Directed Algebraic Topology and Concurrency

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Commutative monoids

- $(M, *, \varepsilon)$ such that for all $a, b, c \in M$,
  - $(ab)c = a(bc)$
  - $\varepsilon a = a = a\varepsilon$
  - $ab = ba$
- For all set X the collection MX of multisets over X i.e. maps $\phi : X \to \mathbb{N}$ s.t. $\{x \in X \mid \phi(x) \neq 0\}$ is finite forms a commutative monoid with pointwise addition
- A commutative monoid is said to be free when it is isomorphic with some MX
- Functor $M : \mathcal{Set} \to \mathcal{Cmon}$
  - A multiset ϕ can be written as

$$\sum_{x \in X} \phi(x)x$$

  - In particular, if $f : X \to Y$ is a set map, then

$$M(f)(\phi) = \sum_{x \in X} \phi(x)f(x)$$

Prime *vs* irreducible

- d divides x, denoted by $d|x$, when there exists x' such that $x = dx'$
- u unit: exists u' s.t. $uu' = \varepsilon$ then write $x \sim y$ when $y = ux$ for some unit u
- i irreducible: i nonunit and $x|i$ implies $x \sim i$ or x unit
- p prime: p nonunit and $p|ab$ implies $p|a$ or $p|b$
- If M contains nontrivial units, then one can consider the quotient monoid $M/\sim$ where $x \sim y$ stands for: there exists a unit u s.t. $y = ux$

Examples

monoid	irreducibles	primes	units
$\mathbb{N} \setminus \{0\}, \times, 1$	{prime numbers}		$\{1\}$
$\mathbb{N}, +, 0$	$\{1\}$		$\{0\}$
$\mathbb{R}_+, +, 0$	$\emptyset$		$\{0\}$
$\mathbb{R}_+, \vee, 0$	$\emptyset$	$\mathbb{R}_+ \setminus \{0\}$	$\{0\}$
$\mathbb{Z}_6, \times, 1$	$\emptyset$	$\{0, 2, 3, 4\}$	$\{1, 5\}$

Graded commutative monoid

- $(M, *, \varepsilon)$ graded: there is a morphism $g : (M, *, \varepsilon) \to (\mathbb{N}, +, 0)$
  s.t. $g^{-1}(\{0\}) = \{\text{units of } M\}$
- If M is graded then
  - $\{\text{irreducibles of } M\}$ generates M
  - $\{\text{primes of } M\} \subseteq \{\text{irreducibles of } M\}$

Irreducible that are not prime

$M = (\{a + b\sqrt{10} \mid a, b \in \mathbb{Z}; a \neq 0 \text{ or } b \neq 0\}, \times, 1)$

- $N : M \to (\mathbb{Z} \setminus \{0\}, \times, 1)$; $N(a + b\sqrt{10}) = a^2 - 10b^2$
  $N(uv) = N(u)N(v)$
  u unit iff $N(u) \in \{\pm 1\}$
  $N(a + b\sqrt{10}) \bmod 10 \in \{1, 4, 5, 6, 9\}$
  therefore $N(a + b\sqrt{10}) \notin \{\pm 2, \pm 3\}$

uv	N(uv)	N(u)
2	4	$\pm 1, \pm 2, \pm 4$
3	9	$\pm 1, \pm 3, \pm 9$
$4 \pm \sqrt{10}$	6	$\pm 1, \pm 2, \pm 3, \pm 6$

- 2, 3, and $4 \pm \sqrt{10}$ are irreducible but not prime
  since $2 \cdot 3 = (4 + \sqrt{10}) \cdot (4 - \sqrt{10})$
- $\{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\} \setminus \{0\}$ is graded by the number of prime factors of $N(u)$

$\mathbb{N}[X]$ polynomials with coefficients in $\mathbb{N}$

On Direct Product Decomposition of Partially Ordered Sets. Junji Hashimoto

Annals of Mathematics **2**(54), pp 315-318 (1951)

$X^5 + X^4 + X^3 + X^2 + X + 1 =$

$$\begin{cases} (X+1)(X^4+X^2+1) = (X^3+1)(X^2+X+1) & \text{in } \mathbb{N}[X] \\ (X+1)(X^2+X+1)(X^2-X+1) & \text{in } \mathbb{Z}[X] \end{cases}$$

- therefore $X+1$, X^2+X+1, X^3+1, and X^4+X^2+1 are irreducible but not prime
- $\mathbb{N}[X] \setminus \{0\}$ is graded by the degree

Characterization of the free commutative monoids

Unique factorization

- The following are equivalent:
  - M is free commutative
  - any element of M can be written as a product of irreducibles in a unique way up to reordering
  - $\{\text{primes of } M\} = \{\text{irreducibles of } M\}$ and generates M
  - M is graded and $\{\text{irreducibles of } M\} \subseteq \{\text{primes of } M\}$
- Standard examples:
  - $(\mathbb{N} \setminus \{0\}, \times, 1)$
  - $(\mathbb{N}, +, 0)$ and its finite products in the category of commutative monoids. Indeed $(\mathbb{N}, +, 0)^n \cong M(\{1, \ldots, n\})$
  - $(\mathbb{Z}[X] \setminus \{0\}, \times, 1)$ (if F is a factorial ring, then so is $F[X]$)
    Algebra, Serge Lang. Springer (2002)
  - Note that two free commutative monoids are isomorphic in $\mathcal{Cmon}$ iff their set of prime elements have the same cardinality
    e.g. $(\mathbb{N} \setminus \{0\}, \times, 1) \cong (\mathbb{Z}[X] \setminus \{0\}, \times, 1)$ in $\mathcal{Cmon}$

Connected sum of manifolds

A less common example

In differential geometry, the compact, connected, oriented, smooth n-dimensional manifolds without boundary equipped with the connected sum $\#$ form a commutative monoid $\mathcal{M}_n$ whose neutral element is the n-sphere.

tom Dieck, T. Algebraic Topology. European Mathematical Society 2008. p.390

$\mathcal{M}_2$ is freely generated by the torus T^2.

