defined as follows:
  - $Sing(X) := \left(\cdots \ \mathcal{C}(C(n), X) \underset{C(\sigma_k)\circ -}{\overset{C(\delta_k^{\varepsilon})\circ -}{\leftrightarrows}} \mathcal{C}(C(n+1), X) \ \cdots \right)$
  - $Sing(\ X \xrightarrow{f} Y \ ) := \left(\cdots \ \mathcal{C}(C(n), X) \xrightarrow{- \circ f} \mathcal{C}(C(n), Y) \ \cdots \right)$
- Then we have $|\_| \dashv Sing$

Theorem

Nonabelian Algebraic Topology, Brown, R., Higgins, P. J., and Sivera R., EMS, 2011. Proposition 11.1.17, p.372

For any "topological space" X, the counit at X

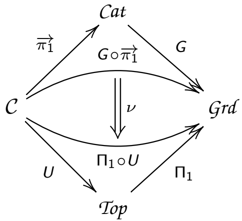
$$\varepsilon_X : |Sing\ X| \to X$$

of the adjunction $\text{“}\mathcal{T}\!op\text{”} \underset{|-|}{\overset{\text{Sing}}{\rightleftarrows}} c\mathcal{S}et$ is a weak homotopy equivalence.

("topological space" maybe mean compactly generated space here.)

The canonical natural transformation $G \circ \overrightarrow{\pi_1} \to \Pi_1 \circ U$

- If X is undirected (i.e. all its paths are directed) then $\overrightarrow{\pi_1}X$ is actually $\Pi_1 \circ U(X)$ the fundamental groupoid of UX
- Denote by $G : \mathcal{Cat} \to \mathcal{Grd}$ the left adjoint to the inclusion functor $\mathcal{Grd} \hookrightarrow \mathcal{Cat}$
- Let X be an object of $\mathcal{C}$
- $\Pi_1 \circ U(X)$ is the fundamental groupoid of the underlying space of X
- $G \circ \overrightarrow{\pi_1}X$ is the enveloping groupoid of the fundamental category of X
- There exists a unique functor $\nu_X : G \circ \overrightarrow{\pi_1}(X) \to \Pi_1 \circ U(X)$,



we would like to known when it is an isomorphism.

Examples

The functor ν_X is an isomorphism when

- all the paths on UX are directed since $\overrightarrow{\pi_1}(X) = \Pi_1(X)$
- XU is totally disconnected since both $G \circ \overrightarrow{\pi_1}(X)$ and $\Pi_1 \circ U(X)$ are discrete
- X is the directed circle: just replace $\mathbb{N}$ by $\mathbb{Z}$ in the description of $\overrightarrow{\pi_1}\mathbb{S}^1$
- X is the directed complex plane or Riemann sphere: precompose by α_x

The functor ν_X is an not isomorphism when

- the direction on X is discrete while UX has a non constant path

Conjecture: η_X is an isomorphism when $X \cong \rceil K \lfloor_{\mathcal{L}po}$ for some precubical set K.

Conjecture: η_X is an isomorphism when X is an isothetic region.