Massey, W.S. A Basic Course in Algebraic Topology. Springer 1991. Chapter 1.

$\mathcal{M}_3$ is freely generated by countably many elements.

Hempel, J. 3-Manifolds. American Mathematical Society 1976. Chapter 3.

Jaco, W. Lectures on Three-Manifold Topology. American Mathematical Society 1980. Chapter 2.

- existence of the decomposition is due to Hellmuth Kneser (1929)
  Kneser, H. Geschlossene Flchen in dreidimensionalen Mannigfaltigkeiten.
  Jahresbericht der Deutschen Mathematiker-Vereinigung 38:248259 1929.
- uniqueness of the decomposition is due to John W. Milnor (1962)
  Milnor, J. A Unique Decomposition Theorem for 3-Manifolds.
  American Journal of Mathematics 84(1):17 1962.

In particular $\mathcal{M}_2 \cong (\mathbb{N}, +, 0)$ and $\mathcal{M}_3 \cong (\mathbb{N} \setminus \{0\}, \times, 1)$

The noncommutative monoid of languages

- $\mathbb{A}^*$ (non commutative) monoid of words on the alphabet $\mathbb{A}$.
  Let ε denotes the empty word
- A language is a set of words on $\mathbb{A}$. Let D and D' be languages
  - define $D \cdot D' := \{w \cdot w' \mid w \in D; w' \in D'\}$
  - one has $\emptyset \cdot D = D \cdot \emptyset = \emptyset$ and $\{\varepsilon\} \cdot D = D \cdot \{\varepsilon\} = D$
  - The monoid of nonempty languages is $\mathcal{D}(\mathbb{A})$
  - $\mathcal{D}(\mathbb{A})$ is commutative iff $\mathrm{Card}(\mathbb{A}) \leqslant 1$. Note that $\mathcal{D}(\emptyset) \cong \{0\}$
  - however $\mathcal{D}(\{a\})$ is not free commutative

The noncommutative monoid of homogeneous languages

- $H \in \mathcal{D}(\mathbb{A})$ is homogeneous when all the words in H have the same length
- Define $\dim(H)$ as the length common to all the words of H.
  It is well defined since H is nonempty.
- $H \cdot H' = \{w \cdot w' \mid w \in H \ ; \ w' \in H'\}$ is homogeneous iff so are H and H'
- $\mathcal{D}_h(\mathbb{A}) \subseteq \mathcal{D}(\mathbb{A})$ the submonoid of homogeneous languages.
- $H \in \mathcal{D}_h(\mathbb{A}) \mapsto \dim(H) \in (\mathbb{N}, +, 0)$ is a morphism of monoid
- $\dim(H) = 0$ iff $H = \{\varepsilon\}$
- $\mathcal{D}_h(\mathbb{A})$ is commutative iff $\mathrm{Card}(\mathbb{A}) \leqslant 1$
- $\mathcal{D}_h(\{a\}) \cong (\mathbb{N}, +, 0)$

Action of the symmetric groups

on the left of the homogeneous languages

- The n^{th} symmetric group $\mathfrak{S}_n$ acts on the left of the set of words of length n i.e. mappings from $\{1, \ldots, n\}$ to $\mathbb{A}$, by $\sigma \cdot \omega := \omega \circ \sigma^{-1}$
- Then $\mathfrak{S}_n$ acts on the left of the homogeneous languages of dimension n
- Write $H \sim H'$ when $\dim(H) = \dim(H')$ and $H' = \sigma \cdot H$ for some $\sigma \in \mathfrak{S}_{\dim(H)}$
- If $\sigma \in \mathfrak{S}_n$ and $\sigma' \in \mathfrak{S}_{n'}$ then define $\sigma \otimes \sigma' \in \mathfrak{S}_{n+n'}$ as:

$$\sigma \otimes \sigma'(k) := \begin{cases} \sigma(k) & \text{if} \quad 1 \leqslant k \leqslant n \\ (\sigma'(k-n)) + n & \text{if} \quad n+1 \leqslant k \leqslant n+n' \end{cases}$$

- A Godement exchange law is satisfied, which ensures that $\sim$ is actually a congruence:

$$(\sigma \cdot H) \cdot (\sigma' \cdot H') = (\sigma \otimes \sigma') \cdot (H \cdot H')$$

i.e. $H \sim K$ and $H' \sim K'$ implies $HH' \sim KK'$

The commutative monoid of homogeneous languages

- The commutative monoid of homogeneous languages is $\mathcal{H}(\mathbb{A}) = (\mathcal{D}_h(\mathbb{A}), \cdot, \{\varepsilon\})/\sim$
- The monoid $\mathcal{H}(\mathbb{A})$ is graded by $H \in \mathcal{H}(\mathbb{A}) \mapsto \dim(H) \in (\mathbb{N}, +, 0)$

The commutative monoid $\mathcal{H}(\mathbb{A})$ is free

- For any homogeneous language H and $\sigma \in \mathfrak{S}_{\dim(H)}$, $\mathrm{card}(H) = \mathrm{card}(\sigma \cdot H)$ so we can define the cardinality of any element of $\mathcal{H}(\mathbb{A})$

The commutative monoid of finite homogeneous languages

- $M' \subseteq M$ is said to be pure when for all $x, y \in M$, $xy \in M'$ implies $x, y, \in M'$
- A pure submonoid of a free commutative monoid is free
- The submonoid $\mathcal{H}_f(\mathbb{A}) \subseteq \mathcal{H}(\mathbb{A})$ of finite languages is pure, therefore it is free
- $H \in \mathcal{H}_f(\mathbb{A}) \mapsto \mathrm{Card}(H) \in (\mathbb{N} \setminus \{0\}, \times, 1)$ is a morphism of monoid
- The primality of Card(H) does not imply that of H
  e.g. $H = \{ab, ac\} = \{a\} \cdot \{b, c\}$ though card(H) $= 2$
- The primality of H does not imply that of Card(H)
  e.g. $H = \{a, b, c, d\}$ is prime though card(H) $= 4$

The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

Theory

Given $w \in \mathbb{A}^n$ and $I \subseteq \{1, \ldots, n\}$, we write $w_{|_I}$ for the subword of w consisting of letters with indices in I.

Given a homogeneous language H of dimension n, we write

$$H_{|_I} \quad = \quad \{w_{|_I} \mid w \in H\}$$

Denoting I^c for $\{1, \ldots, n\} \setminus I$, we have

$$[H] \quad = \quad [H_{|_I}] \cdot [H_{|_{I^c}}]$$

in $\mathcal{H}_f(\mathbb{A})$ if and only if for all words $u, v \in H$ there exists a word $w \in H$ such that

$$w_{|_I} \quad = \quad u_{|_I} \qquad \text{and} \qquad w_{|_{I^c}} \quad = \quad v_{|_{I^c}}$$

The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

Practice

For $I \subseteq \{1,\ldots,n\}$ let $\pi_{|_I}$ be the "projection" that sends $w \in H$ to $w_{|_I} \in \mathbb{A}^{\mathrm{card(I)}}$.

1. choose $I \subseteq \{1,\ldots,n\}$ of cardinality $k \leqslant n/2$
2. if $\pi_{|_{I^c}}(\pi^{-1}_{|_I}(u))$ does not depend on $u \in H_{|_I}$, then we have the factorization

   $$[H] \quad = \quad [H_{|_I}] \cdot [H_{|_{I^c}}]$$

   and we are done
3. otherwise check whether there are still subsets of $\{1,\ldots,n\}$ to check:
   3.1. yes: go to step 1
   3.2. no: $[H]$ is prime

The preorder $\preccurlyeq$ over $\mathcal{H}(\mathbb{A})$

inherited from a preorder $\preccurlyeq$ over $\mathbb{A}$

- Let $\preccurlyeq^n$ by the product preorder on the words of length n
- Given $H, H' \in \mathcal{D}_h(\mathbb{A})$ of the same dimension n, write $H \preccurlyeq H'$ when for all $\omega \in H$ there exists $\omega' \in H'$ such that $\omega \preccurlyeq^n \omega'$
- Given $X, X' \in \mathcal{H}(\mathbb{A})$ of the same dimension n write $X \preccurlyeq X'$ when there exist $H \in X$ and $H' \in X'$ such that $H \preccurlyeq H'$
- $X \preccurlyeq Y$ and $X' \preccurlyeq Y'$ implies $X \cdot X' \preccurlyeq Y \cdot Y'$
  i.e. $(\mathcal{H}(\mathbb{A}), \preccurlyeq)$ is a preordered commutative monoid
- If $\preccurlyeq$ is actually a partial order on $\mathbb{A}$, then so is $\preccurlyeq$ on $\mathcal{H}(\mathbb{A})$
- If $\preccurlyeq$ is the equality relation, then $X \preccurlyeq Y$ amounts to $H_X \subseteq H_Y$ for some representatives H_X and H_Y of X and Y.

Homogeneous languages

over the alphabets $\rceil G \lfloor$ and $\mathcal{R}_1 G \setminus \{\emptyset\}$ with G being a finite graph

- $\mathbb{A} = \rceil G \lfloor$ is the geometric realization of a finite graph:
  - a point of $\rceil G \lfloor^n$ can be seen as a word of length n on $\mathbb{A}$
  - a nonempty subset of $\rceil G \lfloor^n$ is thus a homogeneous language on $\mathbb{A}$
  - the product of the monoid $\mathcal{D}_h(\mathbb{A})$ corresponds to the cartesian product of isothetic regions
- $\mathbb{A} = \mathcal{R}_1 G \setminus \{\emptyset\}$ is the collection of nonempty finite unions of connected subsets of $\rceil G \lfloor$:
  - an n-block is an n-fold product of nonempty elements of $\mathcal{R}_1 G$ i.e. a word of length n on $\mathbb{A}$
  - a nonempty family of n-blocks is thus an homogeneous language on $\mathbb{A}$ (of dimension n)
  - the concatenation of words on $\mathbb{A}$ corresponds to the cartesian product of blocks

The canonical morphism of monoids $\gamma : \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\}) \to \mathcal{H}(\upharpoonleft G \downharpoonright)$

- Let γ be the map sending an homogeneous language on $\mathcal{R}_1 G \setminus \{\emptyset\}$ to the union of its elements
  - γ is a morphism of monoids from $\mathcal{D}_h(\mathcal{R}_1 G \setminus \{\emptyset\})$ to $\mathcal{D}_h(\upharpoonleft G \downharpoonright)$
  - γ is compatible with the action of the symmetric groups in the sense that $H' = \sigma \cdot H \;\Rightarrow\; \bigcup H' = \sigma \cdot (\bigcup H)$
  - γ induces a morphism of monoids from $\mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$ to $\mathcal{H}(\upharpoonleft G \downharpoonright)$
- The induced morphism γ does not preserve the prime elements e.g. consider a covering of $[0,1]^2$ with 3 disctinct rectangles

The canonical morphism of monoids $\alpha : \mathcal{H}(\upharpoonleft G \downharpoonleft) \to \mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$

- Define $\alpha(X)$ as the collection of maximal blocks of X:
  - given $X, Y \subseteq \upharpoonleft G \downharpoonleft^n$, the collection of maximal blocks of $X \times Y$ is $\{C \times D \mid C \text{ and } D \text{ are maximal blocks of } X \text{ and } Y\}$
  - the unique maximal block of the unique nonempty subset of $\upharpoonleft G \downharpoonleft^0$ is ε
  - α is a morphism of monoids form $\mathcal{D}_h(\upharpoonleft G \downharpoonleft)$ to $\mathcal{D}_h(\mathcal{R}_1 G \setminus \{\emptyset\})$
  - if C is a maximal block of $X \subseteq \upharpoonleft G \downharpoonleft^n$ then $\sigma \cdot C$ is a maximal block of $\sigma \cdot X$.
  - α induces a morphism of monoids from $\mathcal{H}(\upharpoonleft G \downharpoonleft)$ to $\mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$
  - $\mathrm{im}(\alpha)$ is a submonoid of $\mathcal{H}(\mathcal{R}_1 G \setminus \{\emptyset\})$
- the morphisms γ and α induce isomorphisms of ordered monoids between $\mathrm{im}(\alpha)$ and $\mathcal{H}(\upharpoonleft G \downharpoonleft)$, the order relation being inherited from inclusion over $\mathcal{R}_1 G \setminus \{\emptyset\}$ and equality over $\upharpoonleft G \downharpoonleft$.
- therefore $\mathrm{im}(\alpha)$ is commutative free

The free commutative monoids of isothetic regions

- By definition, an isothetic region is a finite union of blocks of $X \subseteq \upharpoonleft G \downharpoonright^n$.
- We have seen that an isothetic region has finitely many maximal blocks .
- For $X, Y \in \mathcal{H}(\upharpoonleft G \downharpoonright)$, $\alpha(X \cdot Y)$ is finite iff $\alpha(X)$ and $\alpha(Y)$ are so:
  - then $\{X \in \text{im}(\alpha) \mid \text{card}(X) \text{ is finite}\}$ is a pure submonoid of $\text{im}(\alpha)$
  - this commutative monoid is thus free and isomorphic to the monoid of isothetic regions, the latter being defined as

$$\boxed{\gamma(\{X \in \text{im}(\alpha) \mid \text{card}(X) \text{ is finite}\})}$$

  - The monoid of isothetic regions is thus free commutative.

A better factoring algorithm

by Nicolas Ninin

Let $X \subseteq |G|^n$ be an isothetic region and $\mathcal{F}$ be a finite block covering of X^c

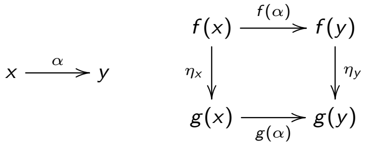
- For each $\omega_1 \times \cdots \times \omega_n \in \mathcal{F}$ define the word $(\beta_1, \ldots, \beta_n)$ over $\{0, 1\}$ as follows:
  for all $k \in \{1, \ldots, n\}$
  - $\beta_k = 0$ if $\omega_k = |G|$;
  - $\beta_k = 1$ otherwise.
- Denote by $\sim$ the equivalence relation generated by $\beta \sim \beta'$ when $\beta \wedge \beta'$ is not identically null.
- For each $\sim$-equivalence class $\mathcal{E}$ denotes $\bigvee \mathcal{E}$ by $\beta_{\mathcal{E}}$
- Then the subsets $\beta_{\mathcal{E}}^{-1}(\{1\})$ with $\mathcal{E}$ ranging through the $\sim$-classes provides a partition of $\{1, \ldots, n\}$ which is a factorization of X.

if $\mathcal{F} = \alpha(X^c)$ then we obtain the prime factorization of X

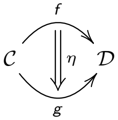
Natural Transformations

morphisms of functors from $f : \mathcal{C} \to \mathcal{D}$ to $g : \mathcal{C} \to \mathcal{D}$

A natural transformation $\eta : f \to g$ is a collection of morphisms $(\eta_x)_{x \in \mathrm{Ob}(\mathcal{C})}$ where $\eta_x \in \mathcal{D}[f(x), g(x)]$ and such that for all $\alpha \in \mathcal{C}[x, y]$ we have $\eta_y \circ f(\alpha) = g(\alpha) \circ \eta_x$ i.e. the following diagram commute



This description is summarized by the following diagram

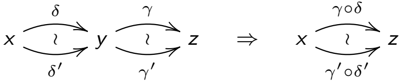


Congruences on small categories

A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\mathrm{Mo}(\mathcal{C})$ such that:

- $\gamma \sim \gamma' \quad \Rightarrow \quad \partial^{-}\gamma = \partial^{-}\gamma'$ and $\partial^{+}\gamma = \partial^{+}\gamma'$
- $\gamma \sim \gamma'$, $\delta \sim \delta'$ and $\partial^{-}\gamma = \partial^{+}\delta \quad \Rightarrow \quad \gamma \circ \delta \sim \gamma' \circ \delta'$

In diagrams we have



Hence the $\sim$-equivalence class of $\gamma \circ \delta$ does not depend on the $\sim$-equivalence classes of γ and δ and we have a quotient category $\mathcal{C}/\sim$ in which the composition is given by

$$[\gamma] \circ [\delta] \quad = \quad [\gamma \circ \delta]$$

Moreover the set-theoretic quotient map $q : \gamma \in \mathrm{Mo}(\mathcal{C}) \mapsto [\gamma] \in \mathrm{Mo}(\mathcal{C})/\sim$ induces a functor $q : \mathcal{C} \to \mathcal{C}/\sim$

Natural congruences

Let $P : \mathcal{C} \to \mathcal{C}at$ be a functor

- Suppose we are given, for each object X of $\mathcal{C}$, a congruence $\sim_X$ on PX
- Suppose that for all morphisms $f : X \to Y$ of $\mathcal{C}$, for all $\alpha, \beta \in PX$,
  if $\alpha \sim_X \beta$, then $P(f)(\alpha) \sim_Y P(f)(\beta)$
- Then we can define the functor $\overrightarrow{\pi_1}{:}\mathcal{C} \to \mathcal{C}at$ as follows:
  - for all $X \in \mathcal{C}$, $\pi_1(X) = P(X)/\sim_X$
  - for all $f : X \to Y$ in $\mathcal{C}$

$$X \xrightarrow{f} Y$$

$$\begin{array}{ccc} PX & \xrightarrow{Pf} & PY \\ {\scriptstyle q_Y}\downarrow & & \downarrow{\scriptstyle q_y} \\ \overrightarrow{\pi_1}X & \xrightarrow[\overrightarrow{\pi_1}f]{} & \overrightarrow{\pi_1}Y \end{array}$$

- The collection of quotient functors q_X, for X ranging through the objects of $\mathcal{C}$, provides a natural transformation from P to $\overrightarrow{\pi_1}$
- The collection $\sim_X$ is called a natural congruence.

The category of directed Moore paths of a local pospace

Let X be a locally ordered space.

- The objects of $\mathcal{M}X$ are the points of X.
- The homset $\mathcal{M}X(a,b)$ is

$$\bigcup_{r \geqslant 0} \{\gamma \in \mathcal{L}po([0,r],X) \mid \gamma(0) = a \text{ and } \gamma(r) = b\}$$

- For $\delta : [0,r] \to X$ and $\gamma : [0,r'] \to X$ with $\delta(r) = \gamma(0)$, define the concatenation

$$\begin{aligned} \gamma \cdot \delta : \; [0, r+r'] & \longrightarrow X \\ t & \longmapsto \begin{cases} \delta(t) & \text{if } t \leqslant r \\ \gamma(t-r) & \text{if } t \geqslant r \end{cases} \end{aligned}$$

The directed Moore path functor over $\mathcal{Lpo}$

The (Moore) path category construction gives rise to a functor $\mathcal{M}$ from $\mathcal{Lpo}$ to $\mathcal{Cat}$ since for all $f \in \mathcal{Lpo}(X, Y)$ and all paths γ on X, the composite $f \circ \gamma$ is a path on Y.

$$\mathcal{M} : \mathcal{Lpo} \longrightarrow \mathcal{Cat}$$

$$\begin{array}{ccc} X & & \mathcal{M}X \\ \downarrow f & \longmapsto & \downarrow \mathcal{M}f \\ Y & & \mathcal{M}Y \end{array}$$

with

$$\mathcal{M}f : \mathcal{M}X \longrightarrow \mathcal{M}Y$$

$$\begin{array}{ccc} p & & f(p) \\ \downarrow \gamma & \longmapsto & \downarrow f\circ\gamma \\ q & & f(q) \end{array}$$

Juxtaposition of homotopies

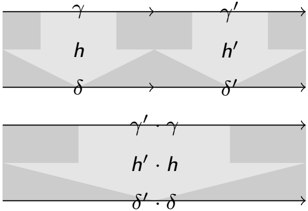
horizontal composition

Let $h : [0,r] \times [0,q] \to X$ and $h' : [0,r'] \times [0,q] \to X$ be homotopies from γ to δ and from γ' to δ' with $\partial^+\gamma = \partial^-\gamma'$.

The mapping $h' * h : [0, r + r'] \times [0,q] \to X$ defined by

$$h' * h(t,s) \quad = \quad \begin{cases} h(t,s) & \text{if } 0 \leqslant t \leqslant r \\ h'(t-r,s) & \text{if } r \leqslant t \leqslant r + r' \end{cases}$$

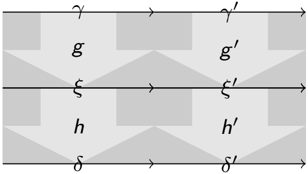
is a homotopy from γ to δ.



If h and h' are ((weakly) directed) homotopies, then so is $h' \cdot h$.

Godement exchange law for homotopies

Suppose we have

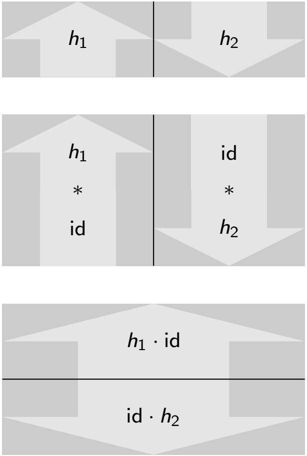


then it comes

$$(g' * h') \cdot (g * h) = (g' \cdot g) * (h' \cdot h)$$

- Given $\gamma : [0, r] \to X$ and $\gamma' : [0, r'] \to X$ write $\gamma \sim_X \gamma'$ when there exists an elementary homotopy between $t \in [0, 1] \mapsto \gamma(rt) \in X$ and $t \in [0, 1] \mapsto \gamma'(r't) \in X$
- The relations $\sim_X$ form a natural congruence on the categories $\mathcal{M}X$.
- The fundamental category functor $\overrightarrow{\pi_1} : \mathcal{Lpo} \to \mathcal{Cat}$ is defined accordingly.
- The fundamental groupoid functor $\Pi_1 : \mathcal{Top} \to \mathcal{Grd}$ is obtained by substituting "paths" and "homotopies" to "directed paths" and "elementary homotopies".

Applying Godement exchange law



Basic properties and computations

- The fundamental category of locally ordered real line is the corresponding partial order.
- For all local pospaces X and Y we have

$$\overrightarrow{\pi_1}(X \times Y) \quad \cong \quad \overrightarrow{\pi_1}X \times \overrightarrow{\pi_1}Y$$

- Given a pospace X, $\overrightarrow{\pi_1}X$ is loop-free i.e.

$$\overrightarrow{\pi_1}X(x,y) \neq \emptyset \text{ and } \overrightarrow{\pi_1}X(y,x) \neq \emptyset \qquad \Rightarrow \qquad x = y \text{ and } \overrightarrow{\pi_1}X(x,x) = \{\mathrm{id}_x\}$$

- The fundamental category of a local pospace has no nontrivial null homotopic directed paths i.e. any directed loop that is related to a constant path by an elementary homotopy is actually a constant.
- In particular the fundamental category of a local pospace has no isomorphism but its identities.

The fundamental category of the locally ordered circle

- Given x, y, $\widehat{xy}$ is the anticlockwise arc from x to y.
  It is a singleton if $x = y$.
- $\vec{\pi}_1\mathbb{S}^1(x, y) = \{x\} \times \mathbb{N} \times \{y\}$
- the identities are the tuples $(x, 0, x)$
- the composition is given by
  - $(y, p, z) \circ (x, n, y) = (x, n + p, z)$ if $\widehat{xy} \cup \widehat{yz} \neq \mathbb{S}^1$
  - $(y, p, z) \circ (x, n, y) = (x, n + p + 1, z)$ if $\widehat{xy} \cup \widehat{yz} = \mathbb{S}^1$

Abstracting locally ordered spaces

- Combine proximity and causality in a single mathematical structure that generalizes local pospaces.
- In this way of thinking note that given a local pospace X:
    - the collection of directed paths on X contains the constant morphisms $[0, r] \to X$,
    - the collection of directed paths on X is stable under concatenation, and
    - if θ is a directed path on $[0, r]$ and $\gamma : [0, r] \to X$ is a directed path, then so is $\gamma \circ \theta$.

d-spaces

Directed Homotopy Theory I, Cah. Top. Géom. Diff. Cat., Marco Grandis (2003)

- A direction dX on a Hausdorff space X is a collection of paths on X such that
  - any constant path belongs to dX,
  - the collection dX is stable under concatenation, and
  - if $\gamma \in dX$, $\mathrm{dom}(\gamma) = [0, r]$ and $\theta : [0, r'] \to [0, r]$ is continuous and increasing, then $\gamma \circ \theta \in dX$
- a d-space is a Hausdorff space together with a direction
- A d-space morphism from (X, dX) to (Y, dY) is a continuous map $f : X \to Y$ s.t. $f \circ dX \subseteq dY$
- The category of d-spaces is denoted by $\mathscr{dTop}$.
- The collection of directed paths on a local pospace X is a direction on the underlying space of X. Hence there is a canonical functor from $\mathscr{Lpo}$ to $\mathscr{dTop}$ sending a local pospace X to the collection of directed paths on X.

Properties of $\mathcal{Lpo} \hookrightarrow \mathcal{dTop}$

The canonical functor from $\mathcal{Lpo}$ to $\mathcal{dTop}$ is

- faithful,
- not full $(\mathbb{Q}, \leqslant)$,
- and not injective on objects $(\mathbb{Q}, \leqslant$ *vs* $\mathbb{Q}, =)$.

Directed paths on a d-space

- The directed real line i.e. $\mathbb{R}$ with increasing paths is a d-space.
- Any interval of $\mathbb{R}$ is a d-subspace of the preceding one.
- A directed path on a d-space (X, dX) is a morphism from some nonempty directed compact interval to (X, dX).
- By the 3$^{\text{rd}}$ property, the collection of directed paths on (X, dX) is precisely dX.

Basic facts abouts directions

- The collection of constant paths on X is a direction.
- The collection of all paths on X is a direction.
- An intersection of directions on X is still a direction on X.
- The collection of directions on X is thus a complete lattice.
- There is a Galois connection $\{\text{directions on } X\} \rightleftarrows \{\text{collections of paths on } X\}$.

Examples

Some d-spaces coming from local pospaces:

- the directed compact unit cube $[0,1]^n$ as the n-fold product of the preceding one, with $n \in \mathbb{N}$.
- the directed circle i.e. $\mathbb{S}^1$ with anticlockwise paths i.e. $t \mapsto \exp^{i\theta(t)}$ with $\theta : [0,r] \to \mathbb{R}$ increasing.

Some d-space having a vortex:

- the directed complex plane i.e. $\mathbb{C}$ with paths of the form $t \mapsto \rho(t) \cdot \exp^{i\theta(t)}$ with $\theta, \rho : [0,r] \to \mathbb{R}_+$ increasing.
- the directed Riemann sphere Σ i.e. $\mathbb{S}^2 \cong \mathbb{C} \cup \{\infty\}$ with paths of the form $t \mapsto \rho(t) \cdot \exp^{i\theta(t)}$ with $\theta : [0,r] \to \mathbb{R}_+$ and $\rho : [0,r] \to \mathbb{R}_+ \cup \{\infty\}$ increasing.

The category of directed Moore paths of a d-space

Let X be a d-space.

- The objects of $\mathcal{M}X$ are the points of X.
- The homset $\mathcal{M}X(a,b)$ is

$$\bigcup_{r \geqslant 0} \{\gamma \in d\mathcal{T}op([0,r],X) \mid \gamma(0) = a \text{ and } \gamma(r) = b\}$$

  that is to say the collection of elements of dX starting at a and finishing at b.
- For $\delta : [0,r] \to X$ and $\gamma : [0,r'] \to X$ with $\delta(r) = \gamma(0)$, define the concatenation

$$\gamma \cdot \delta : \; [0, r + r'] \longrightarrow X$$

$$t \longmapsto \begin{cases} \delta(t) & \text{if } t \leqslant r \\ \gamma(t-r) & \text{if } t \geqslant r \end{cases}$$

  because dX is stable under concatenation.

The directed Moore path functor over dTop

The (Moore) path category construction gives rise to a functor $\mathcal{M}$ from dTop to $\mathcal{C}at$ since for all $f \in \mathit{dTop}(X, Y)$ and all paths $\gamma \in dX$, the composite $f \circ \gamma$ belongs to dY.

$$\mathcal{M} : \mathit{dTop} \longrightarrow \mathcal{C}at$$

$$\begin{array}{ccc} X & & \mathcal{M}X \\ \downarrow f & \longmapsto & \downarrow \mathcal{M}f \\ Y & & \mathcal{M}Y \end{array}$$

with

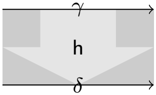
$$\mathcal{M}f : \mathcal{M}X \longrightarrow \mathcal{M}Y$$

$$\begin{array}{ccc} p & & f(p) \\ \downarrow \gamma & \longmapsto & \downarrow f\circ\gamma \\ q & & f(q) \end{array}$$

Directed homotopy on a d-space

Let $\gamma, \delta \in d\mathcal{T}op([0,r], X)$ such that $\partial^- \gamma = \partial^- \delta$ and $\partial^+ \gamma = \partial^+ \delta$.

- A directed homotopy from γ to δ is a homotopy of paths $h : [0,r] \times [0,q] \to X$ that induces a d-space morphism.



- A weakly directed homotopy from γ to δ is a homotopy of paths $h : [0,r] \times [0,q] \to X$ whose intermediate paths $h(\_, s)$, for $s \in [0,q]$, are directed (i.e. belong to dX).
- Any directed homotopy is a weakly directed homotopy. The converse is false.

Fundamental categories of d-spaces

Concatenation and juxtaposition of directed homotopies are directed homotopies.

Elementary homotopies are finite concatenations of directed and anti-directed homotopies.

Given $\gamma : [0, r] \to X$ and $\gamma' : [0, r'] \to X$ write $\gamma \sim_X \gamma'$ when there exists an elementary homotopy between $t \in [0, 1] \mapsto \gamma(rt) \in X$ and $t \in [0, 1] \mapsto \gamma'(r't) \in X$

The relations $\sim_X$ form a natural congruence on the categories $\mathcal{M}X$.

The fundamental category functor $\overrightarrow{\pi_1} : \mathit{dTop} \to \mathcal{C}at$ is defined accordingly.

Comparing fundamental category functors

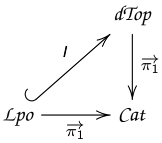
locally ordered spaces *vs* d-spaces

Let $\mathcal{U}$ be an atlas on a Hausdorff space X such that for all chart $U \in \mathcal{U}$, we have $x \sqsubseteq_U y$ iff there exists a directed path on U from x to y. Then for all locally ordered space Y we have

$$\mathcal{L}po(X, Y) \quad \cong \quad d\mathcal{T}op(IX, IY)$$

where I denotes the canonical functor from $\mathcal{L}po$ to $d\mathcal{T}op$.

In particular, for all locally ordered spaces Y, the map $h : [0, r] \times [0, q] \to Y$ is a directed homotopy on Y iff it is a directed homotopy on IY. The following diagram thus commutes



The fundamental category of the directed complex plane

- Let $\theta : \mathbb{C} \setminus \{0\} \to \mathbb{S}^1$ be the radial projection
- $\overrightarrow{\pi_1}\mathbb{C}(x, y)$ is
  - the singleton $\{\alpha_y\}$ if $x = 0$;
  - empty if $|x| > |y|$;
  - $\{x\} \times \mathbb{N} \times \{y\}$ otherwise
- The identities are $(x, 0, x)$
- If $x \neq 0$ the composition is given by
  - $(y, p, z) \circ (x, n, y) = (x, n + p, z)$ if $\theta\widehat{(x)\theta(}y) \cup \theta\widehat{(y)\theta(}z) \neq \mathbb{S}^1$
  - $(y, p, z) \circ (x, n, y) = (x, n + p + 1, z)$ if $\theta\widehat{(x)\theta(}y) \cup \theta\widehat{(y)\theta(}z) = \mathbb{S}^1$
- One can deduce the fundamental category of the directed Riemann sphere i.e. $\mathbb{C} \cup \{\infty\}$

Quotient of d-spaces

Let $\sim$ be an equivalence relation on the underlying set of a d-space X, and $q : X \to X/\sim$ be the quotient map.

The greatest topology on $X/\sim$ making q continuous is equipped with the least direction on $X/\sim$ turning q into a d-space.

The relation $\sim$ on $\mathbb{R}_+ \times \mathbb{S}^1$ identifying (r, θ) and (r', θ') when $r = r' = 0$ gives rise to a d-space that is not isomorphic with the directed complex plane.

Indeed the downward spiral spins infinitely many times in any neighbourhood of the origin, thus crossing $\mathbb{R}_+$ infinitely many times.

Vector fields and directions on smooth manifolds

- Let $\mathcal{M}$ be a C^∞ manifold and $V_1, \ldots, V_n$ be vector fields on it.
- Let $d_{V_1,\ldots,V_n}\mathcal{M}$ be the direction on $\mathcal{M}$ generated by the curves γ such that for all $t \in \mathrm{dom}(\gamma)$

$$\dot{\gamma}(t) = \sum_{k=1}^{n} \lambda_k(t) \cdot V_k(\gamma(t)) \text{ with } \lambda_k(t) \geqslant 0 \text{ for all } k \in \{1, \ldots, n\}$$

- One has $d_{V_1}\mathcal{M} \vee \cdots \vee d_{V_n}\mathcal{M} \subseteq d_{V_1,\ldots,V_n}\mathcal{M}$

The case of the directed plane

- Let $\mathcal{M}$ be $\mathbb{R}^2$ and for all $p \in \mathcal{M}$, $V_1(p) = (1,0)$ and $V_2(p) = (0,1)$
- Then $d_{V_1}\mathcal{M} \vee d_{V_2}\mathcal{M}$ is the collection of increasing staircases i.e. finite concatenation of vertical and horizontal paths
- while $d_{V_1,V_2}\mathcal{M}$ contains all the diagonals
- Hence $d_{V_1}\mathcal{M} \vee d_{V_2}\mathcal{M} \subsetneq d_{V_1,V_2}\mathcal{M}$
- The same phenomenon can be observed with
  - $\mathcal{M} = \mathbb{C}$, $V_1(z) = z$ and $V_2(z) = e^{i\frac{\pi}{2}}z$.
  - $\mathcal{M} = \Sigma$, $V_1(z) = e^{-|z|}z$ and $V_2(z) = e^{-|z|+i\frac{\pi}{2}}z$.

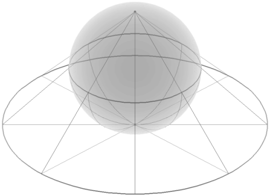
Compactification

Figure from Wikipedia

A compactification of the Hausdorff space X is an embedding of X into a compact Hausdorff space K such that X is dense in K (i.e. the closure of X in K is K)

$$\mathbb{R}_+ \hookrightarrow \mathbb{R}_+ \cup \{\infty\}$$

The inverse of the stereographic projection $\sigma : \Sigma \setminus \{\infty\} \to \mathbb{C}$.



However, given a d-space X, if $X \hookrightarrow K$ is a compactification of X, then the least direction on K containing dX does not contain any directed path arriving at $K \setminus X$ but the constant ones.

In particular, the directed one-point the half-line is not isomorphic with a compact interval of the d-space $\mathbb{R}$, and that of the directed plane is not the directed sphere.